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一些新的 Gronwall-Bellmen-Ou-Iang 型式的離散不等式及其應用

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Contents

ABSTRACT

In this work, we establish some new discrete Gronwall-Bellman-Ou-Iang-type inequalities with explicit bounds, which on one hand generalize some existing results and on the other hand furnish a convenient tool in the study of qualitative as well as quantitative properties of solutions of certain classes of difference equations. We illustrate this by applying these new inequalities to study the boundedness, uniqueness, and continuous dependence of the solutions of some boundary value problems for difference equations.

1. INTRODUCTION

It is well known that the inequalities have always been of great importance for the development of many branches of mathmatics. The inequalities of various types have been widely studied in most subjects involving mathmatical analysis. They are particulary useful for approximation theory and numerical analysis in which estimates of approximation errors are involved. In the past years, the application of inequalities has greatly expanded and they are now used not only in mathematics but also in other areas.

The theory of finite difference equations has been rapidly developed in recent years and has proved to be of fundamental importance in its applications to many different disciplines. In the meantime, finite difference inequalities which exhibit explicit bounds on unknown functions in general provide a very useful and important tool in the development of the theory of finite difference equations. During the past years, motivated and inspired by their applications in various branches of finite difference equations, many such inequalities have been established.

Among varies types of inequalities, the so-called Gronwall-Bellman-Ou-Iang-type inequalities is particularly useful which over the years has proven to be extremely useful in the study of the existence, uniqueness, stability, boundedness, and many other properties of the solutions of a wide range of differential equations. (see [2, 4, 5, 12, 14, 15]).

This work consists of 4 Section and references. Section 1 presents some important integral and discrete inequalities involving functions of one and two independent variables over the years, respectively. Section 2 deals with some new discrete Gronwall-Bellman-Ou-Iang type inequalities involving functions of two independent variables. Section 3 contains some applications in the study of some finite difference equations. Section 4 we make some conclusion and conjecture about our results.

Throughout, we shall use the following notations and defintions: $I := [m_0, M) \cap \mathbb{Z}$ and $J := [n_0, N) \cap \mathbb{Z}$ are two fixed lattices of integral points in $\mathbb{R}$, where $m_0, n_0 \in \mathbb{Z}$, $M, N \in \mathbb{Z} \cup \{\infty\}$. Let $\Omega := I \times J \subset \mathbb{Z}^2$, $\mathbb{R}_+ := (0, \infty)$, $\mathbb{R}_0 := [0, \infty)$, $\mathbb{R}_1 := [1, \infty)$ and for any $(s, t) \in \Omega$, the sub-lattice $[m_0, s] \times [n_0, t] \cap \Omega$ of Ω will be denoted as $\Omega_{(s.t)}$. $I_1 := [x_0, X)$ and $I_2 := [y_0, Y)$ are two given intervals of $\mathbb{R}$, where $x_0, y_0 \in \mathbb{R}$, and $\chi = I_1 \times I_2$.

If U is a lattice in $\mathbb{Z}$ (respectively, $\mathbb{Z}^2$), the collections of all $\mathbb{R}$-valued, $\mathbb{R}_+$-valued, and $\mathbb{R}_0$-valued function on U are denoted by $\mathcal{F}(U)$, $\mathcal{F}_+(U)$, and $\mathcal{F}_0(U)$ respectively. For convenience, we extend the domain of each function in $\mathcal{F}(U)$, $\mathcal{F}_+(U)$, and $\mathcal{F}_0(U)$ trivially to the ambient space $\mathbb{Z}$ (respectively, $\mathbb{Z}^2$). So for example, a function in $\mathcal{F}(U)$ is regarded as a function defined on $\mathbb{Z}$ (respectively, $\mathbb{Z}^2$) with support in U. As usual, the collection of all continuous functions of a topological space X into a topological space Y will be denoted by $C(X, Y)$.

If U is a lattice in $\mathbb{Z}$, the difference operator $\triangle$ on $f \in \mathcal{F}(\mathbb{Z})$ or $\mathcal{F}_+(\mathbb{Z})$ is defined as

$$\triangle f(n) := f(n+1) - f(n), n \in U,$$

and if V is a lattice in $\mathbb{Z}^2$, the partial difference operators $\triangle_1$ and $\triangle_2$ on $u \in \mathcal{F}(\mathbb{Z}^2)$ or $\mathcal{F}_+(\mathbb{Z}^2)$ are defined as

$$\begin{aligned} \triangle_1 u(m, n) &:= u(m+1, n) - u(m, n), (m, n) \in V, \\ \triangle_2 u(m, n) &:= u(m, n+1) - u(m, n), (m, n) \in V. \end{aligned}$$

2. RETARTED INTEGRAL AND DISCRETE INEQUALITIES

Among various branches of Gronwall-Bellman-type inequalities, a very useful one is orginated from Ou-Iang. In his study of the boundedness of certain second order differential equations, he established the following results which is generally known as Ou-Iang's inequality:

Theorem 2.1. $(Ou - Iang[8])$ *If u and f are non-negative functions on $[0, \infty)$ satisfying*

$$u^2(x) \leq k^2 + 2\int_0^x f(s)\, u(s)\, ds$$

for all $x \in [0, \infty)$, where $k \geq 0$ is a constant, then

$$u(x) \leq k + \int_0^x f(s)\, ds$$

for all $x \in [0, \infty)$.

Recently, Pachpatte established the following further generalizations of Ou-Iang inequality:

Theorem 2.2. $(Pachpatte[10])$ *Suppose u, f, g are continuous non-negative functions on $[0, \infty)$ and w a continuous non-decreasing function on $[0, \infty)$ with $w(r) > 0$ for $r > 0$. If*

$$u^2(x) \leq k^2 + 2\int_0^x \left(f(s)\, u(s) + g(s)\, u(s)\, w(u(s))\right) ds$$

for all $x \in [0, \infty)$, where $k \geq 0$ is a constant, then

$$u(x) \leq \Omega^{-1}\left[\Omega\left(k + \int_0^x f(s)\, ds\right) + \int_0^x g(s)\, ds\right]$$

for all $x \in [0, \infty)$, where

$$\Omega(r) := \int_1^r \frac{ds}{w(s)}, r > 0.$$

Ω^{-1} is the inverse of Ω, and $x_1 \in [0, \infty)$ is chosen in such a way that $\Omega\left(k + \int_0^x f(s)\, ds\right) + \int_0^x g(s)\, ds \in \mathrm{Dom}\left(\Omega^{-1}\right)$ for all $x \in [0, x_1]$.

On the other hand, Lipovan oberved the following Gronwall-Bellman-Ou-Iang-type inequality which a handy tool in the study of the global existence of solutions to certain integral equations and functional differential equations.

Theorem 2.3. (*Lipovan*[6]) *Suppose u, f are continuous non-negative functions on $[x_0, X)$, w a continuous non-decreasing function on $[0,\infty)$ with $w(r) > 0$ for $r > 0$, and $\alpha : [x_0, X) \to [x_0, X)$ a continuous non-decreasing function with $\alpha(x) \le x$ on $[x_0, X)$. If*

$$u(x) \le k + \int_{\alpha(x_0)}^{\alpha(x)} f(s)\, w(u(s))\, ds$$

for all $x \in [x_0, X)$, where $k \ge 0$ is a constant, then

$$u(x) \le \Omega^{-1}\left[\Omega(k) + \int_{\alpha(x_0)}^{\alpha(x)} f(s)\, ds\right]$$

for all $x \in [0,\infty)$, where

$$\Omega(r) := \int_1^r \frac{ds}{w(s)}, r > 0.$$

Ω^{-1} is the inverse of Ω, and $x_1 \in [x_0, X)$ is chosen in such a way that $\Omega(k) + \int_{\alpha(x_0)}^{\alpha(x)} f(s)\, ds \in \mathrm{Dom}(\Omega^{-1})$ for all $x \in [x_0, x_1)$.

