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Stabilization for High Order Nonlinear Critical Systems

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Abstract

The aim of the thesis is to study the stabilization of the system $\dot{x} = f(x) + bu$. Here the linearization system contains a high order controllable mode. First, by center manifold theory, we can reduce the dynamics of the given system (higher order system) to the dynamics of the center manifold system(lower order system). Second, using normal forms or Lyapunov function, we can find a sufficient condition such that the center manifold system is locally asymptotically stable. That is, we can control $u(x)$ such that the given system is stabilized.

Keywords: Nonlinear critical system, Nonlinear feedback, Controllable pair, Center manifold, Normal forms.

1 Introduction

We are trying to study the stablization of the nonlinear control system

$$\dot{x} = f(x) + bu(x), \tag{1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $b \in \mathbb{R}^n$. Without loss of generality, we assume the origin is a rest point of (1.1). Our main goal is to find, if possible, a smooth input $u(x)$ such that system (1.1) is asymptotically stable. To change a basis (1.1) is transformed into

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ 0 \end{pmatrix} u + \mathcal{O}(2) \tag{1.2}$$

with (A_{11}, b_1) is a controllable pair. Thus there exists a linear feedback control $u = kx_1 + v$ such that all eigenvalues of $A_{11} + b_1 k$ have negative real parts. Hence the system (1.2) can be written as

$$\begin{pmatrix} \dot{y} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} y \\ \eta \end{pmatrix} + \begin{pmatrix} w(y,\eta) \\ g(y,\eta) \end{pmatrix}, \tag{1.3}$$

where $\lambda(B) = \lambda(A_{11} + b_1 k)$, $\lambda(Q) = \lambda(A_{22})$ and $w((y,\eta) = b_1 v + \mathcal{O}(2)$.

The system (1.3) is stabilizable by a linear feedback if all eigenvalues of Q have negative real parts. And, if Q has at least one eigenvalue with positive real parts, then the system (1.3) can not be stabilized at all [5]. Thus we only study the case which all eigenvalues of Q have non-positive real parts in the thesis.

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In Aeyels [1], the case of a pair of imaginary eigenvalues (that is, $\lambda(Q) = \{i, -i\}$) was treated. In [2], S. Behtash and S. Sastry treated double zero eigenvalues with nonzero jordan form and pair of imaginary and a simple zero eigenvalues. They only discuss those systems with one order controllable system (That is, B is scalar). However, we are trying to study the system with high order controllable system. (That is, $B \in M_{m\times m}$.) In the thesis, the center manifold theorem is introduced in the section 2. In section 3, we describe the normal forms. The main result is given in the section 4.

2 Center manifold theorem

Consider the system

$$\begin{aligned} \dot{y} &= By + w(y,\eta), \\ \dot{\eta} &= Q\eta + g(y,\eta), \end{aligned} \qquad (y,\eta) \in \mathbb{R}^m \times \mathbb{R}^l, \tag{2.1}$$

where

$$w(0,0) = 0, \quad Dw(0,0) = 0,$$

$$g(0,0) = 0, \quad Dg(0,0) = 0,$$

all eigenvalues of B have negative real parts and all eigenvalues of Q have zero real parts. Before stating the center manifold theorem, we need the following definitions.

Definition 2.1 [4] *Let B have m (generalized) eigenvectors $v_1, \ldots, v_m$ and Q have l (generalized) eigenvectors $w_1, \ldots, w_l$. Then the stable and center subspaces, (denoted E^s and E^c), are linear subspaces spanned by $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_l\}$ respectively; that is*

$$E^s = \mathrm{Span}\{v_1, \ldots, v_m\},$$

$$E^c = \mathrm{Span}\{w_1, \ldots, w_l\}.$$

Definition 2.2 [4] *A set S is an invariant set for a flow ϕ_t if S is a subset on $\mathbb{R}^n$ and $x \in S$ such that $\phi_t(x) \in S$ for all $t \in \mathbb{R}$.*

Definition 2.3 [4] *$W^c(0)$ is called a center manifold if and only if $W^c(0)$ is an invariant set and tangent to E^c at 0.*

We know that the center manifold is defined by

$$W^c = \{(y,\eta) | y = h(\eta), \quad h(0) = Dh(0) = 0\ \},$$

where h is a smooth function defined on some neighborhood $U \subset \mathbb{R}^k$ of the origin.

Remark 1 $W^c(0)$ is not unique.

The first result on center manifold is an existence theorem.

Theorem 2.4 [3] *There exists a C^r center manifold for* (2.1). *The dynamics of* (2.1) *restricted to the center manifold is, for η sufficiently small, given by the following vector field*

$$\dot{\eta} = Q\eta + g(h(\eta), \eta). \tag{2.2}$$

The dynamics of the system (2.2) near $\eta = 0$ determine the dynamics of (2.1) near $(y, \eta) = (0, 0)$.

Theorem 2.5 [3] *If the origin $\eta = 0$ of* (2.2) *is locally asymptotically stable* (*unstable*), *then the origin of* (2.1) *is local asymptotically stable* (*unstable*).

We now introduce how $h(\eta)$ can be calculated, or at least approximated. Substituting $y = h(\eta)$ in the second component of (2.1) and using the chain rule, we obtain

$$N(h(\eta)) \equiv Dh(\eta)[Qh(\eta) + g(h(\eta), \eta)] - Bh(\eta) - w(h(\eta), \eta) = 0 \tag{2.3}$$

with boundary conditions

$$h(0) = Dh(0) = 0.$$

Theorem 2.6 [3] *If a function $\phi(\eta)$, with $\phi(0) = D\phi(0) = 0$, can be found such that $N(\phi(\eta)) = \mathcal{O}(|\eta|^p)$ for some $p > 1$ as $|\eta| \to 0$ then it follows that*

$$h(\eta) = \phi(\eta) + \mathcal{O}(|\eta|^p) \quad \text{as} \quad |\eta| \to 0.$$

Hence the system $\dot{\eta} = Q\eta + g(h(\eta), \eta)$ is asymptotically stable whenever the system $\dot{\eta} = Q\eta + g(\phi(\eta), \eta)$ is asymptotically stable.

3 Normal forms

The center manifold theorem tells us that the dynamics of the system (2.2) near $\eta = 0$ determine the dynamics of (2.1) near $(y, \eta) = (0, 0)$. Thus it is enough to understand the dynamics of the system (2.2) whenever we only restrict our attention to the flow within the center manifold. For the purpose, we are trying to simplify the vector field on the center manifold such that it is easily to understand the dynamics of the center manifold system (2.2). The resulting "simplified" vector field are called *normal forms.*

Let H_k be the real vector space of vector fields whose coefficients are homogeneous polynomials of degree k. Using the transformation

$$x = y + p^{(2)}(y). \tag{3.1}$$

We can transform the system

$$\dot{x} = Ax + f^{(2)}(x) + \cdots \tag{3.2}$$

to

$$\dot{y} = Ay + g^{(2)}(y) + \cdots, \tag{3.3}$$

where $f^{(2)}(y), p^{(2)}(y) \in H_2$. Equations (3.1), (3.2) and (3.3) tell us that

$$\begin{aligned} Ax + f^{(2)}(x) + \cdots &= \dot{x} \\ &= \dot{y} + D(p^{(2)}(y))\dot{y} \\ &= \Big[I + D(p^{(2)}(y))\Big]\,\dot{y}. \end{aligned}$$

That is,

$$\begin{aligned} \dot{y} &= \Big(I + D(p^{(2)}(y))\Big)^{-1}\dot{x} \\ &= \Big\{I - D(p^{(2)}(y)) - \Big[D(p^{(2)}(y))\Big]^2 - \cdots\Big\}\,\dot{x} \\ &= \Big\{I - D(p^{(2)}(y)) - \Big[D(p^{(2)}(y))\Big]^2 - \cdots\Big\}\Big\{Ay + Ap^{(2)}(y) + f^{(2)}(y) + \cdots\Big\} \\ &= Ay + \Big[Ap^{(2)}(y) - D[p^{(2)}(y)]Ay + f^{(2)}(y)\Big] + \cdots \end{aligned}$$

Hence

$$g^{(2)}(y) = Ap^{(2)}(y) - D[p^{(2)}(y)]Ay + f^{(2)}(y).$$

Let $T_A^{(2)} : H_2 \to H_2$ define by

$$T_A^{(2)}\Big[p^{(2)}(y)\Big] = Ap^{(2)}(y) - D\Big(p^{(2)}(y)\Big)\,Ay.$$

Theorem 3.1 [4] *Let $\dot{x}$ be a C^r system of differential equations with $f(0) = 0$. Choose a complement G_k for $R(T_A^{(k)})$ in H_k. Then there is an analytic change of coordinates in a neighborhood of the origin which transforms the system $\dot{x} = f(x)$ to $\dot{y} = g(y) = g^{(1)}(y) + g^{(2)}(y) + \cdots + g^{(r)}(y) + R_r$ with $A = g^{(1)}(y)$ and $g^{(k)}(y) \in G_k$ for $2 \le k \le r$ and $R_r = o(|y|^r)$.*

Proof. A proof can be found in [4].

