# Balanced Realization for Stable Nonlinear Input-Affine Systems

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## Abstract

In this paper, we are concerned with the balancing for stable nonlinear time invariant input-affine system. Firstly, the system's Hankel norm is computed via game theoretical approach incorporating with the parameter optimization technique. This computation algorithm is derived based on the continuity of the costate vector at present time, which leads to a benefit to avoid solving the controllability and observability functions through a set of partial differential equations. Afterward, the balanced realization is conducted. Certain numerical examples are used to demonstrate the computational technique for Hankel norm and balanced realization.

**Keywords:** Hankel norm, nonlinear system, game theory, parameter optimization, balancing

## 1   Introduction

In many engineering applications, processes are described by complex models which are difficult to analyze and difficult to control. Reduction of the order of the model may overcome some of these difficulties, but it is quite possible that model reduction incurs a significant loss of accuracy. Therefore, the system has to be analyzed in a manner that is useful for the application purpose. Simplification of the model based on this analysis usually results in a model if lower complexity which is easier to handle, and in a corresponding simplification of synthesis procedures for control and filtering problems. Furthermore, the simplification decreases the computational effort. Every application has its own demands, and different model reduction methods have different properties. For example, we refer to van Woerkom [18], where a survey of order reduction approaches for flexible spacecraft dynamics is given.

For similar reasons, it is also desirable to have methods available for designing lower-order controllers for high-order systems. The implementation of lower-order controller is simpler, since

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there are fewer things to go wrong in the hardware or bugs to fix in the software, they are easy to understand, an the computational requirements are lower. Controller reduction methods may be divided into a direct and indirect class. The direct methods are based on some optimization or other procedure by which lower-order controllers is obtained, and the indirect methods rely upon the higher-order controller which is first found, and then simplified to a lower-order controller (see e.g. Anderson and Liu [3]).

In practice, model reduction approximation is often based on trial and error methods, knowledge about the physical properties of the system, or other intuitive methods. Since this is not every satisfying (it is for instance for obvious reasons better to obtain immediately the approximation that is the best from an analysis and control purpose point of view), formalization of model reduction has been studied by a number of people. For instance, Glover [5] investigated the optimality of model approximations in the Hankel norm, and gave a formal characterization of all optimal Hankel norm approximations.

Moore [10] introduced the balancing for stable minimal linear systems. The balancing method offers a tool to measure the contribution of the different state components to the past input and future output energy of the system, which are measures of controllability and observability. This analysis yield a methodology for model reduction. Since its introduction, balancing theory for stable linear systems has been formalized in several directions. Balancing as a model reduction method has been formalized by Glover [5], and Enns [4], who obtained an upper-bound for the error in the Hankel and $L^\infty$ norm, respectively. Furthermore, open-loop balancing theory has been generalized to a balancing method for unstable linear systems(e.g. Meyer [8], Ober and McFarlane [12]), a balancing method for mechanical system (e.g. Van der Schaft and Oeloff [17]), and to closed-loop balancing methods (e.g. Jonckheere and Silverman [6], Opdenacker and Jonckheere [13], Mustafa and Glover [11]).

The Hankel norm is defined as the supremum of the ratio of the future output energy to the minimal input control energy and it has the same value as the ratio of the controllability function to the observability function. Hankel norm approximation method is one of the most popular approach to find a suitable approximation for a given high-order model, which is a crucial issue in the controller design process. The key step in Hankel norm model reduction is to find the balanced realization of the system. Hence, balancing theory is closely related to Hankel norm approximation. Hankel norm model reduction in state-space approach and in frequency-domain approach has been addressed in many literatures. Readers have interested in this topic can refer to Scherpen [16] and the references therein.

In this study, we are concerned with the balancing for nonlinear system. Firstly, the Hankel norm of nonlinear, time-invariant systems is computed through game theory together with parameter optimization technique which can avoid to solve a set of partial differential equations proposed by Scherpen [16]. Afterward, we consider the balanced realization for stable nonlinear system. In

Scherpen [16], the Morse lemma was used to guarantee the existence of the coordinate transformation such that Lemma 4.2 in this paper holds, but the method to find this transformation is not provided. Here, the construction of this transformation is derived.

This paper is organized as follows. In section 2, we summary certain matrix calculus and some important definitions and lemmas for later use. The computation algorithm for the Hankel norm of nonlinear input-affine system is then established in section 3 and we also demonstrates that the calculation of the Hankel norm of a linear system can be treated as a special case of the nonlinear system. Section 4 is devoted to the construction of balanced realization for the nonlinear system. Finally, some numerical examples are used to illustrate this analytic computation of nonlinear Hankel norm in section 5. In section 6, some concluding remarks are made.

## 2  Mathematical preliminary

In this section, we summarize some important mathematical results including matrix calculus. A useful form of partitioned matrix is obtained by defining the *Kronecker product* of two matrices $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ and $B = [b_{ij}] \in \mathbb{R}^{p \times q}$ by

$$A \otimes B = [a_{ij}B] = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \tag{2.1}$$

Thus $A \otimes B$ is an $mp \times nq$ matrix and is partitioned into the $mn$ blocks shown in (2.1).

The word *smooth* and *differentiable* will be used interchangeably to mean differentiable of class $C^{\infty}$.

Let $x = [x_1 \ x_2 \ \cdots \ x_n]^T \in \mathbb{R}^n$ be a vector, $s(x) \in \mathbb{R}$ be a scalar real-value function, and $f(x) \in \mathbb{R}^m$ be a vector field, defined on an open subset $W$ of $\mathbb{R}^n$. We describe now the differential operation involving these real-value functions. The *gradient* of $s$ with respect to $x$ is the *column* vector

$$s_x \triangleq \frac{\partial s}{\partial x} = \begin{bmatrix} \frac{\partial s}{\partial x_1} \\ \frac{\partial s}{\partial x_2} \\ \vdots \\ \frac{\partial s}{\partial x_n} \end{bmatrix}, \quad s_x^T = \left( \frac{\partial s}{\partial x} \right)^T = \frac{\partial^T s}{\partial x} \tag{2.2}$$

The *Hessian* of $s$ with respect to $x$ is the second derivative

$$s_{xx} \triangleq \frac{\partial^2 s}{\partial x^2} = \left[ \frac{\partial^2 s}{\partial x_i \partial x_j} \right]$$

which is a symmetric $n \times n$ matrix. The *Jacobian* of $f$ with respect to $x$ is the $m \times n$ matrix

$$
\begin{aligned}
f_x \quad &\triangleq \quad \frac{\partial f}{\partial x} = \left[ \frac{\partial f}{\partial x_1} \ \frac{\partial f}{\partial x_2} \ \cdots \ \frac{\partial f}{\partial x_n} \right] \\
&= \quad \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}
\end{aligned}
\tag{2.3}
$$

We shall use the shorthand notation

$$
\frac{\partial^T f}{\partial x} = \left( \frac{\partial f}{\partial x} \right)^T \in \mathbb{R}^{n \times m}
$$

And the chain rule is denoted by

$$
\frac{\partial}{\partial x}(sf) = \frac{\partial}{\partial x}(fs) = sf_x + fs_x^T
$$

Sometime, we use $\frac{\partial}{\partial x}$ to denote the differential operator, i.e.

$$
\frac{\partial}{\partial x} = \left[ \frac{\partial}{\partial x_1} \ \ \frac{\partial}{\partial x_2} \ \ \cdots \ \ \frac{\partial}{\partial x_n} \right]^T
$$

**Lemma 2.1** *Let L be a smooth function in a convex neighborhood W of 0 in $\mathbb{R}^n$, with $L(0) = 0$. Then*

$$
L(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n x_i a_i (x_1, x_2, \ldots, x_n)
$$

*for some suitable smooth function $a_i$ defined on W, with $a_i(0) = \frac{\partial L}{\partial x_i}(0)$.*

**Proof:** See Lemma 2.1 of Milnor(1969). □

From this lemma, if we let

$$
A(x) = \begin{bmatrix} a_1(x) \\ a_2(x) \\ \vdots \\ a_n(x) \end{bmatrix}
$$

then, the function $L$ can be expressed as

$$
L(x) = A(x)x
$$

with $A(0) = \frac{\partial L}{\partial x}(0)$.

