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Studies on the Deficiency Problems of Graceful Labeling over Eulerian Graphs

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所提論文　STUDIES ON THE DEFICIENCY PROBLEMS OF GRACEFUL LABELING OVER EULENIAN GRAPHS

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Abstract

The following condition (due to A. Rosa) is known to be necessary for an Eulerian graph G admitting a graceful valuation: $|E(G)| \equiv 0$ or $3 \pmod 4$. The condition is thus sufficient if G is a cycle C_n on n vertices. In 1994 J. Abrham and A. Kotzig proved that the 2-regular graph kC_4, the disjoint union of k copies of 4-cycles, admits graceful labeling for every positive integer k. In 1996 they also showed that the 2-regular graph $C_p \cup C_q$, the disjoint union of C_p and C_q, admits a graceful valuation if $p + q \equiv 0$ or $3 \pmod 4$. In this thesis we study the notion graceful deficiency, which measures how far a graph is away from being graceful. We completely determine the graceful deficiency for cycles C_n and windmill graphs, and conjecture that the graceful deficiency of the 2-regular graph $C_p \cup C_q$ is 1 with $p + q \equiv 1$ or $2 \pmod 4$.

Acknowledgements

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Chapter 1

Background and Motivation

1.1 Introduction

In this dissertation, by a graph we mean an undirected finite graph without multiple edges and loops. All terminologies and notations on graph theory not mentioned or defined here can be referred to the textbook by D. West[15].

Definition 1.1.1. *Let $G = (V, E)$ be a finite simple undirected graph with $|V| = m$ and $|E| = n$. A one-to-one function $f \colon V \to \{0, 1, ..., n\}$ from the vertex set V (if any) is said to be a* **graceful labeling** *of G, if the absolute value $|f(u) - f(v)|$ is assigned to the edge uv as its label and the resulting edge labels are pairwise distinct (see Figure 1.1).*

This is equivalent to requiring the set of induced edge labels is exactly $\{1, 2, ..., n\}$. A graph admitting such a graceful labeling is called a graceful graph.

In particular, graceful labeling originated as a means of attacking the conjecture of Ringel that K_{2n+1} can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with n edges. For this reason A. Rosa raised the **Graceful Tree Conjecture** (which implies the conjecture of Ringel) that every tree is graceful, which is one of most challenging problems in graph theory and remains wide open until today. Among the trees known to be graceful are: caterpillars (a caterpillar is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices; trees with diameter at most 5;

symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree); rooted trees where the roots have odd degree and the lengths of the paths from the root to the leaves differ by at most one and all the internal vertices have the same parity; rooted trees with diameter D where every vertex has even degree except for one root and the leaves in level $\lfloor \frac{D}{2} \rfloor$; rooted trees with diameter D where every vertex has even degree except for one root and the leaves, which are in level $\lfloor \frac{D}{2} \rfloor$; rooted trees with diameter D where every vertex has even degree except for one root, the vertices in level $\lfloor \frac{D}{2} \rfloor - 1$, and the leaves which are in level $\lfloor \frac{D}{2} \rfloor$, etc.

We focus in this thesis on the graceful labeling of Eulerian graphs in particular for cycles and windmill graphs, and also their associated deficiency problems.

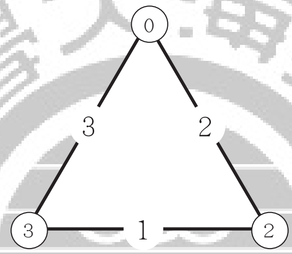

Figure 1.1: Graceful labeling for C_3

The following condition (due to A. Rosa) is known to be necessary for a 2-regular graph G admitting a graceful valuation: $|E(G)| \equiv 0$ or 3 (mod 4). We give a proof here for completeness:

Theorem 1.1.2. (Rosa[12]) *If a graph G with n edges is Eulerian and admits a graceful labeling, then $n \equiv 0$ or 3 (mod 4).*

Proof: The graph $G = (V, E)$ is an Eulerian graph and thus every vertex in the graph has even degree. Note that G admits a graceful labeling, hence there exists a vertex labeling f: $V \to \{0, 1, ..., n\}$ so that edge labels are exactly $1, ..., n$, where the induced edge label for the edge uv is $|f(u) - f(v)|$ which is either $f(u) - f(v)$ or $f(v) - f(u)$ depending on the values of end point labels. Therefore we observe the following while adding all the induced edge labels:

$$\sum_{uv \in E} |f(u) - f(v)| = \sum_{w \in V} \sigma(w) \dot{f}(w)$$

where $\sigma(w)$ is the total contribution of the vertex label $f(w)$ for the above sum. Suppose there are a positive $+f(w)$'s, and b negative $-f(w)$'s. Without loss of generality, may assume $a \geq b$. Hence $\sigma(w) = a - b = (a + b) - 2b$, which is even since $a + b$ is even, as the degree of the vertex w. On the other hand, the sum of all its induce edge labels is

$$\sum_{uv \in E} |f(u) - f(v)| = 1 + 2 + ... + n = \frac{n(n+1)}{2}$$

which must be even, due to the fact that $\sigma(w)$ is even for each vertex w. Therefore $n \equiv 0$ or 3 $(mod\ 4)$ and we are done. □

Remark 1.1.3. *The above Theorem is true in general for disconnected graceful graph whose components are Eulerian.*

1.2 Graceful Labeling

In addition to graceful labeling, which Rosa called β-valuation, we introduce one more special classes of graphs admitting graceful valuations.

Definition 1.2.1. *The α-labeling of a graph G with n vertices and m edges, is a one-to-one mapping f from the set of vertices of G to the set $\{0, 1, 2, \ldots, m\}$, such that all induced edge labels are pairwise distinct, where the induced edge label of the edge uv is $|f(u) - f(v)|$, and in addition there exists a number $x \in \{0, 1, 2, \ldots, m\}$ such that for arbitrary edge uv either $f(u) \leq x < f(v)$ or $f(v) \leq x < f(u)$.*

Note that in 1972 S. W. Golomb independently introduced the same β-labeling and called it graceful.

A natural generalization of graceful graphs is the notion of k-graceful graphs, introduced by Slater[14] in 1982 and independently by Maheo and Thuillier[9] in 1982.

Definition 1.2.2. *A graph G with q edges is k-graceful if there is labeling f from the vertices of G to $\{0, 1, 2, ..., q + k - 1\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{k, k + 1, ..., q + k - 1\}$.*

Obviously, 1-graceful is graceful and it is readily shown that any graph that has an α-labeling is k-graceful for all k. See Figure 1.2 for an example of 2-graceful labeling for C_9.

