東 海 大 學 應 用 數 學 系

碩 士 論 文

雙時滯純量系統之極點配置

Pole Placement for a scalar system with Two Delay

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中 華 民 國 一 〇 七 年 一 月 十 七 日

致謝

從第一次走進數學系教室至今，整整過了十年。

離開大學校園二十年，經歷了逐漸熟悉身爲學生應該有的感覺，期間甚至也歷經許多人生的巨變與轉折，十年雖不算漫長，但回首點點滴滴，白駒過隙，卻有看不到盡頭的長廊。最後支撐我走完全程的動機，只是想完成一件事，不要一個有缺陷，或者未來可能會徒留遺憾的過程。

感謝黃皇男老師，從第三年修習複變數函數論開始，就成爲推動我繼續前進的力量，甚至在最後階段包容我的“在職進修”身分，協助完成論文。

也感謝陳淑珍老師，讓我經歷了難忘的分析課程的淬鍊，成爲在工作上進步的基礎。

當然，對於曾經指導過我的老師，以及系上的每一位老師，助教，同樣心存感謝，由於你們的專業及辛勞，使得我對於能成爲東海應數系的一員而感到驕傲。

凡走過必留下痕跡，發生過必永遠存在。感謝黃老師，以及東海應數。

Abstract

The Lambert W function has many applications in the fields of pure and applied mathematics as well as physics and engineering. In particular, differential equations which represent time delay systems are employed to stability analysis and controller synthesis in the modern control theory.

The main target of this study is to probe the stability of time delay systems and then to place the system's poles to desire locations. Firstly, we discuss how to solute the characteristic equation generated from a single delay system via Lambert W function, and expand further to two-lag linear delay differential equations. Since the positions of eigenvalues influence stability, the problem of delay systems with single or two delays via eigenvalue assignment are then considered. Finally, the pole placement problem is then solved with considerable controller to drive the delay system to have desire response implied by the location of system poles.

Keywords: Lambert W function, delay system, characteristic equation, eigenvalue, pole placement

中文摘要

Lambert W 函數不論在純粹數學、應用數學以及物理、工程等領域都有諸多的應用。特別是在近代的控制理論中，以常微分方程表示的時滯系統常被使用作爲討論系統穩定性分析以控制律合成的例子。

本文的主要目標是以探討時滯系統的穩定性作爲基礎，進行極點配置問題的研究。首先討論單一時滯系統如何透過 Lambert W 函數求解特徵方程，並且進一步延伸討論雙時滯線性微分方程系統。因爲特徵值的位置影響穩定性，我們考慮了單一或兩個時滯的時滯系統, 經由特徵值分配，決定系統的響應符合想要配置系統極點的要求。

關鍵詞： Lambert W 函數，時滯系統，特徵方程，特徵值，極點配置

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Chapter 1

Introduction

Time delays often arise in dynamical control systems, both from delays in the process itself and from delays in the processing of sensed signals. In other words, such systems arise from internal time-delay in the components of the systems or from a external introduction of time-delay into the systems for the purposes of control.

Time delay widely exists in many fields such as industry processes, ecological groups, and so on. There is also a small time delays in any networked control systems due to the cycle time of the computer and the data transmissions. Many examples with delay effects can be found in [1–3] and reference therein.

There are many stability criteria and performance measures studied. Bellman and Cooke [4] has a very complex coverage on the distribution of characteristic roots for differential-difference equations. Kolmanovskii and Nosov [5] has a wide overview of various methods of stability analysis including both frequency domain and time domain methods.

Time delay systems representing by delay differential equations which have infinitely many solutions were introduced by Condorcet and Laplace in the eighteenth century. An approach to obtain the analytic solution of delay systems based on the concept of Lambert W function is developed by Corless et al. [6] and Asl and Ulsoy [7] recently. Shih [8] consists in studying the singular behaviour of each $W(-k, z)$ at the branch point $z = -e^{-1}$.

The addition of two delay significant increases the difficulty of the stability analysis. An economic model with two delays has a region of stability that is larger than one with delays nearby that are irrationally related. Control loops in optics [9] has been modeled with multiple delays. Huang [10] has developed the technique to compute the characteristic roots of a scalar system with multiple delays.

In this thesis, a brief introduction of linear time-delay systems and Lambert W functions as a preliminary of this study are given in the Chapter 2. In Chapter 3, we discuss the stability of linear time-delay system with respect to extreme point results, and introduce the conception of two delays DDE. In Chapter 4, the pole assignment via the Lambert W function is developed. Conversely, by adjusting the parameters, we try to get the desired poles of the characteristic equations of linear time-delay systems.

Chapter 2

Linear Time-Delay Systems

Time-delay systems can be represented by delay differential equations, and have been extensively studied during the past decades. In order to solve the differential equations , we bring it to the frequency domain by the Laplace transform. Furthermore, we introduce the Lambert W function, represented by W(z), and its importance property for stability of linear time-delay system in this chapter.

2.1 Time-Delay Systems

A linear time-delay system is defined by

$$\dot{x}(t) = \alpha x(t) + \beta x(t-h), t > 0 \tag{2.1}$$

$$x(0) = \phi(0), t = 0$$

$$x(t) = \phi(t), t \in [-h, 0)$$

where α and β are scalars . Futhermore, $x(t) = \phi(t)$ is a preshape function, and $\phi(0)$ is an initial state. By using the Laplace transform,

$$\begin{aligned}
sX(s) - x(0) &= \alpha X(s) + \beta \int_0^\infty x(t-h)e^{-st}dt \\
&= \alpha X(s) + \beta \left[\int_0^h \phi(t-h)e^{-st}dt + \int_h^\infty x(t-h)e^{-st}dt \right] \\
&= \alpha X(s) + \beta \left[\int_0^h \phi(t-h)e^{-st}dt + \int_0^\infty x(t-h)e^{-s(u+h)}du \right] \\
&= \alpha X(s) + \beta \left[\int_0^h \phi(t-h)e^{-st}dt + e^{-st} \int_0^\infty x(t-h)e^{-su}du \right] \\
&= \alpha X(s) + \beta e^{-st} X(s) + \beta \int_0^h \phi(t-h)e^{-st}dt
\end{aligned}$$

which implies

$$(sI - \alpha - \beta e^{-sh})X(s) = \phi(0) + \beta \int_0^h \phi(t-h)e^{-st}dt$$

or equivalently,

$$X(s) = (sI - \alpha - \beta e^{-sh})^{-1} \left[\phi(0) + \beta \int_0^h \phi(t-h)e^{-st}dt \right].$$

For this result, we get the characteristic equation

$$\det(s - \alpha - \beta e^{-sh}) = 0. \tag{2.2}$$

Refer to the paper by Coreless et. al. [6], it is suggested by using Lambert W function to describe the corresponding solution as

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{S_k t}$$

for some ck such that the sum makes sense.

Definition 2.1. The linear time delay system in (2.1) is *stable* if

$$\lim_{t \to \infty} x(t) = 0. \tag{2.3}$$

Definition 2.2. [12] The linear time-delay system(2.1) is *stable* if all the roots of (2.2) lie in the complex left half-plane $\mathbb{C}^-$. Furthermore, the real part of the rightmost root is called *stability exponent*, which represents the effect of the most dominant characteristic root on the system behavior.

In this thesis, we will try to solve the characteristic equation in order to find the positions of the poles and expand the result to probe the time delay systems which have multiple delays terms.

2.2 Lambert W function

Introduced by Lambert and Euler in the 1700s, the Lambert W function is defined as the solutions $w \in \mathbb{C}$ of $w(z)e^{w(z)} = z$ for $z \in \mathbb{C}$ and denoted by $w = W(z)$.

The Lambert W function is a complex multivalued function which has infinite number of branches, $W_k(x)$, where $k = 0, \pm 1, \pm 2, \cdots, \pm\infty$. For ease of argument, we make a sort of compactification, i.e., to regard both W_∞ and $_{-\infty}$ as fixed mappings. As seen in Figure 2.1, there are two possible real values of $W_k(x)$, when $-\frac{1}{e} \leq x < 0$ for any $x \in \mathbb{R}$. We denote the branch which satisfies $W(x) \geq -1$ by $W_0(x)$, or $W(x)$ if there is no any confusion, and the branch satisfies $W(x) \leq -1$ by $W_{-1}(x)$, especially W_0 is called the *principal branch*.

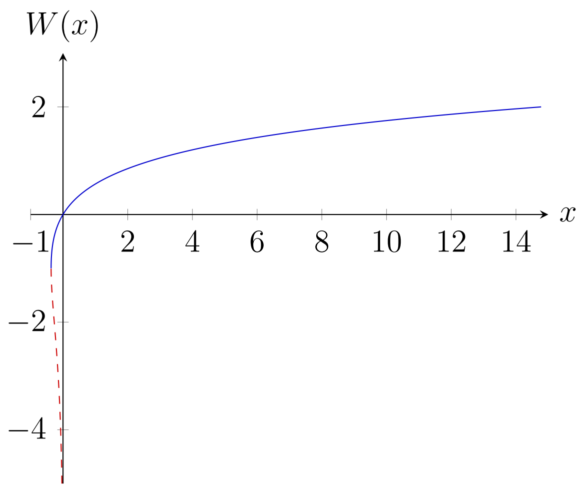


Figure 2.1: Two real branches of $W_k(x)$: - - - -, $W_{-1}(x)$; ——, $W_0(x)$.

By partitioning the z-plane with horizontal boundaries $z = j(2k+1)\pi$ for $k \in \mathbb{Z}$, the ranges of branches of $W_k(z)$ are images of the z between branch cuts in the z-plane, as shown in Figure 2.2.

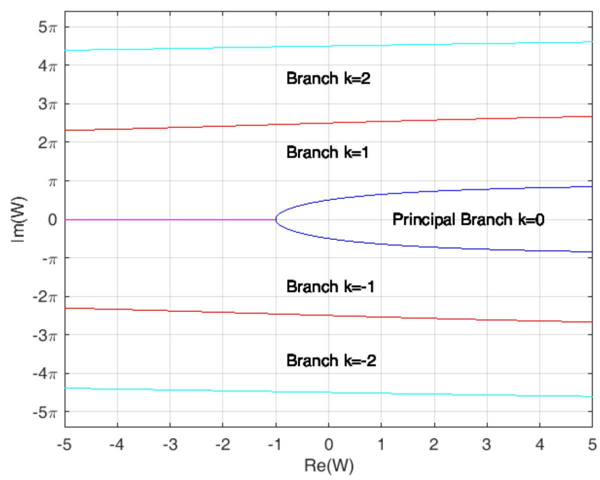


Figure 2.2: The ranges of $W_k(x)$, $k = -2, -1, 0, 1, 2$.

Lemma 2.3. *The range of the Lambert W function is symmetric with respect to the real axis.*

Proof. Let $z = re^{j\phi} \in \mathbb{C}$ such that $\phi \in ((2k-1)\pi, (2k+1)\pi]$ for certain k. Then there is a pair of numbers u and v such that $W_k(z) = u + jv$, i.e., by definition $z = W_k(z)e^{W_k(z)}$ which implies

$$\begin{aligned} re^{j\phi} &= (u+jv)e^{u+jv} = (u+jv)e^u(\cos v + j\sin v) \\ &= e^u(u\cos v - v\sin v) + je^u(v\cos v + u\sin v) \end{aligned}$$

and give us

$$\begin{aligned} r &= e^u\sqrt{(u\cos v - v\sin v)^2 + (v\cos v + u\sin v)^2} \\ \phi &= \tan^{-1}\frac{v\cos v + u\sin v}{u\cos v - v\sin v}. \end{aligned}$$

Now consider the term $\overline{W_k(z)} = u - jv$, then

$$\begin{aligned} (u-jv)e^{u-jv} &= (u-jv)e^u(\cos v - j\sin v) \\ &= e^u(u\cos v - v\sin v) - je^u(v\cos v + u\sin v) \\ &= re^{-j\phi}, \end{aligned}$$

and

$$-\phi \in [(-2k-1)\pi, (-2k+1)\pi) \in [2(-k)-1)\pi, 2(-k)+1\pi)$$

thus, the term $u - jv$ belongs to the $-k$ branch of the Lambert W function acting on $\bar{z}$. That is, $W_{-k}(\bar{z}) = \overline{W_k(z)}$. □

Lemma 2.3 is also referred that the Lambert W function has the conjugate symmetricity.

