東海大學統計研究所

碩士論文

在雙截、相依設限資料下
估計三個持續時間的聯合存活函數

Estimation of the joint survival function
for three successive duration times
under double truncation and dependent censoring

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論文口試委員審定書

統計學系碩士班王　賀君所提之論文

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經本委員會審議，認為符合碩士資格標準。

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王賀謹致
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Abstract

In incident cohort studies, it is common to include subjects who have experienced a certain event within a calendar time window. For all the included individuals, the time of the previous events is retrospectively confirmed and the occurrence of subsequent events is observed during the follow-up periods. During the follow-up periods, subjects may undergo three successive events. Since the second/third duration process becomes observable only if the first/second event has occurred, the data is subject to double truncation and dependent censoring. We consider two cases: the case when the first event time is subject to double truncation and the case when the second event time is subject to double truncation. Using the inverse-probability-weighted (IPW) approach, we propose nonparametric and semiparametric estimators for the estimation of the joint survival function of three successive duration times. We establish the asymptotic properties of the proposed estimators and conduct a simulation study to investigate the finite sample properties of the proposed estimators.

Key Words: double truncation; dependent censoring; inverse-probability-weighted; successive duration times.

Contents

1. Introduction

In natural history studies of diseases, each subject can experience a series of successive events. In many applications, the investigators are interested in the duration times between two successive events. Let E_0, E_1, E_2 and E_3 denote the calendar times of the initiation, first, second and third events respectively. Define $T_1^* = E_1 - E_0$, $T_2^* = E_2 - E_1$ and $T_3^* = E_3 - E_2$ as the first duration time between E_0 and E_1, the second duration time between E_1 and E_2, and the third duration time between E_2 and E_3, respectively. One may be interested in estimating the joint survival function of the three duration times, denoted by $S(t_1, t_2, t_3) = P(T_1^* > t_1, T_2^* > t_2, T_3^* > t_3)$. In incident cohort studies, survival data often include subjects who have experienced a first event within a calendar time window, denoted by $[\tau_0, \tau_1]$. As pointed out in Zhu and Wang (2012,2014,2015), in disease surveillance systems or registries, it is common to collect data with a first event, such as diagnosis of disease, occurring within a calendar time interval and then the time of the initiating event can be retrospectively confirmed and the occurrence of the second/third failure event is observed subject to right censoring. This type of sampling scheme is referred to "interval sampling". For instance, in HIV progression through successive stages, birth (E_0) is the initial event, diagnosed with HIV seroconversion is the first event (E_1), the development of AIDS is the second event (E_2) and death is the third event E_3. Define $T_1^* = E_1 - E_0$, $T_2^* = E_2 - E_1$ and $T_3^* = E_3 - E_2$ as the first duration time between E_0 and E_1, the second duration time between E_1 and E_2 and the third duration time between E_2 and E_3 for a subject, respectively. Suppose that a prevalent cohort is defined as a sample of subjects who have been infected with HIV (E_1) within $[\tau_0, \tau_1]$, i.e., $\tau_0 \leq E_0 + T_1^* \leq \tau_1$, or $U^* = \tau_0 - E_0 \leq T_1^* \leq \tau_1 - E_0 = U^* + d_0$. Hence, observation of the first failure time T_1^* is doubly truncated by U^* and $U^* + d_0$. Let D^* denote the time from E_1 to the right censoring, i.e., the residual censoring time. Note that D^* can be written as $D^* = \min(D_1^*, D_2^*)$, where $D_1^* = U^* + d_0 - T_1^*$ denotes the time from E_1 to the end of study and D_2^* denotes the time from E_1 to drop-out or death due to other causes. In such HIV-prevalent cohort, the time T_1^* from infection of HIV to the development of AIDS is doubly truncated by U^* and

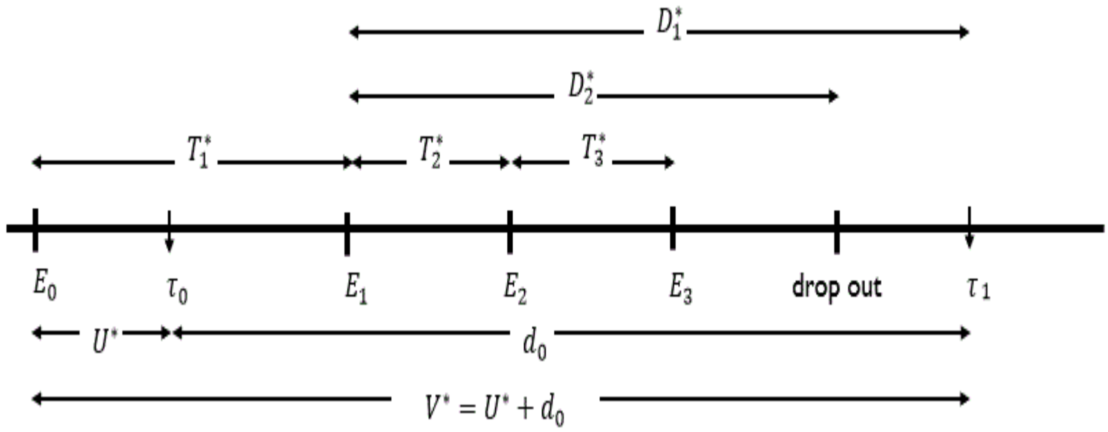

Figure 1. Schematic depiction of doubly-truncated and dependent censored data

$U^* + d_0$. Since the second duration time T_2^* becomes observable only if the first event has occurred, the length of T_1^* affects the probability of T_2^* being censored. Furthermore, the length of $T_1^* + T_2^*$ affects the probability of T_3^* being censored. Dependent censoring arises if T_1^*, T_2^* and T_3^* are not independent, which is often the case. Hence, the data is subject to double truncation and dependent censoring. Figure 1 highlights all the different times for doubly-truncated and dependent censored (DTDC) data described above.

For DTDC data with the first event subject to doubly truncation, one observes nothing if $T_1^* < U^*$ or $T_1^* > U^* + d_0$ and observe $(T_1^*, X_2^*, X_3^*, U^*, \delta_1^*, \delta_2^*, \delta_3^*)$ if $U^* \leq T_1^* \leq U^* + d_0$, where $X_2^* = \min(T_2^*, D^*)$, $X_3^* = \delta_1^* \min(T_3^*, D^* - T_2^*)$, $\delta_1^* = I_{[T_2^* \leq D^*]}$, $\delta_2^* = \delta_1^* I_{[T_3^* \leq D^* - T_2^*]}$ and $\delta_3^* = (1 - \delta_1^*) I_{[D^* = D_2^*]}$. We assume that $(T_1^*, T_2^*, T_3^*, U^*, D_2^*)$ is continuous and U^*, D_2^* and (T_1^*, T_2^*, T_3^*) are independent. Note that as pointed out in Zhu and Wang (2015), the independence assumption between truncation time and failure times may not hold if there exists shift in factors related to disease progression, such as availability of new therapy in the context of HIV infection.