**Definition 2.2** *The* tangent space *of a smooth manifold M at a point p will be denoted by $TM_p$. If $g: M \rightarrow N$ is a smooth map with $g(p) = q$, then the induced linear map of tangent spaces will be denoted by $g_* : TM_p \rightarrow TN_q$*

**Definition 2.3** *Let f be a smooth real valued function on a manifold M. A point p is called a* critical point *of f if*

$$f_x(p) = 0$$

*where x denotes a local coordinate system in a neighborhood of U of p. The real number $f(p)$ is called a* critical values. *Moreover, a critical point p of f is called* non-degenerate *if the matrix*

$$f_{xx} = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right]$$

*is nonsingular.*

**Definition 2.4** *Let p be a critical point of f. The* Hessian *of f at p is a symmetric bilinear functional, $f_{**}$, on $TM_p$ which is defined by*

$$f_{**}(v, w) = \sum_{i,j=1}^{n} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

*where $(x_1, x_2, \dots, x_n)$ is the local coordinate system, and v and w are elements of $TM_p$ given by*

$$v = \sum_{i=1}^{n} a_i \left. \frac{\partial}{\partial x_i} \right|_p, \quad w = \sum_{i=1}^{n} b_i \left. \frac{\partial}{\partial x_i} \right|_p$$

*with $a_i$ and $b_i$ are constant functions.*

**Definition 2.5** *The* index *of a bilinear functional H, on a vector field V, is defined to be the maximal dimension of a subspace of V on which H is negative definite; the* nullity *is the dimension of the null-space, i.e. the subspace consisting of all $v \in V$ such that $H(v, w) = 0$ for every $w \in V$.*

**Lemma 2.6** [Lemma of Morse] *Let p be non-degenerate critical point for f. Then there is a local coordinate system $(y_1, y_2, \dots, y_n)$ in a neighborhood U of p with $y_i(p) = 0$ for all i and such that the identity*

$$f(x) = f(p) - (y_1)^2 - \cdots - (y_\lambda)^2 + (y_{\lambda+1})^2 + \cdots + (y_n)^2$$

*holds throughout U, where $\lambda$ is the index of f at p.*

**Proof:** See Lemma 2.2 of Milnor(1969). $\square$

**Lemma 2.7** *If there exists a neighborhood $V$ of $0$ where the number of distinct eigenvalues of $M(x)$ is constant for $x \in V$, then on $V$ the eigenvalues $\lambda_i(x)$, $i = 1, 2, \ldots, n$, are smooth functions of $x$, as well as the associated eigenvectors.*

**Proof:** Follows from Theorem 5.13a in Kato(1982). □

# 3 Computation of Hankel norm of nonlinear system

Consider a smooth, i.e. $C^\infty$, nonlinear input-affine system of the form

$$G: \quad \begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t), \quad x(0) = x_0 \\ y(t) &= h(x(t)) \end{aligned}$$

(3.1a)

(3.1b)

with $t \in (-\infty, +\infty)$, $u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T \in \mathbb{R}^m$, $y(t) = [y_1(t), y_2(t), \ldots, y_p(t)]^T \in \mathbb{R}^p$, and $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ are local coordinates for a smooth state space manifold denoted by $M$. Furthermore, $f, g_1, g_2, \ldots, g_m$ are smooth vector fields on $M$, isomorphically embedded in $\mathbb{R}^n$, where $g = [g_1, g_2, \ldots, g_m]$, and $h = [h_1, h_2, \ldots, h_p]^T$, $h_i(x) \in \mathbb{R}$ is the smooth output map of the system. Without loss of generality, we assume that the system has an equilibrium in $0$, i.e. $f(0) = 0$ and $h(0) = 0$.

**Remark 3.1** The function $g(x)$ may have the following component form

$$g(x) = [g_1 \ g_2 \ \cdots \ g_m] = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & g_{22} & \cdots & g_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nm} \end{bmatrix}$$

Assume the solution of the system (3.1a) is

$$x(t) = \varphi(t, t_0, x_0, u), \qquad \varphi(t_0, t_0, x_0, u) = x_0$$

under the influence of the control input function $u$. Then we can define the system's reachability and observability as following:

**Definition 3.2** *[van der Schaft(1992)]*

1. *The system (4) is* reachable *from $x_0$ if for any $\tilde{x} \in M$ there exists a $\tilde{t} \geq 0$, and an input $u$ such that $\tilde{x} = \varphi(\tilde{t}, t_0, x_0, u)$.*

2. *The system (4) is* zero-state observable *if $u \equiv 0$, $y \equiv 0$ implies $x(t) \equiv 0$, i.e. for all $x \in M$, $h(\varphi(\tilde{t},t_0,x_0,u)) = 0$, for all $t \geq t_0 \Rightarrow \varphi(\tilde{t},t_0,x_0,u) = 0$, for all $t \geq t_0$.*

3. *The system (4) is* locally zero-state observable *at 0, if there exists a neighborhood $W \subset M$ of 0 such that for all $x \in W$, $h(\varphi(\tilde{t},t_0,x_0,u)) = 0$, for all $t \geq t_0 \Rightarrow \varphi(\tilde{t},t_0,x_0,u) = 0$, for all $t \geq t_0$.*

## 3.1 The controllability function and observability function

Define the controllability and obeservability functions, $L_c(x_0)$ and $L_o(x_0)$, respectively, of the nonlinear system (3.1) as:

$$L_c(x(t)) \triangleq \min_{\substack{u \in L^2_- \\ x(-\infty)=0}} \frac{1}{2} \int_{-\infty}^{t} u(\tau)^T u(\tau) \, d\tau \tag{3.2}$$

$$L_o(x(t)) \triangleq \frac{1}{2} \int_{t}^{\infty} y(\tau)^T y(\tau) \, d\tau \quad (u(\tau) \equiv 0 , t \leq \tau < \infty) \tag{3.3}$$

where $L^2_- \triangleq L^2((-\infty,0])$. Obviously, these functions do not necessarily exist, i.e. are not necessarily finite. In particular, $L_c(x_0)$ may be infinite if the state $x_0$ cannot be asymptotically reached form 0 in backward time, i.e. there exists no input $u \in L^2_-$ such that the system state $x(t)$ is transferred from the rest $x(-\infty) = 0$ to the current state $x(0) = x_0$ and the function $L_o(x_0)$ may also be infinite if the system is unstable. Also, we observe that $L_c(0) = 0$ and $L_o(0) = 0$ due to the reason that when $x_0 = 0$, there is no input needed to drive the system state $x(t)$ from a rest to a rest and hence zero system output will be generated.

**Assumptions:** [Scherpen(1994)] In order to have meaningful derivations in the rest of this paper, we make the following standing assumptions:

1. $f(x)$ is asymptotically stable on some convex neighborhood $W \subset M$ of 0.

2. The system is zero-state observable on $W$.

3. $L_c$ and $L_o$ exist and are smooth on $W$.

4. $\frac{\partial^2 L_c}{\partial x^2}(0) > 0$ and $\frac{\partial^2 L_o}{\partial x^2}(0) > 0$

## 3.2 Variational approach

If the system state $x(t)$ is initially relaxed in the infinitely remote pass, i.e. $\lim_{t \to -\infty} x(t) = 0$, we can denote the solution of the system (3.1a) in a simpler form as

$$x(t) = \varphi(t,u), \qquad \varphi(-\infty,u) = 0 \tag{3.4}$$

under control by the input function $u$.

Based on the assumptions , we define the Hankel norm of this nonlinear system as

**Definition 3.3** *The Hankel norm of the nonlinear input-affine system G is defined as*

$$\|G\|_H \triangleq \sup_{\substack{u \in L_-^2 \\ u \neq 0}} \frac{\|y\|_{L_+^2}}{\|u\|_{L_-^2}} \tag{3.5}$$

*where $L_+^2 \triangleq L^2([0,+\infty))$,*

$$\|y\|_{L_+^2} \triangleq \left[ \int_0^\infty y(t)^T y(t) \, dt \right]^{\frac{1}{2}}, \quad and \quad \|u\|_{L_-^2} \triangleq \left[ \int_{-\infty}^0 u(t)^T u(t) \, dt \right]^{\frac{1}{2}}$$

Let the collection of all $u \in L_-^2$ such that the current state $x_0$ can be reachable from rest $x(-\infty) = 0$ be denoted by $U_0$, i.e.

$$U_0 = \{ u \in L_-^2 \, | \, \varphi(0, u) = x_0 \}, \tag{3.6}$$

and hence, $x_0$ must belong to the reachable set of 0, i.e. $x_0 \in W$. Thus we have the following simple theorem:

**Theorem 3.4** *Suppose the nonlinear system G satisfies the standing assumptions, then*

$$\|G\|_H = \sup_{\substack{x_0 \in W \\ x_0 \neq 0}} \mu(x_0) \tag{3.7}$$

*where*

$$\mu(x_0) = \max_{u \in U_0} \frac{\|y\|_{L_+^2}}{\|u\|_{L_-^2}} \tag{3.8}$$

**Proof:** Since the controllability function $L_c(x_0)$ of the system $G$ exists, hence $U_0$ is nonempty and there exists $u \in L_-^2$ such that the trajectory of $x$ transfers from $x(-\infty) = 0$ to $x(0) = x_0$ for arbitrary $x_0$. Similarly, the observability function $L_o(x_0)$) exists. Therefore we can express the Hankel norm of the system $G$ in (3.5) as

$$\|G\|_H = \sup_{\substack{x_0 \in W \\ x_0 \neq 0}} \max_{u \in U_0} \frac{\|y\|_{L_+^2}}{\|u\|_{L_-^2}}$$

for a given $x_0$, then after defining $\mu(x_0)$ as given by (3.8), the desired result is followed. □

**Remark 3.5** The computation of $\|G\|_H$ can be done in two sequential steps as follows:

**Step 1:** For a given nonzero $x_0 \in W$, find $\mu(x_0)$.

**Step 2:** Find the supremum of $\mu(x_0)$ for all $x_0 \in W$.

Suppose $\gamma$ is an upper bound of $\mu(x_0)$, then

$$\mu^2(x_0) \leq \gamma^2 \quad \text{if and only if} \quad \gamma^2 \|u\|_{L^2_-}^2 - \|y\|_{L^2_+}^2 \geq 0 \ \forall \, u \in U_0 \tag{3.9}$$