Remark 2 (a) *If $T_A^{(k)}$ is nonsingular, then we can take $g^{(k)}(y) = 0$.*

(b) *If $T_A^{(k)}$ is singular and diagonalizable, then $G_k = \ker(T_A^{(k)})$.*

(c) *If A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $T_A^{(k)}$ is diagonalizable with eigenvalues*

$$\{\lambda_i - \sum_{j=1}^{n} a_j\lambda_j \;\mid\; \text{for all } i \text{ and for all } a\},$$

where $a = (a_1, a_2, \ldots, a_n)$ is a vector of non-negative integers with $a_1 + a_2 + \cdots + a_n = k$.

Example 3.1 In equation (3.2), let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. That is,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{3.4}$$

where $\lambda_1 = i$, $\lambda_2 = -i$ are eigenvalues of A and A is diagonalizable. By Remark 2(c), $T_A^{(2)}$ is diagonalizable with eigenvalues $\{3i, -i, -i, -3i, i, -i\}$. Hence $T_A^{(2)}$ is nonsingular. Thus we can take $g^{(2)}(y) = 0$ by Remark 2(a). That is, we can transform (3.4) to

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \mathcal{O}(3). \tag{3.5}$$

Furthermore, $T_A^{(3)}$ is diagonalizable with eigenvalues $\{4i, 2i, -2i, 4i, 2i, 0, 0, -2i\}$. Hence $T_A^{(3)}$ is singular. We know

$$\begin{aligned} H^3 \; = \; \operatorname{span} \Bigg\{ & \begin{pmatrix} z_1^3 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1 z_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} z_2^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1^3 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1^2 z_2 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ z_1 z_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2^3 \end{pmatrix} \Bigg\} \end{aligned}$$

and we can show that

$$\ker(T_A^{(3)}) = \operatorname{span} \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} (z_1^2 + z_2^2), \begin{pmatrix} -z_2 \\ z_1 \end{pmatrix} (z_1^2 + z_2^2) \right\}.$$

That is,

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} (az_1 - bz_2)(z_1^2 + z_2^2) \\ (az_2 + bz_1)(z_1^2 + z_2^2) \end{pmatrix} + \mathcal{O}(4). \tag{3.6}$$

In fact, a is satisfied with the following equation [4]

$$\begin{aligned} a \; = \; & \frac{1}{16} \left[D_{z_1}^3 f_1 + D_{z_1 z_2 z_2}^3 f_1 + D_{z_1 z_1 z_2}^3 f_2 + D_{z_2}^3 f_2 + D_{z_1 z_2}^2 f_1 (D_{z_1}^2 f_1 + D_{z_2}^2 f_1) \right. \\ & \left. - D_{z_1 z_2}^2 f_2 (D_{z_1}^2 f_2 + D_{z_2}^2 f_2) - D_{z_1}^2 f_1 D_{z_1}^2 f_2 + D_{z_2}^2 f_1 D_{z_2}^2 f_2 \right]. \end{aligned}$$

4 Main results

Consider the system

$$\dot{x} = f(x) + bu, \tag{4.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, $b \in \mathbb{R}^n$ and 0 is a rest point of the system (4.1) (that is, $f(0) = 0$). we can rewrite the system (4.1), by changing a basis, into

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ 0 \end{pmatrix} u + \mathcal{O}(2) \tag{4.2}$$

with (A_{11}, b_1) is a controllable pair. Thus there exists a linear feedback control $u = kx_1 + v$ such that all eigenvalues of $A_{11} + b_1 k$ are negative and distinct. Hence the system (4.2) can be written as

$$\begin{aligned}
\begin{pmatrix} \dot{y} \\ \dot{\eta} \end{pmatrix} &= \begin{pmatrix} B & 0 \\ 0 & Q \end{pmatrix}\begin{pmatrix} y \\ \eta \end{pmatrix} + \begin{pmatrix} \bar{b} \\ 0 \end{pmatrix} v + \begin{pmatrix} \bar{f}(y,\eta) \\ g(y,\eta) \end{pmatrix} \qquad (4.3) \\
&= \begin{pmatrix} -k_1 & 0 & \cdots & 0 & 0 \\ 0 & -k_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -k_m & 0 \\ 0 & 0 & \cdots & 0 & Q \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ \eta \end{pmatrix} + \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_m \\ 0 \end{pmatrix} v + \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \vdots \\ \bar{f}_m \\ g \end{pmatrix},
\end{aligned}$$

where $-k_i \in \lambda(A_{11} + \bar{b}_1 k)$ and $\lambda(Q) = \lambda(A_{22})$. Hence by the Hartman-Grobman theorem, the system (4.1) is stabilizable by taking a suitable linear feedback if all eigenvalues of B have negative real parts. And, if B has eigenvalue with positive real part, then the system (4.1) can not be stabilized at all. Hence we only study the case with all eigenvalues of Q have non-positive real parts.

(Notation : In this thesis $\dfrac{\partial^k f}{\partial \eta^k} = \dfrac{\partial^k f}{\partial \eta^k}(0) = D_\eta^k f$)

Case I. $Q = 0_{1\times 1}$

In the case the system (4.3) become

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_m \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -k_1 & 0 & \cdots & 0 & 0 \\ 0 & -k_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -k_m & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ \eta \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \\ g \end{pmatrix}, \qquad (4.4)$$

where $w_1 = \bar{b}_1 v + \bar{f}_1$ and $w_i = \dfrac{\bar{b}_i}{\bar{b}_1}(w_1 - \bar{f}_1) + \bar{f}_i$ for $i = 2, 3, \cdots, m$. Let

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} h_1(\eta) \\ h_2(\eta) \\ \vdots \\ h_m(\eta) \end{pmatrix},$$

where $h_i(\eta) = a_i \eta^2 + \mathcal{O}(3)$ and $w_1 = \alpha \eta^2$. By (2.3) we have

$$\begin{pmatrix} 2a_1\eta + \mathcal{O}(2) \\ 2a_2\eta + \mathcal{O}(2) \\ \vdots \\ 2a_m\eta + \mathcal{O}(2) \end{pmatrix} \cdot g = \begin{pmatrix} -k_1 a_1 \eta^2 + \alpha\eta^2 + \mathcal{O}(3) \\ -k_2 a_2 \eta^2 + \dfrac{\bar{b}_2}{\bar{b}_1}(\alpha - \dfrac{1}{2}\dfrac{\partial^2 \bar{f}_1}{\partial \eta^2})\eta^2 + \dfrac{1}{2}\dfrac{\partial^2 \bar{f}_2}{\partial \eta^2}\eta^2 + \mathcal{O}(3) \\ \vdots \\ -k_m a_m \eta^2 + \dfrac{\bar{b}_m}{\bar{b}_1}(\alpha - \dfrac{1}{2}\dfrac{\partial^2 \bar{f}_1}{\partial \eta^2})\eta^2 + \dfrac{1}{2}\dfrac{\partial^2 \bar{f}_m}{\partial \eta^2}\eta^2 + \mathcal{O}(3) \end{pmatrix}.$$