In some situations, a prevalent cohort is defined as a sample of subjects who have experienced the second event within a calendar time window. For example, in Alzheimer's disease through successive stages, birth (E_0) is the initial event, diagnosed with Alzheimer's disease is the first event (E_1), the development of mild decline in abilities is the second event (E_2) and the development of severe

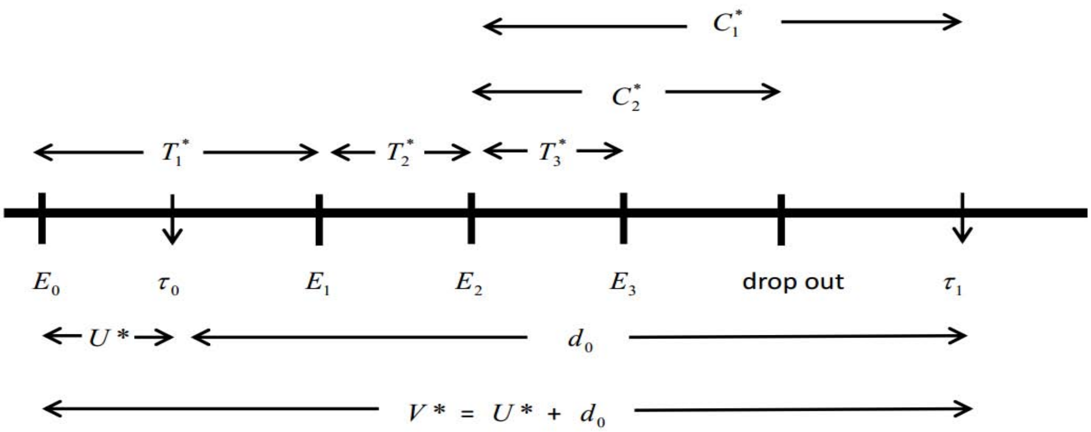

Figure 2. Schematic depiction of doubly-truncated and censored data

decline in abilities/death is the third event (E_3). Suppose that a prevalent cohort is defined as a sample of subjects who have been diagnosed with Alzheimer's disease and experienced mild decline in abilities within a calendar time window, i.e., $\tau_0 \leq E_0 + T_1^* + T_2^* \leq \tau_1$ or $U^* \leq T_1^* + T_2^* \leq U^* + d_0$. Hence, $Y_2^* = T_1^* + T_2^*$ is doubly truncated by U^* and $U^* + d_0$. In this case, the censoring time C^* denotes the time from E_2 to the right censoring and C^* can be written as $C^* = \min(C_1^*, C_2^*)$, where $C_1^* = U^* + d_0 - Y_2^*$ denotes the time from E_2 to the end of study and C_2^* denotes the time from E_2 to drop-out or death due to other causes. One observe nothing if $U^* > Y_2^*$ or $Y_2 > U^* + d_0$ and observes $(T_1^*, T_2^*, X_3^*, U^*, \gamma_1^*, \gamma_2^*)$ if $U^* \leq Y_2^* \leq U^* + d_0$, where $X_3^* = \min(T_3^*, C^*)$, $\gamma_1^* = I_{[T_3^* \leq C^*]}$ and $\gamma_2^* = (1 - \gamma_1^*) I_{[C^* = C_2^*]}$. We assume that U^*, C_2^* and (T_1^*, T_2^*, T_3^*) are independent. Figure 2 highlights all the different times for this type data described above.

In literature, for data without truncation, several nonparametric methods for estimating the joint distribution function of successive duration times have been developed in literature (see Visser (1996), Wang and Wells (1998), and Lin et al. (1999)). For left-truncated and dependent censored (LTDC) data, Chang and Tzeng (2006) proposed an inverse-probability-weighted (IPW) approach for estimating the joint probability function of two successive duration times. Shen and Yan (2008) proposed an alternative estimator of the joint distribution function of T_1^* and T_2^*. Shen (2010a) proposed two IPW estimators of the joint survival function of T_1^* and T_2^* based on the approaches of Chang and Tzeng

(2006) and Wang and Wells (1998). Using the IPW approach, Shen (2017) proposed nonparametric estimators for the estimation of the joint survival function of three successive duration times. For DTDC data, Zhu and Wang (2012,2015) considered semiparametric association estimation of (T_1^*, T_2^*) based on a copula model. Zhu and Wang (2014) proposed nonparametric estimation of the association between T_1^* and T_2^* based on Kendall's tau and developed a nonparametric test of quasi-independence. Using the IPW approach, Shen (2016a) propose nonparametric estimator of the joint survival function of T_1^* and T_2^*.

In this article, we consider the estimation of the joint survival function of three successive duration times for DTDC data. We consider two cases: the case when the first event time is subject to double truncation and the case when the second event time is subject to double truncation. In Section 2, when the distribution of the truncation time is unspecified, using the IPW approach, we propose nonparametric estimators of the joint survival function $S(t_1, t_2, t_3) = P(T_1^* > t_1, T_2^* > t_2, T_3^* > t_3)$. The asymptotic properties of the proposed estimators are established. In Section 3, under the assumption that the distribution of U^* is known up to a finite-dimensional parameter vector, we propose semiparametric estimators of $S(t_1, t_2, t_3)$. In Section 4, a simulation study is conducted to investigate finite sample performance of the proposed estimators.

2. The Nonparametric Estimators

2.1. When the first event time is subject to double truncation

Let $F_k(x) = P(T_k^* \le x)$ $(k = 1, 2, 3)$ denote the distribution function of T_k^*. Let $G(x) = P(U^* \le x)$ and $Q(x) = P(D^* \le x)$ denote the distribution function of U^* and D^*, respectively. Let a_{F_k} and b_{F_k} denote the left and right endpoints of F_k. Similarly, define (a_G, b_G) and (a_Q, b_Q) for U^* and D^*, respectively. For identifiablities of $S(t_1, t_2, t_3)$, we assume that

$$a_G = a_{F_1} = a_{F_2} = a_{F_3} = 0,\ b_G \le b_{F_1} \le b_G + d_0.$$

Then $S(t_1, t_2, t_3)$ is identifiable for $t_1 \le b_{F_1}$, $t_2 \le \min(b_{F_2}, b_Q)$ and $t_3 \le \min(b_{F_3}, b_Q)$. Let $(T_{1i}, X_{2i}, \delta_{1i}X_{3i}, U_i, \delta_{1i}, \delta_{2i}, \delta_{3i})$ $(i = 1, \ldots, n)$ denote the truncated sample. Let $p = P(U^* \le T_1^* \le U^* + d_0)$ denote the untruncated probability. Define the indicator

$$I_i(t_1, t_2, t_3) = I_{[T_{1i}>t_1, X_{2i}>t_2, X_{3i}>t_3, \delta_{1i}=1]}.$$

Consider the function

$$\tilde{S}(t_1, t_2, t_3) = P(T_{1i} > t_1, X_{2i} > t_2, X_{3i} > t_3, \delta_{1i} = 1)$$

$$\tilde{S}(t_1, t_2, t_3) = P(T_{1i}^* > t_1, X_{2i}^* > t_2, X_{3i}^* > t_3, \delta_{1i}^* = 1 | U^* \le T_1^* \le U^* + d_0)$$

$$= p^{-1}P(U^* < T_1^* < U^* + d_0, T_1^* > t_1, T_2^* > t_2, T_3^* > t_3, \min(D_1^*, D_2^*) - T_2^* > t_3, \min(D_1^*, D_2^*) > T_2^*)$$

$$= p^{-1}P(T_1^* + T_2^* + t_3 - d_0 < U^* < T_1^*, T_1^* > t_1, T_2^* > t_2, T_3^* > t_3, D_2^* - T_2^* > t_3).$$

For $x < y$, define $K(x, y) = [G(x) - G(x + y - d_0)]S_{D_2}(y)$, where $S_{D_2}(y) = P(D_2^* > y)$. Then the expected value of $I_i(t_1, t_2, t_3)/K(T_{1i}, X_{2i} + t_3)$ is

$$E[I_i(t_1, t_2, t_3)/K(T_{1i}, X_{2i} + t_3)]$$

$$= \int_{t_3}^{\infty}\int_{t_2}^{\infty}\int_{t_3}^{\infty} \frac{p^{-1}P(u_1 + u_2 + t_3 - d_0 < U^* < u_1)P(D_2^* > u_2 + t_3)}{K(u_1, u_2 + t_3)} dF(u_1, u_2, u_3)$$

$$= p^{-1}S(t_1, t_2, t_3),$$

where $F(u_1,u_2,u_3)$ denotes the joint distribution function of T_1^*, T_2^* and T_3^*.