In the recently, some new nonlinear retarded inequalities of Gronwall-Ou-Iang type are established, which can be used as effective tools in the study of integral and differential equations. Cheung establish the following:

Theorem 2.4. (*Cheung*[1]) *Let $a, b \in C(\chi, \mathbb{R}_0)$, $\alpha_i \in C^1(I_1, I_1)$, $\beta_i \in C^1(I_2, I_2)$ be nondecreasing with $\alpha_i \le x$ on I_1, $\beta_i \le y$ on I_2, $i = 1, 2$, $w \in C(\mathbb{R}_0, \mathbb{R}_0)$ be a nondecreasing function with $w(u) > 0$ for $k \ge 0$ be a constant.*

If $u \in C(\triangle, \mathbb{R}_0)$ and

$$u(x,y) \le k + \int_{\alpha_1(x_0)}^{\alpha_1(x)} \int_{\beta_1(y_0)}^{\beta_1(y)} a(s,t)\, u(s,t)\, dt ds + \int_{\alpha_2(x_0)}^{\alpha_2(x)} \int_{\beta_2(y_0)}^{\beta_2(y)} b(s,t)\, w(u(s,t))\, dt ds$$

for any $(x,y) \in \chi$*, then*

$$u(x,y) \leq G^{-1}\left\{G^{-1}\left[G\left(kexpA_1(x,y)\right)\right] + B_1(x,y)\,A_1(x,y)\right\} \tag{2.1}$$

for all $x_0 \leq x \leq x_1$*,* $y_0 \leq y \leq y_1$*, where*

$$\begin{aligned} A_1(x,y) &:= \int_{\alpha_1(x_0)}^{\alpha_1(x)} \int_{\beta_1(y_0)}^{\beta_1(y)} a(s,t)\,dtds, \\ B_1(x,y) &:= \int_{\alpha_2(x_0)}^{\alpha_2(x)} \int_{\beta_2(y_0)}^{\beta_2(y)} b(s,t)\,dtds, \\ G(r) &:= \int_{r_0}^{r} \frac{ds}{w(s)}, r \geq r_0 > 0. \end{aligned}$$

G^{-1} *denotes the inverse function of* G*, and real numbers* $x_1 \in I_1$*,* $y_1 \in I_2$ *are chosen so that the quantity in the curly brackets of* (1) *is in the range of* G*.*

Theorem 2.5. (*Cheung*[1]) *Let* $a, b, \alpha_i, \beta_i\,(i = 1,2)$*,* w *and* k *be as in* $Theorem 1.4$*. Let* $\varphi \in C^1(\mathbb{R}_0, \mathbb{R}_0)$ *and* $\varphi'' \in C^1(\mathbb{R}_0, \mathbb{R}_0)$ *with* $\varphi' > 0$ *for* $u > 0$*. If* $u \in C(\chi, \mathbb{R}_0)$ *and for any* $(x,y) \in \chi$

$$\begin{aligned} \varphi(u(x,y)) \leq\ & k + \int_{\alpha_1(x_0)}^{\alpha_1(x)} \int_{\beta_1(y_0)}^{\beta_1(y)} a(s,t)\,\varphi'(u(s,t))\,u(s,t)\,dtds \\ & + \int_{\alpha_2(x_0)}^{\alpha_2(x)} \int_{\beta_2(y_0)}^{\beta_2(y)} b(s,t)\,\varphi'(u(s,t))\,w(u(s,t))\,dtds, \end{aligned}$$

then for any $x_0 \leq x \leq x_2$*,* $y_0 \leq y \leq y_2$*,*

$$u(x,y) \leq G^{-1}\left\{G\left[\varphi^{-1}(k)\,expA_1(x,y)\right] + B_1(x,y)\,expA_1(x,y)\right\}, \tag{2.2}$$

where $A_1(x,y)$ *and* $B_1(x,y)$ *are defined in* $Theorem 1.4$*,* G *and* G^{-1}*are as in Theorem 1.4,* φ^{-1} *is the inverse function of* φ *and* $x_2 \in I_1$*,* $y_2 \in I_2$ *are chosen so that the quantity in the curly brackets of* (2) *is in the range of* G*.*

Among various generalizations of Ou-Iang's inequality, discretization is also an interesting direction. Smilar to the contributions of the continuous versions of the inequality to the study of differential equations, one naturally expects that discrete versions of the inequality should also play an important role in the study of difference equations.

One of the earlier versions of discrete Ou-Iang-type inequalities was obtained by Pachpatte.

Theorem 2.6. (*Pachpatte*[13]) *Let $u(t)$, $a(t)$, $b(t)$, $h(t)$ be real-valued nonnegative functions defined for $t \in N_0 = \{0, 1, 2, \cdots\}$ and let c be a nonnegative constant. If*

$$u^2(t) \le c^2 + \sum_{s=0}^{t-1} (u(s+1) + u(s)) [a(s) u(s) + h(s)]$$

for all $t \in N_0$, then

$$u(t) \le p(t) \prod_{s=0}^{t-1} [1 + a(s)]$$

for all $t \in N_0$, where

$$p(t) := c + \sum_{s=0}^{t-1} h(s)$$

for all $t \in N_0$.

Very recently, in the process of studying the boundedness, uniqueness, and continuous dependence of the solutions of some boundary value problems, Cheung establish the following:

Theorem 2.7. (*Cheung*[5]) *Suppose $u \in \mathcal{F}_+(\Omega)$. If $k \ge 0$, $p > 1$ are constants and $a, b \in \mathcal{F}_0(\Omega)$, $\varphi \in C(\mathbb{R}_0, \mathbb{R}_0)$ are functions satisfying*

(i) φ is non-decreasing with $\varphi(r) > 0$ for $r > 0$; and

(ii) for any $(m, n) \in \Omega$,

$$u^p(m,n) \le k + \sum_{s=m_\circ}^{m-1} \sum_{t=n_\circ}^{n-1} a(s,t) u(s,t) + \sum_{s=m_\circ}^{m-1} \sum_{t=n_\circ}^{n-1} b(s,t) u(s,t) \varphi(u(s,t)),$$

then

$$u(m,n) \le \left\{ \Phi_{p-1}^{-1} \left[\Phi_{p-1} \left(k^{1-\frac{1}{p}} + A(m,n) \right) + B(m,n) \right] \right\}^{\frac{1}{p-1}}$$

for all $(m,n) \in \Omega(m_1, n_1)$, where

$$\begin{aligned} A(m,n) &:= \sum_{s=m_\circ}^{m-1} \sum_{t=n_\circ}^{n-1} a(s,t), \\ B(m,n) &:= \sum_{s=m_\circ}^{m-1} \sum_{t=n_\circ}^{n-1} b(s,t), \end{aligned}$$

and $(m_1, n_1) \in \Omega$ is chosen such that $\Phi_{p-1}\left(k^{1-\frac{1}{p}} + A(m,n)\right) + B(m,n) \in \mathrm{Dom}\left(\Phi_{p-1}^{-1}\right)$ for all $(m,n) \in \Omega_{(m_1,n_1)}$.