That is,

$$\alpha \;=\; k_1 a_1$$

and

$$a_i \;=\; \frac{k_1 \bar{b}_i}{k_i \bar{b}_1} a_1 + c_i,$$

where

$$c_i \;=\; \frac{1}{2k_i}\left(\frac{\bar{b}_i}{\bar{b}_1}\frac{\partial^2 \bar{f}_1}{\partial \eta^2} + \frac{\partial^2 \bar{f}_i}{\partial \eta^2}\right) \qquad i = 2, 3, \cdots, m.$$

Theorem 4.1 *The system* (4.4) *is stabilizable by a control law* $w_1 = \alpha\eta^2$ *if* $\dfrac{\partial^2 g}{\partial \eta^2} = 0$ *and* $\displaystyle\sum_{i=1}^{m} \frac{k_1 \bar{b}_i}{k_i \bar{b}_1} \frac{\partial^2 g}{\partial \eta \partial y_i} \neq 0.$

Proof. By theorem 2.5 and theorem 2.6, we want to show that there exists a control law $w_1 = \alpha\eta^2$ such that the system

$$\dot{\eta} \;=\; Q\eta + g(h_1(\eta), h_2(\eta), \cdots, h_m(\eta), \eta)$$

is asymptotically stable. Let $V = \frac{1}{2}\eta^2$, then

$$\begin{aligned} \dot{V} &= \eta\dot{\eta} \\ &= \frac{1}{2}\frac{\partial^2 g}{\partial \eta^2}\eta^3 + \left(\sum_{i=1}^{m} a_i \frac{\partial^2 g}{\partial \eta \partial y_i} + \frac{1}{3!}\frac{\partial^3 g}{\partial \eta^3}\right)\eta^4 + \mathcal{O}(5) \\ &= \frac{1}{2}\frac{\partial^2 g}{\partial \eta^2}\eta^3 + a_1\left(\sum_{i=1}^{m} \frac{k_1 \bar{b}_i}{k_i \bar{b}_1}\frac{\partial^2 g}{\partial \eta \partial y_i}\right)\eta^4 + C\eta^4 + \mathcal{O}(5), \end{aligned}$$

where

$$C = c_2 + c_3 + \cdots + c_m + \frac{1}{3!}\frac{\partial^3 g}{\partial \eta^3}.$$

Thus the system (4.4) is unstable whenever $\dfrac{\partial^2 g}{\partial \eta^2}\eta^3 \neq 0$. On the other hand, if

$$\frac{\partial^2 g}{\partial \eta^2}\eta^3 = 0, \quad \sum_{i=1}^{m} \frac{k_1 \bar{b}_i}{k_i \bar{b}_1}\frac{\partial^2 g}{\partial \eta \partial y_i} \neq 0,$$

then there exists real number a_1 such that

$$a_1\left(\sum_{i=1}^{m} \frac{k_1 \bar{b}_i}{k_i \bar{b}_1}\frac{\partial^2 g}{\partial \eta \partial y_i}\right) + C < 0$$

(that is, $\dot{V} < 0$) for all $\eta \in N_\delta(0) - \{0\}$ and some neighborhood $N_\delta(0)$ at the origin. Hence the system (4.4) is asymptotically stable whenever $w_1 = \alpha\eta^2 = -k_1 a_1 \eta^2$, that is, the system (4.4) is stabilizable by a control law $w_1 = -k_1 a_1 \eta^2$.

Corollary 4.2 *Suppose Q is a Jordan form with only one zero eigenvalue and the other has negative real part. Then the same result as the above theorem is obtained.*

Proof. If Q is a Jordan form with only one zero eigenvalue and the other has negative real part, then (4.3) can be written

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_m \\ \dot{\zeta} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -k_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -k_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -k_m & 0 & 0 \\ 0 & 0 & \cdots & 0 & Q^* & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ \zeta \\ \eta \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \\ \Psi \\ g \end{pmatrix},$$

where $\zeta \in \mathbb{R}^{l-1}$ and

$$Q^* = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_l \end{pmatrix}$$

with

$$D_j = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \quad , \quad \lambda < 0$$

or

$$D_j = \begin{pmatrix} D_j^* & I & 0 & \cdots & 0 \\ 0 & D_j^* & I & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & D_j^* & I \\ 0 & 0 & \cdots & 0 & D_j^* \end{pmatrix},$$

where

$$D_j^* = \begin{pmatrix} s & -t \\ t & s \end{pmatrix}, \quad s < 0.$$

Let

$$Y = \begin{pmatrix} y_1) \\ \vdots \\ y_m \\ \zeta_1 \\ \vdots \\ \zeta_{l-1} \end{pmatrix} = \begin{pmatrix} h_1(\eta) \\ \vdots \\ h_m(\eta) \\ h_{m+1}(\eta) \\ \vdots \\ h_{m+l-1}(\eta) \end{pmatrix},$$

where

$$h_i(\eta) = a_i \eta^2 + \mathcal{O}(3) \quad i = 1, 2, \ldots, m+k-1.$$

By (2.3) we have

$$\alpha = k_1 a_1$$

and

$$a_i = \frac{k_1 \bar{b}_i}{k_i \bar{b}_1} a_1 + c_i,$$

where

$$c_i = \frac{1}{2k_i}\left(\frac{\bar{b}_i}{\bar{b}_1}\frac{\partial^2 \bar{f}_1}{\partial \eta^2} + \frac{\partial^2 \bar{f}_i}{\partial \eta^2}\right) \quad i = 2, 3, \cdots, m$$

and a_{m+j} is a linear combination of

$$\frac{\partial^2 \Psi_1}{\partial \eta^2}, \frac{\partial^2 \Psi_2}{\partial \eta^2}, \ldots, \frac{\partial^2 \Psi_{k-1}}{\partial \eta^2}.$$

Hence a_{m+j} is constant for $j = 1, 2, \ldots, k-1$. Let $V = \frac{1}{2}\eta^2$, then

$$\begin{aligned} \dot{V} &= \eta\dot{\eta} \\ &= \frac{1}{2}\frac{\partial^2 g}{\partial \eta^2}\eta^3 + \left(\sum_{i=1}^{m} a_i \frac{\partial^2 g}{\partial \eta \partial y_i} + \sum_{j=1}^{k-1} a_{m+j}\frac{\partial^2 g}{\partial \eta \partial \zeta_i} + \frac{1}{3!}\frac{\partial^3 g}{\partial \eta^3}\right)\eta^4 + \mathcal{O}(5) \\ &= \frac{1}{2}\frac{\partial^2 g}{\partial \eta^2}\eta^3 + a_1\left(\sum_{i=1}^{m} \frac{k_1 \bar{b}_i}{k_i \bar{b}_1}\frac{\partial^2 g}{\partial \eta \partial y_i}\right)\eta^4 + C\eta^4 + \mathcal{O}(5), \end{aligned}$$

where

$$C = \sum_{i=2}^{m} c_i + \sum_{j=1}^{k-1} a_{m+j}\frac{\partial^2 g}{\partial \eta \partial \zeta_i} + \frac{1}{3!}\frac{\partial^3 g}{\partial \eta^3}.$$

Hence we have the same result as the above theorem.