Since $E[I_i(0,0,0)/K(T_{1i},X_{2i})] = p^{-1}$, given $K(x,y)$, p can be estimated by

$$\hat{p}(K) = n\left[\sum_{i=1}^{n} \frac{\delta_{1i}}{K(T_{1i},X_{2i})}\right]^{-1}.$$

Thus, given $K(x,y)$, we can estimate $S(t_1,t_2,t_3)$ by

$$\hat{S}_n(t_1,t_2,t_3;K) = n^{-1}\hat{p}(K)\sum_{i=1}^{n} \frac{I_i(t_1,t_2,t_3)}{K(T_{1i},X_{2i}+t_3)}$$

$$= \left[\sum_{i=1}^{n} \frac{\delta_{1i}}{K(T_{1i},X_{2i})}\right]^{-1} \sum_{i=1}^{n} \frac{I_i(t_1,t_2,t_3)}{K(T_{1i},X_{2i}+t_3)}. \qquad (2.1)$$

Now, we consider the estimation of $K(x,y)$. Under the assumption that D_2^* is independent of U^* and (T_1^*,T_2^*,T_3^*), $\hat{S}_{D_2}$ can be estimated using the Kaplan-Meier (1958) estimate $\hat{S}_{D_2}(t)$ as follows:

$$\hat{S}_{D_2}(t) = \prod_{X_{2i}\le t}\left(1 - \frac{\delta_{3i}}{n_i}\right),$$

where $n_i = \sum_{j=1}^{n} I_{[X_{2j}\ge X_{2i}]}$. Next, we can estimate $G(x)$ using the argument of Shen (2010b,2016) as follows. Let $S_1(x) = P(T_1^* > x)$ denote the survival function of T_1^*. Consider the distribution function of U_i's.

$$\tilde{G}(t) = P(U_i \le t) = p^{-1}P(U^* \le t, U^* \le T_1^* \le U^* + d_0)$$

$$= p^{-1}\int_0^t [S_1(u-) - S_1(u+d_0)]G(du),$$

Hence, given p and S_1, G can be estimated by an IPW estimator as follows

$$\hat{G}(t;S_1,p) = n^{-1}p\sum_{i=1}^{n} \frac{I_{[U_i\le t]}}{S_1(U_i-) - S_1(U_i+d_0)}.$$

Let $t \to \infty$, it follows that p can be estimated by

$$\hat{p}(S_1) = n\left[\sum_{i=1}^{n} \frac{1}{S_1(U_i-) - S_1(U_i+d_0)}\right]^{-1}$$

and G can be estimated by

$$\hat{G}(t;S_1) = \left[\sum_{i=1}^{n} \frac{1}{S_1(U_i-) - S_1(U_i+d_0)}\right]^{-1} \sum_{i=1}^{n} \frac{I_{[U_i \le t]}}{S_1(U_i-) - S_1(U_i+d_0)}.$$

Similarly, consider the survival function of T_{1i}'s.

$$\tilde{S}_1(t) = P(T_{1i} > t) = p^{-1} P(T_1^* > t, U^* \le T_1^* \le U^* + d_0)$$

$$= p^{-1} \int_0^t [G(u) - G((u-d_0)-] F_1(du),$$

where $F_1(t) = P(T_1^* \le t)$ is the distribution function of T_1^*. Hence, given p and G, S_1 can be estimated by an IPW estimator as follows

$$\hat{S}_1(t;G,p) = n^{-1} p \sum_{i=1}^{n} \frac{I_{[T_{1i} > t]}}{G(T_{1i}) - G((T_{1i} - d_0)-)}.$$

Let $t \to 0$, it follows that p can be estimated by

$$\hat{p}(G) = n \left[\sum_{i=1}^{n} \frac{1}{G(T_{1i}) - G((T_{1i} + d_0)-)}\right]^{-1}$$

and S_1 can be estimated by

$$\hat{S}_1(t;G) = \left[\sum_{i=1}^{n} \frac{1}{G(T_{1i}) - G((T_{1i} + d_0)-)}\right]^{-1} \sum_{i=1}^{n} \frac{I_{[T_{1i} > t]}}{G(T_{1i}) - G((T_{1i} - d_0)-)}.$$

By the above arguments, the IPW estimators of S_1 and G can be obtained by simultaneous solving the following two equations:

$$\hat{S}_{1n}(t) = \left[\sum_{i=1}^{n} \frac{1}{\hat{G}_n(T_{1i}) - \hat{G}_n((T_{1i} - d_0)-)}\right]^{-1} \sum_{i=1}^{n} \frac{I_{[T_{1i} > t]}}{\hat{G}_n(T_{1i}) - \hat{G}_n((T_{1i} - d_0)-)},$$

$$\hat{G}_n(u) = \left[\sum_{i=1}^{n} \frac{1}{\hat{S}_{1n}(U_i-) - \hat{S}_{1n}(U_i + d_0))}\right]^{-1} \sum_{i=1}^{n} \frac{I_{[U_i \le u]}}{\hat{S}_{1n}(U_i-) - \hat{S}_{1n}(U_i + d_0)}.$$

Hence, $K(x,y)$ can be estimated by $\hat{K}_n(x,y) = [\hat{G}_n(x) - \hat{G}_n(y-d_0)]\hat{S}_{D_2}(y)$.

Given $\hat{K}_n(x,y)$, by (2.1) an IPW estimate of $S(t_1,t_2,t_3)$ is given by

$$\hat{S}_n(t_1,t_2,t_3) = \left[\sum_{i=1}^{n} \frac{\delta_{1i}}{\hat{K}_n(T_{1i},X_{2i})}\right]^{-1} \sum_{i=1}^{n} \frac{I_{[T_{1i}>t_1,X_{2i}>t_2,X_{3i}>t_3,\delta_{1i}=1]}}{\hat{K}_n(T_{1i},X_{2i}+t_3)}.$$

For the special case of $D_2^* = \infty$,

$$\tilde{S}(t_1,t_2,t_3) = p^{-1}P(T_1^* - d_0 < U^* < T_1^*, T_1^* > t_1, T_2^* > t_2, T_3^* > t_3, D_1^* - T_2^* > t_3)$$

$$= p^{-1}P(T_1^* + T_2^* + t_3 - d_0 < U^* < T_1^*, T_1^* > t_1, T_2^* > t_2, T_3^* > t_3). \qquad (2.2)$$

By (2.2)

$$E[I_i(t_1,t_2,t_3)/[G(T_{1i}) - G(T_{1i}+X_{2i}+t_3-d_0)]$$

$$= \int_{t_3}^{\infty}\int_{t_2}^{\infty}\int_{t_3}^{\infty} \frac{p^{-1}P(u_1+u_2+t_3-d_0 < U^* < u_1)}{G(u_1) - G(u_1+u_2+t_3-d_0)} dF(u_1,u_2,u_3) = p^{-1}S(t_1,t_2,t_3)$$

and $E[I_i(0,0,0)/[G(T_{1i}) - G(T_{1i}+X_{2i}-d_0)] = p$. Thus, given $G(x)$, p can be estimated by

$$\hat{p}(G) = n\left[\sum_{i=1}^{n} \frac{\delta_{1i}}{G(T_{1i}) - G(T_{1i}+X_{2i}-d_0)}\right]^{-1}$$

and (2.1) is reduced to

$$\hat{S}_n(t_1,t_2,t_3) = \left[\sum_{i=1}^{n} \frac{\delta_{1i}}{\hat{G}_n(T_{1i}) - \hat{G}_n(T_{1i}+X_{2i}-d_0)}\right]^{-1} \sum_{i=1}^{n} \frac{I_{[T_{1i}>t_1,X_{2i}>t_2,X_{3i}>t_3,\delta_{1i}=1]}}{\hat{G}_n(T_{1i}) - \hat{G}_n(T_{1i}+X_{2i}+t_3-d_0)}.$$

In the following Theorem, we show the consistency of $\hat{S}_n(t_1,t_2,t_3)$.