Theorem 2.8. (*Cheung*[4]) *Suppose* $u \in \mathcal{F}_+(\Omega)$. *If* $k > 0$ *,is a constant and* $a, b \in \mathcal{F}_0(\Omega)$, $\varphi, h \in C(\mathbb{R}_+, \mathbb{R}_+)$ *are functions satisfying*

(i) $h(t)$ *and* $H(t) := \frac{h(t)}{t}$, $t > 0$, *are strictly increasing with* $H(t) \to \infty$ *as* $t \to \infty$;

(ii) φ *is non-decreasing; and*

(iii) for any $(m,n) \in \Omega$,

$$h(u(m,n)) \le k + \sum_{s=m_\circ}^{m-1} \sum_{t=n_\circ}^{n-1} a(s,t)\, u(s,t) + \sum_{s=m_\circ}^{m-1} \sum_{t=n_\circ}^{n-1} b(s,t)\, u(s,t)\, \varphi(u(s,t)),$$

then

$$u(m,n) \le H^{-1} \left\{ \Phi_H^{-1} \left[\Phi_H \left(\frac{k}{h^{-1}(k)} + A(m,n) \right) + B(m,n) \right] \right\}$$

for all $(m,n) \in \Omega(m_1, n_1)$, *where*

$$A(m,n) := \sum_{s=m_\circ}^{m-1} \sum_{t=n_\circ}^{n-1} a(s,t),$$

$$B(m,n) := \sum_{s=m_\circ}^{m-1} \sum_{t=n_\circ}^{n-1} b(s,t),$$

and $(m_1, n_1) \in \Omega$ *is chosen such that* $\Phi_H\left(\frac{k}{h^{-1}(k)} + A(m,n)\right) + B(m,n) \in Dom\left(\Phi_H^{-1}\right)$ *for all* $(m,n) \in \Omega_{(m_1,n_1)}$.

3. MAIN RESULTS

The main aim here is to establish some new nonlinear discrete inequalities involving functions of two independent variables, which discretize some results in Cheung [1].

Theorem 3.1. (*Cheung*[2]) *Suppose* $u \in \mathcal{F}_0(\Omega)$. *If* $c \geq 0$ *is a constant and* $a, b \in \mathcal{F}_0(\Omega)$, $w \in C(\mathbb{R}_0, \mathbb{R}_0)$ *are function satisfying*

(*i*) w *is non-decreasing with* $w(r) > 0$ *for* $r > 0$; *and*

(*ii*) *for any* $(m, n) \in \Omega$,

$$u(m,n) \leq c + \sum_{s=m_\circ}^{m-1} \sum_{t=n_\circ}^{n-1} b(s,t)\, w(u(s,t)), \tag{3.1}$$

then

$$u(m,n) \leq G^{-1}\{G(c) + B(m,n)\} \tag{3.2}$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$, *where*

$$\begin{aligned} B(m,n) &:= \sum_{s=m_\circ}^{m-1} \sum_{t=n_\circ}^{n-1} b(s,t), \\ G(v) &:= \int_1^v \frac{ds}{w(s)}, v > 0, \\ G(0) &:= \lim_{v \to 0^+} G(v), \end{aligned}$$

G^{-1} *is the inverse of* G, *and* $(m_1, n_1) \in \Omega$ *is chosen such that* $G(c) + B(m,n) \in Dom(G^{-1})$ *for all* $(m,n) \in \Omega_{(m_1,n_1)}$.

Proof. It suffices to consider the case $c > 0$, for then the case $c = 0$ can be arrived at by continuity argument. Let $c > 0$ and define a positive non-decreasing function

$$z(m,n) = c + \sum_{s=m_\circ}^{m-1} \sum_{t=n_\circ}^{n-1} b(s,t)\, w(u(s,t)),$$

and $z(m_0,n)=z(m,n_0)=c$. Hence, for any $(m,n)\in\Omega$,

$$\begin{aligned}\triangle_1 z(m,n) &= z(m+1,n)-z(m,n)\\ &= \sum_{t=n_0}^{n-1} b(m,t)\,w(u(m,t))\end{aligned}$$

$$\begin{aligned}&\le \sum_{t=n_0}^{n-1} b(m,t)\,w(z(m,t))\\ &\le w(z(m,n-1))\sum_{t=n_0}^{n-1} b(m,t)\,.\end{aligned}$$

Therefore, by the Mean Value Theorem for Integrals, for each $(m,n)\in\Omega$, there exists ξ with $z(m,n)\le\xi\le z(m+1,n)$, such that

$$\begin{aligned}\triangle_1(G\circ z)(m,n) &= G(z(m+1,n))-G(z(m,n))\\ &= \int_{z(m,n)}^{z(m+1,n)}\frac{ds}{w(s)}\\ &= \frac{1}{w(\xi)}\triangle_1 z(m,n)\,.\end{aligned}$$

Since w is non-decreasing, $w(\xi)\ge w(z(m,n))$ and so

$$\begin{aligned}\triangle_1(G\circ z)(m,n) &\le \frac{1}{w(z(m,n))}\triangle_1 z(m,n)\\ &\le \frac{w(z(m,n-1))}{w(z(m,n))}\sum_{t=n_0}^{n-1} b(m,t)\\ &\le \sum_{t=n_0}^{n-1} b(m,t)\end{aligned}$$

for all $(m,n)\in\Omega$. Therefore,

$$\sum_{s=m_0}^{m-1}\triangle_1(G\circ z)(s,n) \le \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)=B(m,n)\,.$$

It follows that

$$\begin{aligned} G(z(m,n)) &\leq G(z(m_0,n)) + B(m,n) \\ &= G(c) + B(m,n). \end{aligned}$$

Since G^{-1} is increasing on $\mathrm{Dom}G^{-1}$, we have

$$z(m,n) \leq G^{-1}\{G(c) + B(m,n)\},$$

and thus

$$u(m,n) \leq G^{-1}\{G(c) + B(m,n)\}.$$

□

Theorem 3.2. *Suppose* $u \in \mathcal{F}_0(\Omega)$. *If* $c \geq 0$ *is a constant and* $a, b \in \mathcal{F}_0(\Omega)$, $w \in C(\mathbb{R}_0, \mathbb{R}_0)$ *are function satisfying*

(i) w *is non-decreasing with* $w(r) > 0$ *for* $r > 0$; *and*

(ii) for any $(m,n) \in \Omega$,

$$u(m,n) \leq c + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\,u(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\,w(u(s,t)), \tag{3.3}$$

then

$$u(m,n) \leq G^{-1}\{G(cK(m,n)) + K(m,n)\,B(m,n)\} \tag{3.4}$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$, *where*

$$\begin{aligned} B(m,n) &:= \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t), \\ K(m,n) &:= \prod_{s=m_0}^{m-1}\left[1 + \sum_{t=n_0}^{n-1} a(s,t)\right], \end{aligned}$$

G *is defined in Theorem 2.1, and* $(m_1,n_1) \in \Omega$ *is chosen such that*

$G\left(cK\left(m,n\right)\right)+K\left(m,n\right)B\left(m,n\right)\in Dom\left(G^{-1}\right)$ *for all* $\left(m,n\right)\in\Omega_{\left(m_1,n_1\right)}$.