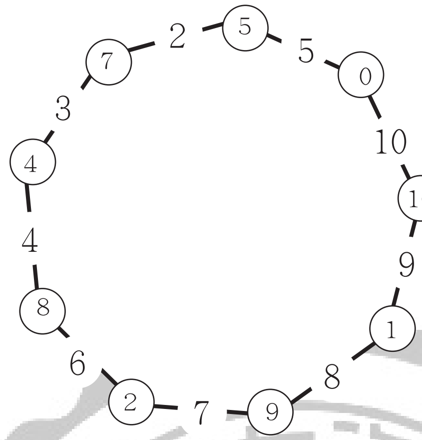

Figure 1.2: A 2-graceful labeling for C_9

1.3 Golomb Ruler and Graceful Deficiency

In order to measure how far for a graph away from being graceful, one may study the following notion:

Definition 1.3.1. *Let G be a graph with $|E(G)| = n$ and $|V(G)| = m$. The minimum value of the integer $r = g(G)$ is said to be a graceful deficiency of G, such that there is a one-to-one function $f\colon V(G) \to \{0, 1, \cdots, n, n+1, \cdots, n+r\}$ which yields pairwise distinct induced edge labels, where the induced edge label for the edge uv is the absolute value $|f(u) - f(v)|$. Obviously if G is graceful, then its graceful deficiency is 0.*

It is well known that the concept graceful deficiency is closely related to that of a Golomb ruler, which has a lot of engineering applications. The Golomb ruler was first described by Solomon W. Golomb, a professor of Mathematics and Electrical Engineering in the University of Southern California[5]. The Golomb ruler measures more discrete lengths than the number of marks it carries. It does not measure the same distance twice. For example, we can get 4 marks and 6 lengths (0,1,4,6), it is called a perfect Golomb ruler. It has been proven that no perfect Golomb ruler exists for five or more marks. A Golomb ruler is optimal if no shorter Golomb ruler of the same order exists, for example, 5 marks and 11 lengths (0,1,4,9,11).

On the other hand, we know that complete graph K_4 has a graceful labeling, so the 4 marks is a perfect Golomb ruler(see Figure 1.3).

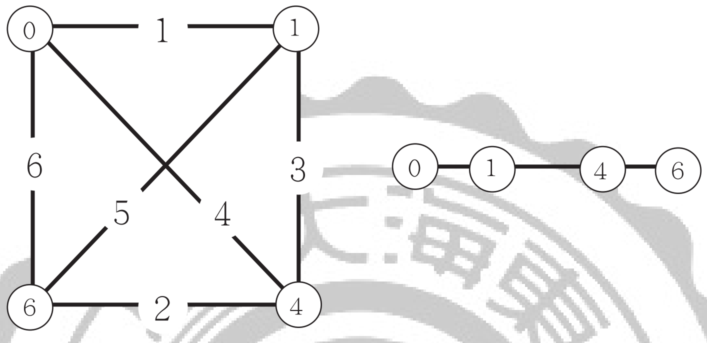

Figure 1.3: Graceful K_4 and perfect Golomb ruler with 4 marks

In fact one knows that a complete graph K_n is graceful if and only if $n \leq 4$. Therefore K_5 is not admitting a graceful labeling and we can calculate the graceful deficiency $d(K_5) = 1$ for K_5, and it is a optimal Golomb ruler with 5 marks(see Figure 1.4).

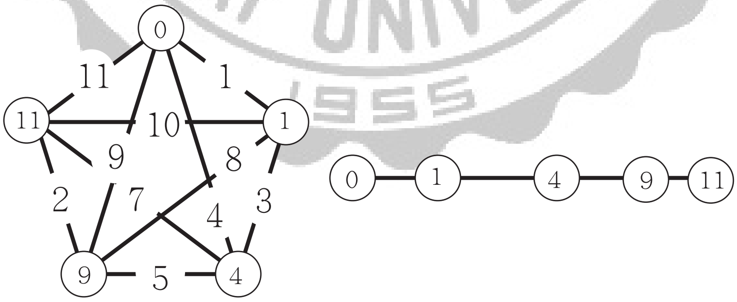

Figure 1.4: K_5 with deficiency 1 and optimal Golomb ruler with 5 marks

1.4 Skolem Sequence and Its Variant

We will make use of Skolem sequences and hooked Skolem sequences for gracefully labeling Eulerian graphs in later sections. We introduce backgrounds for Skolem sequences here.

Definition 1.4.1. *A* **Skolem sequence** *of order n is a sequence of first $2n$ positive integers, so that it is possible to distribute the numbers $1, 2, ..., 2n$ into n pairs (a_r, b_r) such that we have the differences $b_r - a_r = r$ for $r = 1, 2, ..., n$.*

In the following, a set of pairs for the Skolem sequence is called a **1, +1 system**, because the differences $b_r - a_r$ begin with 1 and increase by 1 whenever r increases by 1. For example, for $n = 4$ there is such a system, namely (6,7), (1,3), (2,5), (4,8). It was well-known that, the Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$. For readers' reference, we put the proof here.

Theorem 1.4.2. *The Skolem sequence of order n exists if and only if $n \equiv\ 0, 1 (mod\ 4)$.*

Proof: If the pairs (a_r, b_r), $r = 1, 2, ..., n$, constitute a 1, +1 system of the numbers $1, 2, ..., 2n$, then we have the equations

$$b_r - a_r = r, r = 1, 2, ..., n.$$

Note that the sum of all the above equations is

$$\sum_{r=1}^{n} b_r - \sum_{r=1}^{n} a_r = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

On the other hand, since the collection of the numbers a_r and b_r is the set of integers $1, 2, ..., 2n$, we also have

$$\sum_{r=1}^{n} b_r + \sum_{r=1}^{n} a_r = 1 + 2 + \cdots + 2n = n(2n+1).$$

Adding the last two equations yields

$$\sum_{r=1}^{n} b_r = \frac{1}{4}n(5n+3)$$

which is an integer only when $n \equiv\ 0, 1 (mod\ 4)$.

Conversely we construct the Skolem sequences of proper orders as follows. First let $n \equiv 0\ (mod\ 4)$. It suffices to give a general description of a 1, +1 system for any arbitrary $n = 4m$. Such a system of pairs consists of:

1. all pairs $(4m+r, 8m-r)$ for $r = 0, 1, \cdots, 2m-1$;

2. the pairs $(2m+1, 6m)$ and $(2m, 4m-1)$;

3. the pairs $(r, 4m-1-r)$ for $r = 1, 2, \cdots, m-1$;

4. the pair $(m, m+1)$;

5. the pairs $(m+2+r, 3m-1-r)$ for $r = 0, 1, \cdots, m-3$.

Secondly let $n \equiv 1$ $(mod\ 4)$. It will suffice to give a general description of a 1, +1 system for any arbitrary $n = 4m+1$. Such a system of pairs consists of:

1. the pairs $(4m+2+r, 8m+2-r)$ for $r = 0, 1, \cdots, 2m-1$;

2. the pairs $(2m+1, 6m+2)$ and $(2m+2, 4m+1)$;

3. the pairs $(r, 4m+1-r)$ for $r = 1, 2, \cdots, m$;

4. the pair $(m+1, m+2)$;

5. the pairs $(m+2+r, 3m+1-r)$ for $r = 1, \cdots, m-2$.

Therefore in terms of the above construction we are done.

$\square$

As for the cases $n \equiv 2, 3$ $(mod\ 4)$, the natural alternative is so-called a **Hooked Skolem sequence**.

Definition 1.4.3. *A Hooked Skolem sequence of order n is a sequence of the first $2n+1$ positive integers, skipping $2n$, such that it is possible to distribute the numbers $1, 2, \cdots, 2n-1, 2n+1$ into n pairs (a_r, b_r) such that we have $b_r - a_r = r$ for $r = 1, 2, \cdots, n$.*

It is well-known[11] that a Hooked Skolem sequence of order n exists if and only if $n \equiv 2,\ 3$ (mod 4).