Case II. $Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

For the case the system (4.3) can be written

$$\begin{pmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_m \\ \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} -k_1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -k_m & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ \eta \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_m \\ g_1 \\ g_2 \end{pmatrix}, \tag{4.5}$$

where $w_1 = \bar{b}_1 v + \bar{f}_1$ and $w_i = \frac{\bar{b}_i}{\bar{b}_1}(w_1 - \bar{f}_1) + \bar{f}_i$ for $i = 2, 3, \cdots, m$. Let

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} h_1(\eta_1, \eta_2) \\ h_2(\eta_1, \eta_2) \\ \vdots \\ h_m(\eta_1, \eta_2) \end{pmatrix},$$

where $h_i(\eta_1,\eta_2) = a_{i1}\eta_1^2 + a_{i2}\eta_1\eta_2 + a_{i3}\eta_2^2 + \mathcal{O}(3)$ and define a control law $w_1 = \alpha\eta_1^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2$. By (2.3) we have

$$\begin{pmatrix} -k_1(a_{11}\eta_1^2 + a_{12}\eta_1\eta_2 + a_{13}\eta_2^2) + \alpha\eta_1^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2 + \mathcal{O}(3) \\ -k_1(a_{21}\eta_1^2 + a_{22}\eta_1\eta_2 + a_{23}\eta_2^2) + \dfrac{\bar{b}_2}{\bar{b}_1}(\alpha\eta_1^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2 - \bar{f}_1) + \bar{f}_2 + \mathcal{O}(3) \\ \vdots \\ -k_1(a_{m1}\eta_1^2 + a_{m2}\eta_1\eta_2 + a_{m3}\eta_2^2) + \dfrac{\bar{b}_m}{\bar{b}_1}(\alpha\eta_1^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2 - \bar{f}_1) + \bar{f}_m + \mathcal{O}(3) \end{pmatrix}$$
$$= \begin{pmatrix} 2a_{11}\eta_1\eta_2 + a_{12}\eta_2^2 + \mathcal{O}(3) \\ 2a_{21}\eta_1\eta_2 + a_{22}\eta_2^2 + \mathcal{O}(3) \\ \vdots \\ 2a_{m1}\eta_1\eta^2 + a_{m2}\eta_2^2 + \mathcal{O}(3) \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} k_1 & 0 & 0 \\ 2 & k_1 & 0 \\ 0 & 1 & k_1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix}$$

and

$$\begin{pmatrix} k_i & 0 & 0 \\ 2 & k_i & 0 \\ 0 & 1 & k_i \end{pmatrix} \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{pmatrix} = \frac{\bar{b}_i}{\bar{b}_1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} t_{i1} \\ t_{i2} \\ t_{i3} \end{pmatrix},$$

where

$$\begin{aligned} t_{i1} &= -\frac{\bar{b}_i}{2\bar{b}_1}\frac{\partial^2 \bar{f}_1}{\partial\eta_1^2} + \frac{1}{2}\frac{\partial^2 \bar{f}_i}{\partial\eta_1^2}, \\ t_{i2} &= -\frac{\bar{b}_i}{\bar{b}_1}\frac{\partial^2 \bar{f}_1}{\partial\eta_1\eta_2} + \frac{\partial^2 \bar{f}_i}{\partial\eta_1\eta_2}, \\ t_{i3} &= -\frac{\bar{b}_i}{2\bar{b}_1}\frac{\partial^2 \bar{f}_1}{\partial\eta_2^2} + \frac{1}{2}\frac{\partial^2 \bar{f}_i}{\partial\eta_2^2}. \end{aligned}$$

Thus

$$\begin{aligned} \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{pmatrix} &= \frac{\bar{b}_i}{\bar{b}_1 k_i^3} \begin{pmatrix} k_i^2 & 0 & 0 \\ -2k_i & k_i^2 & 0 \\ 2 & -k_i & k_i^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{pmatrix} \\ &= \frac{\bar{b}_i}{\bar{b}_1 k_i^3} \begin{pmatrix} k_1k_i^2 & 0 & 0 \\ -2k_1k_i + 2k_i^2 & k_1k_i^2 & 0 \\ 2k_1 - 2k_i & k_i^2 - k_1k_i & k_1k_i^2 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} + \begin{pmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{pmatrix}, \end{aligned}$$

where

$$\begin{pmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{pmatrix} = \frac{1}{k_i^3} \begin{pmatrix} k_i^2 & 0 & 0 \\ -2k_i & k_i^2 & 0 \\ 2 & -k_i & k_i^2 \end{pmatrix} \begin{pmatrix} t_{i1} \\ t_{i2} \\ t_{i3} \end{pmatrix}.$$

Theorem 4.3 *If*

$$\frac{\partial^2 g_2}{\partial\eta_1^2} = 0\ ,\ \ \frac{\partial^2 g_2}{\partial\eta_1\partial y_2} + \frac{\partial^2 g_1}{\partial\eta_1^2} = 0$$

and

$$\sum_{i=1}^{n} \frac{\bar{b}_i k_1}{\bar{b}_1 k_i} \frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} \neq 0,$$

then the system (4.5) can be stabilizable by a control law $w_1 = a_{11}\eta_1^2 + a_{12}\eta_1\eta_2 + a_{13}\eta_2^2$.

Theorem 4.4 (Invariant Set Theorem) [6] *Consider a system* $\dot{x} = f(x)$*, with* f *continuous, and let* $V \in C^1$*. Assume that for some* $l > 0$*, the region* $\Omega_l = \{x | V(x) < l\}$ *and* $\dot{V} \leq 0$ *for all* x *in* Ω_l*. Let* R *be the set of all points within* Ω_l*, where* $\dot{V} = 0$*, and* M *be the largest invariant set in* R*. Then, every solution* $x(t)$ *originating in* Ω_l *tends to* M *as* $t \to \infty$*.*

Proof of Theorem 4.3. We want to show that the center manifold system

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{pmatrix} \tag{4.6}$$

is asymptotically stable, where

$$\tilde{g}_i = \tilde{g}_i(\eta_1, \eta_2) \equiv g_i(h_1(\eta_1, \eta_2), \ldots, h_m(\eta_1, \eta_2), \eta_1, \eta_2).$$

Using normal form, we have

$$H_2 = \text{span}\left\{ \begin{pmatrix} \eta_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} \eta_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} \eta_1\eta_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta_1\eta_2 \end{pmatrix} \right\}$$

and

$$T^{(2)}\left(p^{(2)}(\eta)\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} p^{(2)}(\eta) - D\left[p^{(2)}(\eta)\right] \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \eta,$$

where $p^{(2)}(\eta) \in H_2$. Then

$$R\left(T^{(2)}\right) = \text{span}\left\{ \begin{pmatrix} -\eta_1\eta_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \eta_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} \eta_1^2 \\ -2\eta_1\eta_2 \end{pmatrix}, \begin{pmatrix} \eta_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} \eta_1\eta_2 \\ -\eta_2^2 \end{pmatrix} \right\}.$$

Since $H_2 = R\left(T^{(2)}\right) \oplus G_2$, we can choose

$$G_2 = \text{span}\left\{ \begin{pmatrix} 0 \\ \eta_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta_1\eta_2 \end{pmatrix} \right\}.$$

Hence the system (4.6) can be written as

$$\begin{pmatrix} \dot{s}_1 \\ \dot{s}_2 \end{pmatrix} = \begin{pmatrix} s_2 \\ \delta s_1^2 + \varepsilon s_1 s_2 \end{pmatrix} + \mathcal{O}(3), \tag{4.7}$$

where

$$\delta = \frac{1}{2}\frac{\partial^2 \tilde{g}_2}{\partial \eta_1^2},$$
$$\varepsilon = \frac{\partial^2 \tilde{g}_2}{\partial \eta_1 \partial \eta_2} + \frac{\partial^2 \tilde{g}_1}{\partial \eta_1^2}.$$

If $\delta \neq 0$ or $\varepsilon \neq 0$ then the system (4.7) is unstable. Hence we assume $\delta = \varepsilon = 0$ and using the same method, the system (4.7) can be written as

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ \lambda z_1^3 + \mu z_1^2 z_2 \end{pmatrix} + \mathcal{O}(4), \tag{4.8}$$

where

$$\begin{aligned} \lambda &= \frac{1}{6}\frac{\partial^3 \tilde{g}_2}{\partial \eta_1^3} + \frac{1}{2}\left(\frac{\partial^2 \tilde{g}_1}{\partial \eta_1^2}\right)^2, \\ \mu &= \frac{1}{2}\left(\frac{\partial^3 \tilde{g}_2}{\partial \eta_1^2 \partial \eta_2} + \frac{\partial^3 \tilde{g}_1}{\partial \eta_1^3}\right) - \frac{1}{2}\frac{\partial^2 \tilde{g}_1}{\partial \eta_1^2}\left(\frac{\partial^2 \tilde{g}_1}{\partial \eta_1 \partial \eta_2} + \frac{\partial^2 \tilde{g}_2}{\partial \eta_2^2}\right). \end{aligned}$$