Let

$$B_{C0} \triangleq \left\{a + j0 \middle| -\infty < a \leq -\frac{1}{e}\right\}, \quad B_{C1} \triangleq \left\{a + j0 \middle| -\frac{1}{e} < a \leq 0\right\}$$

and

$$B_C = B_{C0} \cup B_{C1}.$$

The curve B_{C0} is the branch cut whose images linking the boundary of W_0 to W_1 and W_{-1} as depicted in Figure 2.2. Also the images of branch cuts B_{C1} and B_C adjoin W_1 to W_{-1} and W_2 branches, respectively. The other branches W_k, $k = \pm 2, \pm 3, \ldots, \pm\infty$ have their boundaries as the image of the branch cut B_C in both of the lower and upper sides. It is obvious that W_k maps the whole complex plane into the region between two images of consecutive branch cuts.

The derivative of the Lambert W function is computed as following. Since

$$W(z)e^{W(z)} = z$$

it follows that

$$W'(z)e^{W(z)} + W(z)e^{W(z)}W'(z) = 1$$

or equivalently,

$$W'(z) = e^{-W(z)}\frac{1}{W(z)+1}$$

which shows that the Lambert W function is not differentiable at $W(z) = -1$, i.e., $z = -1/e$. Hence this function is differentiable on the complex plane except at the branch cuts B_{C0} and B_C.

Lemma 2.4. *Let L be a curve which has no intersection to the branch cuts in the z-plane. The mapping W_k acting on the curve L is both continuous and bijective where $k = 0, \pm 1, \pm 2, \cdots, \pm\infty$.*

Proof. Since $L \cap B_C = \varnothing$, then $L \cap B_{C0} = \varnothing$ and $L \cap B_{C1} = \varnothing$. Then $W_k(L)$ is differentiable, and it is obvious that $W_k(L)$ is continuous.

Let $z_1,\ z_2 \in L$ where $z_1 = a_1 + jb_1,\ z_2 = a_2 + jb_2$. Suppose $W_k(z_1) = W_k(z_2) = u + jv$, then by the definition

$$\begin{cases} a_1 + jb_1 = W_k(z_1)e^{W_k(z_1)} = e^u[(u\ \cos\ v - v\ \sin\ v) + (v\ \cos\ v + u\ \sin\ v)j] \\ a_2 + jb_2 = W_k(z_2)e^{W_k(z_2)} = e^u[(u\ \cos\ v - v\ \sin\ v) + (v\ \cos\ v + u\ \sin\ v)j] \end{cases}$$

It is clear that

$$a_1 = e^u(u\ \cos\ v - v\ \sin\ v) = a_2, \quad b_1 = e^u(u\ \cos\ v - v\ \sin\ v) = b_2$$

which implies $z_1 = z_2$. Since $W_k(z)$ are onto, therefore W_k are bijective.

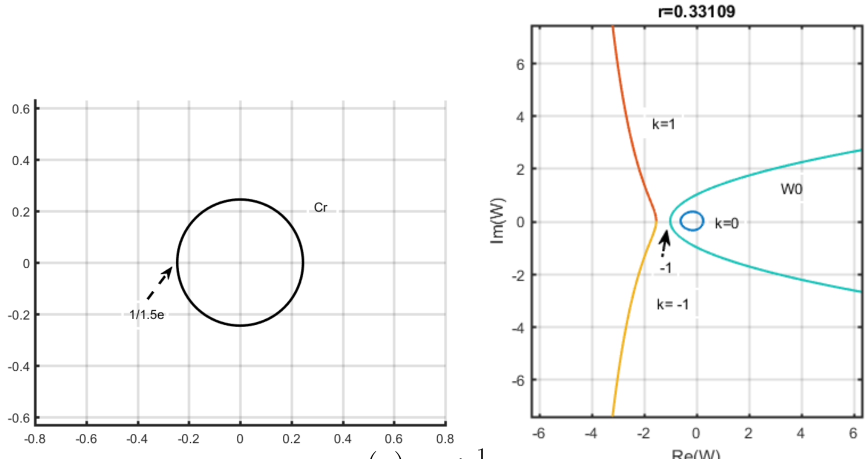
□

Lemma 2.5. [12, Lemma 2.3] *The following statements hold:*

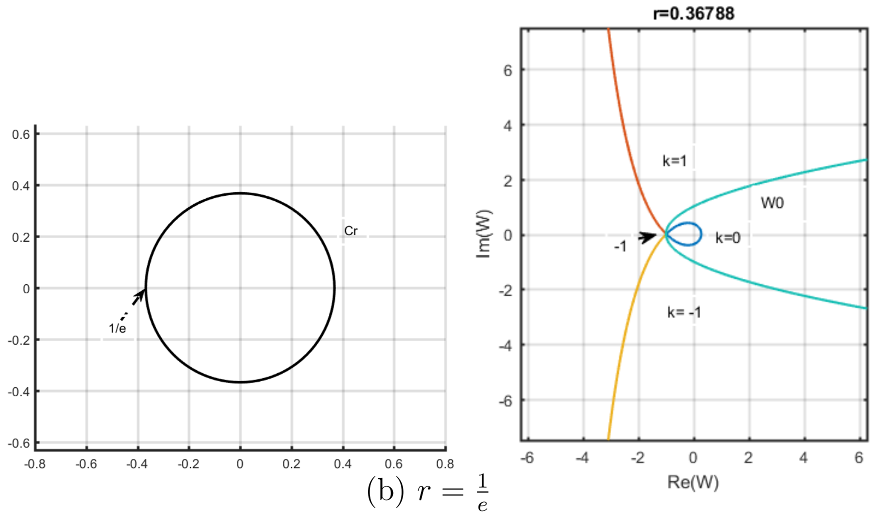
(i) *For* $z \notin B_{C0}$, $\max_{k=0,\pm1,\pm2,\cdots,\pm\infty} \mathrm{Re}[W_k(z)] = \mathrm{Re}[W_0(z)]$.

(ii) *For* $z \in B_{C0}$, $\max_{k=0,\pm1,\pm2,\cdots,\pm\infty} \mathrm{Re}[W_k(z)] = \mathrm{Re}[W_0(z)] = \mathrm{Re}[W_{-1}(z)]$.

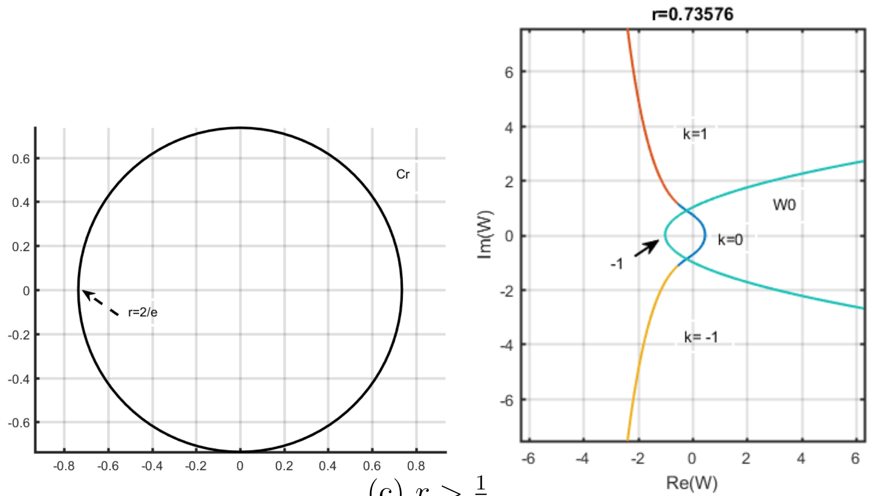
From Figure 2.3, Lemma 2.5 can be observed intuitively. Let $C_r = \{re^{j\theta}|\theta \in (-\pi, \pi]\}$ be a circle centered at origin with radius r in the z-plane. In Figure 2.3(a), the curve C_r with $r < \frac{1}{e}$ has no intersection with B_{C0}, then $W_0(C_r)$ is a closed curve separated from the other branches in the w-plane. In Figure 2.3(b), C_r with $r = \frac{1}{e}$ contacts B_{C0} at the point $z = -\frac{1}{e}$, then $W_0(C_r)$ is also a closed curve but connected to $W_1(C_r)$ and $W_{-1}(C_r)$. In Figure 2.3(c), C_r with $r > \frac{1}{e}$ intersects to B_{C0} at the point $z = -r$, then $W_0(C_r)$ is an open curve which connect to the image of $W_1(C_r)$ and $W_{-1}(C_r)$. For all the cases, it is explicit that the curve $W_0(C_r)$ is placed in the rightmost among all the branches. Let the intersection of C_r and B_{C0} be denoted by P. As shown in the cases of Figure 2.3 (b) and (c), $W_1(P)$ and $W_{-1}(P)$ connect to the upper and lower boundaries of W_0, respectively. These facts just illustrate Lemma 2.5.



(a) $r < \frac{1}{e}$



(b) $r = \frac{1}{e}$



(c) $r > \frac{1}{e}$

Figure 2.3: The left graph shows C_r in the z-plane with its image $W_k(C_r)$, $k = -1, 0, 1$, given in the right graph for (a) $r = 0.33109 < \frac{1}{e}$, (b) $r = \frac{1}{e}$, and (c) $r = 0.73576 > \frac{1}{e}$. The red color line denotes the image of W_1, the yellow color line for W_{-1}, and the blue color line for W_0.

Chapter 3

Stability Analysis

During recent decades, the stabilization of time delays systems of linear DDEs using feedback control has been studied extensively. Originating from the stability analysis of the scalar time-delay systems, we extend it to a system with feedback control and two delays. For the purpose of the robust stability conditions, the discussion of extreme point results are elucidated in this chapter.

3.1 Single Delay DDEs

The most significant use of the Lambert W function in time-delay systems is in the solutions of the characteristic equations, because its roots can be associated with a particular branch of the Lambert W function.

Consider a DDE with a single delay

$$\dot{x}(t) = \alpha x(t) + \beta x(t-h) \tag{3.1}$$

where $h > 0$. In order to solve this equation, we guess $x = e^{st}$ being a solution for some value of s, and then $\dot{x} = se^{st}$. Substituting these relations into (3.1) leads to

$$se^{st} = \alpha e^{st} + \beta e^{s(t-h)}$$

and since $e^{st} \neq 0$ for all s and t it follows that

$$s - \alpha = \beta e^{-hs}.$$

One can deduces some algebraic operations as following:

$$
\begin{aligned}
(s-\alpha)he^{(s-\alpha)h} &= \beta h e^{-\alpha h} \\
\Longrightarrow s-\alpha &= \frac{1}{h}W(\beta h e^{-\alpha h}) \\
\Longrightarrow s &= \alpha + \frac{1}{h}W(\beta h e^{-\alpha h})
\end{aligned}
$$

Since the Lambert W function has infinite many branches, thus the form as

$$
s_k = \alpha + \frac{1}{h}W_k(\beta h e^{-\alpha h}), \quad k = 0, \pm 1, \pm 2, \cdots, \pm\infty, \tag{3.2}
$$

is the solution of the equation which describes the characteristic spectrum of (3.1) expressing by the k-th branch of the Lambert W function. Furthermore, this time delay system is stable if and only if the roots of the characteristic equation, s_k, all lie in the complex open left half-plane by Definition 2.2. According to the explicit expression , Lemma 2.5 offers the stability condition for the DDE (3.1) as follow:

Lemma 3.1. [12] *The linear time delay system* (3.1) *is stable if and only if*

$$
S_W(\alpha,\beta,h) = \mathrm{Re}\left[\alpha + \frac{1}{h}W_0(\beta h e^{-\alpha h})\right] < 0. \tag{3.3}
$$

Example 3.1. Consider the differential equation with single delay

$$
\dot{x} = \alpha x(t) + \beta x(t-h)
$$

with the given data set $\alpha = -1$, $h = 1$ and various β, i.e., $\beta \in \{2, 1, -1\}$. We want to calculate the characteristic roots according to (3.2) such that the rightmost pole can then be used to check the stability of this delay equation. Table 3.1 shows that characteristic roots s_k with $k = 0, \pm 1, \pm 2$ corresponding to the various β. And characteristic roots s_k, $k = 0, \pm 1, \ldots, \pm 10$ with their rightmost pole corresponding to different value of β are also shown in Figure 3.1. It is obviously that, this delay equation is stable only for the case when $\beta = -1$ since $\mathrm{Re}(s_0) = -0.60502 < 0$. We can also conclude that the stability of this equation $\beta < 1$ for the fixed parameters $\alpha = -1$ and $h = 1$.

