Tunghai Science Vol. 8: 33−61
July, 2006

Local and Global Stability for a Predator-Prey Model of Modified Leslie-Gower and Holling-Type II with Time-Delay

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Abstract

In this paper, we analyze the dynamic behavior of a predator-prey model of modified Leslie-Gower and Holling-Type II with time delay. Firstly, we discuss the local and global stability for this system without time delay. Secondly, we find out some sufficient conditions for local and global stability of the unique positive equilibrium point of this system with time delay by constructing different Lyapunov function, respectively. Finally, we illustrate our results by some examples.

Keywords: Leslie-Gower, Holling-Type II, Lyapunov function, time-delay, local stability, global stability, uniform persistence.

1 Introduction

Predator-prey model has been studied by many authors. Most of them are interesting in the stability of the unique positive equilibrium point of the predator-prey systems. The global stability for the unique positive equilibrium point of predator-prey system without time delay has been done by some authors. They usually employ following methods to analyze the stability of a predator-prey system without time delay. The first one is constructing a Lyapunov function [1,2,7,8,10,13,17,18,25,28,33]. The second one is using the Dulac's criterion plus Poincaré−Bendixson Theorem to analyze the global stability of the unique positive equilibrium point of the predator-prey system [6,12,23,24,30,31,32,35,36]. The third one is the limit cycle stability analysis [4,5,13,14,16,21]. The fourth one is the comparison method [5,13,14,21,26].

Many researchers neglect the delay in the predator-prey model when they study them. But more realistic models should include some of the past states of the population systems. In other

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words, real systems should be modified by time delays. In [6,12,23,24,30,31,32,35,36], authors analyzed the global stability of the system with time delay by constructing a Lyapunov functional.

In this paper, we concern the Leslie-Gower predator-prey system with a single delay and the functional response of the predator, $p(x)$ in Holling-Type II. For this system without time delay, reference [2] analyzes the global stability of the unique positive equilibrium point by constructing a Lyapunov function. In [15], authors analyze the global stability for Leslie-Gower system with a single delay and the functional response of the predator, $p(x)$ in Holling-Type I by the same method. In the Lotka-Volterra Model, the carrying capacity of the predator population is independent of the prey population, but in the Leslie-Gower Model, the carrying capacity of the predator population is dependent on the prey population. The main purpose of this paper is to establish local and global stability of the Leslie-Gower Holling-Type II predator-prey system with a single delay. In section 2, we analyze the local and global stability of the Leslie-Gower Holling-Type II system without time delay by using Dulac's Criterion plus Poincaré− Bendixson Theorem and stable limit cycle analysis. And in section 3, we analyze the local and global stability of the Leslie-Gower Holling-Type II system with a single delay by constructing different Lyapunov functional, respectively. In section 4, we illustrate our results by some examples.

2 The Model without Time Delay

Consider the Holling-Type II Leslie-Gower predator-prey system without time delay modelled by

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[r_1 - b_1x_1(t) - \frac{a_1x_2(t)}{x_1(t)+k_1}\right] \equiv x_1 f_1(x_1,x_2) \equiv g_1(x_1,x_2), \\ \dot{x}_2(t) &= x_2(t)\left[r_2 - \frac{a_2x_2(t)}{x_1(t)+k_2}\right] \equiv x_2 f_2(x_1,x_2) \equiv g_2(x_1,x_2), \end{aligned} \tag{2.1}$$

with the initial conditions

$$x_1(0) > 0 \, , \, x_2(0) > 0. \tag{2.2}$$

Here $a_1,\ b_1,\ r_1,\ k_1,\ a_2,\ r_2$ and k_2, are all positive constants. x_1 and x_2 represent the population densities of prey and predator populations, respectively.

In this chapter, firstly, we use Hartman-Grobman Theorem to analyze the local stability of the equilibrium points E^* of the system (2.1); secondly, we analyze the global stability of the unique positive equilibrium point E^* of the system (2.1). This completes the analysis of the behavior of the system (2.1).