Proof. It suffices to consider the case $c>0$, for then the case $c=0$ can be arrived at by continuity argument. Let $c>0$ and define a positive non-decreasing function

$$p\left(m,n\right)=c+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b\left(s,t\right)w\left(u\left(s,t\right)\right) \tag{3.5}$$

for $\left(m,n\right)\in\Omega$, then from (3.3)

$$u\left(m,n\right)\leq p\left(m,n\right)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}a\left(s,t\right)u\left(s,t\right). \tag{3.6}$$

Since $p\left(m,n\right)>0$ is non-decreasing, from (3.6) we have

$$\frac{u\left(m,n\right)}{p\left(m,n\right)}\leq q\left(m,n\right):=1+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}a\left(s,t\right)\frac{u\left(s,t\right)}{p\left(s,t\right)}, \tag{3.7}$$

and $q\left(m_0,n\right)=q\left(m,n_0\right)=1$. Hence,

$$\triangle_1 q\left(m,n\right)=\sum_{t=n_0}^{n-1}a\left(m,t\right)\frac{u\left(m,t\right)}{p\left(m,t\right)}, \tag{3.8}$$

and from 3.8 we have

$$q\left(m+1,n\right)-q\left(m,n\right)=\sum_{t=n_0}^{n-1}a\left(m,t\right)\frac{u\left(m,t\right)}{p\left(m,t\right)}, \tag{3.9}$$

and thus

$$q\left(m+1,n+1\right)-q\left(m,n+1\right)=\sum_{t=n_0}^{n}a\left(m,t\right)\frac{u\left(m,t\right)}{p\left(m,t\right)}. \tag{3.10}$$

From (3.9) and (3.10) we have

$$\triangle_1 q\left(m,n+1\right)-\triangle_1 q\left(m,n\right) \quad\leq\quad a\left(m,n\right)q\left(m,n\right).$$

Then,

$$\frac{\triangle_1 q(m,n+1)}{q(m,n)} - \frac{\triangle_1 q(m,n)}{q(m,n)} \;\; \leq \;\; a(m,n)\,.$$

By the monotonicity of q it follow that

$$\frac{\triangle_1 q(m,n+1)}{q(m,n+1)} - \frac{\triangle_1 q(m,n)}{q(m,n)} \;\; \leq \;\; a(m,n)\,.$$

This implies that

$$\triangle_2 \left(\frac{\triangle_1 q(m,n)}{q(m,n)} \right) \;\; \leq \;\; a(m,n)\,,$$

and thus

$$\frac{\triangle_1 q(m,n)}{q(m,n)} \;\; \leq \;\; \sum_{t=n_0}^{n-1} a(m,t)\,.$$

Therefore, we have

$$\frac{q(m+1,n)}{q(m,n)} - \frac{q(m,n)}{q(m,n)} \;\; \leq \;\; \sum_{t=n_0}^{n-1} a(m,t)\,,$$

which implies that

$$\frac{q(m+1,n)}{q(m,n)} \;\; \leq \;\; 1 + \sum_{t=n_0}^{n-1} a(m,t)\,.$$

Now keeping n fixed, set $m = s$ and substitute $s = m_0, m_0 + 1, \cdots, m - 1$, we get

$$\prod_{s=m_0}^{m-1} \frac{q(s+1,n)}{q(s,n)} \;\; \leq \;\; \prod_{s=m_0}^{m-1} \left[1 + \sum_{t=n_0}^{n-1} a(s,t) \right],$$

which implies that

$$\frac{q(m,n)}{q(m_0,n)} \;\; \leq \;\; \prod_{s=m_\circ}^{m-1} \left[1 + \sum_{t=n_0}^{n-1} a(s,t) \right],$$

and thus

$$q(m,n) \leq \prod_{s=m_0}^{m-1}\left[1+\sum_{t=n_0}^{n-1} a(s,t)\right].$$

By (3.9), (3.7) and the last inequality, we have

$$u(m,n) \leq cK(m,n) + K(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\, w(u(s,t)). \tag{3.11}$$

Fixing any numbers $\overline{m}_1\,(m_0 < \overline{m}_1 - 1 \leq m_1 - 1)$ and $\overline{n}_1\,(n_0 < \overline{n}_1 - 1 \leq n_1 - 1)$, from (3.11) we have

$$u(m,n) \leq cK(\overline{m}_1,\overline{n}_1) + K(\overline{m}_1,\overline{n}_1)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\, w(u(s,t)) \tag{3.12}$$

for $m_\circ \leq m-1 \leq \overline{m}_1 - 1$, $n_\circ \leq n-1 \leq \overline{n}_1 - 1$. Define a positive function

$$r(m,n) = cK(\overline{m}_1,\overline{n}_1) + K(\overline{m}_1,\overline{n}_1)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\, w(u(s,t)) \tag{3.13}$$

with $r(m_0,n) = r(m,n_0) = cK(\overline{m}_1,\overline{n}_1)$.

For $m_\circ \leq m-1 \leq \overline{m}_1 - 1$, $n_0 \leq n-1 \leq \overline{n}_1 - 1$, it follows from (3.13) that

$$u(m,n) \leq r(m,n), \tag{3.14}$$

and

$$\begin{aligned}\triangle_1 r(m,n) &\leq K(\overline{m}_1,\overline{n}_1)\sum_{t=n_0}^{n-1} b(m,t)\, w(u(m,t)) \\ &\leq K(\overline{m}_1,\overline{n}_1)\, w(r(m,n-1))\sum_{t=n_0}^{n-1} b(m,t).\end{aligned}$$

Then

$$\frac{\triangle_1 r(m,n)}{w(r(m,n-1))} = K(\overline{m}_1,\overline{n}_1)\sum_{t=n_0}^{n-1} b(m,t). \tag{3.15}$$

Since

$$\begin{aligned} \triangle_1 (G \circ r)(m,n) &= \int_{r(m,n)}^{r(m+1,n)} \frac{ds}{w(s)} \\ &= \frac{1}{w(\xi)} \triangle_1 r(m,n) \end{aligned}$$

for $r(m,n) \leq \xi \leq r(m+1,n)$

$$\begin{aligned} \triangle_1 (G \circ r)(m,n) &\leq \frac{1}{w(\xi)} K(\overline{m}_1, \overline{n}_1) w(r(m,n-1)) \sum_{t=n_0}^{n-1} b(m,t) \\ &\leq \frac{w(r(m,n-1))}{w(r(m,n))} K(\overline{m}_1, \overline{n}_1) \sum_{t=n_0}^{n-1} b(m,t) \\ &\leq K(\overline{m}_1, \overline{n}_1) \sum_{t=n_0}^{n-1} b(m,t). \end{aligned}$$

This implies that

$$(G \circ r)(m,n) - (G \circ r)(m_\circ, n) \leq K(\overline{m}_1, \overline{n}_1) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t),$$

and thus

$$\begin{aligned} G(r(m,n)) &\leq G(r(m_\circ, n)) + K(\overline{m}_1, \overline{n}_1) B(m,n) \\ &\leq G(cK(\overline{m}_1, \overline{n}_1)) + K(\overline{m}_1, \overline{n}_1) B(m,n) \end{aligned}$$

for $m_0 \leq m-1 \leq \overline{m}_1 - 1$, $n_0 \leq n-1 \leq \overline{n}_1 - 1$. Taking $m = \overline{m}_1$, $n = \overline{n}_1$ in the above inequality, we obtain

$$G(r(\overline{m}_1, \overline{n}_1)) \leq G(cK(\overline{m}_1, \overline{n}_1)) + K(\overline{m}_1, \overline{n}_1) B(\overline{m}_1, \overline{n}_1).$$

Since $m_0 < \overline{m}_1 - 1 \leq m_1 - 1$, $n_0 < \overline{n}_1 - 1 \leq n_1 - 1$ are arbitary, from the last relation, it follows that

$$G(r(m,n)) \leq G(cK(m,n)) + K(m,n) B(m,n).$$

Since G^{-1} is increasing on $DomG^{-1}$, we have

$$r(m,n) \le G^{-1}\{G(cK(m,n)) + K(m,n)B(m,n)\} \tag{3.16}$$

for $m_0 < m-1 \le m_1 - 1$, $n_0 < n-1 \le n_1 - 1$. Hence, by (3.14) and (3.16), we get

$$u(m,n) \le G^{-1}\{G(cK(m,n)) + K(m,n)B(m,n)\}.$$

□

Remark 3.3. (i)Theorem 2.2 cannot derive from Theorem 1.7, for $\alpha > 1$ is necessary in Theorem 1.7.