Theorem 1. (a) Let $[0,b] = [0,b_1]\times[0,b_2]\times[0,b_3]$ such that $[G(x) - G((x+y-d_0)-)]S_{D_2}(y) > 0$ for $x \in [0,b_1]$ and $y \in [0,b_2]$. (b) $\int_{a_G}^{b_G} G(dx)/[S_1(x) - S_1(x+d_0)] < \infty$, (c) $F_1(dx)/G(dx)$ is uniformly bounded on $[a_G,b_G]$. Then $\hat{S}_n(t_1,t_2,t_3)$ is uniformly consistent on $[0,b]$.

Proof:

Write $\hat{S}_n(t_1, t_2, t_3)$ as $\hat{S}_n(t_1, t_2, t_3; \hat{K}_n)$. Thus

$$\hat{S}_n(t_1, t_2, t_3; \hat{K}_n) - S(t_1, t_2, t_3) = \phi_1(t_1, t_2, t_3) + \phi_2(t_1, t_2, t_3),$$

where

$$\phi_1(t_1, t_2, t_3) = \hat{S}_n(t_1, t_2, t_3; \hat{K}_n) - \hat{S}_n(t_1, t_2, t_3; K)$$

and

$$\phi_2(t_1, t_2, t_3) = \hat{S}_n(t_1, t_2, t_3; K) - S(t_1, t_2, t_3).$$

Let $\tilde{S}_n$ denote the empirical function of $\tilde{S}$ and

$$\hat{p}(\hat{K}_n) = n\left[\sum_{i=1}^{n} \frac{\delta_{2i}}{\hat{K}_n(T_{1i}, X_{2i})}\right]^{-1}.$$

Then

$$\phi_1(t_1, t_2, t_3) = \hat{p}(\hat{K}_n)\int_{t_3}^{\infty}\int_{t_2}^{\infty}\int_{t_1}^{\infty} \frac{\tilde{S}_n(du_1, du_2, du_3)}{\hat{K}_n(u_1, u_2 + t_3)} - \hat{p}(K)\int_{t_2}^{\infty}\int_{t_1}^{\infty} \frac{\tilde{S}_n(du_1, du_2, du_3)}{K(u_1, u_2 + t_3)}$$

$$= \int_{t_3}^{\infty}\int_{t_2}^{\infty}\int_{t_1}^{\infty} \frac{\tilde{S}_n(du_1, du_2, du_3)}{\hat{K}_n(u_1, u_2 + t_3)}[\hat{p}(\hat{K}_n) - \hat{p}(K)]$$

$$-\hat{p}(K)\int_{t_3}^{\infty}\int_{t_2}^{\infty}\int_{t_1}^{\infty}\left[\frac{1}{\hat{K}_n(u_1, u_2 + t_3)} - \frac{1}{K(u_1, u_2 + t_3)}\right]\tilde{S}_n(du_1, du_2, du_3). \quad (2.3)$$

By assumptions (a) and (b), it follows by Shen (2010b) that $\hat{G}_n$ is uniformly consistent. Furthermore, by uniform consistency of $\hat{S}_1$, $\hat{K}_n$ is uniformly consistent. Thus,

$$(2.3) = p^{-1}S(t_1, t_2, t_3)[\hat{p}(\hat{K}_n) - \hat{p}(K)]$$

$$-p\int_{t_3}^{\infty}\int_{t_2}^{\infty}\int_{t_1}^{\infty} \frac{1}{K^2(u_1, u_2 + t_3)}[\hat{K}_n(u_1, u_2+t_3) - K(u_1, u_2+t_3)]\tilde{S}(du_1, du_2, du_3) + o_p(n^{-1/2}).$$

Hence, $\phi_1(t_1, t_2, t_3) \xrightarrow{p} 0$. Next,

$$\phi_2(t_1, t_2, t_3) = pn^{-1}\sum_{i=1}^{n} \zeta(T_{1i}, X_{2i}, X_{3i}, \delta_{1i}, t_1, t_2, t_3) + o_p(n^{-1/2}),$$

where

$$\zeta(T_1, X_2, X_3, \delta_1, t_1, t_2, t_3) = [I_{[T_1 \geq t_1, X_2 \geq t_2, X_3 \geq t, \delta_2 = 1]} - \delta_1 S(t_1, t_2, t_3)]/K(T_1, X_2 + t_3).$$

Hence, $\phi_2(t_1, t_2, t_3) \xrightarrow{p} 0$. The proof is compete.

It is difficult to establish the asymptotic normality of $n^{1/2}(\hat{S}_n(t_1, t_2) - S(t_1, t_2))$ since we have difficulty expressing $\hat{G}_n(t)$ as a sum of i.i.d. random processes.

2.2. When the second event is subject to double truncation

Let $(T_{1i}, T_{2i}, X_{3i}, U_i, \gamma_{1i}, \gamma_{2i})$ $(i = 1, \ldots, n)$ denote the truncated sample. Let $p_2 = P(Y_2^* - d_0 \leq U^* \leq Y_2^*)$ denote the untruncated probability. Consider the function

$$\tilde{S}^Y(t_1, t_2, t_3) = P(T_{1i} > t_1, T_{2i} > t_2, X_{3i} > t_3, \gamma_{1i} = 1)$$

$$= p_2^{-1} P(U^* < Y_2^* < U^* + d_0, T_1^* > t_1, T_2^* > t_2, T_3^* > t_3, C^* > T_3^*)$$

$$= p_2^{-1} P(Y_2^* + T_3^* - d_0 < U^* < Y_2^*, T_1^* > t_1, T_2^* > t_2, T_3^* > t_3, C_2^* > T_3^*).$$

For $x < y$, define $H(x, y) = [G(x) - G(x + y - d_0)] S_{C_2}(y)$, where $S_{C_2}(y) = P(C_2^* > y)$. Define the indicator

$$I_i^Y(t_1, t_2, t_3) = I_{[T_{1i} > t_1, T_{2i} > t_2, X_{3i} > t_3, \gamma_{1i} = 1]}.$$

Then the expected value of $I_i^Y(t_1, t_2, t_3)/H(Y_{2i}, X_{3i})$ is

$$E[I_i^Y(t_1, t_2, t_3)/H(Y_{2i}, X_{3i})]$$

$$= \int_{t_3}^{\infty} \int_{t_2}^{\infty} \int_{t_3}^{\infty} \frac{p_2^{-1} P(u_1 + u_2 + t_3 - d_0 < U^* < u_1) P(C_2^* > u_3)}{H(u_1 + u_2, u_3)} dF(u_1, u_2, u_2)$$

$$= p_2^{-1} S(t_1, t_2, t_3).$$

Since $E[I_i^Y(0,0,0)/H(Y_{2i}, X_{3i})] = p_2^{-1}$, given $H(x,y)$, p_2 can be estimated by

$$\hat{p}_2(H) = n\left[\sum_{i=1}^{n} \frac{\gamma_{1i}}{H(Y_{2i}, X_{3i})}\right]^{-1}.$$

Thus, given $H(x,y)$, we can estimate $S(t_1,t_2,t_3)$ by

$$\hat{S}_n(t_1,t_2,t_3;H) = \left[\sum_{i=1}^{n} \frac{\gamma_{1i}}{H(Y_{2i}, X_{3i})}\right]^{-1} \sum_{i=1}^{n} \frac{I_{[T_{1i}>t_1, T_{2i}>t_2, X_{3i}>t_3, \gamma_{1i}=1]}}{H(Y_{2i}, X_{3i})}.$$