If $\lambda < 0$ and $\mu < 0$, then we can choose a Lyapunov function

$$V = -\frac{1}{4}\lambda z_1^4 + \frac{1}{2} z_2^2 \geq 0$$

such that

$$\begin{aligned} \dot{V} &= -\lambda z_1^3 \dot{z}_1 + z_2 \dot{z}_2 \\ &= -\lambda z_1^3 z_2 + \lambda z_1^3 z_2 + \mu z_1^2 z_2^2 \\ &= \mu z_1^2 z_2^2 \\ &\leq 0. \end{aligned}$$

And by theorem 4.4, we have Ω_l is bounded, $R = \{(z_1, z_2) \mid z_1 = 0 \text{ or } z_2 = 0\}$ and $M = \{(0,0)\}$. Hence the system (4.8) is locally asymptotically stable whenever $\lambda < 0$ and $\mu < 0$. Therefore we must show that there exists a control law w_1 such that $\lambda < 0$ and $\mu < 0$ whenever $\sum_{i=1}^{n} \frac{\bar{b}_i k_1}{\bar{b}_1 k_i} \frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} \neq 0$. Since

$$\begin{aligned} \tilde{g}_j(\eta_1, \eta_2) &= \frac{1}{2}\frac{\partial^2 g_j}{\partial \eta_1^2}\eta_1^2 + \frac{\partial^2 g_j}{\partial \eta_1 \partial \eta_2}\eta_1\eta_2 + \frac{1}{2}\frac{\partial^2 g_j}{\partial \eta_2^2}\eta_2^2 \\ &\quad + \frac{1}{6}\frac{\partial^3 g_j}{\partial \eta_1^3}\eta_1^3 + \frac{1}{2}\frac{\partial^3 g_j}{\partial \eta_1^2 \partial \eta_2}\eta_1^2\eta_2 + \frac{1}{2}\frac{\partial^3 g_j}{\partial \eta_1 \partial \eta_2^2}\eta_1\eta_2^2 + \frac{1}{6}\frac{\partial^3 g_j}{\partial \eta_2^3}\eta_2^3 \\ &\quad + \sum_{i=1}^{m} \frac{\partial^2 g_j}{\partial \eta_1 \partial y_j}\eta_1 h_j(\eta_1, \eta_2) + \sum_{i=1}^{m} \frac{\partial^2 g_j}{\partial \eta_2 \partial y_j}\eta_2 h_j(\eta_1, \eta_2) + \cdots \\ &= \frac{1}{2}\frac{\partial^2 g_j}{\partial \eta_1^2}\eta_1^2 + \frac{\partial^2 g_j}{\partial \eta_1 \partial \eta_2}\eta_1\eta_2 + \frac{1}{2}\frac{\partial^2 g_j}{\partial \eta_2^2}\eta_2^2 + \left(\frac{1}{6}\frac{\partial^3 g_j}{\partial \eta_1^3} + \sum_{i=1}^{m} a_{i1}\frac{\partial^2 g_j}{\partial \eta_1 \partial y_i}\right)\eta_1^3 \\ &\quad + \left(\frac{1}{2}\frac{\partial^3 g_j}{\partial \eta_1^2 \partial \eta_2} + \sum_{i=1}^{m} a_{i2}\frac{\partial^2 g_j}{\partial \eta_1 \partial y_i} + \sum_{i=1}^{m} a_{i1}\frac{\partial^2 g_j}{\partial \eta_2 \partial y_i}\right)\eta_1^2\eta_2 \\ &\quad + \left(\frac{1}{2}\frac{\partial^3 g_j}{\partial \eta_1 \partial \eta_2^2} + \sum_{i=1}^{m} a_{i3}\frac{\partial^2 g_j}{\partial \eta_1 \partial y_i} + \sum_{i=1}^{m} a_{i2}\frac{\partial^2 g_j}{\partial \eta_2 \partial y_i}\right)\eta_1\eta_2^2 \\ &\quad + \left(\frac{1}{6}\frac{\partial^3 g_j}{\partial \eta_2^3} + \sum_{i=1}^{m} a_{i3}\frac{\partial^2 g_j}{\partial \eta_2 \partial y_i}\right)\eta_2^3 + \cdots \end{aligned} \tag{4.9}$$

Thus

$$\lambda = \frac{1}{6}\frac{\partial^3 \tilde{g}_2}{\partial \eta_1^3} + \frac{1}{2}\left(\frac{\partial^2 \tilde{g}_1}{\partial \eta_1^2}\right)^2$$

$$
\begin{aligned}
&= \frac{1}{6}\frac{\partial^3 g_2}{\partial \eta_1^3} + \sum_{i=1}^{m} a_{i1}\frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} + \frac{1}{2}\left(\frac{\partial^2 g_1}{\partial \eta_1^2}\right)^2 \\
&= \frac{1}{6}\frac{\partial^3 g_2}{\partial \eta_1^3} + a_{11}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{\bar{b}_1 k_i}\frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} + \sum_{i=1}^{m} c_{i1}\frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} + \frac{1}{2}\left(\frac{\partial^2 g_1}{\partial \eta_1^2}\right)^2 \\
&= a_{11}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{\bar{b}_1 k_i}\frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} + C_1.
\end{aligned}
$$

Hence we can choose a_{11} such $\lambda < 0$ whenever $\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{\bar{b}_1 k_i}\frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} \neq 0$. And

$$
\begin{aligned}
\mu &= \frac{1}{2}\left(\frac{\partial^3 \tilde{g}_2}{\partial \eta_1^2 \partial \eta_2} + \frac{\partial^3 \tilde{g}_1}{\partial \eta_1^3}\right) - \frac{1}{2}\frac{\partial^2 \tilde{g}_1}{\partial \eta_1^2}\left(\frac{\partial^2 \tilde{g}_1}{\partial \eta_1 \partial \eta_2} + \frac{\partial^2 \tilde{g}_2}{\partial \eta_2^2}\right) \\
&= \left(\frac{1}{2}\frac{\partial^3 g_2}{\partial \eta_1^2 \partial \eta_2} + \sum_{i=1}^{m} a_{i2}\frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} + \sum_{i=1}^{m} a_{i1}\frac{\partial^2 g_2}{\partial \eta_2 \partial y_i}\right) \\
&\quad + \left(\frac{1}{6}\frac{\partial^3 g_1}{\partial \eta_1^3}\eta_1^3 + \sum_{i=1}^{m} a_{i1}\frac{\partial^2 g_1}{\partial \eta_1 \partial y_i}\right)\eta_1^4 - n\frac{1}{2}\frac{\partial^2 g_1}{\partial \eta_1^2}\left(\frac{\partial^2 g_1}{\partial \eta_1 \partial \eta_2} + \frac{\partial^2 g_2}{\partial \eta_2^2}\right) \\
&= a_{12}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{\bar{b}_1 k_i}\frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} + a_{11}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{\bar{b}_1 k_i}\frac{\partial^2 g_2}{\partial \eta_2 \partial y_i} + C_2.
\end{aligned}
$$

Since a_{11} has been fixed, hence we can find a_{12} such that $\mu < 0$. That is, there exists a control law $w_1 = \alpha\eta^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2$ such that system (4.5) is asymptotically stable when it is satisfied with the assumption of theorem 4.3.

Corollary 4.5 *If* $Q = \begin{pmatrix} Q^* & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, *where* Q^* *is a Jordan form with all eigenvalues have negative real parts. Then we have the same result as theorem* 4.3.

Proof. Using the proofs of corollary 4.2 and theorem 4.3, we can easily prove the corollary.