Let $\gamma_*$ be the maximal value of $\mu(x_0)$, then there exists the optimal output function $y_*$ and the control input function $u_*$ satisfying the following equation:

$$\gamma_*^2 = \mu^2(x_0) = \frac{\|y_*\|_{L^2_+}^2}{\|u_*\|_{L^2_-}^2} \tag{3.10}$$

Therefore, for any given $x_0 \in W$, the computation of $\mu(x_0)$ is equivalent to solve the following minimal energy problem:

Given

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad x(0) = x_0, \ x(-\infty) = 0 \\ y &= h(x) \end{aligned}$$

Find

$$\min_{u \in U_0} \left\{ J(x(t), u(t)) = \frac{\gamma^2}{2} \int_{-\infty}^{0} u(t)^T u(t) dt - \frac{1}{2} \int_{0}^{\infty} y(t)^T y(t) dt \right\} \tag{3.11}$$

The associated the Hamiltonian function for this minimal problem is

$$H(x, u, \lambda) = \begin{cases} \frac{\gamma^2}{2} u^T u + \lambda^T (f(x) + g(x)u) & \text{if} \quad t \in (-\infty, 0] \\ \lambda^T f(x) - \frac{1}{2} h(x)^T h(x) & \text{if} \quad t \in [0, \infty) \end{cases}$$

where $\lambda(t)$, the Lagrange multiplier, is a continuous function and it must satisfy the condition $\lambda(-\infty) = \lambda(\infty) = 0$. Since the Hamiltoninan function of dynamic system must satisfy

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial u}\dot{u} + \frac{\partial H}{\partial \lambda}\dot{\lambda} = \frac{\partial H}{\partial t}$$

along the optimal trajectory $(x_*, u_*, \lambda_*)$, and now our system is time-invariant, i.e. $\frac{\partial H}{\partial t} = 0$. Therefore the Hamiltonian function along the optimal trajectory is constant, i.e.

$$H(x_*, u_*, \lambda_*) = 0 \tag{3.12}$$

due to the condition for $\lambda(-\infty) = \lambda(\infty) = 0$.

The necessary condition for this extremal problem is computed as follows:

(I) when $t \in (-\infty, 0]$:

The optimal control law $u_*$ satisfies $\frac{\partial H}{\partial u} = 0$, and this gives

$$\gamma^2 u_* + g(x)^T \lambda_* = 0 \quad \text{or} \quad u_* = -\frac{1}{\gamma^2} g(x)^T \lambda_* \tag{3.13}$$

Since

$$
\begin{aligned}
\frac{\partial \lambda^T g(x) u}{\partial x} &= \sum_i \sum_j \lambda_i \frac{\partial g_{ij}(x)}{\partial x} u_j \\
&= \sum_i \sum_j \lambda_i \begin{bmatrix} \frac{\partial g_{ij}(x)}{\partial x_1} \\ \frac{\partial g_{ij}(x)}{\partial x_2} \\ \vdots \\ \frac{\partial g_{ij}(x)}{\partial x_n} \end{bmatrix} u_j = \sum_i \sum_j \begin{bmatrix} \lambda_i \frac{\partial g_{ij}(x)}{\partial x_1} u_j \\ \lambda_i \frac{\partial g_{ij}(x)}{\partial x_2} u_j \\ \vdots \\ \lambda_i \frac{\partial g_{ij}(x)}{\partial x_n} u_j \end{bmatrix} \\
&= (I_n \otimes \lambda^T)(\frac{\partial}{\partial x} \otimes g) u = (I_n \otimes u^T)(\frac{\partial}{\partial x} \otimes g^T)\lambda
\end{aligned}
$$

where $I_n$ denotes an $n \times n$ identity matrix, then the adjoint equation $\dot{\lambda} = -\frac{\partial H}{\partial x}$ becomes

$$\dot{\lambda}_* = -\frac{\partial^T f}{\partial x}\lambda_* + \frac{1}{\gamma_*^2}(I_n \otimes \lambda^T g)(\frac{\partial}{\partial x} \otimes g^T)\lambda_* \tag{3.14}$$

Hence we can consider $\lambda_*$ as an explicit function of $x$. Since from (3.1a) and (3.15)

$$\dot{\lambda} = \frac{\partial^T \lambda}{\partial x}\frac{dx}{dt} = \frac{\partial^T \lambda}{\partial x}\left(f(x) - \frac{1}{\gamma^2}g(x)g(x)^T\lambda\right)$$

and, after some algebraic operator, we have the following equation for the optimal costate vector:

$$\frac{\partial}{\partial x}\left(f(x)^T\lambda - \frac{1}{2\gamma_*^2}\lambda^T g(x)g(x)^T\lambda\right)\Big|_{(x_*, \lambda_*)} = 0$$

This leads to the same equation obtained by using the fact that the Hamiltonian function along the optimal trajectory is constant and equal to zero. Let the solution of (3.13) be in the form

$$\lambda_*(x(t)) = -\gamma^2 \Psi_-(x(t)) \quad \text{with } \Psi_-(0) = 0 \tag{3.15}$$

which satisfies the requirement $\lambda(-\infty) = \lambda(x(-\infty)) = 0$ and the function $\Psi_-(x)$ satisfies the following equation:

$$\Psi_-(x)^T f(x) + \frac{1}{2}\Psi_-(x)^T g(x)g(x)^T \Psi_-(x) = 0 \tag{3.16}$$

The substitution of (3.16) into (3.13) yields

$$u_* = +g(x)^T \Psi_-(x) \tag{3.17}$$

Thus (3.1a), for the optimal state $x_*(t)$, becomes

$$\dot{x} = f(x) + g(x)g(x)^T \Psi_-(x) \tag{3.18}$$

In order to have a stable solution, we need that $-(f(x) + g(x)g(x)^T \Psi_-(x))$ is asymptotically stable on $W$. Let the solution of this equation be

$$x(t) = \Phi_-(t; x_0) \quad \text{with } \Phi_-(-\infty; x_0) = 0 \tag{3.19}$$

It is noted that $\Phi_-(0; x)$ has fixed point at $x = x_0$ and $\Phi_-(-\infty; x) = 0$ for all $x \in W$. Therefore, the optimal control input is

$$u_*(t) = g(\Phi_-(t; x_0))^T \Psi_-(\Phi_-(t; x_0)) \tag{3.20}$$

and optimal costate vector is

$$\lambda_*(t) = -\gamma_*^2 \Psi_-(\Phi_-(t; x_0)) \tag{3.21}$$

Thus

$$\lambda_*(0) = -\gamma_*^2 \Psi_-(x_0) \tag{3.22}$$

(II) when $t \in [0, \infty)$:

Since no control will be input to the system, the optimal control law is $u_*(t) = 0$. The state equation becomes as

$$\dot{x}(t) = f(x), \quad y(x) = h(x) \tag{3.23}$$

Since $f(x)$ is asymptotically stable on $W$, the explicit form of the optimal state vector may take the form

$$x_*(t) = \Phi_+(t; x_0) \quad \text{with } \Phi_+(\infty; x_0) = 0 \tag{3.24}$$

The optimal output is

$$y_*(t) = h(\Phi_+(t; x_0)) \tag{3.25}$$

The corresponding adjoint equation gives

$$\dot{\lambda}_* = -\frac{\partial^T f}{\partial x} \lambda_* + \frac{\partial^T h}{\partial x} h(x)$$

Using the similar idea to solve the costate vector for $t \in (-\infty, 0)$, we may consider $\lambda_*$ as an explicit function of $x_*$ and since

$$\dot{\lambda} = \frac{d\lambda}{dt} = \left(\frac{\partial \lambda}{\partial x}\right)^T \frac{dx}{dt} = \frac{\partial^T \lambda}{\partial x} f(x)$$

then it follows

$$\left. \frac{\partial}{\partial x}\left(f(x)^T \lambda - \frac{1}{2}h(x)^T h(x)\right)\right|_{x_*, \lambda_*} = 0$$

This is the same as the equation given by the fact of zero-value Hamiltonian function along the optimal trajectory. Let the solution of the adjoint equation be in the form

$$\lambda_*(t) = \lambda_*(x(t)) = -\Psi_+(x(t)) \quad \text{with} \quad \Psi_+(0) = 0 \tag{3.26}$$

with $\lambda(\infty) = 0$, and then the function $\Psi_+(x)$ must satisfy

$$\Psi_+(x)^T f(x) + \frac{1}{2}h(x)^T h(x) = 0 \tag{3.27}$$

Due to the continuity of $\lambda_*$ at $t = 0$, we have

$$\gamma_*^2 \Psi_-(x_0) = \Psi_+(x_0) \tag{3.28}$$

where $\gamma_*$ is the corresponding optimal value of $\mu(x_0)$. Multiplying the both sides of (3.28) with nonzero vector $x_0$, one obtains

$$\gamma_*^2 = \frac{x_0^T \Psi_+(x_0)}{x_0^T \Psi_-(x_0)} \tag{3.29}$$

Therefore, from (3.10) one obtains

$$\mu^2(x_0) = \frac{x_0^T \Psi_+(x_0)}{x_0^T \Psi_-(x_0)} \tag{3.30}$$

The sufficient condition for this minimal problem is given in the book of Athans and Falb(1966) or Anderson and Moor(1990), and it requires the following matrix

$$\begin{bmatrix} \frac{\partial^2 H}{\partial u^2} & \frac{\partial^2 H}{\partial u \partial x} \\ \frac{\partial^2 H}{\partial x \partial u} & \frac{\partial^2 H}{\partial x^2} \end{bmatrix}$$

to be semipositive along the optimal trajectory $(x_*(t), u_*(t), \lambda_*(t))$. Or equivalently, the optimal trajectory must give the cost functional $J$ to be minimal , i.e.