Chapter 2

Graceful Deficiency for Eulerian Graphs

In this chapter, we will calculate some missing values for graceful labeling and garceful deficiency of cycle graphs. We also use Skolem sequence and Hooked Skolem sequence to label the windmill graphs.

2.1 Graceful Labeling for 2-Regular Graphs

A graph G will be termed Eulerian if $|E(G)| > 0$ and if every vertex of G is of even degree. Rosa[12] proved that any graceful Eulerian bipartite graph, and in particular, any Eulerian graph G with an α-valuation satisfies the condition $|E(G)| \equiv 0 \pmod 4$.

Graceful valuations and α-valuations of 2-regular graphs have been studied by A. Rosa[12] and A. Kotzig[6][7]. More recently, the relation between graceful valuations and α-valuation of some 2-regular graphs and some Skolem sequences has been studied by J. Abrham[1].

The following results will be needed later:

1. The cycle C_n is graceful if and only if $n \equiv 0$ or $3 \pmod 4$ (Rosa[12]).

2. The cycle C_n has an α-valuation if and only if $n \equiv 0 \pmod 4$ (Kotzig [8], Rosa[12]).

3. If G is a 2-regular graph with a graceful labeling f then there exists a unique

integer $x(0 \leq x \leq |V(G)|)$ such that $f(v) \neq x$ for all $v \in V(G)$. If f is an α-valuation of G and $|V(G)| = 4k$ then either $x = k$ or $x = 3k$ (Kotzig [6]). This number x will be referred to as the **missing value**.

J. Abrham and A. Kotzig[2] proved that there are two results with the inequality for the missing values on C_n graphs when $n \equiv 0$, 3 (mod 4) as follows:

Theorem 2.1.1. *Let G be a 2-regualr graph on $n = 4k$ vertices $(k \geq 1)$ possessing a graceful labeling f. Then the missing value x satisfies the inequalities $k \leq x \leq 3k$; f is an α-valuation of G if and only if either $x = k$ or $x = 3k$. For any $k > 1$, and for any integer x satisfying $\frac{3}{2}k \leq x \leq \frac{5}{2}k$, there exists a graceful labeling of the 4k-cycle with the missing value x. For $1 < k \leq 6$, and for any integer x satisfying $k \leq x \leq 3k$, there exists a graceful labeling of the 4k-cycle with the missing value x.[2]*

Theorem 2.1.2. *Let G be a 2-regualr graph on $n = 4k - 1$ vertices $(k \geq 1)$ possessing a graceful labeling f. Then the missing value x satisfies the inequalities $k \leq x \leq 3k - 1$.[2]*

There are more results on 2-regular with several component by J. Abrham and A. Kotzig [3] [4].

Theorem 2.1.3. *The graph kC_4(consisting of several C_4 cycles) has an α-valuation (a stronger form of the graceful valuation) for every positive integer $k \neq 3$. The graph $3C_4$ is known to be graceful but it does not have an α-valuation.[3]*

Example 2.1.4. $k = 4, 4C_4$ *have an α-valuation with (0,16,2,15), (1,13,5,11), (3,14,7,12), (6,10,8,9)(see Figure 2.1).*

Theorem 2.1.5. *Let p, q be positive integers, $p \geq 3$, $q \geq 3$. Then the 2-regular graph $C_p \cup C_q$, disjoint union of C_p and C_q, has a graceful valuation if and only if $p + q \equiv$ 0 or 3 (mod 4). Moreover $C_p \cup C_q$ has an α-valuation if and only if both p, q are even and $p + q \equiv$ 0 (mod 4).*[**4**]

We have the following table for graceful labeling result with several graphs.

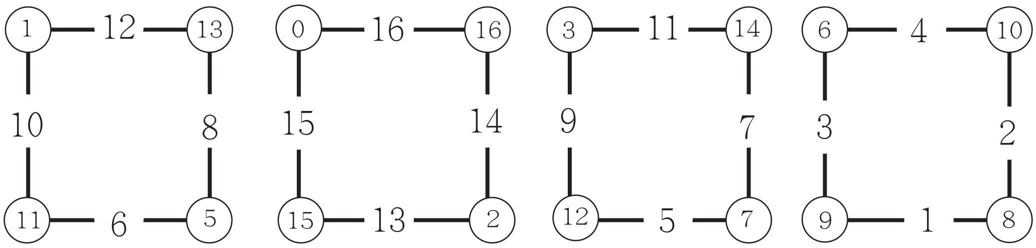

Figure 2.1: $4C_4$ with graceful labeling

Table 2.1: Summary of Graceful results

Graph	Graceful
cycles C_n	G iff $n \equiv 0$, 3(mod 4)
trigular sankes	G iff number of blocks $\equiv 0$, 1 (mod 4)
$C_p \cup C_q$	G iff $p + q \equiv 0$, 3(mod 4)
$C_n^{(t)}$	$n = 3$ G iff $t \equiv$ 0,1 (mod 4)
K_n	G iff $n \leq 4$

2.2 Graceful Labeling of Cycles

In this section we calculate the missing values for cycles C_n with the graceful labeling or the graceful deficiency. We first define related terminologies in order to describe our main results. We also use the result on **Theorem 2.1.1** and **Theorem 2.1.2**, if $n = 4k$ and C_n has a graceful labeling, then the missing vaule x satisfies the inequalities $k \leq x \leq 3k$; and if $n = 4k - 1$ and C_n has a graceful labeling, then the missing value x satisfies the inequalities $k \leq x \leq 3k - 1$. We calculate the missing value $x = k$, $2k$, $3k$ with C_n when $n = 4k$, and calculate the missing value $x = k$, $2k - 1$, $2k$, $3k$ with C_n when $n = 4k - 1$.

Definition 2.2.1. *Assume that a graceful labeling of a graph $G = (V, E)$ with m vertices and n edges is a one-to-one mapping ψ of V into the set $\{0, 1, 2, \ldots, n\}$. We define three subsets U, W, and T of V as follows:*

1. *A vertex $v \in V$ belongs to U if $\psi(v) > \psi(u)$ for any neighbor u of v.*

2. *A vertex $v \in V$ belongs to W if $\psi(v) < \psi(u)$ for any neighbor u of v.*

3. *Moreover* $T = V - (U \cup W)$.

We also define $\bar{\psi}(e) = |\psi(u) - \psi(w)|$ *for the edge* $e = uw$, *where* $u \in U$ *and* $w \in W$.

Note that one can easily verify that $|U| = |W|$, and the common cardinality of these two sets will be denoted by p. Clearly, $p \leq 2k$. Then we have the following: C_n is a graceful labeling if and only in n ≡0 or 3 (mod 4), and we will calculate the missing value x for the cycle graph:

Theorem 2.2.2. C_n *is a graceful if* $n \equiv 0(mod\ 4)$ *and we calculate the missing value* $x = k, 2k, 3k$ *whenever* $n = 4k$.

Proof: With the notations defined as above, we do the following:

1. We calculate $n = 4k$ when $n \equiv 0(mod\ 4)$, and we separate $\{0, 1, 2, ..., 4k\}$ into two sets. The set $\{0, 1, 2, ..., 2k\} = W$, and $\{2k+1, 2k+2, ..., 4k\} = U$. The sets W and U are as we mention before, and $|W| = 2k+1, |U| = 2k$. Clearly, we get the missing value $x \in W$.