Table 3.1: Characteristic roots s_0, s_1 and s_2 corresponding to various β.

s_k	$\beta = 2$	$\beta = 1$	$\beta = -1$
s_2	$-1.70056 + 10.9316i$	$-2.39398 + 10.868i$	$-2.64736 + 14.0202i$
s_1	$-0.863549 + 4.74116i$	$-1.53209 + 4.59716i$	$-2.05283 + 7.71841i$
s_0	0.374823	0	$-0.605021 + 1.78819$
s_{-1}	$-0.863549 - 4.74116i$	$-1.53209 - 4.59716i$	$-0.605021 - 1.78819i$
s_{-2}	$-1.70056 - 10.9316i$	$-2.39398 - 10.868i$	$-2.05283 - 7.71841i$

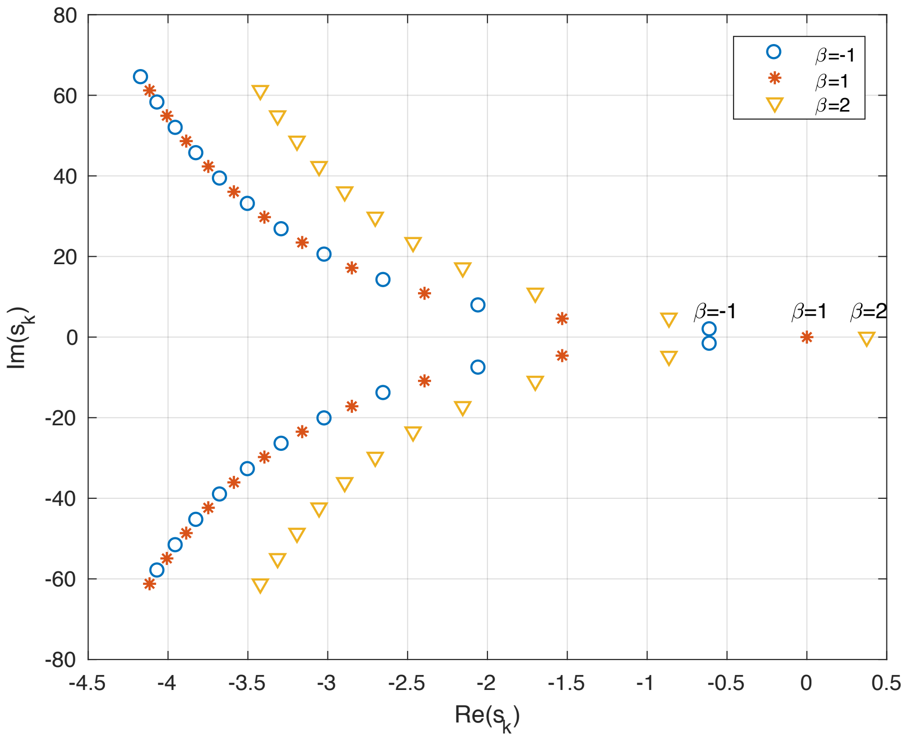


Figure 3.1: The pole distribution corresponding to different β.

3.2 DDE with Two Delays

In this section, we develop our serial expansion for the roots of the nonlinear characteristic equation arising from the DDE with two discrete delays.

Consider a DDE with two delay times h_1 and h_2 ($> h_1$) described by

$$\begin{aligned} y'(\varsigma) &= \alpha_1 y(\varsigma) + \beta_1 y(\varsigma - h_1) + \gamma_1 y(\varsigma - h_2) \\ y(0) &= y_0, \quad y(\tau) = \varphi(\tau), \quad -h_2 \leq \tau < 0, \end{aligned} \tag{3.4}$$

where $y'(\varsigma)$ denotes the derivative of $y(\varsigma)$ with respect to ς; α_1, β_1, and γ_1 are the corresponding coefficients of the equation. Also $x_0 \in \mathbb{R}$ and φ is the initial function to specify the initial condition for the delay state $y(\tau)$, $\tau \in [-h_2, 0)$. We re-scale the variable ς by letting $\varsigma = th_1$, $x(t) \triangleq y(th_1) = y(\varsigma)$, and $\phi(t) \triangleq \varphi(th_1) = \varphi(\varsigma)$, so that

$$\dot{x}(t) = \frac{dx(t)}{dt} = \frac{dy(\varsigma)}{dt} = y'(\varsigma)h_1$$

by using the Chain Rule. Here t is a dimensionless variable for ς which is generally regarded as the "time" variable of the differential equation. Based on these relationships, (3.4) becomes as

$$\dot{x}(t) = \alpha_1 h_1 x(t) + \beta_1 h_1 x(t-1) + \gamma_1 h_1 x(t - \frac{h_2}{h_1}). \tag{3.5}$$

Moreover, substitute $\alpha = \alpha_1 h_1$, $\beta = \beta_1 h_1$, $\gamma = \gamma_1 h_1$, $x_0 = y_0$, and $h = \frac{h_2}{h_1}$ (> 1), then the dimensionless form of the DDE (3.4) is expressed as

$$\begin{aligned} \dot{x}(t) &= \alpha x(t) + \beta x(t-1) + \gamma x(t-h), \\ x(0) &= x_0, \quad x(\tau) = \phi(\tau), \quad -h \leq \tau < 0. \end{aligned} \tag{3.6}$$

Thus the characteristic equation is then given by

$$s - \alpha - \beta e^{-s} - \gamma e^{-sh} = 0. \tag{3.7}$$

By using the Lambert W function approach for the single delay system, the solution, s_k, inside the k-th branch of the Lambert W function, is obtained as

$$\begin{aligned} s_k &= \alpha + W_k(\ \beta e^{-\alpha} + \gamma e^{-s_k(h-1)-\alpha}) \\ &= \alpha + W_k(\ \beta e^{-\alpha} + \gamma e^{-h\alpha} e^{-(s_k-\alpha)(h-1)}). \end{aligned} \tag{3.8}$$

When $h \in \mathbb{N}$ then there are h numbers of characteristic roots located inside the branch W_k, $k \neq 0$. Since the characteristic roots are complex pair, thus only the values of s_k for $k \geq 0$. Thus when $h = 2$ the characteristic roots are computed numerically as following:

1. Compute $s_{0,i}$ by using MATLAB `fsolve` command such that

$$s_{0,i} = \alpha + W_0(\ \beta e^{-\alpha} + \gamma e^{-s_{0,i}(h-1)-\alpha})$$

   with the initial guess

$$s_{0,i}^{(0)} = \alpha + W_k(\beta e^{-\alpha} + \gamma e^{-h*\alpha}) - j(i-1)\pi,$$

   and $i = 1, 2$. When $s_{0,1} \in \mathbb{R}$, then $s_{0,2} = s_{0,1}$; otherwise $s_{0,1}$ and $s_{0,2}$ form a complex pair.

2. Using MATLAB `fsolve` command to find $s_{k,i}$ with $k \neq 0$ and $i = 1, 2$ such that

$$s_{k,i} = \alpha + W_k(\ \beta e^{-\alpha} + \gamma e^{-s_{k,i}(h-1)-\alpha})$$

   The associated initial guess $s_{k,i}^{(0)}$ is constructed as following. When $s_{0,1}$ is real, then

$$s_{k,i}^{(0)} = \alpha + W_k(\beta e^{-\alpha} + \gamma e^{-h*\alpha}) - \begin{cases} j(2-i)\pi, & k > 0, \\ j(i-2)\pi, & k < 0; \end{cases}$$

   and when $s_{0,1}$ is complex, then

$$s_{k,i}^{(0)} = \alpha + W_k(\beta e^{-\alpha} + \gamma e^{-h*\alpha}) - \begin{cases} j(2-i)\pi, & k > 0, \\ j(i-1)2\pi, & k < 0, \end{cases}$$

   with $i = 1, 2$.

We note that if the value of $s_0 = s_{0,1}$ is real, then there is only one root for the principal branch W_0; otherwise it is a complex conjugate $s_{0,2} = \bar{s}_{0,1}$ with $s_{0,1}$ having positive imaginary part. Renumber the characteristic roots presented in (3.8) as S_n; for example, when $s_0 \in \mathbb{R}$,

$$\begin{cases} S_{2n-i} = s_{n,i}, & n > 0 \\ S_0 = s_0 & \\ S_{2n+2-i} = s_{n,i}, & n < 0, \end{cases}$$

otherwise,

$$\begin{cases} S_{2n-i} = s_{n,i}, & n > 0 \\ S_0 = s_{0,1}, S_{-1} = s_{0,2} & \\ S_{2n+1-i} = s_{n,i}, & n < 0. \end{cases}$$

Thus the solution of the scalar homogeneous DDE (3.6) is presented as

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{S_k t} \tag{3.9}$$

where C_k is computed such that the sum makes sense and satisfying the initial condition $x(0) = x_0$ and $x(\tau) = \phi(\tau)$ with $\tau \in [h, 0)$. Once the function $x(t)$ is computed, we then can obtain the solution of (3.6) as $y(\varsigma) = x(\varsigma/h_1)$, i.e.,

$$y(\varsigma) = \sum_{k=-\infty}^{\infty} C_k e^{(S_k/h_1)\varsigma}$$

which depicts that the characteristic roots of the original delay equation is obtained by scaling those corresponding roots of the dimensionless delay equation with a factor $1/h_1$.

Example 3.2. Consider the DDE in (3.6) whose characteristic roots are given by

$$s_{k,i} = \alpha + W_k(\ \beta e^{-\alpha} + \gamma e^{s_{k,i}(1-h)-\alpha}\ ), i = 1, 2. \tag{3.10}$$

When $k = 0$, if the characteristic root $s_0 \in \mathbb{R}$ then there is only one root, otherwise there are two roots for s_0 which also denoted by $s_{0,1}$ and $s_{0,2}$. Consider the following two parameter set:

(a) $\alpha = -1$, $\beta = -1$, $\gamma = -\frac{1}{2}$, and $h = 2$;

(b) $\alpha = -1$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{4}$, and $h = 2$,

and the `fsolve` command in Matlab is used to compute $s_{k,i}$ as the solutions of the equation (3.8). Tables 3.2 and 3.3 show that computation result for $s_{k,i}$ for these two parameter sets, respectively.

3.3 Extreme Point Results

Since the rightmost eigenvalues determine system stability, to determine it in the infinite eigenspectrum is important. However this is difficult, because one cannot be sure that the rightmost eigenvalue is included in a finite set.