$$J(x(t), u(t)) \geq J(x_*(t), u_*(t)) \quad \forall u(t) \neq u_*(t)$$

It is noted that the following simple lemma holds

**Lemma 3.6**

$$\frac{L_o(x_0)}{L_c(x_0)} = \frac{x_0^T \Psi_+(x_0)}{x_0^T \Psi_-(x_0)}$$

(3.31)

**Proof:** The minimal value of the cost function $J(x_*, u_*)$ is given by

$$
\begin{aligned}
J(x_*, u_*) &= \min_{u \in U_0} \frac{\gamma^2}{2} \int_{-\infty}^0 u(t)^T u(t) dt - \frac{1}{2} \int_0^\infty y(t)^T y(t) dt, \\
&= \gamma_*^2 L_c(x_0) - L_o(x_0) = 0,
\end{aligned}
$$

or equivalently,

$$\gamma_*^2 = \frac{L_o(x_0)}{L_c(x_0)}$$

(3.32)

By (3.29) and (3.32), the equality in (3.31) holds. $\qquad\qquad\square$

**Remark 3.7** Substituting (3.19) into (3.2), we obtain

$$
\begin{aligned}
L_c(x_0) &= \frac{1}{2} \int_{-\infty}^0 \Psi_-(x)^T g(x) g(x)^T \Psi_-(x) dt \\
&= \int_{-\infty}^0 \left( \Psi_-(x)^T f(x) + \Psi_-(x)^T g(x) g(x)^T \Psi_-(x) \right) dt \quad \text{(by (3.16))} \\
&= \int_0^{x_0} \Psi_-(x)^T dx \quad \text{( by (3.18))}
\end{aligned}
$$

hence,

$$\frac{\partial L_c}{\partial x}(x_0) = \Psi_-(x_0)$$

(3.33)

According to assumptions that $L_c$ exists and $\frac{\partial^2 L_c}{\partial x^2}(0) > 0$, it can be shown that have $L_c(x) > 0$, $\forall x(\neq 0) \in W$ (see Theorem 3.1.8, Scherpen(1994)).

**Remark 3.8** From (3.3) it follows

$$
\begin{aligned}
L_o(x_0) &= \frac{1}{2} \int_0^\infty h(x)^T h(x) dt \\
&= -\int_0^\infty \Psi_+(x)^T f(x) dt \quad \text{(by (3.27))} \\
&= -\int_{x_0}^0 \Psi_+(x)^T dx \quad \text{(by (3.23))}
\end{aligned}
$$

Therefore

$$\frac{\partial L_o}{\partial x}(x_0) = \Psi_+(x_0)$$

(3.34)

Similarly, with assumptions that this system is locally zero-state observable, $L_o$ exists and $\frac{\partial^2 L_o}{\partial x^2}(0) > 0$, hence, it can be prove that $L_o(x) > 0$, $\forall\, x(\neq 0) \in W$ (see Theorem 3.1.12, Scherpen(1994)).

The following theorem has been derived in Scherpen(1994) (Theorem 3.1.2 and 3.1.3) and we provide an alternative proof here.

**Theorem 3.9** *Assume that $f(x)$ is asymptotically stable on a neighborhood $W$ of $0$. $L_o$ exists and is smooth on $W$ if and only if $L_o$ is the unique smooth solution of*

$$\frac{\partial^T L_o}{\partial x} f(x) + \frac{1}{2} h(x)^T h(x) = 0, \quad L_o(0) = 0, \tag{3.35}$$

*for all $x \in W$. Furthermore $L_c$ exists and is smooth on $W$ (and thus the minimum in (3.11) is obtained) if and only if $L_c$ is the unique solution of*

$$\frac{\partial^T L_c}{\partial x} f(x) + \frac{1}{2} \frac{\partial^T L_c}{\partial x} g(x) g(x)^T \frac{\partial L_c}{\partial x} = 0, \quad L_c(0) = 0, \tag{3.36}$$

*for all $x \in W$, such that $-(f(x) + g(x)g(x)^T \frac{\partial L_c}{\partial x})$ is asymptotically stable on $W$*

**Proof:**

$L_o$ **Part :** Suppose that $f(x)$ is asymptotically stable on $W$, then the solution $x$ of $\dot{x}(t) = f(x(t))$ is stable and $\lim_{t \to \infty} x(t) = 0$. Since

$$
\begin{aligned}
L_o(x(t)) &= \frac{1}{2} \int_t^\infty y(t)^T y(t) dt \\
&= \frac{1}{2} \int_t^\infty h(x(t))^T h(x(t)) dt \quad \text{by (3.1b)} \\
&= \int_t^\infty \left( \frac{1}{2} h(x)^T h(x) + \frac{\partial^T L_o}{\partial x}(f(x) - \dot{x}) \right) dt \\
&= \int_t^\infty \left( \frac{1}{2} h(x)^T h(x) + \frac{\partial^T L_o}{\partial x} f(x) \right) dt - \int_{x(t)}^0 \frac{\partial^T L_o}{\partial x} dx \\
&= \int_t^\infty \left( \frac{\partial^T L_o}{\partial x} f(x) + \frac{1}{2} h(x)^T h(x) \right) dt - L_o(0) + L_o(x(t))
\end{aligned}
$$

hence, the $L_o$ exists and is smooth if and only if $L_o$ is the unique smooth solution of (3.35).

$L_c$ **Part :** Since

$$
\begin{aligned}
\frac{1}{2}\int_{-\infty}^{t} u(t)^T u(t)dt &= \frac{1}{2}\int_{-\infty}^{t} u(t)^T u(t)dt + \int_{-\infty}^{t}\left(\frac{\partial^T L_c}{\partial x}(\dot{x} - f(x) - g(x)u)\right)dt \quad \text{by (3.1a)} \\
&= \frac{1}{2}\int_{-\infty}^{t}\left(u - g(x)^T\frac{\partial L_c}{\partial x}\right)^T\left(u - g(x)^T\frac{\partial L_c}{\partial x}\right)dt \\
&\quad - \int_{-\infty}^{t}\left(\frac{\partial^T L_c}{\partial x}f(x) + \frac{1}{2}\frac{\partial^T L_c}{\partial x}g(x)g(x)^T\frac{\partial L_c}{\partial x}\right)dt + \int_{0}^{x(t)}\frac{\partial^T L_c}{\partial x}dx \\
&= \frac{1}{2}\int_{-\infty}^{t}\left(u - g(x)^T\frac{\partial L_c}{\partial x}\right)^T\left(u - g(x)^T\frac{\partial L_c}{\partial x}\right)dt \\
&\quad - \int_{-\infty}^{t}\left(\frac{\partial^T L_c}{\partial x}f(x) + \frac{1}{2}\frac{\partial^T L_c}{\partial x}g(x)g(x)^T\frac{\partial L_c}{\partial x}\right)dt + L_c(x(t)) - L_c(0)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
L_c(x(t)) &= \min_{u\in L_-^2}\frac{1}{2}\int_{-\infty}^{t} u(t)^T u(t)dt \\
&= \min_{u\in L_-^2}\left\{\frac{1}{2}\int_{-\infty}^{t}\left(u - g(x)^T\frac{\partial L_c}{\partial x}\right)^T\left(u - g(x)^T\frac{\partial L_c}{\partial x}\right)dt \right. \\
&\quad \left. - \int_{-\infty}^{t}\left(\frac{\partial^T L_c}{\partial x}f(x) + \frac{1}{2}\frac{\partial^T L_c}{\partial x}g(x)g(x)^T\frac{\partial L_c}{\partial x}\right)dt\right\} + L_c(x(t)) - L_c(0)
\end{aligned}
$$

Hence $L_c$ exists and is smooth if and only if $L_c$ is the unique smooth solution of (3.36) providing the the system

$$
\dot{x}(t) = f(x(t)) + g(x)g(x)^T\frac{\partial L_c}{\partial x}
$$

is antistable, or equivalently, $-(f(x) + g(x)g(x)^T\frac{\partial L_c}{\partial x})$ is asymptotically stable on $W$.