To obtain an estimate of $H(x,y)$, we need to estimate both $G(x) - G(x + y - d_0)$ and $S_{C_2}(y)$. The survival function $S_{C_2}(y)$ can be estimated using the Kaplan-Meier (1958) estimate $\hat{S}_{C_2}(t)$ as follows:

$$\hat{S}_{C_2}(y) = \prod_{X_{3i} \le y} \left(1 - \frac{\gamma_{2i}}{m_i}\right),$$

where $m_i = \sum_{j=1}^{n} I_{[X_{3j} \ge X_{3i}]}$. Next, we consider the estimation of $G(x)$. Let $Y_{2i} = T_{1i} + T_{2i}$ and $L(y) = P(Y_2^* \le y)$. Consider the distribution function of U_i's.

$$\begin{aligned}\tilde{G}(t) = P(U_i \le t) &= p_2^{-1} P(U^* \le t, U^* \le Y_2^*, Y_2^* \le U^* + d_0)\\ &= p_2^{-1} \int_0^t [L(u + d_0) - L(u-)] G(du),\end{aligned}$$

Hence, given p_2 and L, G can be estimated by an IPW estimator as follows

$$\hat{G}(t; L, p_2) = n^{-1} p_2 \sum_{i=1}^{n} \frac{I_{[U_i \le t]}}{L(U_i + d_0) - L(U_i-)}.$$

Let $t \to \infty$, it follows that p_2 can be estimated by

$$\hat{p}_2(L) = n\left[\sum_{i=1}^{n} \frac{1}{L(U_i + d_0) - L(U_i-)}\right]^{-1},$$

and G can be estimated by

$$\hat{G}(t; L) = \left[\sum_{i=1}^{n} \frac{1}{L(U_i + d_0) - L(U_i-)}\right]^{-1} \sum_{i=1}^{n} \frac{I_{[U_i \le t]}}{L(U_i + d_0) - L(U_i-)}.$$

Similarly, consider the function

$$\tilde{L}(t) = P(Y_{2i} \le t) = p_2^{-1} P(Y_2^* \le t, Y_2^* - d_0 \le U^* \le Y_2^*)$$

$$= p_2^{-1} \int_0^t \int_0^{t_2} [G(u) - G((u - d_0)-]L(du).$$

Hence, given p_2 and G, $L(t)$ can be estimated by an IPW estimator as follows

$$\hat{L}(t; G, p_2) = n^{-1} p_2 \sum_{i=1}^n \frac{I_{[Y_{2i} \le t]}}{G(Y_{2i}) - G((Y_{2i} - d_0)-)}.$$

Let $t \to 0$, it follows that p_2 can be estimated by

$$\hat{p}_2(G) = n \left[\sum_{i=1}^n \frac{1}{G(Y_{2i}) - G((Y_{2i} - d_0)-)} \right]^{-1}$$

and L can be estimated by

$$\hat{L}(t; G) = \left[\sum_{i=1}^n \frac{1}{G(Y_{2i}) - G((Y_{2i} - d_0)-)} \right]^{-1} \sum_{i=1}^n \frac{I_{[Y_{2i} \le t]}}{G(Y_{2i}) - G((Y_{2i} - d_0)-)}.$$

By the above arguments, the IPW estimators of L and G can be obtained by simultaneous solving the following two equations:

$$\hat{L}_n(t) = \left[\sum_{i=1}^n \frac{1}{\hat{G}_n^Y(Y_{2i}) - \hat{G}_n^Y((Y_{2i} - d_0)-)} \right]^{-1} \sum_{i=1}^n \frac{I_{[Y_{2i} \le t_2]}}{\hat{G}_n^Y(Y_{2i}) - \hat{G}_n^Y((Y_{2i} - d_0)-)},$$

$$\hat{G}_n^Y(u) = \left[\sum_{i=1}^n \frac{1}{\hat{L}_n(U_i + d_0) - \hat{L}_n(U_i-)} \right]^{-1} \sum_{i=1}^n \frac{I_{[U_i \le u]}}{\hat{L}_n(U_i + d_0) - \hat{L}_n(U_i-)}.$$

Thus, $H(x, y)$ can be estimated by $\hat{H}_n(x, y) = [\hat{G}_n^Y(x, y) - \hat{G}_n^Y(x + y - d_0)]\hat{S}_{C_2}(y)$, and an IPW estimator of $S(t_1, t_2, t_3)$ is given by

$$\hat{S}_n^Y(t_1, t_2, t_3) = \left[\sum_{i=1}^n \frac{\gamma_{1i}}{\hat{H}_n(Y_{2i}, X_{3i})} \right]^{-1} \sum_{i=1}^n \frac{I_{[T_{1i} > t_1, T_{2i} > t_2, X_{3i} > t_3, \gamma_{1i} = 1]}}{\hat{H}_n(Y_{2i}, X_{3i})}.$$

Theorem 2. (a) $[0, c] = [0, c_1] \times [0, c_2] \times [0, c_3]$ such that $H(x, y) > 0$ for $x \in [0, c_1 + c_2]$ and $y \in [0, c_3]$. (b) $\int_{a_G}^{b_G} G(dx)/[L(x + d_0) - L(x-)] < \infty$, (c) $L(dx)/G(dx)$ is uniformly bounded on $[a_G, b_G]$. Then $\hat{S}_n^Y(t_1, t_2, t_3)$ is uniformly consistent on $[0, c]$.

Proof: The proof is similar to that of Theorem 1 and is omitted.

3. Semiparametric Estimators

In some cases, there is sufficient information on the initial event E_0 to assume that the left-truncated time U^* has a parametric density function $g(x;\theta)$ with corresponding distribution function $G(x;\theta)$, where $\theta \in \Theta$, Θ is a known compact set in R^q and θ is a q-dimensional vector. For prevalent data with fixed recruitment time, the truncation distribution G can be interpreted as the distribution of the initial event E_0. For example, when E_0 corresponds to birth, one might parameterize G so that the parameterization reflects the growth of the birth over time. When $G(x)$ is parameterized as $G(x;\theta)$, Moreira and de Uña-Álvarez (2010b) and Shen (2010c) proposed a semiparametric estimator of $F_1(t)$ that is more efficient than the NPMLE $\hat{F}_{1n}(t) = 1 - \hat{S}_{1n}(t)$. Using their approach, we consider the conditional likelihood function for U_i's given T_{1i}'s:

$$L_c(\theta) = \prod_{i=1}^{n} P(U_i^* = U_i | T_{1i} - d_0 \leq U_i^* \leq T_{1i})$$

$$= \prod_{i=1}^{n} \frac{g(U_i;\theta)}{G(T_{1i};\theta) - G(T_{1i} - d_0;\theta)}.$$

Let $\hat{\theta}_n$ denote the MLE by maximizing $L_c(\theta)$. Given $\hat{\theta}_n$, $K(x,y)$ can be estimated by $K_{\hat{\theta}_n}(x,y) = [G(x;\hat{\theta}_n) - G(y - d_0;\hat{\theta}_n)]\hat{S}_{D_2}(y)$. When the first event time is subject to double truncation, given $K_{\hat{\theta}_n}(x,y)$, by (2.1) an IPW estimate of $S(t_1,t_2,t_3)$ is given by

$$\hat{S}_{\hat{\theta}_n}(t_1,t_2,t_3) = \left[\sum_{i=1}^{n} \frac{\delta_{1i}}{K_{\hat{\theta}_n}(T_{1i}, X_{2i})}\right]^{-1} \sum_{i=1}^{n} \frac{I_{[T_{1i}>t_1, X_{2i}>t_2, X_{3i}>t_3, \delta_{1i}=1]}}{K_{\hat{\theta}_n}(T_{1i}, X_{2i}+t_3)}.$$