Case III. $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

For the case, equation (4.3) become

$$
\begin{pmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_m \\ \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} -k_1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -k_m & 0 & 0 \\ 0 & \cdots & 0 & 0 & -1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ \eta \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_m \\ g_1 \\ g_2 \end{pmatrix}, \quad (4.10)
$$

where $w_1 = \bar{b}_1 v + \bar{f}_1$ and $w_i = \dfrac{\bar{b}_i}{\bar{b}_1}(w_1 - \bar{f}_1) + \bar{f}_i$ for $i = 2, 3, \ldots, m$. Let

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} h_1(\eta_1, \eta_2) \\ h_2(\eta_1, \eta_2) \\ \vdots \\ h_m(\eta_1, \eta_2) \end{pmatrix},$$

where $h_i(\eta_1, \eta_2) = a_{i1}\eta_1^2 + a_{i2}\eta_1\eta_2 + a_{i3}\eta_2^2 + \mathcal{O}(3)$ and define a control law $w_1 = \alpha\eta_1^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2$. By (2.3) we have

$$\begin{pmatrix} -k_1(a_{11}\eta_1^2 + a_{12}\eta_1\eta_2 + a_{13}\eta_2^2) + \alpha\eta_1^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2 + \mathcal{O}(3) \\ -k_1(a_{21}\eta_1^2 + a_{22}\eta_1\eta_2 + a_{23}\eta_2^2) + \dfrac{\bar{b}_2}{\bar{b}_1}(\alpha\eta_1^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2 - \bar{f}_1) + \bar{f}_2 + \mathcal{O}(3) \\ \vdots \\ -k_1(a_{m1}\eta_1^2 + a_{m2}\eta_1\eta_2 + a_{m3}\eta_2^2) + \dfrac{\bar{b}_m}{\bar{b}_1}(\alpha\eta_1^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2 - \bar{f}_1) + \bar{f}_m + \mathcal{O}(3) \end{pmatrix}$$
$$= \begin{pmatrix} a_{12}\eta_1^2 + 2(a_{13} - a_{11})\eta_1\eta_2 - a_{12}\eta_2^2 + \mathcal{O}(3) \\ a_{22}\eta_1^2 + 2(a_{23} - a_{21})\eta_1\eta_2 - a_{22}\eta_2^2 + \mathcal{O}(3) \\ \vdots \\ a_{m2}\eta_1^2 + 2(a_{m3} - a_{m1})\eta_1\eta_2 - a_{m2}\eta_2^2 + \mathcal{O}(3) \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} k_1 & 1 & 0 \\ -2 & k_1 & 2 \\ 0 & -1 & k_1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix}$$

and

$$\begin{pmatrix} k_i & 1 & 0 \\ -2 & k_i & 2 \\ 0 & -1 & k_i \end{pmatrix} \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{pmatrix} = \frac{\bar{b}_i}{\bar{b}_1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} t_{i1} \\ t_{i2} \\ t_{i3} \end{pmatrix},$$

where

$$\begin{aligned} t_{i1} &= -\frac{\bar{b}_i}{2\bar{b}_1}\frac{\partial^2 \bar{f}_1}{\partial \eta_1^2} + \frac{1}{2}\frac{\partial^2 \bar{f}_i}{\partial \eta_1^2}, \\ t_{i2} &= -\frac{\bar{b}_i}{\bar{b}_1}\frac{\partial^2 \bar{f}_1}{\partial \eta_1\eta_2} + \frac{\partial^2 \bar{f}_i}{\partial \eta_1\eta_2}, \\ t_{i3} &= -\frac{\bar{b}_i}{2\bar{b}_1}\frac{\partial^2 \bar{f}_1}{\partial \eta_2^2} + \frac{1}{2}\frac{\partial^2 \bar{f}_i}{\partial \eta_2^2}. \end{aligned}$$

Thus

$$\begin{aligned} &\begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{pmatrix} - \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \end{pmatrix} \\ = \;& \frac{\bar{b}_i}{\bar{b}_1(k_i^3 + 4k_i)} \begin{pmatrix} k_i^2 + 2 & -k_2 & 2 \\ 2k_i & k_i^2 & -2k_i \\ 2 & k_i & k_i^2 + 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \\ = \;& \frac{\bar{b}_i}{\bar{b}_1(k_i^3 + 4k_i)} \begin{pmatrix} (2k_1 + 2k_i + k_1k_i^2)a_{11} + (k_i^2 - k_1k_i)a_{12} + (2k_1 - 2k_i)a_{13} \\ (2k_1k_i - 2k_i^2)a_{11} + (2k_1 + 2k_i + k_1k_i^2)a_{12} + (k_i^2 - 2k_1k_i)a_{13} \\ (2k_1 - 2k_i)a_{11} + (k_1k_i - k_i^2)a_{12} + (2k_1 + 2k_i + k_1k_i^2)a_{13} \end{pmatrix}, \end{aligned}$$

where

$$\begin{pmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{pmatrix} = \frac{1}{k_i^3 + 4k_i} \begin{pmatrix} k_i^2 + 2 & -k_2 & 2 \\ 2k_i & k_i^2 & -2k_i \\ 2 & k_i & k_i^2 + 2 \end{pmatrix} \begin{pmatrix} t_{i1} \\ t_{i2} \\ t_{i3} \end{pmatrix}.$$

Theorem 4.6 *If the system* (4.10) *satisfies*

$$\sum_{i=1}^{m} \frac{\bar{b}_i(k_i^2 - k_1 k_i)}{\bar{b}_1(k_i^3 + 4k_i)} \left(\frac{\partial^2 g_1}{\partial \eta_2 \partial y_i} + \frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} \right) \neq 0$$

or

$$\sum_{i=1}^{m} \frac{\bar{b}_i(2k_1 + 2k_i + k_1 k_i^2)}{\bar{b}_1(k_i^3 + 4k_i)} \left(\frac{\partial^2 g_1}{\partial \eta_2 \partial y_i} + \frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} \right) \neq 0,$$

then it can be stabilizable by a control law $w_1 = \alpha \eta^2 + \beta \eta_1 \eta_2 + \gamma \eta_2^2$.

Proof. The center manifold system is

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{pmatrix}, \tag{4.11}$$

where

$$\tilde{g}_i = \tilde{g}_i(\eta_1, \eta_2) \equiv g_i(h_1(\eta_1, \eta_2), \ldots, h_m(\eta_1, \eta_2), \eta_1, \eta_2).$$

Using the result of example 3.1, the system (4.11) can be reduced to

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} (dz_1 - ez_2)(z_1^2 + z_2^2) \\ (dz_2 + ez_1)(z_1^2 + z_2^2) \end{pmatrix} + \mathcal{O}(4). \tag{4.12}$$

Let $z_1 = r\cos\theta$, $z_2 = r\sin\theta$, then the system (4.12) can be put into the normal form

$$\begin{aligned} \dot{r} &= dr^3 + \mathcal{O}(4), \\ \dot{\theta} &= -1 + f(r). \end{aligned} \tag{4.13}$$

We want to show that (4.13) is asymptotically stable. That is, we must find a control law such that d is negative. Equations (3.7) and (4.9) imply that

$$\begin{aligned}
& D^3_{\eta_1}\tilde{g}_1 + D^3_{\eta_1\eta_2\eta_2}\tilde{g}_1 + D^3_{\eta_1\eta_1\eta_2}\tilde{g}_2 + D^3_{\eta_2}\tilde{g}_2 \\
= \; & \frac{1}{6}\frac{\partial^3 g_1}{\partial \eta_1^3} + \sum_{i=1}^{m} a_{i1}\frac{\partial^2 g_1}{\partial \eta_1 \partial y_i} + \frac{1}{2}\frac{\partial^3 g_1}{\partial \eta_1^2 \partial \eta_2} + \sum_{i=1}^{m} a_{i2}\frac{\partial^2 g_1}{\partial \eta_1 \partial y_i} + \sum_{i=1}^{m} a_{i1}\frac{\partial^2 g_1}{\partial \eta_2 \partial y_i} \\
& + \frac{1}{2}\frac{\partial^3 g_2}{\partial \eta_1 \partial \eta_2^2} + \sum_{i=1}^{m} a_{i3}\frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} + \sum_{i=1}^{m} a_{i2}\frac{\partial^2 g_2}{\partial \eta_2 \partial y_i} + \frac{1}{6}\frac{\partial^3 g_2}{\partial \eta_2^3} + \sum_{i=1}^{m} a_{i3}\frac{\partial^2 g_2}{\partial \eta_2 \partial y_i} \\
= \; & \sum_{i=1}^{m} (a_{i1} + a_{i3}) \left(\frac{\partial^2 g_1}{\partial \eta_1 \partial y_i} + \frac{\partial^2 g_2}{\partial \eta_2 \partial y_i} \right) + \sum_{i=1}^{m} a_{i2} \left(\frac{\partial^2 g_1}{\partial \eta_2 \partial y_i} + \frac{\partial^2 g_2}{\partial \eta_1 \partial y_i} \right) \\
& + \frac{1}{6}\frac{\partial^3 g_1}{\partial \eta_1^3} + \frac{1}{2}\frac{\partial^3 g_1}{\partial \eta_1^2 \partial \eta_2} + \frac{1}{2}\frac{\partial^3 g_2}{\partial \eta_1 \partial \eta_2^2} + \frac{1}{6}\frac{\partial^3 g_2}{\partial \eta_2^3},
\end{aligned} \tag{4.14}$$