   The sum of induced edges labeling is

$$\sum_{e \in E(G)} \bar{\psi}(e) = \sum_{i=1}^{4k} i = 2k(4k+1)$$

   The sum of induced edges labeling is equal to 2 times of vertex set in U subtraction 2 times of vertex set in W.

$$\sum_{e \in E(G)} \bar{\psi}(e) = 2\sum_{u \in U} \psi(u) - 2\sum_{w \in W} \psi(w)$$

   The sum of $\psi(u)$ is

$$\sum_{u \in U} \psi(u) = \sum_{i=2k+1}^{4k} i = \frac{1}{2} \times 2k(6k+1)$$

   The sum of $\psi(w)$ and subtraction the missing value x.

$$\sum_{w \in W} \psi(w) = (\sum_{i=0}^{2k} i) - x = \frac{1}{2} \times 2k(2k+1) - x$$

The sum of edges labeling is equal to sum of vertex labeling.

$$2k(4k+1) = 2 \times [\frac{1}{2} \times 2k(8k-2k+1)] - 2 \times [\frac{1}{2} \times 2k(2k+1) - x]$$

Therefore we have $x = k$. $\square$

So we can get that, the $4k$-cycle have a graceful labeling and its missing value is $x = k$. The graceful labeling of $4k$-cycle can be given by

$$(0, 4k, 1, 4k-1, 2, 4k-2, ..., k-1, 3k+1, k+1, 3k, ..., 2k, 2k+1)$$

Example 2.2.3. *$k = 2$, the missing value $x = 2$, then 8-cycle graceful labeling is $(0, 8, 1, 7, 3, 6, 4, 5)$ (see Figure 2.2)*

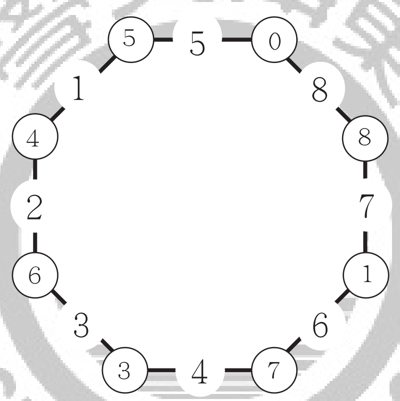

Figure 2.2: Graceful labeling for C_8 with missing value $x = 2$

2. If we separate $\{0, 1, 2, ..., 4k\}$ to two sets. The set $\{0, 1, 2, ..., 2k-1\} \in W$, and $\{2k, 2k+1, ..., 4k\} \in U$.$|W| = 2k$, $|U| = 2k+1$. Clearly, we can get the missing value $x \in U$.
   The sum of edges labeling is equal to sum of vertex labeling.

$$2k(4k+1) = 2 \times [\frac{1}{2} \times (2k+1)(8k-2k) - x] - 2 \times [\frac{1}{2} \times 2k(2k-1)]$$

$$4k^2 + k = \frac{1}{2} \times (2k+1)(6k) - x - \frac{1}{2} \times 2k(2k-1)$$

$$4k^2 + k = 6k^2 + 3k - x - 2k^2 + k$$

$$\therefore x = 3k$$

$\square$

We can get the $4k$-cycle have a graceful labeling and its missing value $x = 3k$. The graceful labeling of $4k$-cycle can be given by

$(0, 4k, 1, 4k-1, 2, 4k-2, ..., k, 3k-1, k+1, 3k-2, ..., 2k-1, 2k)$

Example 2.2.4. *$k = 2$, the missing value $x = 6$, then another 8-cycle graceful labeling is* $(0, 8, 1, 7, 2, 5, 3, 4)$*(see Figure 2.3)*

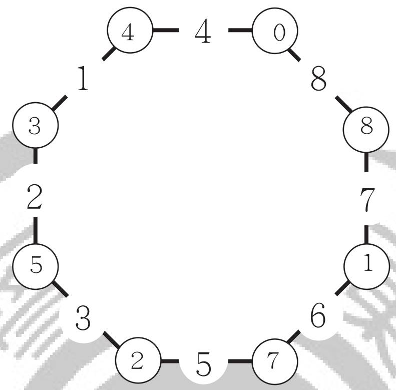

Figure 2.3: Graceful labeling for C_8 with missing value $x = 6$

In the **Example 2.1.1** and **Example 2.1.2**, the missing value $x = k$ and $x = 3k$ can be present a **graceful of α-valuation**, but when we consider the missing $x = 2k$, the sum of edges labeling is

$$\sum_{e \in E(G)} \bar{\psi}(e) = \sum_{i=1}^{4k} i = 2k(4k+1)$$

On the other hand, the sum of $\psi(u)$ with 2 times

$$2 \times \sum_{u \in U} \psi(u) = 2 \times \sum_{2k+1}^{4k} i = \frac{1}{2}[(2k+1) + 4k] \times 2k = (6k+1) \times 2k$$

The sum of $\psi(w)$ with 2 times

$$2 \times \sum_{w \in W} \psi(w) = 2 \times \sum_{0}^{2k-1} i = \frac{1}{2}[0 + (2k-1)] \times 2k = (2k-1) \times 2k$$

If the cycle graph has a graceful labeling, then

$$\sum_{e \in E(G)} \bar{\psi}(e) = 2\sum_{u \in U} \psi(u) - 2\sum_{w \in W} \psi(w)$$

But

$$2k(4k+1) = 8k^2 + 2k \neq (12k^2 + 2k) - (4k^2 - 2k) = 8k^2 + 4k$$

so,

$$\sum_{e \in E(G)} \bar{\psi}(e) \neq 2\sum_{u \in U} \psi(u) - 2\sum_{w \in W} \psi(w)$$

So if the **missing value** $x = 2k$, and the set $\{0, 1, 2, ..., 2k\} \in W$, and the set $\{2k+1, 2k+2, ..., 4k\} \in U$, then the cycle graph has no graceful labeling.
We put the vertices value $2k-1$ and $3k-1$ to the set of T, and we find a system of a graceful labeling with the missing value of $x = 2k$. It can be given by

$$(0, 4k, 1, 4k-1, 2, 4k-2, ..., 3k, 3k-1, k+1, 3k-2, ..., 2k-1)$$

and it is **not a α-labeling**.

Example 2.2.5. *$k = 2$, the missing value $x = 4$, then another 8-cycle graceful labeling is* $(0, 8, 1, 7, 2, 6, 5, 3)$*(see Figure 2.4), and this is a β-valuation*

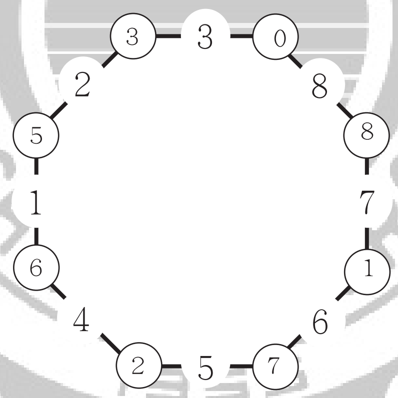

Figure 2.4: Graceful labeling for C_8 with missing value $x = 4$

Theorem 2.2.6. *C_n is graceful if $n \equiv 3(mod\ 4)$ and the missing value $x = k, 2k-1, 2k, 3k-1$ when $n = 4k-1$.*

Proof:

1. We calculate $n = 4k-1$ when $n \equiv 3(mod\ 4)$, and we separate $\{0, 1, 2, ..., 4k-1\}$ to three sets. The set $\{0, 1, 2, ..., 2k-1\} \in W$, and $\{2k+1, 2k+2, ..., 4k-1\}$ $\in U$, and $|W| = 2k$ $|U| = 2k-1$, also we put $2k$ into the set $\in T$. We can

clearly find the missing value $x \in W$.