Similarly, when the second event time is subject to double truncation, $H(x,y)$ can be estimated by $H_{\hat{\theta}_n}(x,y) = [G(x,y;\hat{\theta}_n) - G(x+y-d_0;\hat{\theta}_n)]\hat{S}_{C_2}(y)$, and an IPW estimator of $S(t_1,t_2,t_3)$ is given by

$$\hat{S}^{Y}_{\hat{\theta}_n}(t_1,t_2,t_3) = \left[\sum_{i=1}^{n} \frac{\gamma_{1i}}{H_{\hat{\theta}_n}(Y_{2i}, X_{3i})}\right]^{-1} \sum_{i=1}^{n} \frac{I_{[T_{1i}>t_1, T_{2i}>t_2, X_{3i}>t_3, \gamma_{1i}=1]}}{H_{\hat{\theta}_n}(Y_{2i}, X_{3i})}.$$

Theorem 3. Under condition (a) in Theorem 1, assume that (b) $G(x;\theta)$ is continuous in $x \in [a_G, b_G]$ for each $\theta \in \Theta$ and (c) $\hat{\theta}_n \xrightarrow{a.s.} \theta$ implies that $G(x;\hat{\theta}_n) \xrightarrow{a.s.} G(x;\theta)$ for each $x \in [a_G, b_G]$. Then $S_{\hat{\theta}_n}(t_1, t_2, t_3)$ is uniformly consistent on $[0, b]$.

Proof:

By Anderson (1970), $\hat{\theta}_n$ converges to θ with probability one. By assumption, $G(x;\hat{\theta}_n) \xrightarrow{a.s.} G(x;\theta)$. By the continuity of $G(x;\theta)$ in $x \in [a_G, b_G]$ and the monotonicity property of $G(x;\theta)$, it follows that with probability one, $\sup_{x\in[a_G,b_G]} |G(x;\hat{\theta}_n) - G(x;\theta)| \to 0$ as $n \to \infty$. The rest of proof is similar to that of Theorem 1 and is omitted.

Theorem 4. Under condition (a) in Theorem 2 and the assumptions (b) and (c) in Theorem 3, $S^Y_{\hat{\theta}_n}(t_1, t_2, t_3)$ is uniformly consistent on $[0, c]$.

Proof: The proof is similar to that of Theorem 3 and is omitted.

When the parametric information is correct, it is expected the semiparametric estimator $\hat{S}_{\hat{\theta}_n}$ outperforms the estimator $\hat{S}_n$, but may behave badly when the assumed parametric model is incorrect. Moreira et al. (2014) proposed several Kolmogorov-Smirnov and Cramér-von Mises type test statistics, by which we can check if G can be parameterized as $G(t;\theta)$.

4. Simulation Study

Case 1: When the first event is doubly truncated

To investigate the performance of the proposed estimator $\hat{S}_n$ and $\hat{S}_{\hat{\theta}_n}$, we conduct simulations under the recruiting criterion $U^* \leq T_1^* \leq U^* + d_0$. The joint distribution of (T_1^*, T_2^*, T_3^*)'s are generated using Clayton's (1978) bivariate exponential survival function with association parameter β_1 between T_1^* and T_2^*, and association parameter β_2 between T_2^* and T_3^*, i.e. $S_{12}(t_1, t_2) = P(T_1^* > t_1, T_2^* > t_2) = [S_1(t_1)^{1-\beta_1} + S_2(t_2)^{1-\beta_1} - 1]^{1/1-\beta_1}$ and $S_{23}(t_1, t_2) = P(T_2^* > t_1, T_3^* > t_2) = [S_1(t_1)^{1-\beta_2} + S_2(t_2)^{1-\beta_2} - 1]^{1/1-\beta_2}$ with marginal survival functions $S_i(t) = e^{-t/\mu_i}$ $(i = 1, 2, 3)$ with $\mu_1 = 5$, $\mu_2 = 1$ and $\mu_3 = 0.5$. The values of β_1 and β_2 are chosen as $\beta_1 = \beta_2 = 2$ such that the Kendall's tau of (T_1^*, T_2^*), (T_2^*, T_3^*) and (T_1^*, T_3^*) are equal to 0.5, 0.5 and 0.35, respectively. The left truncation time U^* is exponentially distributed with mean μ_g and the right truncation time $V^* = U^* + d_0$. The value of (μ_g, d_0) are chosen as $(4, 8)$ and $(2, 6)$ such that the proportions of left truncation (denoted by q_l) and right truncation (denoted by q_r) are equal to $(q_l, q_r) = (0.45, 0.10)$ and $(0.28, 0.22)$, respectively. The censoring time D_2^* is exponentially distributed with mean μ_D. The values of μ_D are chosen as 2 and 4 such that $(p_1, p_2, p_3) = (0.60, 0.47, 0.31)$ and $(0.71, 0.60, 0.18)$, respectively, for $(q_l, q_r) = (0.28, 0.22)$, and $(p_1, p_2, p_3) = (0.6, 0.48, 0.36)$ and $(0.72, 0.62, 0.20)$, respectively, for $(q_l, q_r) = (0.45, 0.10)$, where $p_1 = P(\delta_{1i} = 1)$, $p_2 = P(\delta_{2i} = 1)$ and $p_3 = P(\delta_{3i} = 1)$. We keep the sample if $U^* \leq T_1^* \leq U^* + d_0$ and obtain the truncated observations $(X_{1i}, X_{2i}, X_{3i}, U_i, \delta_{1i}, \delta_{2i}, \delta_{3i})$. We regenerate a sample if $T_1^* < U^*$ or $T_1^* > U^* + d_0$ such that the untruncated sample size is equal to n. We consider the estimation of $S(t_0)$ for some selected points of $t_0 = (1.0, 0.1, 0.1)$, $(1.0, 0.5, 0.1)$, $(4.0, 1.0, 0.1)$, $(1.0, 0.1, 0.25)$, $(1.0, 0.5, 0.25)$ and $(4.0, 1.0, 0.25)$ with corresponding true values equal to 0.74, 0.58, 0.27, 0.14, 0.56, 0.49, 0.24 and 0.13, respectively. The sample size is chosen as $n = 200, 400$ and the replication is 1000 times. Tables 1 and 2 show the biases, standard deviations (std) and root meas squared errors (rmse) of the two estimators $\hat{S}_n(t)$ and $\hat{S}_{\hat{\theta}_n}(t)$ for the selected points.

Case 2: When the second event is doubly truncated

To investigate the performance of the proposed estimator $\hat{S}_n^Y$ and $\hat{S}_{\hat{\theta}_n}^Y(t)$, we conduct simulations under the recruiting criterion $U^* \leq Y_2^* \leq U^* + d_0$. The distribution of (T_1^*, T_2^*, T_3^*), U^* and V^* are the same as those used in Section 3.1. The value of (μ_g, d_0) are chosen as $(4, 8)$ and $(2, 6)$ such that the proportions of left truncation (denoted by q_l) and right truncation (denoted by q_r) are equal to $(q_l, q_r) = (0.38, 0.15)$ and $(0.28, 0.23)$, respectively. The distribution of censoring time C_2^* is the same as D_2^* is Section 3.1. We keep the sample if $U^* \leq Y_2^* \leq U^* + d_0$ and regenerate a sample if $Y_2^* < U^*$ or $Y_2^* > U^* + d_0$ such that the untruncated sample size is equal to n. We also calculate the simulated proportion $r_1 = P(\gamma_{1i} = 1)$ and $r_2 = P(\gamma_{2i} = 1)$. Tables 3 and 4 show the simulation results for the two estimators $\hat{S}_n^Y(t)$ and $\hat{S}_{\hat{\theta}_n}^Y(t)$.