Lemma 2.1 *All solutions $(x_1(t), x_2(t))$ of the system (2.1) with the initial condition (2.2) are positive, i.e., $x_1(t) > 0$, $x_2(t) > 0$, for all $t \geq 0$.*

Proof : We will show that all solutions $(x_1(t), x_2(t))$ of the system (2.1) with the initial condition (2.2) are positive. That is, if the initial point $(x_1(0), x_2(0))$ is in the first quadrant, then the solution $(x_1(t), x_2(t))$ is also in the first quadrant for all $t > 0$. Since x_1-axis and x_2-axis are the solutions of the system (2.1), then the trajectory of the solutions $(x_1(t), x_2(t))$ with the initial point $(x_1(0), x_2(0))$ in the first quadrant can not cross with x_1-axis and x_2-axis by the uniqueness of the solution. Therefore, we know that all solutions $(x_1(t), x_2(t))$ of the system (2.1) with the initial condition (2.2) are positive.

Lemma 2.2 *All solutions $(x_1(t), x_2(t))$ of the system (2.1) with the initial condition (2.2) are bounded.*

Now, we want to show that all solutions $(x_1(t), x_2(t))$ of the system (2.1) with the initial condition (2.2) are bounded. From (2.1), it follows that

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)\left[r_1 - b_1 x_1(t) - \frac{a_1 x_2(t)}{x_1(t) + k_1}\right] \\
&\leq x_1(t)\left[r_1 - b_1 x_1(t)\right] \\
&= r_1 x_1(t)\left[1 - \frac{x_1(t)}{\frac{r_1}{b_1}}\right],
\end{aligned}$$

therefore,

$$x_1(t) \leq \max\left\{\frac{r_1}{b_1}, x_1(0)\right\} \equiv K_1.$$

Similarly,

$$\begin{aligned}
\dot{x}_2(t) &= x_2(t)\left[r_2 - \frac{a_2 x_2(t)}{x_1(t) + k_2}\right] \\
&\leq x_2(t)\left[r_2 - \frac{a_2 x_2(t)}{K_1 + k_2}\right] \\
&= r_2 x_2(t)\left[1 - \frac{x_2(t)}{\frac{r_2(K_1 + k_2)}{a_2}}\right],
\end{aligned}$$

therefore, we have

$$x_2(t) \leq \max\left\{\frac{r_2(K_1 + k_2)}{a_2}, x_2(0)\right\} \equiv K_2.$$

Hence, by above discussion, we know that the solution $(x_1(t), x_2(t))$ of the system (2.1) with the initial condition (2.2) are bounded.

2.1 Local Stability

Clearly, $\overline{E}=(0,0)$, $\widetilde{E}=(\frac{r_1}{b_1},0)$ and $\widehat{E}=(0,\frac{r_2k_2}{a_2})$ are the equilibrium points and $E^*=(x_1^*,x_2^*)$ is the unique positive equilibrium point in the first quadrant of the system (2.1) with the initial condition (2.2) satisfying the following condition:

$$\frac{r_1k_1}{a_1}-\frac{r_2k_2}{a_2} > 0 \,, \tag{2.3}$$

where

$$x_1^* = \frac{\Delta^{1/2}-(a_1r_2-a_2r_1+a_2b_1k_1)}{2a_2b_1} \;,\; x_2^* = \frac{r_2(x_1^*+k_2)}{a_2} \tag{2.4}$$

and

$$\Delta=(a_1r_2-a_2r_1+a_2b_1k_1)^2-4a_2b_1(a_1r_2k_2-a_2r_1k_1). \tag{2.5}$$

Δ and x_1^* are nonnegative if (2.3) holds.

Now let us start to discuss the local behavior of the system (2.1) of those equilibrium points $\overline{E}=(0,0)$, $\widetilde{E}=(\frac{r_1}{b_1},0)$, $\widehat{E}=(0,\frac{r_2k_2}{a_2})$ and $E^*=(x_1^*,x_2^*)$. The Jacobian matrix of the system (2.1) takes the form

$$J \equiv \begin{bmatrix} r_1-2b_1x_1-\left[\dfrac{a_1k_1x_2}{(x_1+k_1)^2}\right] & -\dfrac{a_1x_1}{x_1+k_1} \\ \dfrac{a_2x_2^2}{(x_1+k_2)^2} & r_2-\dfrac{2a_2x_2}{x_1+k_2} \end{bmatrix}.$$

The Jacobian matrix of the system (2.1) at $\overline{E}=(0,0)$ is

$$\overline{J}=\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix},$$

and the eigenvalues $\lambda_1=r_1$ and $\lambda_2=r_2$ of $\overline{J}$ are positive. Thus, the equilibrium point $\overline{E}$ of the system (2.1) is unstable.