For a scalar time-delay system, the root obtained using the principal branch ($k = 0$) always decides the stability of the system using monotinicity of the real part of the Lambert W function with respect to its other branch $k \neq 0$ [11]. In this section, we

Table 3.2: Characteristic root $s_{k,i}$ with $\alpha=-1$, $\beta=-1$, $\gamma=-\frac{1}{2}$, and $h=2$.

k	i	the initial guess	$S_n = s_{k,i}$
2	2	$-1.78811+14.0813i$	$S_3 = s_{2,2} = -1.50747+13.4657i$
2	1	$-1.78810+7.79810i$	$S_4 = s_{2,1} = -1.66427+10.0261i$
1	2	$-1.19980+7.82850i$	$S_1 = s_{1,2} = -1.14680+7.24010i$
1	1	$-1.19980+1.54530i$	$S_2 = s_{1,1} = -1.27050-3.64510i$
0	1	$0.03177+2.03920i$	$S_0 = s_{0,1} = -0.27495+1.47520i$
0	2	$0.03177-1.10240i$	$S_{-1} = s_{0,2} = -0.27495-1.47520i$
-1	1	$0.03177-2.03920i$	$S_{-3} = s_{-1,1} = -1.27050-3.64510i$
-1	2	$0.03177-8.32240i$	$S_{-2} = s_{-1,2} = -1.14680-7.24000i$
-2	1	$-1.19980-7.82850i$	$S_{-5} = s_{-2,1} = -1.66427-10.0261i$
-2	2	$-1.19979-14.11170i$	$S_{-4} = s_{-2,2} = -1.50747-13.4657i$

Table 3.3: Characteristic roots $s_{k,i}$ with $\alpha=-1$, $\beta=\frac{1}{2}$, $\gamma=\frac{1}{4}$, and $h=2$.

k	i	the initial guess	$S_n = s_{k,i}$
2	2	$-2.22840+10.8832i$	$S_3 = s_{2,2} = -1.82137+11.6390i$
2	1	$-2.22840+4.60000i$	$S_4 = s_{2,1} = -1.89200+8.71330i$
1	2	$-1.37110+4.63240i$	$S_1 = s_{1,2} = -1.37970+5.30450i$
1	1	$-1.37110-1.65070i$	$S_2 = s_{1,1} = -1.36930+2.51760i$
0	-	0.084256	$S_0 = s_0 = -0.11929$
-1	1	$-1.37110+1.65070i$	$S_{-3} = s_{-1,1} = -1.36930-2.51760i$
-1	2	$-1.37110-4.63240i$	$S_{-2} = s_{-1,2} = -1.37970-5.30450i$
-2	1	$-2.22840-4.60000i$	$S_{-5} = s_{-2,1} = -1.89200-8.71330i$
-2	2	$-2.22840-10.8832i$	$S_{-4} = s_{-2,2} = -1.82137-11.6390i$

expound where the eigenvalues of time-delay systems are maximized in order to associated with Lemma 3.1.

Consider the scalar time-delay system with two delays

$$\dot{x}(t)=\alpha x(t)+\beta x(t-1)+\gamma x(t-h)$$

which is induced from (3.4) and described in (3.5). Let $\alpha=c+jd$, $\beta=r_1e^{j\theta_1}$ and $\gamma=r_2e^{j\theta_2}$ with c, d , r_1, r_2, θ_1, and $\theta_2\in\mathbb{R}$. Also $x(t)\in\mathbb{C}$ and $h>0$ is fixed.

We follow the idea proposed by Shinozaki [12] to discuss the robust behavior of Lambert W function. Define Ω^α, Ω^β and Ω^γ as

$$\begin{aligned}
\Omega^\alpha &\triangleq \left\{c+jd \,\middle|\, c\in[\underline{c},\bar{c}], d\in[\underline{d},\bar{d}]\right\},\\
\Omega^\beta &\triangleq \left\{r_1e^{j\theta_1} \,\middle|\, r_1\in[\underline{r_1},\overline{r_1}], \theta_1\in[\underline{\theta_1},\overline{\theta_1}]\right\},\\
\Omega^\gamma &\triangleq \left\{r_2e^{j\theta_2} \,\middle|\, r_2\in[\underline{r_2},\overline{r_2}], \theta_2\in[\underline{\theta_2},\overline{\theta_2}]\right\},
\end{aligned}$$

where $\underline{c}\le\bar{c}$, $\underline{d}\le\bar{d}$ and $0\le\underline{r_1}\le\overline{r_1}$, $0\le\underline{r_2}\le\overline{r_2}$, $\underline{\theta_1}\le\overline{\theta_1}$, $\underline{\theta_2}\le\overline{\theta_2}$. Suppose that the time-delay system (3.6) has uncertainties prescribed by

$$\alpha \in \Omega^{\alpha}, \beta \in \Omega^{\beta}, \gamma \in \Omega^{\gamma}, h \in [\underline{h}, \overline{h}], \tag{3.11}$$

with $1 < \underline{h} \leq \overline{h} \in \mathbb{R}$. Based on the concept proposed by Hiroshi Shinozaki, 2007 [12], we obtain the stability condition as following:

Lemma 3.2. *The linear scalar time-delay system* (3.4) *with the uncertainties prescribed by* (3.9) *is robustly stable if and only if*

$$\max_{\alpha \in \Omega^{\alpha}, \beta \in \Omega^{\beta}, \gamma \in \Omega^{\gamma}, h \in [\underline{h}, \bar{h}]} S_W(\alpha, \beta, \gamma, h) < 0 \tag{3.12}$$

where

$$S_W(\alpha, \beta, \gamma, h) = \mathrm{Re}(S_0) = \mathrm{Re}[\alpha + W_0(\beta e^{-\alpha} + \gamma e^{-S_0(h-1)-\alpha})]$$

In order to discuss the monotonicity of S_W, we refer to the differentiability condition first.

Lemma 3.3. $W_k(z)$ is analytic in $B^c_{C0} = \{a + j0 | -\frac{1}{e} < a < \infty\}$ where $k = 0, \pm 1, \pm 2, \cdots, \pm\infty$.

Proof. Let $Z(w) = we^w$, then $Z(w)$ is an analytic function and

$$\frac{dZ(w)}{dw} = we^w + e^w = (1 + w)e^w \neq 0 \text{ for } w \neq -1.$$

Therefore, there is an analytic inverse function of $Z(w)$ in a neighborhood of $w \neq -1$. Then $W_k(z)$ is the inverse function of $Z(w)$ in $W_k(B^c_{C0})$ for $k = \pm 1, \pm 2, \ldots, \pm\infty$. □

In the following paragraphs, we discuss the monotonicity condition with respect to $c = \mathrm{Re}(\alpha)$, i.e., the other parameters d, r_1, r_2, θ_1, and θ_2 are kept fixed. Although the case for single delay is studied by Hiroshi Shinozaki, 2007 [12], we present an different approach.

Next, we turn to discuss the effect of β and γ, i.e., r_1 and r_2, respectively. Let $z = a + jb, w = u + jv$, and setting to the definition $z = we^w$, then

$$a = e^u(u \cos\ v - v \sin\ v), b = e^u(v \cos\ v + u \sin\ v) \tag{3.13}$$

In order to identify the rightmost point of $S_W(\alpha, \beta, \gamma, h)$ with respect to r_1, r_2, consider W_0, the image of a line segment $S_g := \{p(x_0 + jy_0) | p \in [\ \underline{p}, \bar{p}]\}$, where $x_0, y_0 \in \mathbb{R}$. Then

$$a = x_0 p = e^u(u \cos v - v \sin v), b = y_0 p = e^u(v \cos v + u \sin v).$$

Suppose that $x_0 \neq 0$ and $y_0 \neq 0$ for

$$a = x_0 p = e^u(u\cos v - v\sin v), b = y_0 p = e^u(v\cos v + u\sin v).$$

Then

$$\frac{x_0}{y_0} = \frac{u\cos\ v - v\sin v}{v\cos v + u\sin v}$$

which implies

$$\begin{aligned} u &= v \cdot \frac{x_0\cos v + y_0\sin v}{y_0\cos v - x_0\sin v} \\ &= v \cdot \frac{x_0 + y_0\tan v}{y_0 - x_0\tan v}, \quad v \in (\underline{v}, \bar{v}) \end{aligned} \tag{3.14}$$

where $\underline{v} = \tan^{-1}(\frac{y_0}{x_0}) - \pi$ and $\bar{v} = \tan^{-1}(\frac{y_0}{x_0})$ for $\tan^{-1}(\cdot) \in (0, \pi]$.

Differentiating u with respect to v, we have

$$\frac{du}{dv} = v \cdot \frac{(x_0^2 + y_0^2)(1 + \tan^2 v)}{(y_0 - x_0\tan\ v)^2} + \frac{y_0\tan\ v + x_0}{y_0 - x_0\tan\ v}$$

and

$$\frac{du^2}{d^2v} = \frac{2(x_0^2 + y_0^2)(1 + \tan^2 v)}{(y_0 - x_0\tan v)^2}(u + 1)$$

Since $v = \tan^{-1}(\frac{y_0}{x_0+\varepsilon}) - \pi$ and $\tan^{-1}(\frac{y_0}{x_0-\varepsilon})$ with $\varepsilon > 0$, so $\frac{du^2}{d^2v} > 0$ for $u > -1$, and then $\frac{du}{dv}$ is monotone increasing for $[\underline{v}, \bar{v}\ ]$. By Intermediate Value Theorem, there exist a $v \in [\underline{v}, \bar{v}\ ]$ such that $\frac{du}{dv} = 0$. Hence, the graph of (3.11) is leftward convex.

Let $\underline{z} = \underline{p}(x_0 + jy_0)$ and $\bar{z} = \bar{p}(x_0 + jy_0)$ are points on the segment S_g, and suppose that max $\mathrm{Re}[W_0(Sg)]$ is taken at a point between $\underline{z}$ and $\bar{z}$. Then $W_0(z_0)$ lies on more right than $W_0(\underline{z})$ and $W_0(\bar{z})$ in the w-plane. However, the curve $W_0(S_g)$ is continuous and bijective, and the leftward convexity implies that the graph is concave rightward, then the curve $W_0(S_g)$ must be overlapped in some interval, a contradition to the fact that the mapping is bijection. Hence, the maximal values of $\mathrm{Re}[W_0(S_g)]$ must be taken at $\underline{z}$ or $\bar{z}$ (see Figure 3.2).

Secondly, if $x_0 = 0$ and $y_0 \neq 0$, it is the extreme cases of case(1). Figure (3.3) shows that it leads to the same result with similar argument.

Consider the last case $x_0 \neq 0, y_0 = 0$, as show in Figure 3.4, there exist one singularity at $W_0(-\frac{1}{e})$, and $W_0(S_g)$ is a continuous curve and bijective.

Finally, we transform to discuss d and θ_1, θ_2, the image of α, β and γ, respectively.

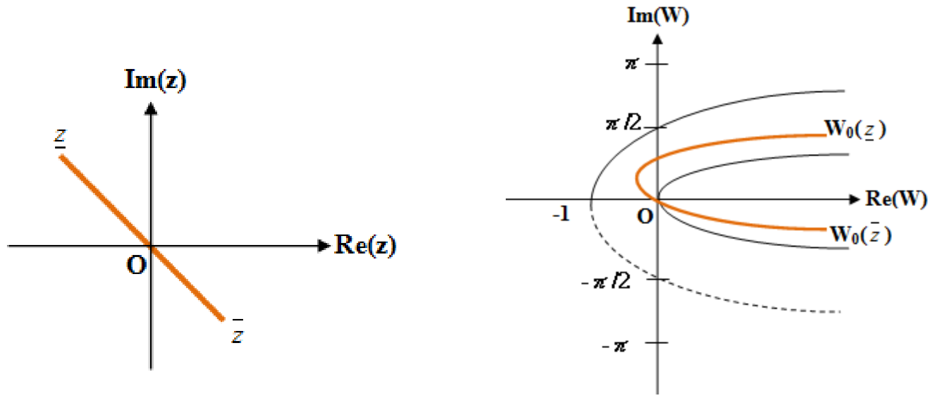


Figure 3.2: The mapping of S_g by W_0 if $x_0 \neq 0$ and $y_0 \neq 0$.