Based on the results of Tables 1 and 4, we conclude that:

(i) For case 1, the biases and standard deviations of both $\hat{S}_n$ and $\hat{S}_{\hat{\theta}_n}$ decrease as sample size increases. When left truncation is mild ($q_l = 0.28$) and right truncation is not light ($q_r = 0.22$), the biases of $\hat{S}_n$ and $\hat{S}_{\hat{\theta}_n}$ can be large for the estimation of $S(t_0)$ at non-early time points $t_0 = (4.0, 1.0, 0.1)$, $(1.0, 0.1, 0.25)$ and $(4.0, 1.0, 0.25)$. This is improved when the proportion of left truncation is increased to $q_l = 0.45$ since more failure times with larger values can be observed. For case 2, the biases of both $\hat{S}_n^Y$ and $\hat{S}_{\hat{\theta}_n}^Y(t)$ are small for most of the cases considered and the standard deviations decrease as sample size increases.

(ii) For both cases, the standard deviations of the semiparametric estimator $\hat{S}_{\hat{\theta}_n}$ ($\hat{S}_{\hat{\theta}_n}^Y(t)$) are smaller than that of the nonparametric estimator $\hat{S}_n$ ($\hat{S}_n^Y$). In terms of rmse, the semiparametric estimator outperforms the nonparametric estimator.

5. Discussion

In this article, for doubly-truncated and dependent censored data, we have proposed inverse-probability weighted estimators for the estimation of the joint survival function of three successive duration times. Simulation results indicate that both nonparametric estimator $\hat{S}_n$ and semiparametric estimator $\hat{S}_{\hat{\theta}_n}$ perform well. Our proposed approach can be extended to the case of more than three successive duration times. In some cases, the calendar times of the initiation E_0 or the subsequent events E_1 and E_2 are only known to fall within intervals, leading to doubly censored data. Further research is needed to extend our approach to deal with such data.

Table 1. Simulation results for $\hat{S}_n$ and $\hat{S}_{\hat{\theta}_n}$ $(q_l = 0.28, q_r = 0.22)$

			$(p_1, p_2, p_3) = (0.60, 0.47, 0.31)$					
				$\hat{S}_n(t)$			$\hat{S}_{\hat{\theta}_n}(t)$	
(t_1, t_2, t_3)	$S(t_0)$	n	bias	std	rmse	bias	std	rmse
(1.0,0.1,0.1)	0.74	200	-0.034	0.129	0.133	-0.031	0.107	0.111
(1.0,0.1,0.1)	0.74	400	-0.013	0.083	0.084	-0.021	0.072	0.075
(1.0,0.5,0.1)	0.58	200	-0.048	0.124	0.133	-0.041	0.111	0.118
(1.0,0.5,0.1)	0.58	400	-0.021	0.099	0.101	-0.034	0.081	0.088
(4.0,1.0,0.1)	0.27	200	-0.068	0.127	0.144	-0.043	0.120	0.127
(4.0,1.0,0.1)	0.27	400	-0.053	0.095	0.108	-0.037	0.092	0.099
(1.0,0.1,0.25)	0.56	200	-0.072	0.122	0.142	-0.067	0.121	0.138
(1.0,0.1,0.25)	0.56	400	-0.050	0.091	0.104	-0.056	0.085	0.102
(1.0,0.5,0.25)	0.49	200	-0.084	0.127	0.152	-0.092	0.119	0.150
(1.0,0.5,0.25)	0.49	400	-0.040	0.095	0.103	-0.064	0.088	0.109
(4.0,1.0,0.25)	0.24	200	-0.064	0.119	0.135	-0.076	0.107	0.131
(4.0,1.0,0.25)	0.24	400	-0.036	0.095	0.102	-0.043	0.086	0.096
			$(p_1, p_2, p_3) = (0.71, 0.60, 0.18)$					
(t_1, t_2, t_3)	$S(t_0)$	n	bias	std	rmse	bias	std	rmse
(1.0,0.1,0.1)	0.74	200	-0.018	0.105	0.107	-0.027	0.083	0.087
(1.0,0.1,0.1)	0.74	400	-0.005	0.079	0.078	-0.019	0.067	0.070
(1.0,0.5,0.1)	0.58	200	-0.033	0.124	0.128	-0.045	0.089	0.100
(1.0,0.5,0.1)	0.58	400	-0.022	0.094	0.097	-0.033	0.073	0.081
(4.0,1.0,0.1)	0.27	200	-0.056	0.130	0.131	-0.055	0.102	0.116
(4.0,1.0,0.1)	0.27	400	-0.045	0.087	0.098	-0.043	0.081	0.092
(1.0,0.1,0.25)	0.56	200	-0.067	0.124	0.141	-0.081	0.085	0.117
(1.0,0.1,0.25)	0.56	400	-0.030	0.092	0.097	-0.067	0.079	0.085
(1.0,0.5,0.25)	0.49	200	-0.070	0.129	0.147	-0.080	0.098	0.127
(1.0,0.5,0.25)	0.49	400	-0.032	0.096	0.101	-0.052	0.081	0.096
(4.0,1.0,0.25)	0.24	200	-0.059	0.121	0.135	-0.074	0.097	0.122
(4.0,1.0,0.25)	0.24	400	-0.036	0.083	0.090	-0.052	0.075	0.083

Table 2. Simulation results for $\hat{S}_n$ and $\hat{S}_{\hat{\theta}_n}$ $(q_l = 0.45, q_r = 0.10)$

			$(p_1, p_2, p_3) = (0.60, 0.48, 0.36)$					
				$\hat{S}_n(t)$			$\hat{S}_{\hat{\theta}_n}(t)$	
(t_1, t_2, t_3)	$S(t_0)$	n	bias	std	rmse	bias	std	rmse
(1.0,0.1,0.1)	0.74	200	-0.021	0.122	0.124	0.005	0.082	0.082
(1.0,0.1,0.1)	0.74	400	-0.008	0.093	0.093	-0.006	0.070	0.070
(1.0,0.5,0.1)	0.58	200	-0.022	0.127	0.129	-0.002	0.078	0.078
(1.0,0.5,0.1)	0.58	400	-0.009	0.094	0.094	-0.008	0.069	0.069
(4.0,1.0,0.1)	0.27	200	-0.024	0.118	0.120	-0.020	0.078	0.080
(4.0,1.0,0.1)	0.27	400	-0.012	0.085	0.086	-0.008	0.067	0.067
(1.0,0.1,0.25)	0.56	200	-0.054	0.126	0.137	-0.051	0.077	0.092
(1.0,0.1,0.25)	0.56	400	-0.018	0.097	0.099	-0.046	0.065	0.080
(1.0,0.5,0.25)	0.49	200	-0.052	0.119	0.130	-0.058	0.072	0.092
(1.0,0.5,0.25)	0.49	400	-0.010	0.092	0.093	-0.045	0.066	0.080
(4.0,1.0,0.25)	0.24	200	-0.041	0.113	0.120	-0.040	0.073	0.083
(4.0,1.0,0.25)	0.24	400	-0.012	0.087	0.088	-0.037	0.059	0.070
			$(p_1, p_2, p_3) = (0.72, 0.62, 0.20)$					
(t_1, t_2, t_3)	$S(t_0)$	n	bias	std	rmse	bias	std	rmse
(1.0,0.1,0.1)	0.74	200	-0.011	0.107	0.108	-0.009	0.083	0.083
(1.0,0.1,0.1)	0.74	400	-0.009	0.095	0.095	-0.002	0.061	0.061
(1.0,0.5,0.1)	0.58	200	-0.012	0.112	0.113	-0.019	0.081	0.083
(1.0,0.5,0.1)	0.58	400	-0.005	0.088	0.088	-0.004	0.058	0.058
(4.0,1.0,0.1)	0.27	200	-0.009	0.106	0.106	-0.033	0.066	0.074
(4.0,1.0,0.1)	0.27	400	-0.005	0.078	0.078	-0.010	0.051	0.052
(1.0,0.1,0.25)	0.56	200	-0.013	0.113	0.114	-0.050	0.075	0.090
(1.0,0.1,0.25)	0.56	400	-0.009	0.095	0.095	-0.037	0.058	0.069
(1.0,0.5,0.25)	0.49	200	-0.010	0.111	0.111	-0.051	0.074	0.090
(1.0,0.5,0.25)	0.49	400	-0.005	0.088	0.088	-0.035	0.054	0.064
(4.0,1.0,0.25)	0.24	200	-0.015	0.107	0.108	-0.048	0.062	0.078
(4.0,1.0,0.25)	0.24	400	-0.005	0.078	0.078	-0.026	0.048	0.054