(ii) It is also easy to see that Theorem 2.2 cannot derive from Theorem 1.8.

(iii) Similarly to the previous remark, in many cases $G(\infty) = \infty$ and in these situations, inequality (3.4) holds for all $(m,n) \in \Omega$.

Theorem 3.4. *Let a, b, m, n, w and c be the same as in Theorem 2.2. Let $\varphi \in C^1(\mathbb{R}_0, \mathbb{R}_0)$*

and $\varphi'' \in C(\mathbb{R}_+, \mathbb{R}_0)$ with $\varphi'(u) > 0$ for $u > 0$,

$$\varphi(u(m,n)) \le c + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\varphi'(u(s,t))u(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\varphi'(u(s,t))w(u(s,t)), \tag{3.17}$$

then

$$u(m,n) \le G^{-1}\left\{G\left(\varphi^{-1}(c)K(m,n)\right) + K(m,n)B(m,n)\right\} \tag{3.18}$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$, where $B(m,n)$,$K(m,n)$ are defined as in Theorem 2.2, and $(m_1,n_1) \in \Omega$ is chosen such that $G(\varphi^{-1}(c)K(m,n)) + K(m,n)B(m,n) \in \mathrm{Dom}(G^{-1})$ for all $(m,n) \in \Omega_{(m_1,n_1)}$.

Proof. It suffices to consider the case $c > 0$, for then the case $c = 0$ can be arrived at by continuity argument. Let $c>0$. Define

$$z(m,n) := \varphi^{-1}\left\{c+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}a(s,t)\varphi'(u(s,t))u(s,t)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b(s,t)\varphi'(u(s,t))w(u(s,t))\right\}.$$

Then $z(m_0,n)=z(m,n_0)=\varphi^{-1}(c)$, z is a position nondecreasing function, and $\varphi(u(m,n))\le\varphi(z(m,n))$. Now,

$$\begin{aligned}\triangle_1\varphi(z(m,n)) &= \sum_{t=n_0}^{n-1}a(m,t)\varphi'(u(m,t))u(m,t)+\sum_{t=n_0}^{n-1}b(m,t)\varphi'(u(m,t))w(u(m,t))\\ &= \sum_{t=n_0}^{n-1}\Big[a(m,t)\varphi'(u(m,t))u(m,t)+b(m,t)\varphi'(u(m,t))w(u(m,t))\Big]\\ &\le \sum_{t=n_0}^{n-1}\Big[a(m,t)\varphi'(z(m,t))z(m,t)+b(m,t)\varphi'(z(m,t))w(z(m,t))\Big].\end{aligned}$$

Hence, by the Mean Value Theorem, for each $(m,n)\in\Omega$, there exists ξ with $z(m,n)\le\xi\le z(m+1,n)$, such that

$$\begin{aligned}\triangle_1\varphi(z(m,n)) &= \varphi(z(m+1,n))-\varphi(z(m,n))\\ &= \varphi'(\xi)\triangle_1 z(m,n).\end{aligned}$$

Since w is non-deceasing, $w(\xi)\le w(z(m,n))$, then

$$\begin{aligned}\triangle_1 z(m,n) &= \frac{\triangle_1\varphi(z(m,n))}{\varphi'(\xi)}\\ &\le \frac{1}{\varphi'(\xi)}\varphi'(z(m,n-1))\sum_{t=n_0}^{n-1}[a(m,t)z(m,t)+b(m,t)w(z(m,t))]\\ &\le \sum_{t=n_0}^{n-1}a(m,t)z(m,t)+\sum_{t=n_0}^{n-1}b(m,t)w(z(m,t))\end{aligned}$$

for all $(m,n)\in\Omega$. Therefore,

$$\sum_{s=m_0}^{m-1} \triangle_1 z(s,n) \;\le\; \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\, z(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\, w(z(s,t)).$$

It follows that

$$z(m,n) - z(m_0,n) \;\le\; \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\, z(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\, w(z(s,t)),$$

and thus

$$z(m,n) \;\le\; \varphi^{-1}(c) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\, z(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\, w(z(s,t)).$$

By Theorem 2.2, we get

$$z(m,n) \;\le\; G^{-1}\left\{G\left(\varphi^{-1}(c)\, K(m,n)\right) + K(m,n)\, B(m,n)\right\},$$

which implies that

$$u(m,n) \;\le\; G^{-1}\left\{G\left(\varphi^{-1}(c)\, K(m,n)\right) + K(m,n)\, B(m,n)\right\}.$$

□

Corollary 3.5. *Let a, b, m, n, w and c be the same as in Theorem 2.2. Let $p \ge 1$ be a constant. If $u(m,n) \in C(\Omega, \mathbb{R}_0)$ and for any $(m,n) \in \Omega$,*

$$\begin{aligned} u^p(m,n) \;\le\; & c + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\, u^p(s,t) \log u(s,t) \\ & + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\, u^p(s,t)\, w(\log u(s,t)), \end{aligned} \tag{3.19}$$

then

$$u(m,n) \le G^{-1}\left\{G\left(c^{\frac{1}{p}} K(m,n)\right) + K(m,n)\, B(m,n)\right\} \tag{3.20}$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$*, where* $B(m,n)$*,*$K(m,n)$ *are defined as in Theorem 2.2.*

Proof. This follows immediately from Theorem 2.4 by letting $\varphi(s) = s^p$. □

Corollary 3.6. *Let* a, b, m, n, w *and* c *be the same as in Theorem 2.2. Let* $p \geq 1$ *be a constant. If* $u(m,n) \in C(\Omega, \mathbb{R}_1)$ *and for any* $(m,n) \in \Omega$*,*

$$u^p(m,n) \leq c + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)\, u^p(s,t) \log u(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\, u^p(s,t)\, w(\log u(s,t)), \tag{3.21}$$

then

$$u(m,n) \leq \exp\left(G^{-1}\left\{G\left[\left(\frac{1}{p}\log c\right) K(m,n)\right] + K(m,n)\, B(m,n)\right\}\right) \tag{3.22}$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$*, where* $B(m,n)$*,*$K(m,n)$ *are defined as in Theorem 2.2.*

Proof. It suffices to consider the case $c > 0$, for then the case $c = 0$ can be arrived at by continuity argument. Taking $h(m,n) = \log u(m,n)$, then inequality (3.19) reduces to

$$e^{ph} \leq c + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)\, e^{ph} h(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\, e^{ph} h(s,t),$$

which is a special case of inequality (3.17) which $\varphi = \exp(ph)$. By Theorem 2.4, we get the desired inequality (3.22) directly. □