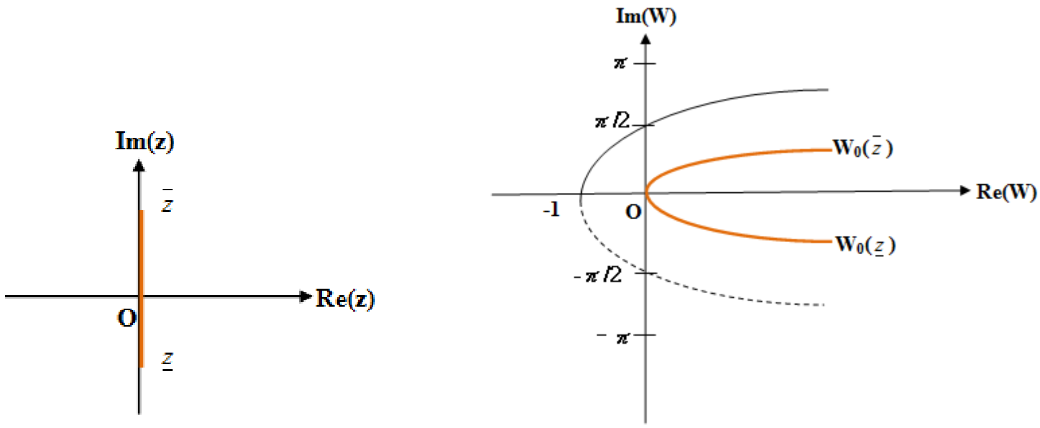


Figure 3.3: The mapping of S_g by W_0 if $x_0 = 0$ and $y_0 \neq 0$.

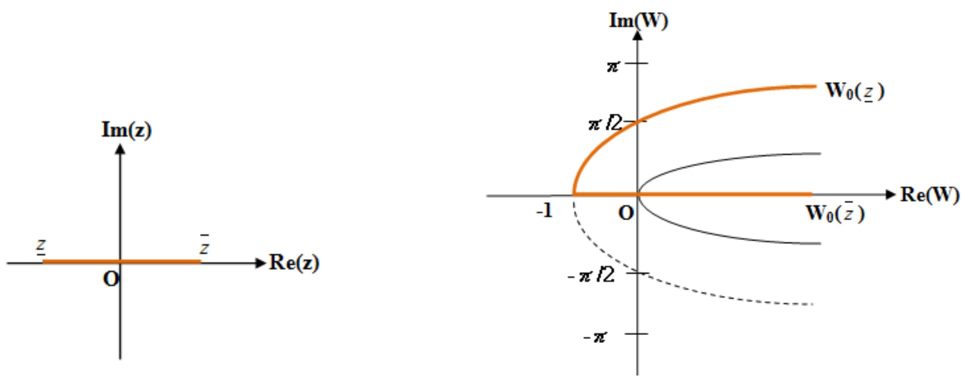


Figure 3.4: The mapping of S_g by W_0 if $x_0 \neq 0$ and $y_0 = 0$.

Lemma 3.4. [12] $\mathrm{Re}[W_0(re^{j\theta})]$ *is a monotone increasing function in* $\theta \in (-\pi, 0]$ *and a monotone decreasing function in* $\theta \in [0, \pi]$. $\mathrm{Re}[W_k(re^{j\theta})]$, $k = -1, \cdots, -\infty$ *are monotone increasing function in* $\theta \in (-\pi, \pi]$. $Re[W_k(re^{j\theta})], k = 1, \cdots, \infty$ *are monotone decreasing function in* $\theta \in (-\pi, \pi]$

Before discuss the monotonicity of the function S_W dependence on the parameters α, β and γ. We need to calculate the partial derivative of the characteristic roots with respect to these parameters. Since the characteristic equation is given by

$$s - \alpha - \beta e^{-s} - \gamma e^{-sh} = 0$$

then the derivative to α is derived as follows:

$$\frac{\partial s}{\partial \alpha} - 1 + \beta e^{-s} \frac{\partial s}{\partial \alpha} + h\gamma e^{-sh} \frac{\partial s}{\partial \alpha} = 0$$

or equivalently,

$$\frac{\partial s}{\partial \alpha} = \frac{1}{1 + \beta e^{-s} + h\gamma e^{-sh}}.$$

Similarly,

$$\frac{\partial s}{\partial \beta} = \frac{e^{-s}}{1 + \beta e^{-s} + h\gamma e^{-sh}},$$
$$\frac{\partial s}{\partial \gamma} = \frac{e^{-sh}}{1 + \beta e^{-s} + h\gamma e^{-sh}}.$$

We note that the following relationship holds:

$$\frac{\partial s}{\partial \alpha} + \beta \frac{\partial s}{\partial \beta} + h\gamma \frac{\partial s}{\partial \gamma} = 1$$

whenever

$$1 + \beta e^{-s} + h\gamma e^{-sh} \neq 0.$$

Also by substituting βe^{-s} with $s - \alpha - \gamma e^{-sh}$, these three derivatives become

$$\begin{aligned} \frac{\partial s}{\partial \alpha} &= \frac{1}{1 + s - \alpha + (h-1)\gamma e^{-sh}} \\ \frac{\partial s}{\partial \beta} &= \frac{e^{-s}}{1 + s - \alpha + (h-1)\gamma e^{-sh}} \\ \frac{\partial s}{\partial \gamma} &= \frac{e^{-sh}}{1 + s - \alpha + (h-1)\gamma e^{-sh}} \end{aligned} \tag{3.15}$$

It is then obviously that the characteristic root s is not differential with respect to parameters α, β and γ at those s such that

$$1 + \beta e^{-s} + h\gamma e^{-sh} = 0$$

but these s must also satisfy

$$s - \alpha - \beta e^{-s} - \gamma e^{-sh} = 0.$$

It turns out that

$$\begin{cases} -(1 + h\gamma e^{-sh}) & = \beta e^{-s} \\ s - \alpha - \gamma e^{-sh} & = \beta e^{-s} \end{cases}$$

i.e.,

$$\begin{aligned} & s - \alpha - \gamma e^{-sh} = -(1 + h\gamma e^{-sh}) \\ \Longrightarrow\ & s - \alpha + 1 = -(h-1)\gamma e^{-sh} \\ \Longrightarrow\ & (s - \alpha + 1)\, h e^{(s-\alpha+1)h} = -h(h-1)\gamma e^{-(\alpha-1)h} \end{aligned}$$

Hence these non-differentiable points for S_0 are given by

$$S_0^* = \alpha - 1 + \frac{1}{h} W\left(-h(h-1)\gamma e^{-(\alpha-1)h}\right). \tag{3.16}$$

Alternatively, we can replace γe^{-sh} instead of βe^{-s}, and after some algebraic operation we obtain another form for non-differentiable s_0:

$$S_0^* = \alpha - \frac{1}{h} + W\left((1 - \frac{1}{h})\beta e^{-(\alpha - \frac{1}{h})}\right).$$

For simplicity, let $z = \beta e^{-\alpha} + \gamma e^{-S_0(h-1)-\alpha}$ and $z^* \triangleq \beta e^{-\alpha} + \gamma e^{-S_0^*(h-1)-\alpha}$, and then

$$W_0(z)e^{W_0(z)} = z, \quad S_0 = \alpha + W_0(z).$$

Suppose $z \neq z^*$, i.e. S_0 is differentiable. Let

$$S_0 - \alpha = W_0(z) = u + jv, \quad u > -1$$

or equivalently,

$$S_0 = (u + c) + j(v + d) \neq S_0^*,$$

and since S_0 must satisfy the implicit equation

$$S_0 - \alpha = \beta e^{-S_0} + \gamma e^{-S_0 h}$$

we have

$$u + jv = r_1 e^{j\theta_1} e^{-(u+c)-j(v+d))} + r_2 e^{j\theta_2} e^{-h(u+c)-jh(v+d)}$$

that is,

$$\begin{aligned} u + jv = {} & r_1 e^{-(u+c)}[\cos(\theta_1 - (v+d)) + j\sin(\theta_1 - (v+d))] \\ & + r_2 e^{-h(u+c)}[\cos(\theta_2 - h(v+d)) + j\sin(\theta_2 - h(v+d))] \end{aligned}$$

or equivalently,

$$\begin{cases} u & = r_1 e^{-(u+c)}\cos(\theta_1 - (v+d)) + r_2 e^{-h(u+c)}\cos(\theta_2 - h(v+d)), \\ v & = r_1 e^{-(u+c)}\sin(\theta_1 - (v+d)) + r_2 e^{-h(u+c)}\sin(\theta_2 - h(v+d)). \end{cases} \tag{3.17}$$

We observe that

$$\gamma e^{-S_0 h} = r_2 e^{-h(u+c)}[\cos(\theta_2 - h(v+d)) + j\sin(\theta_2 - h(v+d))]$$

and hence

$$\begin{aligned} & 1 + S_0 - \alpha + (h-1)\gamma e^{-S_0 h} \\ & = \left[1 + u + r_2 e^{-h(u+c)}\cos(\theta_2 - h(v+d))\right] + j\left[v + r_2 e^{-h(u+c)}\sin(\theta_2 - h(v+d))\right] \end{aligned}$$

In advance to discuss the effect of c with other parameters are fixed, we need to know the partial derivative of S_W w.r.t. c, i.e.,

$$\begin{aligned} \frac{dS_0}{dc} & = \frac{dS_0}{d\alpha}\frac{d\alpha}{dc} = \frac{dS_0}{d\alpha} \\ & = \frac{1}{[1 + u + r_2 e^{-h(u+c)}\cos(\theta_2 - h(v+d))] + j[v + r_2 e^{-h(u+c)}\sin(\theta_2 - h(v+d))]} \end{aligned}$$

and then

$$\begin{aligned} \frac{dS_W}{dc} & = \mathrm{Re}\left(\frac{dS_0}{dc}\right) \\ & = \frac{1 + u + r_2 e^{-h(u+c)}\cos(\theta_2 - h(v+d))}{[1 + u + r_2 e^{-h(u+c)}\cos(\theta_2 - h(v+d))]^2 + [v + r_2 e^{-h(u+c)}\sin(\theta_2 - h(v+d))]^2}. \end{aligned} \tag{3.18}$$

Lemma 3.5. *The following properties of $S_W(\alpha, \beta, \gamma, h)$ with respect to the parameter c hold:*

1. *When $\gamma = 0$, i.e., a single delay system, $S_W(\alpha, \beta, h)$ is a monotone increasing function of c*

2. *When $\gamma \neq 0$, $S_W(\alpha, \beta, \gamma, h)$ is a monotone increasing function of c in the large.*

Proof. Since the partial derivative of S_W w.r.t. c is given by (3.18), then we discuss the monotocity of S_W as following:

1. Suppose $\gamma = 0$ and $\beta = 0$, $S_W(\alpha, \beta, \gamma, h) = S_W(\alpha, h) = \alpha = c + jd$ is an increasing function of c.

   Assume $\gamma = 0$ and $\beta \neq 0$, we have from (3.16) that

   $$S_0^* = \alpha - 1$$

   and this S_0^* must also satisfy (3.7), i.e.,

   $$S_0^* - \alpha = \beta e^{-S_0^*}$$

   thus the corresponding z^* becomes

   $$z^* = \beta e^{-\alpha} = \beta e^{-S_0^*} e^{S_0^* - \alpha} = -\frac{1}{e}.$$

   When $z \neq z^* = -1/e$, the substitution of $r_2 = 0$ into (3.15) leads to

   $$\frac{dS_W}{dc} = \frac{1 + u}{[1 + u]^2 + v^2}$$

   which is positive (since $u > -1$). Since S_0 is a continuous function, i.e., $\frac{dS_W}{dc}$ at the both sides of z^* are also positive by limiting processing. Hence $S_W(\alpha, \beta, h)$ is an increasing function of c.

2. Suppose that $\gamma \neq 0$. Since $z \neq z^*$ (or $S_0 \neq S^*$) then the denominator of $\frac{dS_W}{dc}$, i.e., $1 + S_0 - \alpha + (h-1)\gamma e^{-S_0 h}$, is not equal to zero. From (3.15), it can then be discussed according to the positiveness of the real part of $\frac{dS_W}{dc}$ into the following three cases:

   - Assume $1 + u + r_2 e^{-h(u+c)} \cos(\theta_2 - h(v + d)) > 0$. Note that $1 + u > 0$ holds. When c increasing, then e^{-hc} is decreasing and hence this term remains positive. Hence S_W is an increasing function of c in this case.