The sum of edges labeling

$$\sum_{e \in E(G)} \bar{\psi}(e) = \sum_{i=1}^{4k-1} i = 2k(4k-1)$$

The sum of $\psi(u)$ is

$$\sum_{u \in U} \psi(u) = \sum_{i=2k+1}^{4k-1} i = \frac{1}{2} \times (2k-1)(8k-2k+1-1)$$

The sum of $\psi(w)$ and subtraction the missing value x

$$\sum_{w \in W} \psi(w) = (\sum_{i=0}^{2k-1} i) - x = \frac{1}{2} \times (2k-1)(2k-1+1) - x$$

We calculate that

$$k(4k-1) = \frac{1}{2}(2k-1)(8k-2k+1-1) - [\frac{1}{2}(2k-1)(2k-1+1) - x]$$

$$4k^2 - k = 6k^2 - 3k - 2k^2 + k + x$$

So we get

$$\therefore x = k$$

□

A graceful labeling of a $(4k-1)$-cycle with the missing value $x = k$,$T = 2k$, can be given by

$$(0, 4k-1, 1, 4k-2, 2, 4k-3, ..., k-1, 3k, k+1, 3k-1, ..., 2k+1, 2k)$$

Example 2.2.7. *$k = 2$, the missing value $x = 2$, then 7-cycle graceful labeling is* $(0, 7, 1, 6, 3, 5, 4)$*(see Figure 2.5)*

2. We separate $\{0, 1, 2, ..., 4k-1\}$ to three sets. The set $\{0, 1, 2, ..., 2k-2\} \in W$, and $\{2k, 2k+1, ..., 4k-1\} \in U$, and the element $2k-1 \in T$. $|W| = 2k-1$, $|U| = 2k$, $|T| = 1$, so We can clearly find the missing value $x \in U$.

   The sum of edges labeling

$$\sum_{e \in E(G)} \bar{\psi}(e) = \sum_{i=1}^{4k-1} i = 2k(4k-1)$$

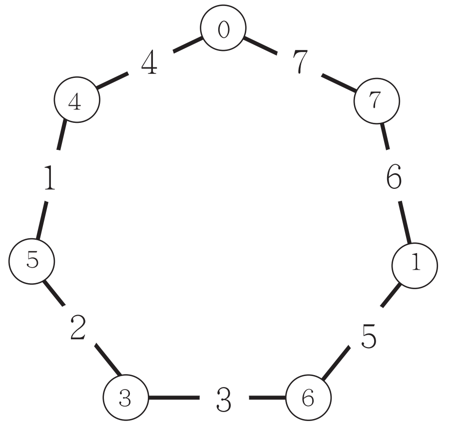

Figure 2.5: Graceful labeling for C_7 with missing value $x = 2$

The sum of $\psi(u)$ and subtraction the missing value x

$$\sum_{u \in U} \psi(u) = (\sum_{i=2k}^{4k-1} i) - x = \frac{1}{2} 2k(2k + 4k - 1) - x$$

The sum of $\psi(w)$

$$\sum_{w \in W} \psi(w) = \sum_{i=0}^{2k-2} i = \frac{1}{2}(2k-2)(2k-1)$$

The sum of edges labeling is equal to the sum of vertex labeling

$$k(4k-1) = \frac{1}{2}(6k-1)(2k) - x - \frac{1}{2}(2k-2)(2k-1)$$

We calculate that

$$4k^2 - k = 6k^2 - k - x - 2k^2 + 3k - 1$$

So we get

$$\therefore x = 3k - 1$$

□

A graceful numbering of a $(4k-1)$-cycle with the missing value $x = 3k - 1$, $T = \{2k - 1\}$, can be given by

$(0, 4k-1, 1, 4k-2, 2, 4k-3, , k, 3k+1, k+1, 3k, , 2k, 2k-1)$

Example 2.2.8. *$k = 2$, the missing value $x = 5$, then 7-cycle graceful labeling is $(0, 7, 1, 6, 2, 4, 3)$(see Figure 2.6)*

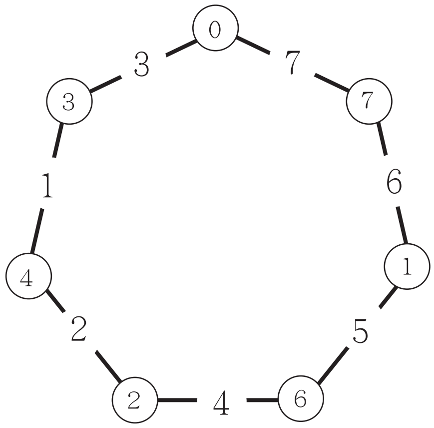

Figure 2.6: Graceful labeling for C_7 with missing value $x = 5$

In fact, we can change the missing value and the T set element and then it has a graceful labeling

A graceful numbering of a $(4k-1)$-cycle with $x = 2k$ and $T = \{k\}$ can be given by

$$(4k-1, 0, 4k-2, 1, 4k-3, ..., k-1, k, 3k-1, k+1, ..., 2k-1)$$

Example 2.2.9. *$k = 2$, the missing value $x = 4$, then 7-cycle graceful labeling is* $(7, 0, 6, 1, 2, 5, 3)$*(see Figure 2.7)*

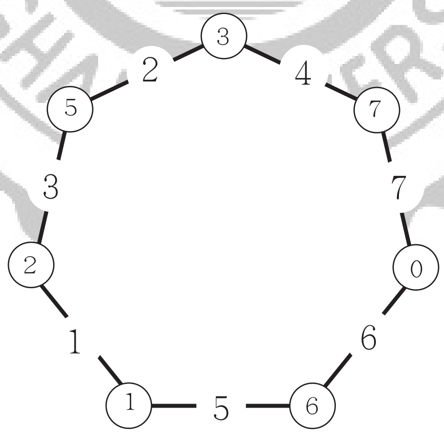

Figure 2.7: Graceful labeling for C_7 with missing value $x = 4$

Another graceful numbering of a $(4k-1)$-cycle with $x = 2k-1$ and $T = \{3k-1\}$ can be given by

$(4k-1, 0, 4k-2, 1, ..., 3k, 3k-1, k-1, 3k-2, ..., 2k-2)$

Example 2.2.10. *$k = 2$, the missing value $x = 3$, then 7-cycle graceful labeling is $(7, 0, 6, 5, 1, 4, 2)$(see Figure 2.8)*

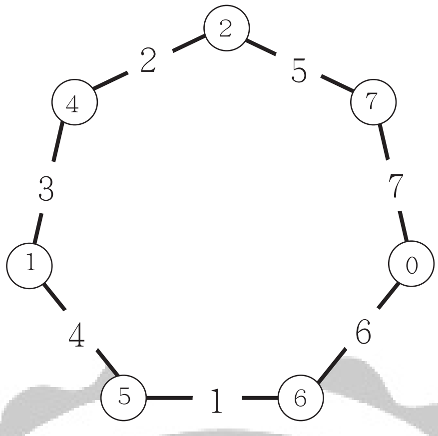

Figure 2.8: Graceful labeling for C_7 with missing value $x = 3$

2.3 Graceful Deficiency of Cycles

In previous chapter it is mentioned that, if a graph G is an Eulerian graph and has a graceful labeling then $n \equiv 0$ or 3 $(mod\ 4)$. Therefore one can try to find a graceful labeling for n-cycle when $n \equiv 0$ or 3 $(mod\ 4)$, however when $n \equiv 1$ or 2 $(mod\ 4)$, it is impossible to find a graceful labeling, thus one may define a new concept called graceful deficiency in this case.