Theorem 3.7. *Let* φ, a, b, m, n, w *and* c *be the same as in Theorem 2.4. Let* $k \in C(\Omega^2, \mathbb{R}_0)$*. If* $u(m,n) \in C(\Omega, \mathbb{R}_0)$ *and for any* $(m,n) \in \Omega$*,*

$$\varphi(u(m,n)) \leq c + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)\, \varphi'(u(s,t))\, u(s,t)$$

$$+ \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\, \varphi'(u(s,t)) \left[\sum_{\sigma=m_0}^{s} \sum_{\eta=n_0}^{t} k(s,t,\sigma,\eta)\, w(u(\sigma,\eta))\right], \tag{3.23}$$

then

$$u(m,n) \leq G^{-1}\left\{G\left(\varphi^{-1}(c)\, K(m,n)\right) + K(m,n)\, \overline{B}(m,n)\right\} \tag{3.24}$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$*, where*

$$\overline{B}(m,n) = \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\left[\sum_{\sigma=m_0}^{s}\sum_{\eta=n_0}^{t} k(s,t,\sigma,\eta)\right],$$

G, G^{-1} *and* $K(m,n)$ *are defined as in Theorem 2.2, and* $(m_1,n_1) \in \Omega$ *is chosen such that* $G(\varphi^{-1}(c)K(m,n)) + K(m,n)\overline{B}(m,n) \in Dom(G^{-1})$ *for all* $(m,n) \in \Omega_{(m_1,n_1)}$*.*

Proof. It suffices to consider the case $c > 0$, for then the case $c = 0$ can be arrived at by continuity argument. Let $c > 0$. Define

$$\begin{aligned} z(m,n) \quad := \quad & \varphi^{-1}\left\{c + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\varphi^{'}(u(s,t))u(s,t) + \right. \\ & \left. \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\varphi^{'}(u(s,t))\left[\sum_{\sigma=m_0}^{s}\sum_{\eta=n_0}^{t} k(s,t,\sigma,\eta)w(u(\sigma,\eta))\right]\right\}. \end{aligned}$$

Then $z(m_0,n) = z(m,n_0) = \varphi^{-1}(c)$, z is a position nondecreasing function, and $\varphi(u(m,n)) \le \varphi(z(m,n))$.

Using similar procedure as in Theorem 2.4's proof, we can obtain

$$z(m,n) \quad \le \quad \varphi^{-1}(c) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)z(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\left[\sum_{\sigma=m_0}^{s}\sum_{\eta=n_0}^{t} k(s,t,\sigma,\eta)w(z(\sigma,\eta))\right].$$

Then use the similar procedure as in Theorem 2.2's proof, we get

$$z(m,n) \quad \le \quad G^{-1}\left\{G\left(\varphi^{-1}(c)K(m,n)\right) + K(m,n)\overline{B}(m,n)\right\}$$

and thus

$$u(m,n) \quad \le \quad G^{-1}\left\{G\left(\varphi^{-1}(c)K(m,n)\right) + K(m,n)\overline{B}(m,n)\right\}.$$

□

Theorem 3.8. *Let φ, a, b, m, n, w and c be the same as in Theorem 2.4, and L, $H \in C\left(\mathbb{R}_0^3, \mathbb{R}_0\right)$ satisfy*

$$0 \leq L(m,n,v) - L(m,n,w) \leq H(m,n,w)(v-w) \tag{3.25}$$

for all $(m,n) \in \mathbb{R}_0^2$, with $v \leq w$. If $u(m,n) \in C(\Omega, \mathbb{R}_0)$ and for any $(m,n) \in \Omega$,

$$\begin{aligned} \varphi(u(m,n)) \leq c &+ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\varphi'(u(s,t))u(s,t) \\ &+ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\varphi'(u(s,t))L(s,t,u(s,t)), \end{aligned} \tag{3.26}$$

then

$$u(m,n) \leq \varphi^{-1}(c)K(m,n) + K(m,n)\mathcal{L}(m,n)\mathcal{H}(m,n), \tag{3.27}$$

where $K(m,n)$ is defined as in Theorem 2.2,

$$\mathcal{L}(m,n) = \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)L\left(s,t,\varphi^{-1}(c)K(s,t)\right), \tag{3.28}$$

and

$$\mathcal{H}(m,n) = \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)K(s,t)H\left(s,t,\varphi^{-1}(c)K(s,t)\right). \tag{3.29}$$

Proof. It suffices to consider the case $c > 0$, for then the case $c = 0$ can be arrived at by continuity argument. Let $c > 0$. Define

$$z(m,n) = \varphi^{-1}\left\{c + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\varphi'(u(s,t))u(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\varphi'(u(s,t))L(s,t,u(s,t))\right\}.$$

Then $z(m_0,n) = z(m,n_0) = \varphi^{-1}(c)$, z is a position nondecreasing function, and $\varphi(u(m,n)) \leq \varphi(z(m,n))$. Now,

$$\triangle_1\varphi(z(m,n)) = \sum_{t=n_0}^{n-1} a(m,t)\varphi'(u(m,t))u(m,t) + \sum_{t=n_0}^{n-1} b(m,t)\varphi'(u(m,t))L(m,t,u(m,t))$$

$$\leq \sum_{t=n_0}^{n-1} \left[a(m,t)\, \varphi^{'}(z(m,t))\, z(m,t) + b(m,t)\, \varphi^{'}(z(m,t))\, L(m,t,z(m,t)) \right].$$

Hence, by the Mean Value Theorem, for each $(m,n) \in \Omega$, there exists ξ with $z(m,n) \leq \xi \leq z(m+1,n)$, such that

$$\begin{aligned} \triangle_1 \varphi(z(m,n)) &= \varphi(z(m+1,n)) - \varphi(z(m,n)) \\ &= \varphi^{'}(\xi)\, \triangle_1 z(m,n). \end{aligned}$$

Since w is non-deceasing, $w(\xi) \leq w(z(m,n))$ and so

$$\triangle_1 z(m,n) = \frac{\triangle_1 \varphi(z(m,n))}{\varphi^{'}(\xi)}$$

$$\begin{aligned} &\leq \frac{1}{\varphi^{'}(\xi)} \varphi^{'}(z(m,n-1)) \sum_{t=n_0}^{n-1} [a(m,t)\, z(m,t) + b(m,t)\, L(s,t,z(m,t))] \\ &\leq \sum_{t=n_0}^{n-1} a(m,t)\, z(m,t) + \sum_{t=n_0}^{n-1} b(m,t)\, L(s,t,z(m,t)) \end{aligned}$$

for all $(m,n) \in \Omega$. Therefore,

$$\sum_{s=m_0}^{m-1} \triangle_1 z(s,n) \leq \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)\, z(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\, L(s,t,z(s,t)).$$

It follows that

$$z(m,n) - z(m_0,n) \leq \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)\, z(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\, L(s,t,z(s,t)),$$

and thus

$$z(m,n) \leq \varphi^{-1}(c) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)\, z(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\, L(s,t,z(s,t)).$$

By Theorem 2.2, we get

$$z(m,n) \le \varphi^{-1}(c)\,K(m,n) + K(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\,L(s,t,z(s,t)).$$

Setting

$$\rho(m,n) = \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\,L(s,t,z(s,t)),$$

then

$$z(m,n) \le \varphi^{-1}(c)\,K(m,n) + K(m,n)\,\rho(m,n). \tag{3.30}$$

Since $L(m,n,v)$ is nondecreasing with respect to v for fixed (m,n), from (3.30) we can obtain that

$$\begin{aligned}
\rho(m,n) &\le \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\,L\big(s,t,\varphi^{-1}(c)\,K(s,t) + K(s,t)\,\rho(s,t)\big) \\
&\le \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\,H\big[s,t,\varphi^{-1}(c)\,K(s,t)\big]\,K(s,t)\,\rho(s,t) \\
&\quad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\,L\big[s,t,\varphi^{-1}(c)\,K(s,t)\big]
\end{aligned}$$

by the condition (3.25). By Theorem 2.2, we get

$$\begin{aligned}
\rho(m,n) &\le \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\,L\big[s,t,\varphi^{-1}(c)\,K(s,t)\big] \\
&\quad \times \left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\,H\big[s,t,\varphi^{-1}(c)\,K(s,t)\big]\,K(s,t)\right\}
\end{aligned} \tag{3.31}$$