where

$$\begin{aligned} a_{i1}+a_{i3} &= \frac{\bar{b}_i}{\bar{b}_1(k_i^3+4k_i)}\left[(k_1k_i^2+4k_1)(a_{11}+a_{13})\right]+c_{i1}+c_{i3}, \\ a_{12} &= \frac{\bar{b}_i}{\bar{b}_1(k_i^3+4k_i)}\left[2(k_1k_i-k_i^2)(a_{11}-a_{13})+(2k_1+2k_i+k_1k_i^2)a_{12}\right]+c_{i2}. \end{aligned}$$

The equation (4.14) is changed into, by taking $a_{13}=-a_{11}$,

$$\begin{aligned} &\sum_{i=1}^{m}\frac{\bar{b}_i}{\bar{b}_1(k_i^3+4k_i)}[4(k_1k_i-k_i^2)a_{11}+(2k_1+2k_i+k_1k_i^2)a_{12}]\left(\frac{\partial^2 g_1}{\partial\eta_2\partial y_i}+\frac{\partial^2 g_2}{\partial\eta_1\partial y_i}\right) \\ &\quad+\sum_{i=1}^{m}(s_{i1}+s_{i3})\left(\frac{\partial^2 g_1}{\partial\eta_1\partial y_i}+\frac{\partial^2 g_2}{\partial\eta_2\partial y_i}\right)+\frac{1}{6}\frac{\partial^3 g_1}{\partial\eta_1^3}+\frac{1}{2}\frac{\partial^3 g_1}{\partial\eta_1^2\partial\eta_2}+\frac{1}{2}\frac{\partial^3 g_2}{\partial\eta_1\partial\eta_2^2}+\frac{1}{6}\frac{\partial^3 g_2}{\partial\eta_2^3} \\ =\;& a_{11}\sum_{i=1}^{m}\frac{4\bar{b}_i(k_1k_i-k_i^2)}{\bar{b}_1(k_i^3+4k_i)}\left(\frac{\partial^2 g_1}{\partial\eta_2\partial y_i}+\frac{\partial^2 g_2}{\partial\eta_1\partial y_i}\right) \\ &\quad+a_{12}\sum_{i=1}^{m}\frac{\bar{b}_i(2k_1+2k_i+k_1k_i^2)}{\bar{b}_1(k_i^3+4k_i)}\left(\frac{\partial^2 g_1}{\partial\eta_2\partial y_i}+\frac{\partial^2 g_2}{\partial\eta_1\partial y_i}\right)+C, \end{aligned}$$

whenever

$$\sum_{i=1}^{m}\frac{\bar{b}_i(k_1k_i-k_i^2)}{\bar{b}_1(k_i^3+4k_i)}\left(\frac{\partial^2 g_1}{\partial\eta_2\partial y_i}+\frac{\partial^2 g_2}{\partial\eta_1\partial y_i}\right)\neq 0$$

or

$$\sum_{i=1}^{m}\frac{\bar{b}_i(2k_1+2k_i+k_1k_i^2)}{\bar{b}_1(k_i^3+4k_i)}\left(\frac{\partial^2 g_1}{\partial\eta_2\partial y_i}+\frac{\partial^2 g_2}{\partial\eta_1\partial y_i}\right)\neq 0.$$

We can take suitable a_{11} and a_{12} such that d is negative. That is, we can find a control law $w_1=\alpha\eta_1^2+\beta\eta_1\eta_2+\gamma\eta_2^2$ such that system (4.10) is asymptotically stable.

Corollary 4.7 *We have the same result as theorem* 4.5 *whenever*

$$Q=\begin{pmatrix} Q^* & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

where Q^* is a Jordan form with all eigenvalues have negative real parts.

Case IV. $Q=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

In the case, the system (4.3) become

$$\begin{pmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_m \\ \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} -k_1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -k_m & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ \eta \end{pmatrix}+\begin{pmatrix} w_1 \\ \vdots \\ w_m \\ g_1 \\ g_2 \end{pmatrix}, \tag{4.15}$$

where $w_1 = \bar{b}_1 v + \bar{f}_1$ and $w_i = \dfrac{\bar{b}_i}{\bar{b}_1}(w_1 - \bar{f}_1) + \bar{f}_i$ for $i = 2, 3, \ldots, m$. Let

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} h_1(\eta_1, \eta_2) \\ h_2(\eta_1, \eta_2) \\ \vdots \\ h_m(\eta_1, \eta_2) \end{pmatrix},$$

where $h_i(\eta_1, \eta_2) = a_{i1}\eta_1^2 + a_{i2}\eta_1\eta_2 + a_{i3}\eta_2^2 + \mathcal{O}(3)$ and a control law be $w_1 = \alpha\eta_1^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2$. By (2.3) we have

$$\begin{pmatrix} -k_1(a_{11}\eta_1^2 + a_{12}\eta_1\eta_2 + a_{13}\eta_2^2) + \alpha\eta_1^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2 + \mathcal{O}(3) \\ -k_1(a_{21}\eta_1^2 + a_{22}\eta_1\eta_2 + a_{23}\eta_2^2) + \dfrac{\bar{b}_2}{\bar{b}_1}(\alpha\eta_1^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2 - \bar{f}_1) + \bar{f}_2 + \mathcal{O}(3) \\ \vdots \\ -k_1(a_{m1}\eta_1^2 + a_{m2}\eta_1\eta_2 + a_{m3}\eta_2^2) + \dfrac{\bar{b}_m}{\bar{b}_1}(\alpha\eta_1^2 + \beta\eta_1\eta_2 + \gamma\eta_2^2 - \bar{f}_1) + \bar{f}_m + \mathcal{O}(3) \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{O}(3) \\ \mathcal{O}(3) \\ \vdots \\ \mathcal{O}(3) \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_1 & 0 \\ 0 & 0 & k_1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix}$$

and

$$\begin{pmatrix} k_i & 0 & 0 \\ 0 & k_i & 0 \\ 0 & 0 & k_i \end{pmatrix} \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{pmatrix} = \frac{\bar{b}_i}{\bar{b}_1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{pmatrix},$$

where

$$\begin{aligned} c_{i1} &= -\frac{\bar{b}_i}{2\bar{b}_1}\frac{\partial^2 \bar{f}_1}{\partial \eta_1^2} + \frac{1}{2}\frac{\partial^2 \bar{f}_i}{\partial \eta_1^2}, \\ c_{i2} &= -\frac{\bar{b}_i}{\bar{b}_1}\frac{\partial^2 \bar{f}_1}{\partial \eta_1 \eta_2} + \frac{\partial^2 \bar{f}_i}{\partial \eta_1 \eta_2}, \\ c_{i3} &= -\frac{\bar{b}_i}{2\bar{b}_1}\frac{\partial^2 \bar{f}_1}{\partial \eta_2^2} + \frac{1}{2}\frac{\partial^2 \bar{f}_i}{\partial \eta_2^2}. \end{aligned}$$