$\square$

**Theorem 3.10** *Assume that $f(x)$ and $-(f(x) + g(x)g(x)^T\Psi_-(x))$ are asymptotically stable on a neighborhood $W$ of $0$, and $\Psi_+(x), \Psi_-(x)$ are the unique smooth solutions of*

$$
\Psi_+(x)^T f(x) + \frac{1}{2}h(x)^T h(x) = 0, \quad \Psi_+(0) = 0, \tag{3.37}
$$

*and*

$$
\Psi_-(x)^T f(x) + \frac{1}{2}\Psi_-(x)^T g(x)g(x)^T\Psi_-(x) = 0, \quad \Psi_-(0) = 0, \tag{3.38}
$$

*for all $x \in W$, respectively. Then the Hankel norm of $G$, $\|G\|_H$, is*

$$
\|G\|_H^2 = \sup_{x_0\in W}\frac{x_0^T\Psi_+(x_0)}{x_0^T\Psi_-(x_0)} \tag{3.39}
$$

*which is an one-parameter optimization problem with $x_0 \in W$ and the optimal state $x_0^*$ must satisfy*

$$\frac{d}{dx_0}\left(\frac{x_0^T\Psi_+(x_0)}{x_0^T\Psi_-(x_0)}\right)\Bigg|_{x_0=x_0^*} = 0 \tag{3.40}$$

*with constraint*

$$\frac{d^2}{dx_0^2}\left(\frac{x_0^T\Psi_+(x_0)}{x_0^T\Psi_-(x_0)}\right)\Bigg|_{x_0=x_0^*} < 0 \tag{3.41}$$

*If the optimal state $x_0^*$, obtained from (3.40), is not located inside $W$ or does not satisfy (3.41), then other numerical methods should be used to solve this one-parameter optimization problem (3.39).*

**Proof:** Since $\Psi_+(x) = \frac{\partial L_o}{\partial x}$ and $\Psi_-(x) = \frac{\partial L_c}{\partial x}$ and due to the assumption that $f(x)$ and $-(f(x) + g(x)g(x)^T\Psi_-(x))$ are asymptotically stable on $W$, the existence of $\Psi_+(x)$ and $\Psi_-(x)$ are followed from Theorem 3.9 and they are the smooth, unique solutions of (3.37) and (3.38), respectively. Then by Theorem 3.4 and Lemma 3.6, the Hankel norm of the system given by (3.39) is followed. Hence, the optimal initial state, $x_0^*$ must satisfy (3.40) and the constraint (3.41) such that the quantity $\frac{x_0^T\Psi_+(x_0)}{x_0^T\Psi_-(x_0)}$ at $x_0^*$ has global supremum. If the optimal state $x_0^*$ of (3.40) does not satisfy the constraint (3.41) or is not located inside $W$, the supremum calculation of (3.39) should be solved by using other optimization techniques. □

From Theorem 3.10, we see that the method for computing the Hankel norm of the nonlinear system is much more difficult than that for the linear case.

**Example 3.11** Consider a simple one-dimensional nonlinear system

$$G: \begin{array}{rcl} \dot{x} & = & f(x)+g(x)u \\ y & = & h(x) \end{array} \qquad x \in \mathbb{R}$$

Suppose this system is asymptotically stable about the equilibrium point 0, and it is weakly controllable and locally zero-state observable, then $f(x) \neq 0$ for all $x \neq 0$ in $W$ of 0. Thus by Theorem 3.10, the functions $\Psi_+$ is given

$$\Psi_+(x) = -\frac{h^2(x)}{2f(x)}$$

Similarly, the function $\Psi_-$ is

$$\Psi_-(x) = -\frac{2f(x)}{g^2(x)}$$

Therefore

$$\|G\|_H^2 = \sup_{x_0 \in W} \frac{x_0^T\Psi_+(x_0)}{x_0^T\Psi_-(x_0)} = \frac{h^2(x_0^*)g^2(x_0^*)}{4f^2(x_0^*)} \tag{3.42}$$

where $x_0^*$, the maximum $x_0$, satisfies

$$\frac{h_x(x_0^*)h(x_0^*)g^2(x_0^*)+h^2(x_0^*)g_x(x_0^*)g(x_0^*)}{2f^2(x_0^*)}-\frac{h^2(x_0^*)g^2(x_0^*)f_x(x_0^*)}{2f^3(x_0^*)}=0$$

or

$$\frac{h_x(x_0^*)}{h(x_0^*)}+\frac{g_x(x_0^*)}{g(x_0^*)}-\frac{f_x(x_0^*)}{f(x_0^*)}=0 \text{ with } f(x_0^*),g(x_0^*),h(x_0^*)\neq 0 \qquad (3.43)$$

and

$$\frac{h^2(x_0^*)g^2(x_0^*)}{2f^2(x_0^*)}\left(\frac{h_{xx}(x_0^*)}{h(x_0^*)}-\frac{h_x^2(x_0^*)}{h^2(x_0^*)}+\frac{g_{xx}(x_0^*)}{g(x_0^*)}-\frac{g_x^2(x_0^*)}{g^2(x_0^*)}-\frac{f_{xx}(x_0^*)}{f(x_0^*)}+\frac{f_x^2(x_0^*)}{f^2(x_0^*)}\right)<0$$

After substituting (3.43), the above equation becomes

$$\frac{h_{xx}(x_0^*)}{h(x_0^*)}+\frac{g_{xx}(x_0^*)}{g(x_0^*)}-\frac{f_{xx}(x_0^*)}{f(x_0^*)}+2\frac{h_x(x_0^*)}{h(x_0^*)}\frac{g_x(x_0^*)}{g(x_0^*)}<0$$

If the optimal state $x_0^*$, obtained through (3.43), is not located inside $W$, we can use other numerical methods to solve this one-parameter optimization problem.

## 3.3  Special case: linear system

In this section, we would like to compute the Hankel norm of a linear, time-invariant system using Theorem 3.10. Let us consider a stable, linear, time-invariant system as given by

$$\begin{aligned}\dot{x} &= Ax+Bu\\ y &= Cx\end{aligned}$$

where $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$ and $C\in\mathbb{R}^{p\times n}$ and $Re\lambda_i(A)<0$. Suppose this system is controllable and observable. Therefore, we take $W=\mathbb{R}$. Substituting $f(x)=Ax$, $g(x)=B$ and $h(x)=Cx$ into Theorem 3.10, we have

$$\Psi_+(x)^T Ax+\frac{1}{2}x^T C^T Cx=0$$

and

$$\Psi_-(x)^T Ax+\frac{1}{2}\Psi_-(x)^T BB^T\Psi_-(x)=0$$

Equivalently,

$$\Psi_+(x)^T Ax+x^T A^T\Psi_+(x)+x^T C^T Cx=0$$

and

$$\Psi_-(x)^T Ax+x^T A^T\Psi_-(x)+\Psi_-(x)^T BB^T\Psi_-(x)=0$$

Let $P$ and $Q$ denote the positive controllability and observability gramians, respectively, i.e. they are the solutions of the following Lyapunov equations:

$$PA^T + AP + BB^T = 0$$
$$QA + A^T Q + C^T C = 0$$

Then the functions $\Psi_-(x)$ and $\Psi_+(x)$ are solved by

$$\Psi_-(x) = P^{-1}x \quad \text{and} \quad \Psi_+(x) = Qx$$

Hence the Hankel norm is given by

$$\|G\|_H^2 = \sup_{x_0 \in \mathbb{R}} \frac{x_0^T Q x_0}{x_0^T P^{-1} x_0} \tag{3.44}$$

and the optimal state $x_0^*$ must satisfy the condition

$$\frac{d}{dx_0} \frac{x_0^T Q x_0}{x_0^T P^{-1} x_0} = 0$$

or

$$2 \frac{Q x_0 x_0^T P^{-1} x_0 - P^{-1} x_0 x_0^T Q x_0}{(x_0^T P^{-1} x_0)^2} = 0 \tag{3.45}$$

Let

$$\sigma^2 = \frac{x_0^T Q x_0}{x_0^T P^{-1} x_0}, \tag{3.46}$$

then (3.45) becomes

$$\sigma^2 P^{-1} x_0^* = Q x_0^*$$

Multiplication of both side with $P$ leads to

$$\sigma^2 x_0^* = PQ x_0^* \tag{3.47}$$

i.e. $\sigma$ is the singular value of the matrix $PQ$ and $x_0^*$ is the corresponding eigenvector and satisfies the constraint

$$\frac{d^2}{dx_0^2} \frac{x_0^T Q x_0}{x_0^T P^{-1} x_0} = 2 \frac{d}{dx_0} \left( \frac{Q x_0}{x_0^T P^{-1} x_0} - \frac{P^{-1} x_0 x_0^T Q x_0}{(x_0^T P^{-1} x_0)^2} \right) \bigg|_{x_0 = x_{0*}} < 0$$

Since

$$\frac{d}{dx} P^{-1} x_0 x_0^T Q x_0 = P^{-1} x_0^T Q x_0 + P^{-1} x_0 x_0^T Q = P^{-1} x_0^T Q x_0 + Q x_0 x_0^T P^{-1}$$

the above constraint can be deduced as

$$\left. \frac{Q}{x_0^T P^{-1} x_0} - \frac{P^{-1} x_0^T Q x_0}{(x_0^T P^{-1} x_0)^2} - \frac{4 Q x_0 x_0^T P^{-1}}{(x_0^T P^{-1} x_0)^2} + \frac{4 P^{-1} x_0 x_0^T P^{-1} x_0^T Q x_0}{(x_0^T P^{-1} x_0)^3} \right|_{x_0 = x_{0*}} < 0$$

and, after using (3.46), further simplification leads to

$$\frac{Q - \sigma^2 P^{-1}}{x_0^{*T} P^{-1} x_0^*} < 0$$

or $Q - \sigma^2 P^{-1}$ must be negative definite i.e. for all $x \neq 0$

$$\frac{x^T Q x}{x^T P^{-1} x} < \sigma^2 \qquad (3.48)$$

Hence a suitable $\sigma^2$, satisfying (3.47), must be an upper bound of the function $\mu^2(x_0)$. Thus the Hankel norm of this system is

$$\|G\|_H^2 = \sup_{x_0 \in \mathbb{R}} \mu^2(x_0) = \sup_{x_0 \in \mathbb{R}} \frac{x_0^T Q x_0}{x_0^T P^{-1} x_0} = \sup_{x_0} \frac{x_0^T \sigma_i^2 P^{-1} x_0}{x_0^T P^{-1} x_0} = \max_i \sigma_i^2 = \sigma_1^2 \qquad (3.49)$$

by choosing $x_0$ to be the $i$th eigenvector. Therefore the Hankel norm is the maximal singular value of the matrix $PQ$, denoted by $\sigma_{max}(PQ)$ which is the same as the result of Glover(1984).