- Assume $1 + u + r_2 e^{-h(u+c)} \cos(\theta_2 - h(v + d)) = 0$ and $v + r_2 e^{-h(u+c)} \sin(\theta_2 - h(v + d)) \neq 0$. When c increasing, then e^{-hc} is decreasing and hence this term will becomes positive. Hence S_W is increasing when c increases.
- Assume $1 + u + r_2 e^{-h(u+c)} \cos(\theta_2 - h(v + d)) < 0$. When c increasing, then e^{-hc} is decreasing and hence this term remains increasing. And it becomes positive eventually. Hence S_W is initially decreasing but increasing when c increases. In this case S_W has a global min w.r.t. c.

Therefore, $S(\alpha, \beta, \gamma, h)$ is not necessary a monotone increasing function of c. Consider the smallest value of $1 + u + r_2 e^{-h(u+c)} \cos(\theta_2 - h(v + d))$, i.e.

$$1 + u - r_2 e^{-h(u+c^*)} = 0 \implies c^* = -u + \frac{1}{h} \ln \frac{r_2}{1+u}$$

And when c is large enough (for example $c \geq c^*$) , it becomes an increasing function of c. Figure (3.3) depicts the result with different d.

$\square$

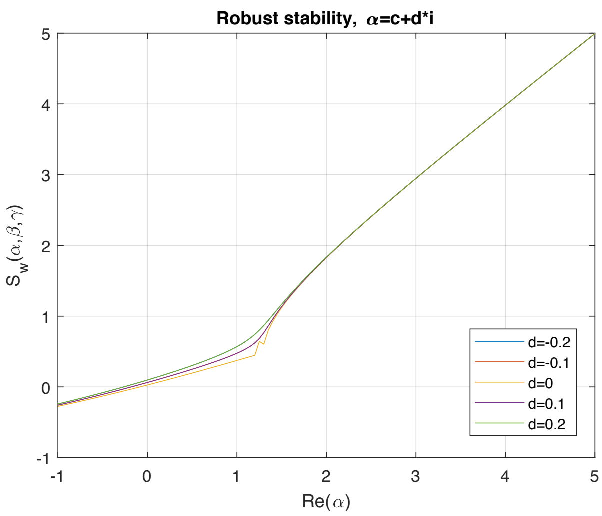


Figure 3.5: Robust stability with $\alpha = c + d^\star j$

The following lemma illustrates where $S_W(\alpha, \beta, \gamma, h)$ is maximized with respect to d, θ_1 and θ_2, respectively.

Lemma 3.6. *Let c , r_1 and r_2 be constant and define*

$$C^{\alpha\beta\gamma} := \{e^{j(\theta_1 - d)} + e^{j[\theta_2 - s_k(h-1) - d]} | d \in [\underline{d} , \bar{d}], \theta_1 \in [\underline{\theta_1} , \overline{\theta_1}], \theta_2 \in [\underline{\theta_2} , \overline{\theta_2} \} \tag{3.19}$$

(1)If $C^{\alpha\beta\gamma}$ crosses the positive real axis, then

$$\max S_W(\alpha,\beta,\gamma) = S_W(c, r_1, r_2)$$

(2)If $C^{\alpha\beta\gamma}$ does not cross the positive real axis, then

$$\max S_W(\alpha,\beta,\gamma) = \max\{S_W(c+j\underline{d}, r_1 e^{j\bar{\theta}_1}, r_2 e^{j\overline{\theta_2}}), S_W(c+j\bar{d}, r_1 e^{j\underline{\theta_1}}, r_2 e^{j\underline{\theta_2}})\}$$

Proof. Base on r_1, r_2, it divides into the following cases:

Case I: If $r_1 = r_2 = 0$, it is obvious since $S_W(\alpha,\beta,\gamma) = S_W(c, 0) = c$.

Case II: Let $r_1, r_2 > 0$. Define

$$C_z^{\alpha\beta\gamma} := \{z = \beta e^{-\alpha} + \gamma e^{-s_k(h-1)-\alpha} \mid d \in [\ \underline{d}, \bar{d}\ ], \theta_1 \in [\ \underline{\theta_1}, \overline{\theta_2}], \theta_2 \in [\ \underline{\theta_2}, \overline{\theta_2}]\}$$

, so that $C^{\alpha\beta\gamma}$ is the argument of $C_z^{\alpha\beta\gamma}$.

(1) $C^{\alpha\beta\gamma}$ crosses the positive real axis.

Since $C_z^{\alpha\beta\gamma}$ is a cylinder, then

$$\max_{d\in[\underline{d},\bar{d}], \theta_1\in[\underline{\theta_1},\overline{\theta_1}], \theta_2\in[\underline{\theta_2},\overline{\theta_2}]} S_W(\alpha,\beta,\gamma) = S_W(c, r_1, r_2)$$

is the crucial point of $C^{\alpha\beta\gamma}$ by Lamma (3.5).

(2) $C^{\alpha\beta\gamma}$ does not cross the positive real axis.

Obviously, by the monotone, $C_z^{\alpha\beta\gamma}$ is an arc in this case, and it has two possible extreme points corresponding to $\{\underline{d}, \overline{\theta_1}, \overline{\theta_2}\}$ and $\{\bar{d}, \underline{\theta_1}, \underline{\theta_2}\}$. This case holds.

□

Figure 3.5 shows the robust stability with respect to β^θ and γ^θ, i.e., θ_1, θ_2, respectively. Here depicts the non-differentiable points for S_0.

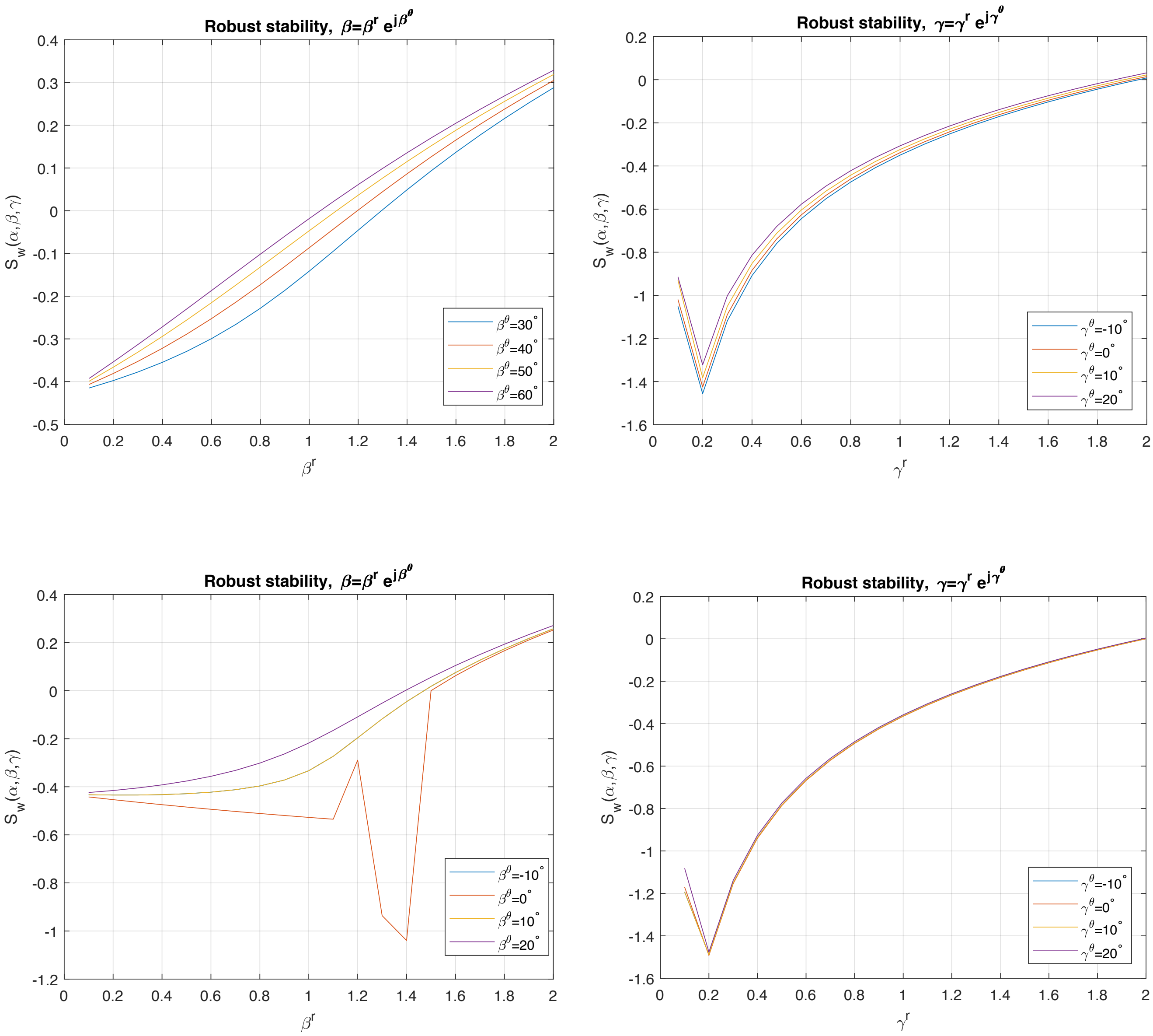


Figure 3.6: Robust stability with respect to β^θ and γ^θ .

Chapter 4

Pole Assignment for the Time-Delay System

The approach to assign eigenvalue using the Lambert W function is used to design robust linear feedback control law. In this chapter, we will discuss the pole assignment with single delay and two delays, respectively, in the framework of the Lambert W function approach. The first section discuss the system with single delay and two delays under the action of linear feedback control law. The prescribed eigenvalue assignments are discuss in §4.2. Conversely, we use the `fsolve` command in Matlab to get the desired poles in the §4.2, and expand it to deal with two delays in §4.3.

4.1 Single Delay Systems

We here consider a scalar delay system with an exogenous input from environment:

$$\begin{aligned} \dot{x} &= ax(t) + a_{1d}x(t-h) + u(t), \quad h > 0, \\ x(0) &= x_0, \quad x(\tau) = \phi(\tau), \quad -h \le \tau < 0, \end{aligned} \tag{4.1}$$

where $x_0,\ h \in \mathbb{R}$, and ϕ is the initial function to specify the initial condition for the delay state $x(\tau)$, $\tau \in [-h, 0)$. Suppose a proportional control is proposed to drive this delay system to a desired state, i.e., a constant state feedback is applied to this system

$$u = kx(t) + k_{1d}x(t-h)$$

where $k,\ k_{1d} \in \mathbb{R}$ are the designed parameters for the state feedback law.