Definition 2.3.1. *Let G be a graph with $|E(G)| = n$ and $|V(G)| = m$. The minimum value of the integer $r = g(G)$ is said to be a graceful deficiency of G, such that there is a one-to-one function $f\colon V(G) \to \{0, 1, \cdots, n, n+1, \cdots, n+r\}$ which yields pairwise distinct induced edge labels, where the induced edge label for the edge uv is the absolute value $|f(u) - f(v)|$. Obviously if G is graceful, then its graceful deficiency is 0.*

Now we are in a position to get $g(C_n) = 1$ if and only n $\equiv$ 1 or 2 (mod 4).

Theorem 2.3.2. *The graceful deficiency $g(C_n) = 1$ if $n \equiv 1(mod\ 4)$, and its missing values are k and $2k$ when n=4k-3.*

Proof: We calculate $n = 4k - 3$ when $n \equiv 1(mod\ 4)$, and we separate $\{0, ..., 4k-2\}$ to three sets. The set $\{0, ..., 2k-2\} \in W$, and $\{2k, ..., 4k-2\} \in U$, and the element $2k-1 \in T.|W| = 2k-1, |U| = 2k-1, |T| = 1$. We can let the missing value $x \in W$, and the missing value $y \in U$. We need to remove the edge of the value 1.

The sum of edges labeling remove 1 is

$$\sum_{e \in E(G)} \bar{\psi}(e) = (\sum_{i=1}^{4k-2} i) - 1 = (4k-1)(2k-1) - 1$$

The sum of edges labeling is equal to sum of vertex labeling.

$$\sum_{e \in E(G)} \bar{\psi}(e) = 2\sum_{u \in U} \psi(u) - 2\sum_{w \in W} \psi(w)$$

The sum of $\psi(u)$ and subtraction the missing value y

$$\sum_{u \in U} \psi(u) = (\sum_{i=2k}^{4k-2} i) - y = (3k-1)(2k-1) - y$$

The sum of $\psi(w)$ and subtraction the missing value x

$$\sum_{w \in W} \psi(w) = (\sum_{i=0}^{2k-2} i) - x = (k-1)(2k-1) - x$$

The sum of edges labeling is equal to the sum of vertex labeling

$$(4k-1)(2k-1) - 1 = 2[(3k-1)(2k-1) - y] - 2[(k-1)(2k-1) - x]$$

We calculate that

$$8k^2 - 6k = 8k^2 - 4k - 2y - 2x$$

$$2y - 2x = 2k$$

So we can get

$$y - x = k$$

we take $y = 2k$ and $x = k$ $\square$

A graceful labeling of a $(4k-3)$-cycle with deficiency $g(Cn) = 1$ and $x = k$, $y = 2k$ can be given by

$(0, 4k-2, 1, 4k-3, ..., k-1, 2k+3, k+1, 2k+2, ..., 2k-1)$

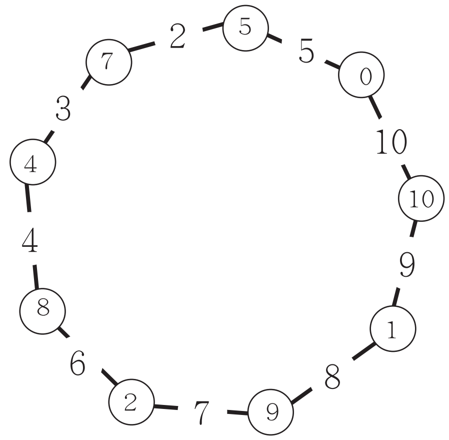

Figure 2.9: C_9 with graceful decificiency $g(C_9) = 1$

Example 2.3.3. *$k = 3$, the missing values $x = 3$, $y = 6$, then 9-cycle graceful deficiency of $g(C_9) = 1$, and we can get a label for C_9 is* $(0, 10, 1, 9, 2, 8, 4, 7, 5)$ *(see Figure 2.9)*

Theorem 2.3.4. *The graceful deficiency $g(C_n) = 1$ if and only if $n \equiv 2(mod\ 4)$, and its missing values are k and $2k + 1$ when $n = 4k - 2$.*

Proof: We calculate $n = 4k - 2$ when $n \equiv 2(mod\ 4)$, and we separate $\{0, ..., 4k - 1\}$ to two sets. The set $\{0, ..., 2k - 1\} \in W$, and $\{2k, ..., 4k - 1\} \in U$. $|W| = 2k, |U| = 2k$. We can let the missing value $x \in W$, and the missing value $y \in U$. We need to remove the edge of the value 2.
The sum of edges labeling remove 2 is

$$\sum_{e \in E(G)} \bar{\psi}(e) = (\sum_{i=1}^{4k-1} i) - 2 = 2k(4k - 1) - 2$$

The sum of edges labeling is equal to sum of vertex labeling.

$$\sum_{e \in E(G)} \bar{\psi}(e) = 2 \sum_{u \in U} \psi(u) - 2 \sum_{w \in W} \psi(w)$$

The sum of $\psi(u)$ and subtraction the missing value y

$$\sum_{u \in U} \psi(u) = (\sum_{i=2k}^{4k-1} i) - y = k(6k - 1) - y$$

The sum of $\psi(w)$ and subtraction the missing value x

$$\sum_{w\in W}\psi(w) = (\sum_{i=0}^{2k-1} i) - x = k(2k-1) - x$$

The sum of edges labeling is equal to the sum of vertex labeling

$$2k(4k-1) - 2 = 2[k(6k-1) - y] - 2[k(2k-1) - x]$$

We calculate that

$$8k^2 - 2k - 2 = 12k^2 - 2k - 2y - 4k^2 + 2k + 2x$$

So we get

$$y - x = k + 1$$

we take $y = 2k+1$ and $x = k$ $\square$

A graceful labeling of a $(4k-2)$-cycle with deficiency $g(Cn) = 1$ and $x = k$, $y = 2k+1$ can be given by

$$(0, 4k-1, 1, 4k-2, ..., k-1, 3k, k+1, 3k-1, ..., 2k)$$

Example 2.3.5. *$k = 3$, the missing values $x = 3$, $y = 7$, then 10-cycle graceful definiciency of $g(C_{10}) = 1$, and we can get a label for C_{10} is $(0, 11, 1, 10, 2, 9, 4, 8, 5, 6)$ (see Figure 2.10)*

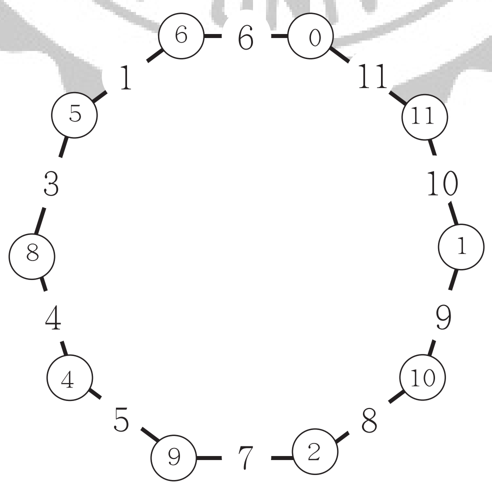

Figure 2.10: C_{10} with graceful deficiency $g(C_{10}) = 1$

2.4 Graceful Labeling on Windmill Graphs

The well-known windmill graph $Wd(k,n)$ is an undirected graph constructed for $k \geq 2$ and $n \geq 2$ by joining n copies of the complete graph K_k at a shared vertex. In this section we will discuss several K_3 graph shared a vertex, and we call it $Wd(3,n)$ graph.