The Jacobian matrix of the system (2.1) at $\widetilde{E}=(r_1/b_1,0)$ is

$$\tilde{J}=\begin{bmatrix} -r_1 & -\dfrac{a_1r_1}{r_1+b_1k_1} \\ 0 & r_2 \end{bmatrix},$$

and the eigenvalues of $\tilde{J}$ are $\lambda_1=-r_1$, $\lambda_2=r_2$. Since $\lambda_1<0$ and $\lambda_2>0$, the equilibrium point $\widetilde{E}$ of the system (2.1) is a saddle point. And we know that

$$\Gamma_1 \;=\; \{(x_1,x_2)\mid x_1>0, x_2=0\}$$

is the stable manifold of the equilibrium point $\widetilde{E}$.

The Jacobian matrix of the system (2.1) at $\widehat{E} = (0, r_2k_2/a_2)$ is

$$\hat{J} = \begin{bmatrix} r_1 - \dfrac{a_1r_2k_2}{a_2k_1} & 0 \\ \dfrac{r_2^2}{a_2} & -r_2 \end{bmatrix},$$

and the eigenvalues of $\hat{J}$ are $\lambda_1 = r_1 - a_1r_2k_2/a_2k_1$ and $\lambda_2 = -r_2$. From (2.3), we know $\lambda_1 > 0$, and since $\lambda_2 < 0$, the equilibrium point $\widehat{E}$ of the system (2.1) is a saddle point. We know that

$$\Gamma_2 \;=\; \{(x_1, x_2) \mid x_1 = 0, x_2 > 0\}$$

is the stable manifold of the equilibrium point $\widehat{E}$.

Lemma 2.3 *Let E^* be the unique positive equilibrium point of the system (2.1) and If*

$$b_1 - \frac{a_1x_2^*}{(x_1^* + k_1)^2} \quad > \quad 0 \tag{2.6}$$

then the unique positive equilibrium point E^* of the system (2.1) is locally asymptotically stable.

Proof : The Jacobian matrix of the system (2.1) at E^* is

$$J^* = \begin{bmatrix} \left[\dfrac{a_1x_2^*}{(x_1^*+k_1)^2} - b_1\right] x_1^* & -\dfrac{a_1x_1^*}{x_1^*+k_1} \\ \dfrac{r_2^2}{a_2} & -r_2 \end{bmatrix}.$$

If (2.6) holds, then

$$\begin{aligned} Det(J^*) &= \left[b_1 - \frac{a_1x_2^*}{(x_1^*+k_1)^2}\right] r_2x_1^* + \frac{a_1x_1^*}{x_1^*+k_1} \cdot \frac{r_2^2}{a_2} \\ &= \left[b_1 - \frac{a_1x_2^*}{(x_1^*+k_1)^2} + \frac{a_1r_2}{a_2(x_1^*+k_1)}\right] r_2x_1^* \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} Trace(J^*) &= \left[\frac{a_1x_2^*}{(x_1^*+k_1)^2} - b_1\right] x_1^* - r_2 \\ &< 0. \end{aligned}$$

Hence, the unique positive equilibrium point E^* of the system (2.1) is locally asymptotically stable.

2.2 Global Stability

Now, we want to analyze the global stability of the unique positive equilibrium point E^* of the system (2.1) by the following two methods:

(i) Dulac's criterion plus Poincaré- Bendixson theorem;

(ii) Stable limit cycle analysis.

Theorem 2.4 *Let E^* be the unique positive equilibrium point of the system (2.1) and if*

$$a_1 \le a_2, \tag{2.7}$$

$$k_2 \le k_1, \tag{2.8}$$

and (2.6) hold, then the unique positive equilibrium point E^* of the system (2.1) is globally asymptotically stable.