The closed-loop system is then given by

$$\begin{aligned} \dot{x} &= (a+k)x(t) + (a_{1d} + k_{1d})x(t-h) \\ &\triangleq \alpha x(t) + \beta x(t-h) \end{aligned}$$

where $\alpha = a + k$ and $\beta = a_{1d} + k_{1d}$. Thus the closed-loop system is obvious in the form of (3.1), and the associated roots (or *poles*, s_k, $k \in \mathbb{Z}$) of the closed-loop system are then defined by (3.2):

$$s_k = \alpha + \frac{1}{h} W_k(\beta h e^{-\alpha h}),$$

with $k = 0, \pm 1, \pm 2, \cdots, \pm\infty$. By using Lemma 3.1, the closed-loop system is stable if the feedback parameters k and k_{1d} are designed such that

$$S_W(a + k, a_{1d} + k_{1d}, h) = Re\left[a + k + \frac{1}{h} W_0\left((a_{1d} + k_{1d})he^{-(a+k)h}\right)\right] < 0. \tag{4.2}$$

4.2 Pole Assignment for Single Delay Systems

As see in §4.1, the roots of the single delay system

$$\dot{x} = \alpha x(t) + \beta x(t - h)$$

are expressed as

$$s_k = \alpha + \frac{1}{h} W_k(\beta h e^{-\alpha h}), \quad k = 0, \pm 1, \pm 2, \cdots, \pm\infty,$$

Furthermore, the root

$$s_0 = \alpha + \frac{1}{h} W_0(\beta h e^{-\alpha h}) \tag{4.3}$$

is always in the rightmost of all the roots by Lemma (2.5). Suppose the desired pole is $s_{0,des} \in \mathbb{C}$, we insert it into s_k, so that the problem is to solve (4.3) for α and β such that the rightmost root $s_0 = s_{0,des}$. Two different approaches are proposed. The first approach is to determine α first and then compute β accordingly. From (4.3) it follows that

$$\alpha = s_0 - \frac{1}{h} W_0(\beta h e^{-\alpha h}),$$

and set $z_W = \beta h e^{-\alpha h}$ to define

$$W_0^\alpha \triangleq \left\{ s_{0,des} - \frac{1}{h} W_0(z_W) \middle| z_W \in \mathbb{C} \right\}. \tag{4.4}$$

The region W_0^α is as shown in Figure 4.1. Select an $\alpha \in W_0^\alpha \cap \mathbb{R}$ and let

$$\beta = (s_{0,des} - \alpha)e^{hs_{0,des}} \tag{4.5}$$

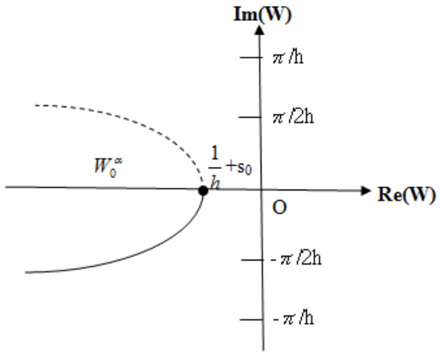


Figure 4.1: Region of W_0^α except for the dashed curves.

that is,

$$h(s_{0,des} - \alpha)e^{h(s_{0,des}-\alpha)} = \beta h e^{-h\alpha}$$

or equivalently,

$$s_{0,des} - \alpha = \frac{1}{h} W_0(\beta h e^{-h\alpha}),$$

then comparing with (4.3) it follows that $s_{0,des}$ must be the rightmost root s_0 of the characteristic equation. We also note that the corresponding value z_W for the selection of the real parameter α must be equal to $\beta h e^{-h\alpha}$, i.e.,

$$z_W = \beta h e^{-h\alpha} = (s_{0,des} - \alpha)e^{hs_{0,des}},$$

which means $z_W \in \mathbb{R}$ to obtain another real parameter β, and thus

$$\beta = \frac{1}{h} z_W e^{h\alpha}.$$

Therefore if there is no $z_W \in \mathbb{R}$ such that $\alpha \in W_0^\alpha \cap \mathbb{R}$ means there is no real parameters α and β such that the rightmost eigenvalue s_0 is assigned to $s_{0,des} \in \mathbb{C}$. Furthermore, W_0 has the range $Re[W_0(z)] \geqslant -1$ so that $Re(s_{0,des}) \leqslant \frac{1}{h} + \alpha$.

Conversely, the second approach is to determine β first, and then to compute the corresponding α. The following two relationships

$$\begin{cases} s_0 = \alpha + \frac{1}{h} W_0(\beta h e^{-\alpha h}) \\ \beta = (s_0 - \alpha)e^{hs_0} \end{cases}$$

induce to

$$\frac{1}{h} W_0(\beta h e^{-\alpha h}) = s_0 - \alpha = \beta e^{-hs_0},$$

so that

$$\beta = \frac{e^{hs_0}}{h} W_0(\beta h e^{-\alpha h}).$$

Define the set

$$\beta \in W_0^\beta \triangleq \left\{ \frac{e^{hs_{0,des}}}{h} W_0(z_W) \middle| z_W \in \mathbb{C} \right\}, \tag{4.6}$$

and hence $\beta \in W_0^\beta \cap \mathbb{R}$. Once β is determined with a selected $z_W = \beta h e^{-h\alpha}$:

$$\beta = \frac{e^{hs_{0,des}}}{h} W_0(z_W)$$

then we can deduce α by the definition of the Lambert W function as follow:

$$\begin{aligned}
& W_0(z_W) = \beta h e^{-hs_{0,des}} \\
\Longrightarrow\ & \beta h e^{-hs_{0,des}} e^{\beta h e^{-hs_{0,des}}} = z_W = \beta h e^{-h\alpha} \\
\Longrightarrow\ & e^{-h\alpha} = e^{-hs_{0,des}} e^{\beta h e^{-hs_{0,des}}} = e^{-hs_{0,des} + \beta h e^{-hs_{0,des}}} \\
\Longrightarrow\ & -h\alpha = -hs_{0,des} + \beta h e^{-hs_{0,des}} + j2\pi k, k \in \mathbb{Z}
\end{aligned}$$

Thus

$$\alpha = s_{0,des} - \beta e^{-hs_{0,des}} + j\frac{2\pi k}{h}, k \in \mathbb{Z} \tag{4.7}$$

But from the original characteristic equation we know

$$s_0 - \alpha = \beta e^{-hs_0}$$

hence $k = 0$ is necessary in (4.7) if β is given as in (4.6), otherwise α deviates from W_0^α. Thus, α ought to be as

$$\alpha = s_{0,des} - \beta e^{-hs_{0,des}} \tag{4.8}$$

or from $z_W = \beta h e^{-h\alpha}$ to be

$$\alpha = \frac{1}{h}(\ln \beta + \ln h - \ln z_W).$$

As we discuss in previous approach, there must be a condition on $s_{0,des}$ such that real parameters α and β can be computed in this approach.

Remark. For arbitrary $s_0 \in \mathbb{C}$, suppose that α is given as in (4.4), and β which is an unknown quantity as in (4.5), or conversely, β is given in (4.6) and α an unknown quantity as in (4.8). Then, the characteristic equation has s_0 as the rightmost root.

Example 4.1. Consider a single delay system

$$\dot{x} = ax(t) + a_{1d}x(t-h) + u(t)$$

under the influence of feedback control law

$$u(t) = kx(t) + k_{1d}x(t-h)$$

then the closed loop system is given by

$$\begin{aligned}\dot{x} &= (a+k)x(t) + (a_{1d}+k_{1d})x(t-h) \\ &= \alpha x(t) + \beta x(t-h)\end{aligned}$$

Consider the following given data set $a = 1$, $a_{1d} = -1$, and $h = 1$. We want to determine the values of k and k_{1d} such that the rightmost closed-loop pole is located at $-0.092484 + 1.9973i$, $-0.60502 + 1.7882i$, and -1, respectively.

Firstly, we compute the set W_0^α and adjust the parameter α in $W_0^\alpha \cap \mathbb{R}$ such that the rightmost pole is assigned to s_{des}. Secondly, we compute the value β by using the equation (4.5). Once these two values are determined, we then calculate the corresponding values for k and k_{1d}, respectively. Table 4.1 shows that three desired rightmost poles are assigned by adjusting the corresponding parameters. Figure 4.2 depicts the region of W_0^α and W_0^β when $s_{0,des} = -0.092484 + 1.97730i$. And the corresponding variation of characteristic roots of the delay system before and after pole assignment is also shown in Table 4.2.

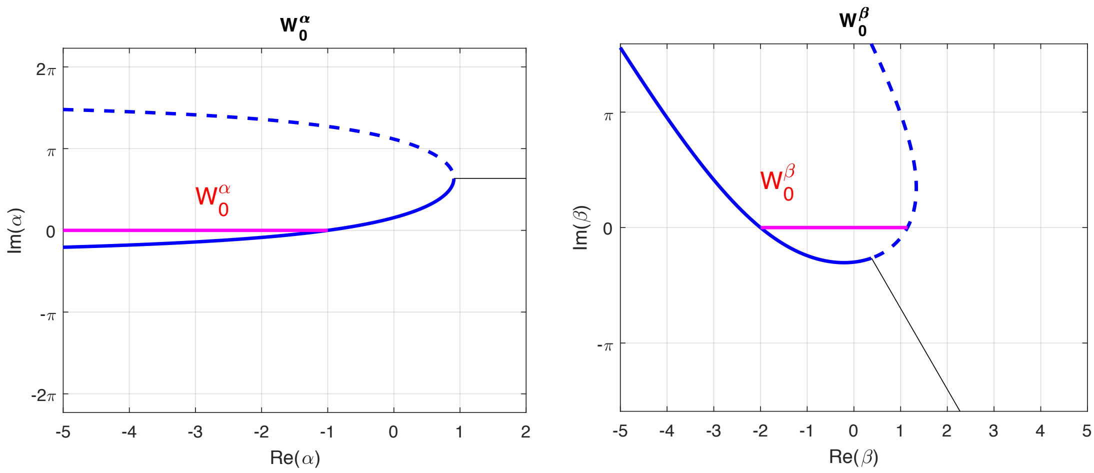


Figure 4.2: Region of W_0^α and W_0^β with $s_{0,des} = -0.092484 + 1.97730i$

.

Table 4.1: The variation of corresponding parameters with respect to three different pole locations.

$s_{0,des}$	$-0.092484 + 1.99730i$	$-0.60502 + 1.78820i$	$-1.0 + 0i$
$s_{0,des} + \frac{1}{h}$	$0.9007516 + 1.99730i$	$0.39498 + 1.78820i$	0
α	-1	-1	-1
β	-2	-1	0
a	1	1	1
a_{1d}	-1	-1	-1
k	-2	-2	-2
k_{1d}	-1	0	1

Table 4.2: The variation of characteristic roots before and after pole placement.

	$a = 1,\, a_{1d} = -1$	$\alpha = -1,\, \beta = -2$	$\alpha = -1,\, \beta = -1$	$\alpha = -1,\, \beta = 0$
s_3	$-3.02630 + 20.22380i$	$-2.32231 + 20.3555i$	$-3.01658 + 20.3214i$	$-$
s_2	$-2.66407 + 13.8791i$	$-1.95315 + 14.0695i$	$-2.64736 + 14.0202i$	$-$
s_1	$-2.08880 + 7.46150i$	$-1.36300 + 7.80750i$	$-2.05280 + 7.71840i$	$-$
s_0	0	$-0.092484 + 1.99730i$	$-0.60502 + 1.78820i$	-1
s_{-1}	$-2.08880 - 7.46150i$	$-0.092484 - 1.99730i$	$-0.60502 - 1.78820i$	$-$
s_{-2}	$-2.66407 - 13.87910i$	$-1.36300 - 7.80750i$	$-2.05280 - 7.71840i$	$-$
s_{-3}	$-3.02630 - 20.22380i$	$-1.95315 - 14.0695i$	$-2.64736 - 14.0202i$	$-$
s_{-4}	$-3.29168 - 26.54320i$	$-2.32231 - 20.3555i$	$-3.01658 - 20.3214i$	$-$

4.3 Linear Systems with Two Delays

In recent decades, great attention has been paid to differential equations with two delays which not only have considerable physical background but also exhibit very abundant in dynamics.