Hence, from (3.30) and (3.31), it follows that

$$z(m,n) \le \varphi^{-1}(c)\,K(m,n) + K(m,n)\,\mathcal{L}(m,n)\,\mathcal{H}(m,n),$$

and thus

$$u(m,n) \le \varphi^{-1}(c)\,K(m,n) + K(m,n)\,\mathcal{L}(m,n)\,\mathcal{H}(m,n).$$

□

4. APPLICATIONS

In this section, we use the results obtained in Section 3 to study the boundedness, uniqueness, and continuous dependence of the solutions of certain boundary value problems for difference equations involving 2 independent variables.

Subsection 1: Consider the boundary value problem (BVP):

$$\Delta_{12} z(m,n) = f(m,n,z(m,n), w(z(m,n)))$$

with

$$z(m,n_0) = p(m)\,,\ z(m_0,n) = q(n)\,,\ p(m_0) = q(n_0) = 0.$$

Here, $f \in \mathcal{F}(\Omega \times \mathbb{R})$, $p \in \mathcal{F}(I)$, and $q \in \mathcal{F}(J)$ are given.

Our first result deals with the boundedness of solutions.

Theorem 4.1. *Consider (BVP), and suppose*

$$|f(m,n,u,\varphi(u))| \leq a(m,n)|u| + b(m,n)\varphi(|u|)\,, \tag{4.1}$$

and

$$|p(m) + q(n)| \leq c \tag{4.2}$$

for some $c \geq 0$, where $a, b \in \mathcal{F}_+(\Omega)$, then all solutions of (BVP) satisfy

$$|z(m,n)| \leq G^{-1}\{G(cK(m,n)) + K(m,n)B(m,n)\}\,,$$

where $B(m,n)$, $K(m,n)$ is defined as in Theorem 2.2. In particular, if $B(m,n)$ is bounded on Ω then every solution of (BVP) is bounded on Ω.

Proof. Observe first that $z = z(m,n)$ solves (BVP) if and only if it satisfies the difference equation

$$z(m,n) = p(m) + q(n) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t,z(s,t),w(z(s,t))). \tag{4.3}$$

Hence, by Inequalities (4.1) and (4.2) it follows that

$$|z(m,n)| \leq c + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)|z(s,t)| + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\, w(|z(s,t)|).$$

By Theorem 2.2, we have

$$|z(m,n)| \leq G^{-1}\{G(cK(m,n)) + K(m,n)\, B(m,n)\}$$

for all $(m,n) \in \Omega$. □

The next result is about the uniqueness of solutions.

Theorem 4.2. *Consider* (BVP)*, and*

$$|f(m,n,u_1,\varphi(u_1)) - f(m,n,u_2,\varphi(u_2))| \leq a(m,n)|u_1-u_2| + b(m,n)\,\varphi(|u_1-u_2|), \tag{4.4}$$

and

$$\varphi(|u_1-u_2|) \leq |u_1-u_2| \tag{4.5}$$

for some $a, b \in \mathcal{F}_+(\Omega)$*, then (BVP) has at most one solution on* Ω*.*

Proof. Let $z(m,n)$ and $\overline{z}(m,n)$ be two solutions of (BVP) on Ω. By Equations (4.4) and (4.5), we have

$$\begin{aligned}
|z(m,n) - \overline{z}(m,n)| &\leq \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} |f(s,t,z(s,t),w(z(s,t))) - f(s,t,\overline{z}(s,t),w(\overline{z}(s,t)))| \\
&\leq \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)|z(s,t)-\overline{z}(s,t)| + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\, w(|z(s,t)-\overline{z}(s,t)|) \\
&\leq \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)|z(s,t)-\overline{z}(s,t)| + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)|z(s,t)-\overline{z}(s,t)|.
\end{aligned}$$

By Theorem 2.2, let w be the identity function, we get

$$|z(m,n) - \overline{z}(m,n)| \leq 0$$

for all $(s,t) \in \Omega$. Hence $z = \overline{z}$ on Ω.

$\square$

Finally, we investigate the continuous dependence of the solutions of (BVP) on the function f and the boundary data p and q. For this we consider the corresponding variation of problem (BVPV):

$$\Delta_{12} z(m,n) = \overline{f}(m,n,z(m,n), w(z(m,n)))$$

with

$$z(m,n_0) = \overline{p}(m)\,,\ z(m_0,n) = \overline{q}(n)\,,\ \overline{p}(m_0) = \overline{q}(n_0) = 0.$$

Here, $\overline{f} \in \mathcal{F}(\Omega \times \mathbb{R})\,,\ \overline{p} \in \mathcal{F}(I)$, and $\overline{q} \in \mathcal{F}(J)$ are given.

Theorem 4.3. *Consider (BVP) and (BVPV).* Let $\varepsilon > 0$. If for all $(m,n) \in \Omega$, $u_1, u_2 \in \mathbb{R}$,

(i)$|f(m,n,u_1,\varphi(u_1)) - f(m,n,u_2,\varphi(u_2))| \leq a(m,n)|u_1 - u_2| + b(m,n)\varphi(|u_1 - u_2|)$ *for some* $a, b \in \mathcal{F}_+(\Omega)$;

(ii) $|(p(m) - \overline{p}(m)) + (q(n) - \overline{q}(n))| \leq \frac{\varepsilon}{2}$, $\varphi(|u_1 - u_2|) \leq |u_1 - u_2|$; *and*

(iii) for all solutions $\overline{z}(m,n)$ *of (BVPV),*

$$\sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left| f(m,n,\overline{z}, w(\overline{z})) - \overline{f}(m,n,\overline{z}, w(\overline{z})) \right| \leq \frac{\varepsilon}{2},$$

then

$$|z(m,n) - \overline{z}(m,n)| \leq \varepsilon K(m,n) \exp\left(K(m,n) B(m,n)\right),$$

where $B(m,n)$, $K(m,n)$ *are defined as in Theorem 2.2. Hence,* z *depends continuously on* f, p, *and* q.