Proof : Firstly, we use the method (i) to analyze the system (2.1). Consider

$$H(x_1,x_2) = \frac{1}{x_1x_2}, \qquad x_1 > 0,\ x_2 > 0.$$

Then

$$\begin{aligned}
\nabla\cdot(Hg) &= \frac{\partial}{\partial x_1}\left\{H\left[\left(r_1 - b_1x_1 - \frac{a_1x_2}{x_1+k_1}\right)x_1\right]\right\} \\
&\quad + \frac{\partial}{\partial x_2}\left\{H\left[\left(r_2 - \frac{a_2x_2}{x_1+k_2}\right)x_2\right]\right\} \\
&= \frac{\partial}{\partial x_1}\left\{\frac{r_1}{x_2} - \frac{b_1x_1}{x_2} - \frac{a_1}{x_1+k_1}\right\} + \frac{\partial}{\partial x_2}\left\{\frac{r_2}{x_1} - \frac{a_2x_2}{x_1(x_1+k_2)}\right\} \\
&= -\frac{b_1}{x_2} + \frac{a_1}{(x_1+k_1)^2} - \frac{a_2}{x_1(x_1+k_2)} \\
&= -\frac{b_1}{x_2} - \frac{1}{x_1}\left[\frac{a_2}{x_1+k_2} - \frac{a_1x_1}{(x_1+k_1)^2}\right] \\
&< -\frac{b_1}{x_2} - \frac{1}{x_1}\left[\frac{a_2}{x_1+k_2} - \frac{a_1}{x_1+k_1}\right] \\
&= -\frac{b_1}{x_2} - \frac{1}{x_1}\left[\frac{(a_2-a_1)x_1 + a_2k_1 - a_1k_2}{(x_1+k_1)(x_1+k_2)}\right] \\
&< 0.
\end{aligned}$$

Hence by the Dulac's criterion, there is no closed orbit in the first quadrant. From Lemma 2.2, we know that the unique positive equilibrium point E^* is locally asymptotically stable. By the Lemma 2.3 and the Poincaré-Bendixson theorem, it suffices to show that the unique positive equilibrium point E^* is globally asymptotically stable in the first quadrant.

Secondly, we analyze the global stability of the system (2.1) by the method (ii). Now, we want to show that the system (2.1) has no closed orbit in the first quadrant. Suppose on the contrary that there is a T-periodic orbit $\Gamma = \{(x_1(t), x_2(t)) \mid 0 \le t \le T\}$ in the first quadrant. Compute

$$\begin{aligned}
\triangle &= \int_\Gamma \left(\frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2}\right) ds \\
&= \int_0^T \left\{ \frac{\partial}{\partial x_1}\left[\left(r_1 - b_1 x_1(t) - \frac{a_1 x_2(t)}{x_1(t)+k_1}\right) x_1(t)\right] \right. \\
&\qquad \left. + \frac{\partial}{\partial x_2}\left[\left(r_2 - \frac{a_2 x_2(t)}{x_1(t)+k_2}\right) x_2(t)\right]\right\} dt \\
&= \int_0^T \left[r_1 - 2b_1 x_1(t) - \frac{a_1 k_1 x_2(t)}{(x_1(t)+k_1)^2} + r_2 - \frac{2a_2 x_2(t)}{x_1(t)+k_2}\right] dt \\
&= \int_0^T \left[\left(r_1 - b_1 x_1(t) - \frac{a_1 x_2(t)}{x_1(t)+k_1}\right) - b_1 x_1(t) + \frac{a_1 x_2(t)}{x_1(t)+k_1} \right. \\
&\qquad \left. - \frac{a_1 k_1 x_2(t)}{(x_1(t)+k_1)^2} + \left(r_2 - \frac{a_2 x_2(t)}{x_1(t)+k_2}\right) - \frac{a_2 x_2(t)}{x_1(t)+k_2}\right] dt \\
&= -\int_0^T \left[b_1 x_1(t) + \frac{a_1 k_1 x_2(t)}{(x_1(t)+k_1)^2} + \frac{a_2 x_2(t)}{x_1(t)+k_2} - \frac{a_1 x_2(t)}{x_1(t)+k_1}\right] dt \\
&\quad + \int_0^T \frac{\dot{x}_1(t)}{x_1(t)} dt + \int_0^T \frac{\dot{x}_2(t)}{x_2(t)} dt \\
&= -\int_0^T \left\{ b_1 x_1(t) + \frac{a_1 k_1 x_2(t)}{(x_1(t)+k_1)^2} + \frac{x_2(t)\left[(a_2 - a_1)x_1(t) + a_2 k_1 - a_1 k_2\right]}{(x_1+k_1)(x_1+k_2)}\right\} dt \\
&\quad + \int_{x_1(0)}^{x_1(T)} \frac{1}{x_1} dx_1 + \int_{x_2(0)}^{x_2(T)} \frac{1}{x_2} dx_2.
\end{aligned}$$