Consider a DDE with two delay times h_1 and h_2 ($> h_1$) in §3.2 described by

$$\begin{aligned} &y'(\varsigma) = \alpha_1 y(\varsigma) + \beta_1 y(\varsigma - h_1) + \gamma_1 y(\varsigma - h_2) \\ &y(0) = y_0, \quad y(\tau) = \varphi(\tau), \quad -h_2 \le \tau < 0. \end{aligned}$$

After re-scaling the variables, the DDE is converted to

$$\begin{aligned} \dot{x}(t) &= \alpha x(t) + \beta x(t-1) + \gamma x(t-h), \\ x(0) &= x_0, \quad x(\tau) = \phi(\tau), \quad -h \le \tau < 0. \end{aligned} \tag{4.9}$$

By the same approach which deals with the single time delay systems, the roots of the characteristic equation

$$s - \alpha - \beta e^{-s} - \gamma e^{-sh} = 0$$

are

$$s_k = \alpha + W_k(\ \beta e^{-\alpha} + \gamma e^{s_k(1-h)-\alpha}). \tag{4.10}$$

with $k = 0, \pm 1, \pm 2, \cdots, \pm\infty$. And we renumber the characteristic roots as S_n as depicted in §3.2. The solution is the form as [7]

$$x = \sum_{k=-\infty}^{\infty} C_k e^{S_k t} + \int_0^t e^{S_k(t-\xi)} bu(\xi) d\xi. \tag{4.11}$$

By using the result of Lemma 3.2, the linear scalar time-delay system (4.9) with the uncertainties prescribed by (3.11) is robustly stable if and only if

$$\max_{\alpha\in\Omega^\alpha, \beta\in\Omega^\beta, \gamma\in\Omega^\gamma, h\in[\underline{h}, \bar{h}]} S_W(\alpha, \beta, \gamma, h) < 0 \tag{4.12}$$

where

$$S_W(\alpha, \beta, \gamma, h) = \mathrm{Re}[S_0] = \mathrm{Re}[\alpha + W_0(\ \beta e^{-\alpha} + \gamma e^{-S_0(h-1)-\alpha})].$$

4.4 Pole Placement for Two-Delay Systems

In this section, we continue to discuss the pole placement for two delays system. Consider a DDE with state feedback and two delay times h_1 and h_2 ($> h_1$) described by

$$\begin{aligned} &y'(\varsigma) = a\ y(\varsigma) + a_{1d}\ y(\varsigma - h_1) + a_{2d}\ y(\varsigma - h_2) + u(\varsigma) \\ &y(0) = y_0, \quad y(\tau) = \varphi(\tau), \quad -h_2 \le \tau < 0, \end{aligned} \tag{4.13}$$

where

$$u(\varsigma) = kx(\varsigma) + k_{1d}x(\varsigma - h_1) + k_{2d}x(\varsigma - h_2).$$

As seen in §3.2, after rescaling the variables, the DDE is converted to

$$\begin{aligned} \dot{x}(t) &= (a+k)h_1 x(t) + (a_{1d} + k_{1d})h_1 x(t-1) + (a_{2d} + k_{2d})h_1 x(t - \frac{h_2}{h_1}) \\ &\triangleq \alpha x(t) + \beta x(t-1) + \gamma x(t-h) \end{aligned}$$

where $(a+k)h_1 = \alpha, (a_{1d} + k_{1d})h_1 = \beta, (a_{2d} + k_{2d})h_1 = \gamma$ and $\frac{h_2}{h_1} = h(> 1)$. Then the roots of $s - \alpha - \beta e^{-s} - \gamma e^{-sh} = 0$ are expressed as

$$s_k = \alpha + W_k(\ \beta e^{-\alpha} + \gamma e^{s_k(1-h)-\alpha}\ ).$$

which has been renumbered as S_n as described in §3.2. By Lemma 2.5, S_0 is always in the rightmost of all the characteristic roots. In this section we try to assign the values of S_n

for $n = 0$ or $n = 0, \pm 1$ into prescribed locations by selecting appropriate real parameters α, β and γ. The following cases are considered:

1. Assign the rightmost root S_0 to $s_{0,des} \in \mathbb{R}$,
2. Assign the rightmost root S_0 to $s_{0,des} \in \mathbb{C}$,
3. Assign the rightmost root S_0 to $s_{0,des} \in \mathbb{R}$ and S_1, S_{-1} to $s_{1,des}$ and $\bar{s}_{1,des}$, respectively, with $s_{1,des} \in \mathbb{C}$.

Firstly, we try to assign the value S_0 to the desired pole $s_{0,des} \in \mathbb{R}$. Since

$$S_0 = \alpha + W_0(\ \beta e^{-\alpha} + \gamma e^{S_0(1-h)-\alpha}\ ) \tag{4.14}$$

must hold, hence the problem is to solve (4.15) for real α, β and γ. Let $z_W = \beta e^{-\alpha} + \gamma e^{S_0(1-h)-\alpha}$ then from (4.15) we obtain

$$\alpha = S_0 - W_0(z_W),$$

or equivalently,

$$\alpha = s_{0,des} - W_0(z_W).$$

Since $s_{0,des}$ and α are real numbers, thus the corresponding $W_0(z_W)$ must also belong to $\mathbb{R}$, i.e., $W_0(z_W) \geq -1$ and the corresponding $z_W \geq -1/e$. Define the following feasible set W_0^α:

$$W_0^\alpha \triangleq \{s_{0,des} - W_0(z_W) \big| z_W \geq -1/e\}. \tag{4.15}$$

We choose α from W_0^α and set the corresponding z_W to an appropriate $z_W^\alpha = (s_{0,des} - \alpha)e^{s_{0,des}-\alpha}$. Then the values of β and γ from the following set of equations:

$$\begin{aligned} \beta e^{-\alpha} + \gamma e^{s_{0,des}(1-h)-\alpha} &= z_W^\alpha, \\ \beta e^{-s_{0,des}} + \gamma e^{-hs_{0,des}} &= s_{0,des} - \alpha. \end{aligned}$$

or equivalently,

$$\beta + \gamma e^{s_{0,des}(1-h)} = (s_{0,des} - \alpha)e^{s_{0,des}} = z_W^\alpha e^\alpha. \tag{4.16}$$

Since $z_W^\alpha \geq -1/e$, we can obtain that

$$(s_{0,des} - \alpha)e^{s_{0,des}} \geq -e^{\alpha-1}$$

which implies that

$$(s_{0,des}-\alpha)e^{s_{0,des}-\alpha} \geq -e^{-1}$$
$$s_{0,des}-\alpha \geq W_0(-e^{-1}) = -1$$

or equivalently,

$$\alpha \leq s_{0,des}+1.$$

There are infinitely many solution pairs for (β,γ).

Secondly, we try to assign the value S_0 to the desired pole $s_{0,des} \in \mathbb{C}$. Since

$$\alpha = s_{0,des} - W_0(z_W),$$

we define the feasible set W_0^α by:

$$W_0^\alpha \triangleq \{s_{0,des} - W_0(z_W)\big|z_W \in \mathbb{C}\}. \tag{4.17}$$

We choose α from $W_0^\alpha \cap \mathbb{R}$ and then the values of β and γ solve from real and imaginary parts of the following equation:

$$\beta + \gamma e^{s_{0,des}(1-h)} = (s_{0,des}-\alpha)e^{s_{0,des}}. \tag{4.18}$$

After some algebraic operation, we obtain

$$\begin{aligned}
\beta &= \mathrm{Re}[(s_{0,des}-\alpha)e^{s_{0,des}}] - \mathrm{Im}[(s_{0,des}-\alpha)e^{s_{0,des}}]\frac{\mathrm{Re}[e^{s_{0,des}(1-h)}]}{\mathrm{Im}[e^{s_{0,des}(1-h)}]},\\
\gamma &= \mathrm{Im}[(s_{0,des}-\alpha)e^{s_{0,des}}].
\end{aligned}$$

The third case, given $s_{0,des} \in \mathbb{R}$ and $s_{1,des} \in \mathbb{C}$, then the following equations hold:

$$\begin{aligned}
s_{0,des} &= \alpha + W_0(\beta e^{-\alpha} + \gamma e^{s_{0,des}(1-h)-\alpha}) \text{ with } \beta e^{-\alpha} + \gamma e^{s_{0,des}(1-h)-\alpha} \geq -1/e\\
s_{1,des} &= \alpha + W_k(\beta e^{-\alpha} + \gamma e^{s_{1,des}(1-h)-\alpha})
\end{aligned}$$

for some $k \neq 0$ and we want to solve for α, β and γ. In advance we select α from W_0^α as given in (4.16). Select a k such that $s_{1,des}-\alpha$ belongs to the range of W_k. And then apply it to find the possible solution for β and γ such that

$$s_{1,des} = \alpha + W_k(\beta e^{-\alpha} + \gamma e^{s_{1,des}(1-h)-\alpha})$$

and satisfy the following condition

$$\beta e^{-\alpha} + \gamma e^{s_{0,des}(1-h)-\alpha} \geq -1/e.$$

Consider the case which the desired pole $s_{0,des} \in \mathbb{R}$. If $s_{0,des}$ is inserted into (4.15), by Lemma 2.3, it is ensured that $s_{0,des}$ exist in the right position than the others.

Example 4.2. Consider a DDE with two time delays and state feedback

$$\begin{aligned}\dot{x}(t) &= (a+k)h_1 x(t) + (a_{1d}+k_{1d})h_1 x(t-1) + (a_{2d}+k_{2d})x(t-h_2) \\ &= \alpha x(t) + \beta x(t-1) + \gamma x(t-h)\end{aligned}$$

with $a = -1$, $a_{1d} = 2$, and $a_{2d} = -1/2$. Then the open-loop characteristic roots are shown in Table 4.3 and since $s_0 = 0.25222 > 0$ this system is unstable.

Table 4.3: The characteristic roots of the open-loop system.

k	i	$s_{k,i}$
2	2	$-1.92417 + 13.0581i$
2	1	$-1.20198 + 10.4954i$
1	2	$-1.71720 + 6.67350i$
1	1	$-0.60716 + 4.42870i$
0	1	0.25222
0	2	0.25222
-1	1	$-0.60716 - 4.42870i$
-1	2	$-1.71720 - 6.67350i$
-2	1	$-1.20198 - 10.4954i$
-2	2	$-1.92417 - 13.0581i$

Now we want to find the feedback control law such that the desired rightmost pole is placed at $s_{0,des} = -0.11929$ or $s_{0,des} = -0.27495 \pm 1.47520$. As seen in Example 4.1, we can calculate W_0^α first, and determine α from the set $W_0^\alpha \cap \mathbb{R}$. Once α is selected, we then compute two other parameters β and γ accordingly. Afterward, parameters k, k_{1d}, and k_{2d} are adjusted such that the $k = \alpha - a$, $k_{1d} = \beta - a_{1d}$, and $k_{2d} = \gamma - a_{2d}$. Two set of solutions are presented in Table 4.4 to two desired close-loop poles $s_{0,des} = -0.11929$ or $s_{0,des} = -0.27495 \pm 1.47520$, respectively, whose closed-loop characteristic roots are also described in Example 3.2. Table 4.5 shows the variation of characteristic roots for two set of designed parameters.

Table 4.4: The computed parameters for two desired pole locations.

$s_{0,des}$	-0.11929	$-0.27495 \pm 1.47520i$
$s_{0,des}+1$	0.88071	$1.27495 \pm 1.47520i$
α	-1	-1
β	$\frac{1}{2}$	-1
γ	$\frac{1}{4}$	$-\frac{1}{2}$
a	-1	-1
a_{1d}	2	2
a_{2d}	$-\frac{1}{2}$	$-\frac{1}{2}$
k	0	0
k_{1d}	$-\frac{3}{2}$	-3
k_{2d}	$\frac{3}{4}$	0

Table 4.5: Variation of characteristic roots.

k	i	open-loop poles $a=-1, a_{1d}=2, a_{2d}=-\frac{1}{2}$	close-loop poles $\alpha=-1, \beta=\frac{1}{2}, \gamma=\frac{1}{4}$ $k=0, k_{1d}=-\frac{3}{2}, k_{2d}=\frac{3}{4}$	close-loop poles $\alpha=-1, \beta=-1, \gamma=-\frac{1}{2}$ $k=0, k_{1d}=-3, k_{2d}=0$
2	2	$-1.92417+13.0581i$	$-1.82137+11.6390i$	$-1.50747+13.4657i$
2	1	$-1.20198+10.4954i$	$-1.89200+8.71330i$	$-1.66427+10.0261i$
1	2	$-1.71720+6.67350i$	$-1.37970+5.30450i$	$-1.14680+7.24010i$
1	1	$-0.60716+4.42870i$	$-1.36930+2.51760i$	$-1.27050+3.64510i$
0	1	0.25222	-0.11929	$-0.27495+1.47520i$
0	2	0.25222	-0.11929	$-0.27495-1.47520i$
-1	1	$-0.60716-4.42870i$	$-1.36930-2.51760i$	$-1.27050-3.64510i$
-1	2	$-1.71720-6.67350i$	$-1.37970-5.30450i$	$-1.14680-7.24010i$
-2	1	$-1.20198-10.4954i$	$-1.89200-8.71330i$	$-1.66427-10.0261i$
-2	2	$-1.92417-13.0581i$	$-1.82137-11.6390i$	$-1.50747-13.4657i$

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