Theorem 2.4.1. *The windmill* $Wd(3,n)$ *has a graceful labeling if and only if n* $\equiv$ *0 or 1 (mod 4)*

In the section 1.3, we introduction, a Skolem sequence of order n is a sequence $S = (s_1, s_2, ..., s_{2n})$ of $2n$ integers if and only if $n \equiv 0$ or 1 (mod 4). We use the Skolem sequence to label the windmill graph.

Case 1: $n = 4m$ (n $\equiv$ 0 mod 4)(n is the number of triangles)

We use Skolem sequence to construction several groups with $\{1, 2, ..., 8m\}$

1. The pairs $(4m + r, 8m - r)$ for $r = 0, 1, , 2m - 1$
2. The pairs $(2m + 1, 6m)$ and $(2m, 4m - 1)$
3. The pairs $(r, 4m - 1 - r)$ for $r = 1, 2, , m - 1$
4. The pair $(m, m + 1)$
5. The pairs $(m + 2 + r, 3m - 1 - r)$ for $r = 0, 1, , m - 3$

Example 2.4.2. $m = 2$, *we use Skolem sequence to divided several groups. All the pairs are* $(8, 16)(9, 15)(10, 14)(11, 13)(5, 12)(1, 6)(2, 3)(4, 7)$.

And we translation every number to 8 units. We get the new pairs are $(16, 24)(17, 23)(18, 22)(19, 21)(13, 20)(9, 14)(10, 11)(12, 15)$.

And we have a graceful labeling for $Wd(3, 8)$.*(see Figure 2.11)*

Case 2: $n = 4m + 1$ ($n \equiv 1$ mod 4)(n is the number of triangles)

We use Skolem sequence to construction several groups with $\{1, 2, ..., 8m + 2\}$

1. The pairs $(4m + 2 + r, 8m + 2 - r)$ for $r = 0, 1, , 2m - 1$
2. The pairs $(2m + 1, 6m + 2)$ and $(2m + 2, 4m + 1)$
3. The pairs $(r, 4m + 1 - r)$ for $r = 1, 2, , m$

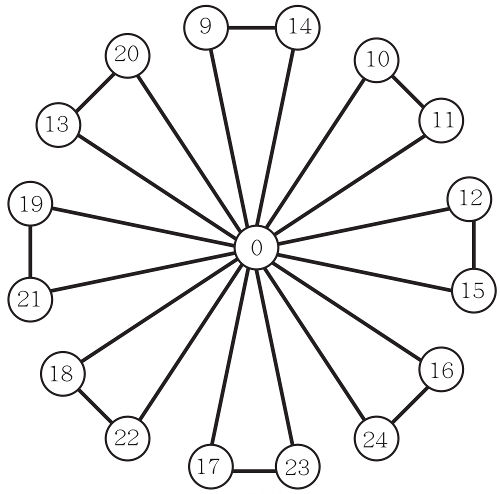

Figure 2.11: $Wd(3,8)$ with graceful labeling

4. The pair $(m+1, m+2)$
5. The pairs $(m+2+r, 3m+1-r)$ for $r = 1, , m-2$

Example 2.4.3. *$m = 2$, we use Skolem sequence to divided several groups. All the pairs are* $(10,18)(11,17)(12,16)(13,15)(5,14)(6,9)(3,4)(1,8)(2,7)$.
And we translation every number to 9 units. We get the new pairs are $(19,27)(20,26)(21,25)(22,24)(14,23)(15,18)(12,13)(10,17)(11,16)$.
And we have a graceful labeling for $Wd(3,9)$.(see Figure 2.12)

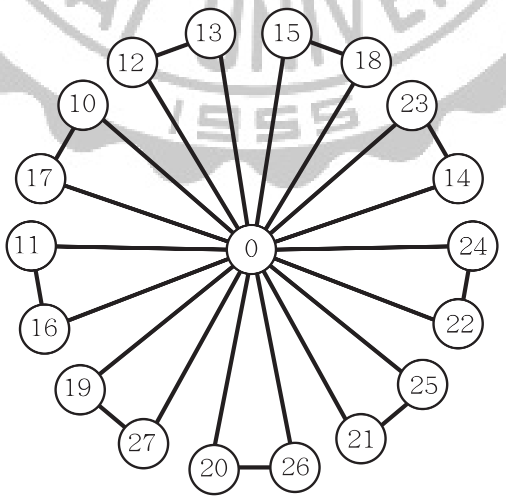

Figure 2.12: $Wd(3,9)$ with graceful labeling

2.5 Graceful Deficiency on Windmill Graphs

The windmill graph $Wd(3,n)$ has no graceful labeling when $n \equiv 2$ or 3.By the **Definition 2.2.1.** we can find a graceful definiciency for the graph.

Theorem 2.5.1. *The graceful definiency* $g(Wd(3,n)) = 1$ *for* $n \equiv$ *2 or 3 (mod 4)*

In **section 1.3**, we know for n $\equiv$ 2 or 3 (mod 4), it has no Skolem sequence, so we use the numbers $1, 2, ..., 2n-1, 2n+1$ can be distributed into n disjoint pairs (a_r, b_r) such that $b_r = a_r + r$ for $r = 1, ..., n$. This is well-known Hooked Skolem sequence.

Case 3: $n = 4m + 2$ (n $\equiv$ 2 mod 4)(n is the number of triangles)

We use Hooked Skolem sequence to construction several groups with $\{1, 2, ..., 8m + 3, 8m + 5\}$

1. The pairs $(r, 4m + 2 - r)$ for $r = 1, , 2m$
2. The pair $(2m + 1, 6m + 2)$
3. The pair $(4m + 2, 6m + 3)$
4. The pair $(4m + 3, 8m + 5)$
5. The pairs $(4m + 3 + r, 8m + 4 - r)$ for $r = 1, , m - 1$
6. The pairs $(5m + 2 + r, 7m + 3 - r)$ for $r = 1, , m - 1$
7. The pair $(7m + 3, 7m + 4)$

Example 2.5.2. $m = 2$*, we use Hooked Skolem sequence to divided several groups. All the pairs are* $((1,9)(2,8)(3,7)(4,6)(5,14)(10,15)(11,21)(12,19)(13,16)(17,18)$. *And we translation every number to 10 units. We get the new pairs are* $(11,19)(12,18)(13,17)(14,16)(15,24)(20,25)(21,31)(22,29)(23,26)(27,28)$. *And we have a graceful deficiency* $g(Wd(3,10)) = 1$*.(see Figure 2.13)*