# 4    Balanced realization

In this section, we will discuss the algorithm to perform the balanced realization for nonlinear system.

**Definition 4.1** *A nonlinear system is in* balanced form *(or simple* balanced*) if there exists a local coordinate system z on the neighborhood U of* 0 *such that the controllability and observability functions are of the form*

$$L_c(0, \ldots, 0, z_i, 0, \ldots, 0) = \frac{1}{2} \sigma(z_i)^{-1} z_i^2$$

$$L_o(0, \ldots, 0, z_i, 0, \ldots, 0) = \frac{1}{2} \sigma(z_i) z_i^2$$

*where* $\sigma_i$*'s are in fact the* Hankel singular values.

We need the following Lemma:

**Lemma 4.2** *There exists a coordinate transformation $x = \phi(\tilde{x})$, $\phi(0) = 0$ (defined on a neighborhood $W$ of 0), such that in the new coordinate $\tilde{x} = \phi^{-1}(x)$ the functions $L_c$ and $L_o$ are of the form*

$$L_c(\tilde{x}) \quad \overset{\triangle}{=} \quad L_c(\phi(\tilde{x})) = \frac{1}{2}\tilde{x}^T\tilde{x}, \tag{4.1}$$

$$L_o(\tilde{x}) \quad \overset{\triangle}{=} \quad L_o(\phi(\tilde{x})) = \frac{1}{2}\tilde{x}^T M(\tilde{x})\tilde{x}, \tag{4.2}$$

*where $M(\tilde{x})$ is an $n \times n$ symmetric positive-definite matrix such that it entries are smooth functions of $\tilde{x}$ on $\phi^{-1}(W)$.*

**Proof:** Since $L_c(0) = 0$ and $\frac{\partial L_c}{\partial x}(0) = 0$, by applying Lemma 2.1 twice, we have

$$L_c(x) = \frac{1}{2}x^T W_c(x) x \quad \text{where} \quad W_c(0) = \frac{\partial^2 L_c}{\partial x^2}(0) \tag{4.3}$$

By Lemma 2.7, we know that there exists a neighborhood $W$ of 0 such that the eigenvalues of $W_c(x)$ and the associated normalized eigenvectors are smooth functions of $x$. This implies that $W_c(x)$ is smoothly diagonalizable on $W$. Indeed, we can write

$$W_c(x) = T_c(x)\Lambda_c(x)T_c^T(x) \quad \text{where} \quad \Lambda_c(x) = \text{diag}(\lambda_{c1}(x), \dots, \lambda_{cn}(x))$$

Here, $\lambda_{ci}(x) \geq 0, i = 1, \dots, n$ are the eigenvalues of $W_c(x)$ and $T_c(x)$ is the corresponding orthogonal matrix of the normalized eigenvectors, i.e. $T_c^T(x)T_c(x) = I_n$, $x \in W$. The non-negativity of $\lambda_{ci}$ is due to **Assumption** 4 that $L_c(x) > 0$. Define a new coordinate transformation $\tilde{x} = v(x) = \Lambda_c(x)^{\frac{1}{2}}T_c^T(x)x$. In these coordinate, we can rewrite (4.3) as:

$$L_c(\tilde{x}) \overset{\triangle}{=} L_c(v^{-1}(\tilde{x})) = \frac{1}{2}\tilde{x}^T\tilde{x},$$

where $\tilde{x} \in \tilde{W} \overset{\triangle}{=} v(W)$. Similarly, $L_o$ can be transformed into

$$L_o(\tilde{x}) \overset{\triangle}{=} L_o(v^{-1}(\tilde{x})) \quad \text{with } L_o(0) = 0, \ \frac{\partial L_o}{\partial x}(0) = 0 \text{ on } \tilde{W}$$

and, after applying Lemma 2.1 twice, we get

$$L_o(\tilde{x}) = \frac{1}{2}\tilde{x}^T M(\tilde{x})\tilde{x}, \quad M(0) = 0 \tag{4.4}$$

Thus, let

$$\phi = v^{-1}, \quad \text{or} \quad \phi^{-1}(x) = \Lambda_c(x)^{\frac{1}{2}}T_c^T(x) x \tag{4.5}$$

then, $x = \phi(\tilde{x})$ and $\tilde{W} = \phi^{-1}(W)$. Thus, the lemma is proved. $\qquad\square$

This lemma has been proved in Scherpen (1994) by using Lemma of Moorse (see Lemma 2.6), and it does not provide any information about the construction of the coordinate transformation $\phi$. In our proof, although the neighborhood $W$ may be smaller than that obtained by using Lemma of Moorse, but we can construct the coordinate transformation $\phi$, which means that the balancing for nonlinear system is realizable.

**Theorem 4.3** *Consider the nonlinear system $G$ and the condition of Lemma 2.7 is fulfilled. On a neighborhood $W$ of $0$ there exists a coordinate transformation $x = \psi(z)$, $\psi(0) = 0$, such that in the new coordinate $z \in V := \psi^{-1}(W)$ the function $L_c$ is of the form*

$$\tilde{L}_c(z) := L_c(\psi(z)) = \frac{1}{2}z^T z, \tag{4.6}$$

*and the function $L_0$ is of the form*

$$\tilde{L}_o(z) := L_o(\psi(z)) = \frac{1}{2}z^T \begin{pmatrix} \tau_1(z) & & 0 \\ & \ddots & \\ 0 & & \tau_n(z) \end{pmatrix} z, \tag{4.7}$$

*Here, $\tau_1(z) \geq \cdots \geq \tau_n(z) > 0$ are smooth functions of $z$, called the* singular value functions *of the system.*

**Proof:** By Lemma 4.2, we can express the functions $L_c$ and $L_o$ as given by (4.1) and (4.2) on $W$ via coordinate transformation $x = \phi(\tilde{x})$. By Lemma 2.7, we know that there exists a neighborhood $\tilde{W}$ of $0$, such that the eigenvalues of $M(\tilde{x})$ and the associated normalized eigenvectors are smooth functions of $\tilde{x}$. This implies that $M(\tilde{x})$ is smoothly diagonalizable on $\tilde{W}$. It is noted that the neighborhood $\phi(\tilde{W})$ may not be the same as the $W$ in Lemma 4.2, we still use $W$ to denote their intersection.

Indeed, we can write

$$M(\tilde{x}) = T_o(\tilde{x})\Lambda_o(\tilde{x})T_o^T(\tilde{x}) \quad \text{where} \quad \Lambda_o(\tilde{x}) = \text{diag}(\lambda_{o1}(\tilde{x}), \dots, \lambda_{on}(\tilde{x}))$$

Here, $\lambda_{oi}(\tilde{x}) \geq 0, i = 1, \dots, n$ are the eigenvalues of $W_c(x)$ and $T_c(x)$ is the corresponding orthogonal matrix of the normalized eigenvectors, i.e. $T_o^T(\tilde{x})T_o(\tilde{x}) = I_n, \tilde{x} \in \tilde{W}$. The non-negativity of $\lambda_{oi}$ is due to **Assumption** 4 that $L_o(\tilde{x}) > 0$. Define a new coordinate transformation $\tilde{z} = w(\tilde{x}) = T_o^T(\tilde{x})\tilde{x}$. In these coordinate, we can rewrite (4.1) and (4.2) as:

$$L_c(w^{-1}(\tilde{z})) = \frac{1}{2}\tilde{z}^T\tilde{z},$$

$$L_o(w^{-1}(\tilde{z})) = \frac{1}{2}\tilde{z}^T\Lambda_o(\tilde{z})\tilde{z},$$

where $\Lambda_o(\tilde{z}) \triangleq \Lambda_o(w^{-1}(\tilde{z}))$ has eigenvalues $\lambda_{oi}(\tilde{z}) \triangleq \lambda_{oi}(w^{-1}(\tilde{z}))$. Then, we can transfer $\Lambda_o(\tilde{z})$ by permutation matrix such that it eigenvalues are arranged in the order of decreasing magnitude. Let