Since Γ is a T-periodic,

$$\int_{x_1(0)}^{x_1(T)} \frac{1}{x_1} dx_1 = 0 \text{ and } \int_{x_2(0)}^{x_2(T)} \frac{1}{x_2} dx_2 = 0.$$

Hence we obtain that

$$\begin{aligned}
\triangle &= -\int_0^T \left\{ b_1 x_1(t) + \frac{a_1 k_1 x_2(t)}{(x_1(t)+k_1)^2} + \frac{x_2(t)\left[(a_2 - a_1)x_1(t) + a_2 k_1 - a_1 k_2\right]}{(x_1+k_1)(x_1+k_2)}\right\} dt \\
&< 0.
\end{aligned}$$

So all closed orbits of the system (2.1) in the first quadrant are orbitally stable. Since every closed orbit is orbitally stable and then there is a unique stable limit cycle in the first quadrant, the unique positive equilibrium point E^* is unstable. However, E^* is locally asymptotically stable by Lemma 2.3, thus there is no periodic orbit in the first quadrant. From Lemma 2.3 and the Poincaré-Bendixson theorem, it suffices to show that the unique positive equilibrium point E^* is globally asymptotically stable in the first quadrant.

Remark 2.5 ([2]) The system (2.1) satisfies the following condition:

$$\begin{aligned}\frac{1}{4a_2b_1}(a_2r_1(r_1+4)+(r_1+2)^2(r_1+b_1k_2)) &< \frac{r_1k_1}{2a_1};\\ k_1 &< 2k_2;\\ 4(r_1+b_1k_1) &< a_1.\end{aligned}$$

Then by the Lyapunov functional

$$\begin{aligned}&V(x_1,x_2)\\ =\ &(x_1^*+k_1)\left[x_1-x_1^*-x_1^*\ln\left(\frac{x_1}{x_1^*}\right)\right]+\frac{a_1(x_1^*+k_2)}{a_2}\left[x_2-x_2^*-x_2^*\ln\left(\frac{x_2}{x_2^*}\right)\right],\end{aligned}$$

theorem 2.1 also holds.

3 The Model with Time Delay

Consider the Holling-Type II Leslie-Gower predator-prey system with time delay τ modelled by

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left[r_1-b_1x_1(t-\tau)-\frac{a_1x_2(t)}{x_1(t)+k_1}\right],\\ \dot{x}_2(t) &= x_2(t)\left[r_2-\frac{a_2x_2(t)}{x_1(t)+k_2}\right],\end{aligned}\tag{3.1}$$

with the initial conditions

$$\begin{aligned}x_1(\theta) &= \phi(\theta)\geq 0\ ,\ \ \theta\in[-\tau,0]\ ,\ \phi\in C([-\tau,0],R)\\ x_1(0) &> 0\ ,\ \ x_2(0)>0\end{aligned}\tag{3.2}$$

Here $a_1,\ b_1,\ r_1,\ k_1,\ a_2,\ r_2$ and k_2 are all positive constants. x_1 and x_2 denote the densities of prey and predator population, respectively.

3.1 Uniform Persistence

Lemma 3.1 *Every solutions of the system (3.1) with the initial conditions (3.2) remains positive, i.e.,* $x_1(t)>0$, $x_2(t)>0$, *for all* $t\geq 0$.