Case 4: $n = 4m - 1$ (n $\equiv$ 3 mod 4)(n is the number of triangles)

We use Hooked Skolem sequence to construction several groups with $\{1, 2, ..., 8m - 3, 8m - 1\}$

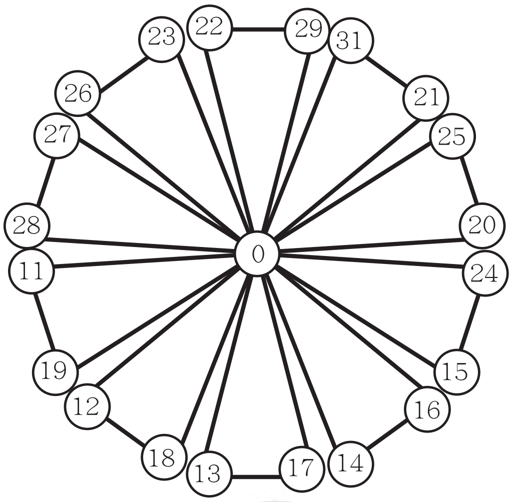

Figure 2.13: $Wd(3,10)$ with graceful deficiency $g(Wd(3,10)) = 1$

1. The pairs $(r, 4m-1-r)$ for $r = 1, , m-1$
2. The pair $(m, m+1)$
3. The pairs $(m+1+r, 3m-r)$ for $r = 1, , m-2$
4. The pair $(2m, 4m-1)$
5. The pair $(4m, 8m-1)$
6. The pairs $(4m+r, 8m-2-r)$ for $r = 1, , 2m-2$
7. The pair $(2m+1, 6m-1)$

Example 2.5.3. $m = 2$, *we use Hooked Skolem sequence to divided several groups. All the pairs are* $(1,6)(2,3)(4,7)(8,15)(9,13)(10,12)(5,11)$.
And we translation every number to 7 units. We get the new pairs are $(8,13)(9,10)(11,14)(15,22)(16,20)(17,19)(12,18)$.
And we have a graceful deficiency $g(Wd(3,7)) = 1$.*(see Figure 2.14)*

2.6 Graceful Labeling of Triangular Snakes

A. Rosa[13] has defined a triangular snake (or $\triangle$-snake) as a connected graph in which all blocks are triangles and the block-cutpoint graph is a path. We call a

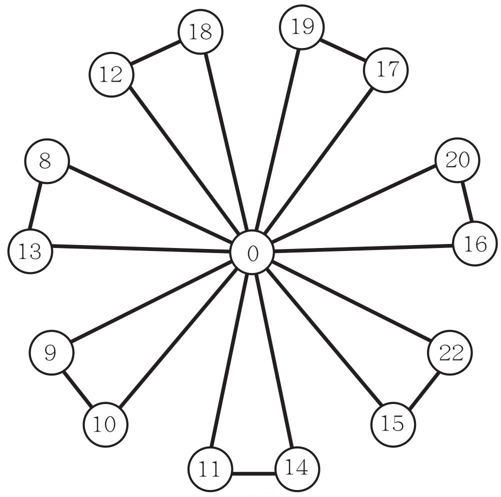

Figure 2.14: $Wd(3,7)$ with graceful deficiency $g(Wd(3,7)) = 1$

$\triangle$-snake with n blocks a $\triangle_n$-snake. Not all $\triangle$-snake are graceful, for as Rosa[12] has shown, an Eulerian graph can only be graceful if it size (number of edges) is congruent to 0 or 3 modulo 4. Hence a $\triangle_n$-snake can only be graceful for n congruent to 0 or 1 modulo 4. See Figure 2.15.

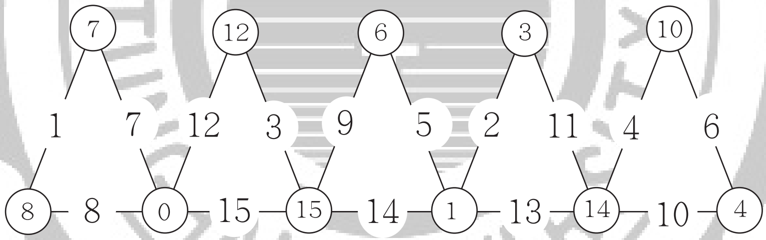

Figure 2.15: $\triangle_5$-snake for graceful labeling

In that paper[10], in order to deal with the other cases, they defined a weaker property than that of being graceful, namely that of being **almost graceful**.

Definition 2.6.1. *Let $G = (V, E)$ be a simple graph and $\varphi : V \to \{0, 1, ..., |E| - 1, x\}$ be an injective map where x is either $|E|$ or $|E| + 1$. Define the induced map $\bar{\varphi} : E \to \{1, 2, ..., x\}$ as in the definition of graceful above. If $\bar{\varphi}$ maps E onto $\{1, 2, ..., |E| - 1, x\}$, then we call φ an almost graceful labeling of G, and we say that G is an almost graceful graph.*

Rosa[13] has introduced a slightly weaker form of almost graceful called **nearly graceful**, which allows the range of φ to be any subset of $\{0, 1, ..., |E| + 1\}$.

Theorem 2.6.2. *Every $\triangle_n$-snake for n congruent to 0 or 1 modulo 4 is* **graceful***, and every $\triangle_n$-snake for n congruent to 2 or 3 modulo 4 is* **almost graceful***.*

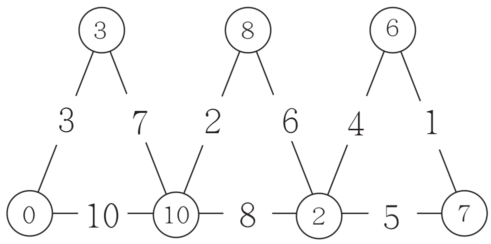

Figure 2.16: $\triangle_3$-snake for almost graceful labeling

In this paper, we define the graceful deficiency, which is congruent to the **almost graceful** labeling or the **nearly graceful** labeling. It is the same labeling for that has no graceful labeling graph.

Chapter 3

Concluding Remarks

3.1 Summary of Results

In this thesis we give a new graceful labeling with cycle graph C_n for n $\equiv$ 0, 3 (mod 4), and if n $\equiv$ 1, 2 (mod 4), there are no graceful labeling, so we define a graceful deficiency to label it. And the windmill graph we also find a graceful labeling for it if it has a graceful labeling. We use the Skolem sequence to label it, and it has no graceful labeling, we find the graceful deficiency to label it. We use the hooked Skolem sequence to label it.

3.2 Further Studies

It would be interesting to explore and identify more related concepts and relationships among them. For example, it is nice trying to find the graceful deficiency for two component of 2-regular graph $C_p \cup C_q$ if $p+q \equiv 1$ or 2 (mod 4). We predict that the graceful deficiency $g(C_p \cup C_q)$=1 with $p+q \equiv 1$ or 2 (mod 4). This is a interesting and open problem.

Conjecture 1: The graceful deficiency for two component of 2-regular graph $d(C_p \cup C_q) = 1$ if $p+q \equiv 1$ or 2 (mod 4).

Conjecture 2: The Eulerian graph G with q edges has no graceful labeling if $q \equiv$ 1 or 2 (mod 4), and the graceful deficiency $d(G) = 1$.

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