$S$ be the corresponding permutation matrix with $S^T S = I_n$. Again, we can apply a new coordinate transformation $z = \zeta(\tilde{z}) \overset{\triangle}{=} S^T \tilde{z}$ such that in these coordinate, the diagonal elements of $S^T \Lambda(\tilde{z}) S$ are in the order of decreasing magnitude. We denote this matrix as $\Lambda(z) = \text{diag}(\tau_1(z), \dots, \tau_n(z))$ with $\tau_1(z) \geq \cdots \geq \tau_n(z) > 0$. Thus, let

$$\psi = \phi \circ w^{-1} \circ \zeta^{-1} \tag{4.8}$$

, then the theorem is proved. □

Finally, the realization of the functions $\tilde{L}_c(z)$ and $\tilde{L}_o(z)$ into balanced form can be done according to the method proposed by Scherpen (1994). It is summarized as follows. We take a smooth transformation $\tilde{z}_i = \eta_i(z_i) := \tau_i(0, \dots, 0, z_i, 0, \dots, 0)^{\frac{1}{4}} z_i$, $i = 1, \dots, n$, and hence $\tilde{z} = \eta(z) = [\eta_1(z_1), \dots, \eta_n(z_n)]^T$ on $\tilde{z} \in \tilde{V} := \eta(V)$. Since $\tilde{L}_o(z) > 0$, we have that $\tau_i(0, \dots, 0, z_i, 0, \dots, 0) > 0$, $i = 1, \dots, n$, for $z \in V, z \neq 0$. Define $\sigma_i(\tilde{z}_i) = \tau_i(0, \dots, 0, \eta_i^{-1}(\tilde{z}_i), 0, \dots, 0)^{\frac{1}{2}}$, $\tilde{L}_c(\tilde{z}) := \tilde{L}_c(\eta^{-1}(\tilde{z}))$ and $\tilde{L}_o(\tilde{z}) := \tilde{L}_o(\eta^{-1}(\tilde{z}))$. Then,

$$\tilde{L}_c(\tilde{z}) = \frac{1}{2} \tilde{z}^T \begin{pmatrix} \sigma_1(\tilde{z}_1)^{-1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n(\tilde{z}_n)^{-1} \end{pmatrix} \tilde{z}, \tag{4.9}$$

$$\tilde{L}_o(\tilde{z}) = \frac{1}{2} \tilde{z}^T \begin{pmatrix} \sigma_1(\tilde{z}_1)^{-1} \tau_1(\tilde{z}) & & 0 \\ & \ddots & \\ 0 & & \sigma_n(\tilde{z}_n)^{-1} \tau_n(\tilde{z}) \end{pmatrix} \tilde{z}, \tag{4.10}$$

where $\tau_i(\tilde{z}) \overset{\triangle}{=} \tau_i(\eta^{-1}(\tilde{z}))$. This means that at the coordinate axes we have

$$\tilde{L}_c(0, \dots, 0, \tilde{z}_i, 0, \dots, 0) = \frac{1}{2} \sigma(\tilde{z}_i)^{-1} \tilde{z}_i^2,$$

$$\tilde{L}_o(0, \dots, 0, \tilde{z}_i, 0, \dots, 0) = \frac{1}{2} \sigma(\tilde{z}_i) \tilde{z}_i^2$$

Hence, the controllability and observability functions are in the balanced form.

Now, consider the system $G$ after the coordinate transformation $x = \chi(\tilde{z}) \overset{\triangle}{=} \psi \circ \eta^{-1}$ where $\psi$ is given in (4.8):

$$\dot{\tilde{z}} = \tilde{f}(\tilde{z}) + \tilde{g}(\tilde{z}) u \tag{4.11}$$

$$y = \tilde{h}(\tilde{z}). \tag{4.12}$$

**Algorithm 41** (Balanced Realization Algorithm)

**1)** *Solve (3.35) and (3.36) for $L_c(x)$ and $L_o(x)$.*

**2)** *Compute $W_c(x)$ according to (4.3).*

**3)** *Decompose $W_c(x) = T_c^T(x)\Lambda_c(x)T_c(x)$ where $\Lambda_c(x)$ is a diagonal matrix formed by the eigen-vectors of $W_c(x)$ and $T_c(x)$ is orthogonal matrix.*

**4)** *Find the function $\phi$ such that $\phi^{-1}(x) = \Lambda_c(x)^{\frac{1}{2}}T_c(x)\,x$.*

**5)** *Compute $L_o(\tilde{x}) = L_o(\phi(\tilde{x}))$*

**6)** *Compute $M(\tilde{x})$ according to (4.4).*

**7)** *Decompose $M(\tilde{x}) = T_o^T(\tilde{x})\Lambda_o(\tilde{x})T_o(\tilde{x})$ where $\Lambda_o(\tilde{x})$ is a diagonal matrix formed by the eigen-vectors of $M(\tilde{x})$ and $T_o(x)$ is orthogonal matrix.*

**8)** *Compute $w^{-1}$ such that $w(\tilde{x}) = T_o^T(\tilde{x})x$.*

**9)** *Find the permutation matrix $S$ such that the matrix $S^T\Lambda_o(w^{-1}(s))S$ has its diagonal elements arranged in the order of decreasing magnitude.*

**10)** *Compute $\zeta^{-1}$ such that $\zeta(s) = S^T s$.*

**11)** *Compute $\Lambda(z) = S^T\Lambda_o(w^{-1}(\zeta^{-1}(z)))S$ and its eigenvalues $\tau_i(z)$, $i = 1,\dots,n$.*

**12)** *Compute $\eta^{-1}$ such that $\eta(z) = [\eta_1(z_1),\dots,\eta_n(z_n)]^T$ with its i-th component*
*$\eta_i = \tau_i(0,\dots,0,z_i,0,\dots,0)^{\frac{1}{4}}z_i$.*

**13)** *Form the coordinate transformation function $x = \chi(\tilde{z})$ where $\chi = \phi \circ w^{-1} \circ \zeta^{-1} \circ \eta^{-1}$.*

# 5   Numerical examples

**Example 5.1**  Consider the following first-order nonlinear system

$$\begin{aligned}\dot{x} &= -x + \sqrt{x}\,u \\ y &= \sin x\end{aligned}$$

Since $f(x) = -x$, $g(x) = \sqrt{x}$, and $h(x) = \sin x$, we know that the linearized model is stable, con-trollable and observable about the equilibrium $(0,0)$.  And it follows that the original system is asymptotically stable, controllable and zero-state observable. We take

$$W = \{x \in \mathbb{R} | 0 \le x < \pi\}$$

The substitution of $f(x) = -x$, $g(x) = \sqrt{x}$, and $h(x) = \sin x$ into (3.49) yields the Hankel norm of this system

$$\|G\|_H = \left| \frac{\sin x_0^*}{2\sqrt{x_0^*}} \right|$$

where $x_0^*$ is the solution of

$$x_0^* = \frac{1}{2}\tan x_0^*, \quad x_0^* \neq 0$$

with the constraint

$$-1 - \frac{1}{4(x_0^*)^2} + \frac{\cos x_0^*}{x_0^* \sin x_0^*} < 0$$

which can be reduced to the condition that

$$\frac{4}{1+4(x_0^*)^2}x_0^* < \tan x_0^* = 2x_0^* < 0$$

or equivalently, $x_0^* > \frac{1}{2}$. The solution is $x_0^* = 1.165561$, which belongs to $W$. Hence the Hankel norm of the system is found as

$$\|G\|_H = 0.425621$$

**Example 5.2** Consider the following first-order nonlinear system

$$\dot{x} = -x^2 + x\,u$$
$$y = \sin x$$

The unique solution of the $\dot{x} = -x^2$ with $x(0) = x_0$

$$x(t) = \frac{x_0}{1 + tx_0}$$

exists over $\left(-\infty, -\frac{1}{x_0}\right) \cup \left(-\frac{1}{x_0}, \infty\right)$. Thus if we take

$$W = \{x \in \mathbb{R} | -\pi < x < \pi\}$$

then the equilibrium point $(0,0)$ is asymptotically stable and this system is zero-state observable. The substitution of $f(x) = -x^2$, $g(x) = x$, and $h(x) = \sin x$ into (3.49), yields the Hankel norm of this system

$$\|G\|_H = \left|\frac{\sin x_0^*}{2x_0^*}\right|$$

where $x_0^*$ is the solution of

$$x_0^* = \tan x_0^*, \quad x_0^* \neq 0$$

with the constraint

$$-1 - \frac{2}{(x_0^*)^2} + 2\frac{\cos x_0^*}{\sin x_0^*}\frac{1}{x_0^*} < 0 \quad \text{or} \quad \frac{2}{2+(x_0^*)^2}x_0^* < \tan x_0^* = x_0^*$$

i.e. $x_0^* > 0$. Although no explicit $x_0^*$ can be found to satisfy the above conditions, we still can compute the Hankel norm of the system by

$$\|G\|_H = \lim_{\varepsilon \to 0} \left| \frac{\sin \varepsilon}{2\varepsilon} \right| = 0.5$$

The following example is adopted from Example 3.2.8 of Scherpen(1994).