Thus

$$\begin{aligned} \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{pmatrix} &= \frac{\bar{b}_i}{\bar{b}_1 k_i} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \frac{1}{k_i} \begin{pmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{pmatrix} \\ &= \frac{\bar{b}_i k_1}{\bar{b}_1 k_i} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} + \frac{1}{k_i} \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \end{pmatrix}. \end{aligned} \tag{4.16}$$

Theorem 4.8 *If the system* (4.15) *satisfies*

$$\frac{\partial^2 g_1}{\partial \eta_1^2} = 0, \quad \frac{\partial^2 g_1}{\partial \eta_1 \partial \eta_2} + \frac{1}{2}\frac{\partial^2 g_2}{\partial \eta_2^2} = 0,$$
$$\frac{1}{2}\frac{\partial^2 g_1}{\partial \eta_2^2} + \frac{\partial^2 g_2}{\partial \eta_1 \partial \eta_2} = 0, \quad \frac{\partial^2 g_2}{\partial \eta_2^2} = 0$$

and

$$2\left(\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{\bar{b}_1 k_i} \frac{\partial^2 g_2}{\partial \eta_2 \partial y_i}\right)\left(\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{\bar{b}_1 k_i} \frac{\partial^2 g_1}{\partial \eta_1 \partial y_i}\right) > \left[\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{\bar{b}_1 k_i}\left(\frac{\partial^2 g_1}{\partial \eta_2 \partial y_i} + \frac{\partial^2 g_2}{\partial \eta_1 \partial y_i}\right)\right]^2 \tag{4.17}$$

then there exists a control law w_1 such that the system is asymptotically stable.

Proof. The center manifold system is

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} \tilde{g}_1(\eta_1, \eta_2) \\ \tilde{g}_2(\eta_1, \eta_2) \end{pmatrix},$$

where $\tilde{g}_i(\eta_1, \eta_2) \equiv g_i(h_1(\eta_1, \eta_2), \ldots, h_m(\eta_1, \eta_2), \eta_1, \eta_2)$.

Using a Lyapunov function $V = \frac{1}{2}\eta_1^2 + \frac{1}{2}\eta_2^2$ and by equations (4.9) and (4.16), we have

$$\begin{aligned}
\dot{V} &= \eta_1\dot{\eta}_1 + \eta_2\dot{\eta}_2 \\
&= \frac{1}{2}\frac{\partial^2 g_1}{\partial \eta_1^2}\eta_1^3 + \left(\frac{\partial^2 g_1}{\partial \eta_1 \partial \eta_2} + \frac{1}{2}\frac{\partial^2 g_2}{\partial \eta_2^2}\right)\eta_1^2\eta_2 + \left(\frac{1}{2}\frac{\partial^2 g_1}{\partial \eta_2^2} + \frac{\partial^2 g_2}{\partial \eta_1 \partial \eta_2}\right)\eta_1\eta_2^2 \\
&\quad + \frac{1}{2}\frac{\partial^2 g_2}{\partial \eta_2^2}\eta_2^3 + \left[a_{11}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{k_i \bar{b}_1}\frac{\partial^2 g_1}{\partial \eta_1 \partial y_i} + C_1\right]\eta_1^4 \\
&\quad + \left[a_{13}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{k_i \bar{b}_1}\frac{\partial^2 g_2}{\partial \eta_2 \partial y_i} + C_2\right]\eta_2^4 \\
&\quad + \left[a_{11}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{k_i \bar{b}_1}\left(\frac{\partial^2 g_1}{\partial \eta_2 \partial y_i} + \frac{\partial^2 g_2}{\partial \eta_1 \partial y_i}\right) + a_{12}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{k_i \bar{b}_1}\frac{\partial^2 g_1}{\partial \eta_1 \partial y_i} + C_3\right]\eta_1^3\eta_2 \\
&\quad + \left[a_{11}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{k_i \bar{b}_1}\frac{\partial^2 g_2}{\partial \eta_2 \partial y_i} + a_{12}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{k_i \bar{b}_1}\left(\frac{\partial^2 g_1}{\partial \eta_2 \partial y_i} + \frac{\partial^2 g_2}{\partial \eta_1 \partial y_i}\right)\right. \\
&\qquad \left. + a_{13}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{k_i \bar{b}_1}\frac{\partial^2 g_1}{\partial \eta_1 \partial y_i} + C_4\right]\eta_1^2\eta_2^2 \\
&\quad + \left[a_{12}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{k_i \bar{b}_1}\frac{\partial^2 g_2}{\partial \eta_2 \partial y_i} + a_{12}\sum_{i=1}^{m} \frac{\bar{b}_i k_1}{k_i \bar{b}_1}\left(\frac{\partial^2 g_1}{\partial \eta_2 \partial y_i} + \frac{\partial^2 g_2}{\partial \eta_1 \partial y_i}\right) + C_5\right]\eta_1\eta_2^3,
\end{aligned}$$

where C_i is constant for all $i = 1, 2, \ldots 5$

If the coefficients of η_1^3, $\eta_1^2\eta_2$, $\eta_1\eta_2^2$, η_2^3, $\eta_1^3\eta_2$, $\eta_1\eta_2^3$ are zero and the coefficients of η_1^4, $\eta_1^2\eta_2^2$, η_2^4 are less then zero, then the system is asymptotically stable. If the system (4.15) satisfies (4.17) then there exists a pair (a_{11}, a_{12}, a_{13}) such that the coefficients of $\eta_1^3\eta_2$ and $\eta_1\eta_2^3$ are zero and the coefficient of $\eta_1^2\eta_2^2$ and η_2^4 are less than zero. That is, we can find a control law such that the system (4.15) is asymptotically stable.

Corollary 4.9 *If* $Q = \begin{pmatrix} Q^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, Q^* *is a Jordan form with all eigenvalues have negative real parts, then we have the same result as theorem* 4.8.

5 Discussion

In the thesis, we study the stabilization problem of the system (4.1). we can rewrite the system (4.1), by changing a basis, into the system (4.2) with (A_{11}, b) is a controllable pair. Hence there is a linear feedback control $u(x) = kx_1 + v$ such that all eigenvalues of $A_{11} + b_1 k$ have negative real parts and are distinct. Thus the system (4.2) can be written as the system (4.3). In the thesis, we only study the case with all eigenvalues of Q have non-positive real parts. In fact, it is enough to study the case with all eigenvalues of Q have zero real parts.

There are two open questions. One is to find a better Lyapunov function than one of the proof of theorem 4.8 to improve the sufficient condition of theorem 4.8. The other one is to study the matrix Q of the system (4.3) whose dimension is greater than two.

References

[1] D. Aeyels *Stabilization of class of Nonlinear systems by a smooth feedback control.* Syst. contr. lett. Vol 5. p289-294. (1985)

[2] S. Behtash and S. Sastry. *Stabilization of Nonlinear System with Uncontrollable Linearization.* IEEE. Trans. on auto. control. Vol 33. no. 6. p585-590, (1988).

[3] J. Carr. *Applications of Center Manifold Theory.* Springer, New York, 1981.

[4] J. Guckenheimer and P. Holvnes. *Nonlinear Oscillations Dynamical Systems and Bifurcations of Vector Fields.* Springer, New York, 1983. p123-156.

[5] Lawrence Porko. *Differential Equations and Dynamical Systems* Springer-Verlag New York, 1993.

[6] Slotine. Li. *Applied Nonlinear Control.* Prentice-Hall, Inc. 1991.

高階非線性臨界系統的穩定化

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摘 要

本篇論文主要討論非線性系統 $\dot{x} = f(x) + bu$ 之穩定化問題。在大部分的文獻中，都 只考慮將系統改寫為 $\dot{y} = Ay + bu + \mathcal{O}(2)$ 之後，其可控制系統為一階系統。而本篇論文在討論可控制系統為高階系統(即 A 可分為可控制部分和不可控制部份，其中可控制部分為多維矩陣)時的穩定化問題。首先利用 Center Manifold 定理將原系統降階，再用 Normal Forms 或 Lyapunov function 去找適當的條件，使得系統能夠被穩定化。

關建字： 非線性臨界系統、非線性回饋、可控制部份、中心流形、正規形式

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