Table 3. Simulation results for $\hat{S}_n^Y$ and $\hat{S}_{\hat{\theta}_n}^Y$ ($q_l = 0.23, q_r = 0.28$)

			$(\gamma_1,\gamma_2) = (0.75, 0.19)$					
			$\hat{S}_n^Y(t)$			$\hat{S}_{\hat{\theta}_n}^Y(t)$		
(t_1,t_2,t_3)	$S(t_0)$	n	bias	std	rmse	bias	std	rmse
(1.0,0.1,0.1)	0.74	200	-0.024	0.108	0.111	-0.026	0.083	0.087
(1.0,0.1,0.1)	0.74	400	-0.009	0.063	0.063	-0.007	0.057	0.057
(1.0,0.5,0.1)	0.58	200	-0.025	0.130	0.132	-0.039	0.098	0.105
(1.0,0.5,0.1)	0.58	400	-0.008	0.091	0.091	-0.016	0.080	0.081
(4.0,1.0,0.1)	0.27	200	-0.017	0.175	0.176	-0.040	0.138	0.144
(4.0,1.0,0.1)	0.27	400	-0.012	0.143	0.144	-0.024	0.112	0.115
(1.0,0.1,0.25)	0.56	200	-0.025	0.126	0.128	-0.037	0.100	0.107
(1.0,0.1,0.25)	0.56	400	-0.010	0.095	0.096	-0.013	0.081	0.082
(1.0,0.5,0.25)	0.49	200	-0.026	0.137	0.139	-0.047	0.104	0.114
(1.0,0.5,0.25)	0.49	400	-0.009	0.099	0.099	-0.017	0.089	0.091
(4.0,1.0,0.25)	0.24	200	-0.021	0.185	0.186	-0.042	0.143	0.149
(4.0,1.0,0.25)	0.24	400	-0.011	0.133	0.133	-0.016	0.114	0.116
			$(\gamma_1,\gamma_2) = (0.84, 0.12)$					
(t_1,t_2,t_3)	$S(t_0)$	n	bias	std	rmse	bias	std	rmse
(1.0,0.1,0.1)	0.74	200	-0.008	0.087	0.087	-0.011	0.082	0.082
(1.0,0.1,0.1)	0.74	400	-0.002	0.065	0.065	-0.008	0.050	0.051
(1.0,0.5,0.1)	0.58	200	-0.010	0.118	0.118	-0.022	0.091	0.094
(1.0,0.5,0.1)	0.58	400	-0.003	0.089	0.089	-0.015	0.064	0.066
(4.0,1.0,0.1)	0.27	200	-0.016	0.180	0.180	-0.035	0.121	0.126
(4.0,1.0,0.1)	0.27	400	-0.007	0.145	0.145	-0.028	0.088	0.092
(1.0,0.1,0.25)	0.56	200	-0.010	0.111	0.111	-0.016	0.087	0.088
(1.0,0.1,0.25)	0.56	400	-0.004	0.098	0.098	-0.022	0.070	0.073
(1.0,0.5,0.25)	0.49	200	-0.014	0.128	0.129	-0.022	0.090	0.093
(1.0,0.5,0.25)	0.49	400	-0.007	0.103	0.103	-0.027	0.071	0.076
(4.0,1.0,0.25)	0.24	200	-0.013	0.176	0.176	-0.027	0.120	0.123
(4.0,1.0,0.25)	0.24	400	-0.011	0.137	0.137	-0.023	0.078	0.083

Table 4. Simulation results for $\hat{S}_n^Y$ and $\hat{S}_{\hat{\theta}_n}^Y$ $(q_l = 0.38, q_r = 0.15)$

			$(\gamma_1, \gamma_2) = (0.75, 0.21)$					
				$\hat{S}_n^Y(t)$			$\hat{S}_{\hat{\theta}_n}^Y(t)$	
(t_1, t_2, t_3)	$S(t_0)$	n	bias	std	rmse	bias	std	rmse
(1.0,0.1,0.1)	0.74	200	0.003	0.128	0.128	-0.013	0.076	0.077
(1.0,0.1,0.1)	0.74	400	0.002	0.083	0.083	0.002	0.061	0.061
(1.0,0.5,0.1)	0.58	200	0.009	0.135	0.135	-0.018	0.073	0.075
(1.0,0.5,0.1)	0.58	400	0.004	0.095	0.095	-0.004	0.065	0.065
(4.0,1.0,0.1)	0.27	200	0.015	0.152	0.152	-0.021	0.093	0.095
(4.0,1.0,0.1)	0.27	400	0.010	0.113	0.113	0.003	0.074	0.074
(1.0,0.1,0.25)	0.56	200	0.004	0.135	0.135	-0.016	0.082	0.084
(1.0,0.1,0.25)	0.56	400	0.002	0.102	0.102	0.001	0.065	0.065
(1.0,0.5,0.25)	0.49	200	0.008	0.147	0.147	-0.012	0.087	0.088
(1.0,0.5,0.25)	0.49	400	0.002	0.107	0.107	0.004	0.067	0.067
(4.0,1.0,0.25)	0.24	200	0.012	0.160	0.160	-0.017	0.096	0.097
(4.0,1.0,0.25)	0.24	400	0.006	0.119	0.119	0.004	0.074	0.074
			$(\gamma_1, \gamma_2) = (0.84, 0.12)$					
(t_1, t_2, t_3)	$S(t_0)$	n	bias	std	rmse	bias	std	rmse
(1.0,0.1,0.1)	0.74	200	0.015	0.111	0.111	0.006	0.071	0.071
(1.0,0.1,0.1)	0.74	400	0.008	0.084	0.084	0.009	0.047	0.048
(1.0,0.5,0.1)	0.58	200	0.010	0.120	0.120	0.008	0.084	0.084
(1.0,0.5,0.1)	0.58	400	0.008	0.097	0.097	0.003	0.050	0.050
(4.0,1.0,0.1)	0.27	200	0.017	0.140	0.140	0.004	0.081	0.081
(4.0,1.0,0.1)	0.27	400	0.012	0.107	0.107	0.008	0.055	0.055
(1.0,0.1,0.25)	0.56	200	0.023	0.123	0.125	0.007	0.075	0.075
(1.0,0.1,0.25)	0.56	400	0.007	0.094	0.094	0.004	0.050	0.050
(1.0,0.5,0.25)	0.49	200	0.028	0.127	0.130	0.010	0.084	0.085
(1.0,0.5,0.25)	0.49	400	0.007	0.097	0.097	0.003	0.049	0.049
(4.0,1.0,0.25)	0.24	200	0.023	0.136	0.135	0.002	0.081	0.081
(4.0,1.0,0.25)	0.24	400	0.018	0.100	0.102	0.006	0.051	0.051

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