**Example 5.3** Consider the following second-order nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\, u \\ y &= h(x) \end{aligned}$$

where the state $x$ is acting on

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, (3x_1 - 4x_2)^2 < 25 \right\}$$

This system fulfills the assumptions of Section 3 with $x = [x_1, x_2]^T$, $u = [u_1, u_2]^T$ and $y = [y_1, y_2]^T$ and $f, g$ and $h$ as follows:

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = -\frac{1}{625} \begin{bmatrix} 625x_1 + 112x_1^3 + 552x_1^2 x_2 + 639x_1 x_2^2 + 216x_2^3 \\ 384x_1^3 + 625x_2 + 464x_1^2 x_2 + 48x_1 x_2^2 - 63x_2^3 \end{bmatrix},$$

$$g(x) = \begin{bmatrix} g_{11}(x) & g_{12}(x) \\ g_{21}(x) & g_{22}(x) \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{2}}{5} & \frac{4\sqrt{2}}{25}\sqrt{25 + 7x_1^2 + 48x_1 x_2 - 7x_2^2} \\ -\frac{4\sqrt{2}}{5} & \frac{3\sqrt{2}}{25}\sqrt{25 + 7x_1^2 + 48x_1 x_2 - 7x_2^2} \end{bmatrix},$$

$$h(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{5}\sqrt{34x_1^2 - 24x_1 x_2 + 41x_2^2} \\ \frac{\sqrt{2}}{25}(4x_1 + 3x_2)^2 \end{bmatrix}.$$

The substitution of $f(x)$, $g(x)$, and $h(x)$ into (3.37), and (3.38) yields

$$\Psi_-(x_0) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \Psi_+(x_0) = \frac{1}{625} \begin{bmatrix} 850x_1 + 288x_1^3 - 300x_2 - 252x_1^2 x_2 - 239x_1 x2^2 + 84x_2^3 \\ -300x_1 - 84x_1^3 + 1025x_2 - 239x_1^2 x_2 + 252x_1 x_2^2 + 288x_2^3 \end{bmatrix}$$

Consequently the Hankel norm of this system is given by

$$\|G\|_H = \left| \frac{1}{625} \frac{850x_1^2 + 144x_1^4 - 600x_1 x_2 - 168x_1^3 x_2 + 1025x_2^2 - 239x_1^2 x_2^2 + 168x_1 x_2^3 + 144x_2^4}{x_1^2 + x_2^2} \right|$$

where $x_1$, $x_2$ must solve the following equations

$$\frac{2(4x_1 + 3x_2)(-3x_1 + 4x_2)(-12x_1^3 - 25x_2 - 36x_1 x_2^2 + 7x_2^3)}{625(x_1^2 + x_2^2)^2} = 0$$

$$\frac{2(4x_1 + 3x_2)(-3x_1 + 4x_2)(25x_1 + 7x_1^3 + 36x_1^2 x_2 + 12x_2^3)}{625(x_1^2 + x_2^2)^2} = 0$$

with the constraint that the Jacobian matrix

$$\frac{1}{625(x_1^2 + x_2^2)^3} \cdot$$

$$\left[\begin{array}{c|c} 2(144x_1^6 - 600x_1^3x_2 + 525x_1^2x_2^2 + 432x_1^4x_2^2 + \\ 432x_1^4x_2^2 + 1800x_1x_2^3 + 336x_1^3x_2^3 - 175x_2^4 + \\ 2013x_1^2x_2^4 - 1008x_1x_2^5 - 383x_2^6) \end{array} \right. \begin{array}{c} 4(150x_1^4 - 42x_1^6 - 175x_1^3x_2 - 900x_1^2x_2^2 - \\ 378x_1^4x_2^2 + 175x_1x_2^3 - 1054x_1^3x_2^3 + 150x_2^4 + \\ 378x_1^2x_2^4 + 42x_2^6) \end{array}$$
$$\left. \begin{array}{c} 4(150x_1^4 - 42x_1^6 - 175x_1^3x_2 - 900x_1^2x_2^2 - \\ 378x_1^4x_2^2 + 175x_1x_2^3 - 1054x_1^3x_2^3 + \\ 150x_2^4 + 378x_1^2x_2^4 + 42x_2^6) \end{array} \right| \begin{array}{c} 2(175x_1^4 - 383x_1^6 + 1800x_1^3x_2 + 1008x_1^5x_2 - \\ 525x_1^2x_2^2 + 2013x_1^4x_2^2 - 600x_1x_2^3 - \\ 336x_1^3x_2^3 + 432x_1^2x_2^4 + 144x_2^6) \end{array}\right]$$

is negative definite. Since all the solutions can not satisfy the requirement that the Jacobian matrix is negative definite, we use the graphical method to analyze the extremal property of the Hankel norm inside $W$. The result shows that the supremum value is found to be located along the boundary of $W$, i.e. $(3x_1 - 4x_2)^2 = 25$, therefore the Hankel norm of the system is

$$\|G\|_H = \sqrt{2}$$

This result is as same as the value of largest singular value of this system in Example 3.2.8 by Scherpen(1994).

The integration of $\Psi_-$ and $\Psi_+$ give us the controllability and observability functions as follows:

$$L_c(x) = \frac{1}{2}x^T x, \quad L_o(x) = \frac{1}{2}x^T \left[\begin{array}{cc} m_{11}(x) & m_{12}(x) \\ m_{21}(x) & m_{22}(x) \end{array}\right] x$$

where

$$m_{11}(x) \quad = \quad \frac{2}{625}(425 + 72x_1^2 - 192x_1x_2 + 128x_2^2)$$

$$m_{12}(x) \quad = m_{21}(x) = \quad \frac{12}{625}(-25 + 9x_1^2 - 24x_1x_2 + 16x_2^2)$$

$$m_{22}(x) \quad = \quad \frac{1}{625}(1025 + 81x_1^2 - 216x_1x_2 + 144x_2^2)$$

We see that this system already has the form of Lemma 4.2, hence, chose the coordinate transformation as $x = \psi(\tilde{x}) = \tilde{x}$ and

$$M(\tilde{x}) = M(x) = \left[\begin{array}{cc} m_{11}(x) & m_{12}(x) \\ m_{21}(x) & m_{22}(x) \end{array}\right].$$

The eigenvalues of $M(x)$ are:

$$\lambda_1(x) \quad = \quad \frac{1}{25}(25 + 9x_1^2 - 24x_1x_2 + 16x_2^2) = 1 + \left(\frac{1}{5}(3x_1 - 4x_2)\right)^2$$

$$\lambda_2(x) \quad = \quad 2.$$

The neighborhood $V$ of 0 where the number of distinct eigenvalues is constant, is

$$V = \{x \mid (3x_1 - 4x_2)^2 < 25\}$$

i.e. $\lambda_1(x) < 2$ for all $x \in V$. Therefore, the largest eigenvalue for all $x \in V$ is $\lambda_2(x)$. The unitary matrix of eigenvectors is

$$T_o(x) = T = \frac{1}{5} \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

Thus, the second coordinate transformation on $V$ is

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = w(x) = T^T x = \frac{1}{5} \begin{bmatrix} 3x_1 - 4x_2 \\ 4x_1 + 3x_2 \end{bmatrix}.$$

We note that the transformation function $\psi(z) = w^{-1}(z)$ and

$$W = \psi^{-1}(V) = w(V) = \{z \mid z_1^2 < 1\}.$$

In the new coordinates, the controllability and observability functions become for $z \in W$

$$\tilde{L}_c(z) = \frac{1}{2}z^T z, \quad \tilde{L}_o(z) = \frac{1}{2}z^T \begin{bmatrix} 2 & 0 \\ 0 & 1+z_1^2 \end{bmatrix} z.$$

The singular value functions are $\tau_1(z) = 2$ and $\tau_2(z) = 1 + z_1^2$. To bring in balanced form, we need to define the transformation $\tilde{z} = \eta(z)$ as follows:

$$\tilde{z} = \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} = \begin{bmatrix} 2^{\frac{1}{4}}z_1 \\ z_2 \end{bmatrix}$$

and

$$\tilde{z} \in \tilde{W} = \eta(W) = \left\{ \tilde{z} \mid \tilde{z}_1^2 < 2^{\frac{1}{2}} \right\}.$$

For $\tilde{z} \in \tilde{W}$, the system is transformed into the following form:

$$\begin{cases} \dot{\tilde{z}}_1 &= -\tilde{z}_1 + \tilde{z}_1 \tilde{z}_2^2 + u_1 8^{\frac{1}{4}} \\ \dot{\tilde{z}}_2 &= -\tilde{z}_1 - \tilde{z}_2^3 + u_2 \sqrt{2 - \sqrt{2}\tilde{z}_1^2 + 2\tilde{z}_2^2} \\ y_1 &= 8^{\frac{1}{4}} \tilde{z}_1 \\ y_2 &= \sqrt{2}\tilde{z}_2 \end{cases}$$

This is the balanced realization of the system (4).

# 6  Conclusion

The distance between stable and unstable system can be measured by Hankel norm between them. This paper establish the analytic method to compute the Hankel norm of a nonlinear, input-affine, time-invariant systems through game-theoretic approach. This computational method can be recasted into two sequential steps: solving the minimal energy problem and then an one parameter-optimization problem. According to the continuity of the costate vector, the Hankel norm is computed by solving two algebraic equations instead of two partial differential equation as other paper does. We also show that the Hankel norm computation for the linear time-invariant system can be regarded as a special case of our results. After all, we derive the algorithm for the balanced realization of nonlinear input-affine system. Numerical examples for nonlinear input-affine systems are used to illustrate our computational procedures.

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Hankel          Lagrange

Hankel

Hankel