Proof. Let $z(m,n)$ and $\overline{z}(m,n)$ be solutions of (BVP) and (BVPV), respectively. Then z satisfies Equation (4.3) and $\overline{z}$ satisfies the corresponding equation

$$\overline{z}(m,n)=\overline{p}(m)+\overline{q}(n)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\overline{f}(s,t,\overline{z}(s,t),w(\overline{z}(s,t))).$$

Hence,

$$\begin{aligned}
|z(m,n)-\overline{z}(m,n)| &\leq |(p(m)-\overline{p}(m))+(q(n)-\overline{q}(n))| \\
&\quad+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|f(m,n,z,w(z))-\overline{f}(m,n,\overline{z},w(\overline{z}))\right| \\
&\leq \frac{\varepsilon}{2}+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}|f(m,n,z,w(z))-f(m,n,\overline{z},w(\overline{z}))| \\
&\quad+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|f(m,n,\overline{z},w(\overline{z}))-\overline{f}(m,n,\overline{z},w(\overline{z}))\right| \\
&\leq \varepsilon+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}(a(m,n)|z-\overline{z}|+b(m,n)w(|z-\overline{z}|)) \\
&\leq \varepsilon+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}(a(m,n)|z-\overline{z}|+b(m,n)|z-\overline{z}|).
\end{aligned}$$

By assumptions (i), (ii), (iii) and Theorem 2.2 to the function $|z(m,n)-\overline{z}(m,n)|$, we have

$$|z(m,n)-\overline{z}(m,n)|\leq \varepsilon K(m,n)(K(m,n)B(m,n)).$$

Now restricted to any compact sub-lattice, $B(m,n)$, $K(m,n)$ is bounded, so

$$|z(m,n)-\overline{z}(m,n)|\leq \varepsilon T$$

for some $T>0$ and for all (m,n) in this compact sub-lattice. Hence, z depends continuously on f, p, and q. □

Subsection 2: Consider the boundary value problem of hyperbolic partial delay equation (BVPH)

$$D_2 D_1 u^p (x,y) = F(x,y,u(x-h_{11}(x), y-h_{12}(y)), u(x-h_{21}(x), y-h_{22}(y))) \quad (4.6)$$

with

$$u(x,y_0) = k_1(x), u(x_0,y) = k_2(n), k_1(x_0) = k_2(y_0) = 0. \quad (4.7)$$

Here, $p \geq 1$ is a constant, $I_1 = [x_0, X)$, $I_2 = [y_0, Y)$, $\triangle = I_1 \times I_2$, $k_1 \in C(I_1, \mathbb{R})$, $k_2 \in C(I_2, \mathbb{R})$, $h_{i1}(x) \in C^1(I_1, \mathbb{R}_0)$, $h_{i2}(y) \in C^1(I_2, \mathbb{R}_0)$ with $x - h_{i1}(x) \geq 0$, $y - h_{i2}(y) \geq 0$, $h'_{i1}(x) < 1$, $h'_{i2}(y) < 1$ and $h_{i1}(x_0) = h_{i2}(y_0) = 0$, $i = 1, 2$.

Consider (BVPH). If

$$|F(x,y,u,v)| \leq |u|^{p-1}(a(x,y)|u| + b(x,y)w(|v|)), \quad (4.8)$$

and

$$|k_1(x) + k_2(y)| \leq c, \quad (4.9)$$

then

$$u(m,n) \leq G^{-1}\left\{G\left(c^{\frac{1}{p}}\overline{K}(m,n)\right) + \overline{K}(m,n)\overline{B}(m,n)\right\},$$

where $a, b \in C(\triangle, \mathbb{R}_0)$, $c \geq 0$ is a constant, w is defined as in Theorem 2.1 and let

$$H_{i1} = \max_{x \in I} \frac{1}{1 - h'_{i1}(x)}, \qquad H_{i2} = \max_{y \in J} \frac{1}{1 - h'_{i2}(y)}, \qquad i = 1, 2. \quad (4.10)$$

$$\begin{aligned} \overline{K}(m,n) &:= \prod_{s=m_0}^{m-1}\left[1 + \sum_{t=n_0}^{n-1} \overline{a}(\sigma,\tau)\right], \\ \overline{B}(m,n) &:= \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \overline{b}(\sigma,\tau). \end{aligned}$$

It is easy to observe that every solution $u(m,n)$ of (4.6)-(4.7) satisfies the equivalent integral equation

$$u^p(x,y) = k_1(x) + k_2(y) + \int_{x_0}^{x}\int_{y_0}^{y} F(s,t,u(s-h_{11}(s),t-h_{12}(t)),u(s-h_{21}(s),t-h_{22}(t)))\,dt\,ds. \quad (4.11)$$

Applying (4.8)-(4.10) to (4.11) and changing the variables we obtain

$$\begin{aligned} |u(x,y)|^p &\le c + \int_{\alpha_1(x_0)}^{\alpha_1(x)}\int_{\beta_1(y_0)}^{\beta_1(y)} \overline{a}(\sigma,\tau)|u(\sigma,\tau)|^p\,d\tau d\sigma \\ &+ \int_{\alpha_1(x_0)}^{\alpha_1(x)}\int_{\beta_1(y_0)}^{\beta_1(y)} \overline{b}(\sigma,\tau)|u(\sigma,\tau)|^{p-1} w(u(\sigma,\tau))\,d\tau d\sigma, \end{aligned} \quad (4.12)$$

where $\alpha_i(x) = x - h_{i1}(x)$, $\beta_i(y) = y - h_{i2}(y)$, $i = 1,2$, $\overline{a}(\sigma,\tau) = H_{11}H_{12}a\left(\alpha_1^{-1}(\sigma),\right.$ $\left.\beta_1^{-1}(\tau)\right)$ $\overline{b}(\sigma,\tau) = H_{21}H_{22}a\left(\alpha_2^{-1}(\sigma),\beta_2^{-1}(\tau)\right)$ and $\tilde{\triangle}$ denote the maximal existent interval of $u(x,y)$.

Now, we discretize the inequalities (4.12), we can obtain

$$|u(m,n)|^p \le c + \sum_{s=m_0}^{m-1}\sum_{s=n_0}^{n-1} \overline{a}(\sigma,\tau)|u(\sigma,\tau)|^p + \sum_{s=m_0}^{m-1}\sum_{s=n_0}^{n-1} \overline{b}(\sigma,\tau)|u(\sigma,\tau)|^{p-1} w(u(\sigma,\tau)). \quad (4.13)$$

Then, by the Theorem 2.4, we get

$$u(m,n) \le G^{-1}\left\{G\left(c^{\frac{1}{p}}\overline{K}(m,n)\right) + \overline{K}(m,n)\overline{B}(m,n)\right\}.$$

Hence we can get the upper bound of $u(m,n)$.

5. CONCLUSIONS

Although our main result is far from the generalization of Theorem A and Theorem B, it is indeed not a special case of them. As stated in Remark 3.3, Theorem 3.2 can not be derived from Theorem 2.7. On one hand, $p > 1$ is one of the conditions in Theorem 2.7. On the other hand, $u(s,t) \cdot \varphi(u(s,t))$ is not identical with $\varphi(u(s,t))$ for both Theorem 2.7 and Theorem 2.8. Hence, in the future we will consider whether we can find a more general pattern to cover these different situations.

In this work, we discuss the case with finite domain. In Chapter 5 of [6], Pachpatte considered some infinite case which is stated as follows:

Theorem. *5.4.1 Let* $u(m,n)$ $a(m,n)$ $b(m,n) \in \mathcal{D}(N_0^2, \mathbb{R}_+)$, *where* $N_0 = \{0, 1, ...\}$.

(a_1) Let $a(m,n)$ be nondecreasing in m and nonincreasing in n. If

$$u(m,n) \leq a(m,n) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} b(s,t)\, u(s,t)$$

for all $(m,n) \in N_0$, then

$$u(m,n) \leq a(m,n) \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{\infty} b(s,t)\right]$$

for all $(m,n) \in N_0$.

(a_2) Let $a(m,n)$ be nonincreasing in each variable m and n. If

$$u(m,n) \leq a(m,n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} b(s,t)\, u(s,t),$$

for all $(m,n) \in N_0$, then

$$u(m,n) \leq a(m,n) \prod_{s=m+1}^{\infty} \left[1 + \sum_{t=n+1}^{\infty} b(s,t)\right]$$

for all $(m,n) \in N_0$.

We conjecture that our main results can also be extended to infinity.

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