Proof : It is true because

$$\begin{aligned} x_1(t) &= x_1(0)exp\left\{\int_0^t \left[r_1 - b_1x_1(s-\tau) - \frac{a_1x_2(s)}{x_1(s)+k_1}\right]ds\right\} \\ x_2(t) &= x_2(0)exp\left\{\int_0^t \left[r_2 - \frac{a_2x_2(s)}{x_1(s)+k_2}\right]ds\right\} \end{aligned}$$

and $x_i(0) > 0$ for $i = 1,2$. Therefore, we obtain that all solutions $(x_1(t), x_2(t))$ of the system (3.1) with the initial conditions (3.2) are positive.

Lemma 3.2 *Let $(x_1(t), x_2(t))$ denote the solution of (3.1) with the initial conditions (3.2), then*

$$0 < x_i(t) \leq M_i\ , \quad i = 1,2, \tag{3.3}$$

eventually for all large t, where

$$M_1 \equiv \frac{r_1}{b_1}e^{r_1\tau}, \tag{3.4}$$

$$M_2 \equiv \frac{3r_2(M_1+k_2)}{2a_2}, \tag{3.5}$$

Proof : Now we want to show that there exists a $T > 0$ such that $x_1(t) \leq M_1$ for $t > T$. By Lemma 3.1, we know the solution of the system (3.1) with the initial conditions (3.2) are positive, by (3.1)

$$\begin{aligned} \dot{x_1}(t) &= x_1(t)\left[r_1 - b_1x_1(t-\tau) - \frac{a_1x_2(t)}{x_1(t)+k_1}\right] \\ &\leq x_1(t)\left[r_1 - b_1x_1(t-\tau)\right]. \end{aligned} \tag{3.6}$$

Taking $M_1^* = \frac{r_1(1+B)}{b_1}$, $0 < B < e^{r_1\tau} - 1$. Suppose that x_1 is not oscillatory about M_1^*. That is, there exists a $T_0 > 0$ such that either

$$x_1(t) > M_1^* \quad for \quad t > T_0, \tag{3.7}$$

or

$$x_1(t) \leq M_1^* \quad for \quad t > T_0. \tag{3.8}$$

If (3.8) holds, then for $t > T_0$

$$x_1(t) \leq M_1^* = \frac{r_1(1+B)}{b_1} < \frac{r_1}{b_1}e^{r_1\tau} = M_1.$$

That is, (3.3) holds. Suppose (3.7) holds. Equation (3.6) implies that for
$t > T_0 + \tau$,

$$\begin{aligned} \dot{x}_1(t) &\leq x_1(t)[r_1 - b_1x_1(t-\tau)] \\ &< x_1(t)[r_1 - b_1M_1^*] \\ &= -r_1Bx_1(t). \end{aligned}$$

It follows that

$$\int_{T_0+\tau}^{t} \frac{\dot{x}_1(s)}{x_1(s)} ds \;<\; \int_{T_0+\tau}^{t} -Br_1 ds \;=\; -Br_1(t-T_0-\tau).$$

Then $0 < x_1(t) < x_1(T_0+\tau)e^{-Br_1(t-T_0-\tau)} \to 0$ as $t \to \infty$. By the Squeeze Theorem, we have $\lim_{t\to\infty} x_1(t) = 0$. It contradicts to (3.7). Therefore, there must exist a $T_1 > T_0$ such that $x_1(T_1) \le M_1^*$. If $x_1(t) \le M_1^*$ for all $t \ge T_1$, then (3.3) follows. If not, then there must exist a $T_2 > T_1$ such that T_2 be first time which $x_1(T_2) > M_1^*$. Therefore, there exists a $T_3 > T_2$ such that T_3 be the first time which $x_1(T_3) < M_1^*$ by above discussion. By above, we know that $x_1(T_1) \le M_1^*$, $x_1(T_2) > M_1^*$ and $x_1(T_3) \le M_1^*$ where $T_1 < T_2 < T_3$. Then, by the Intermediate Value Theorem, there exists T_4 and T_5 such that

$$x_1(T_4) = M_1^* \;\;,\;\; T_1 \le T_4 < T_2,$$

$$x_1(T_5) = M_1^* \;\;,\;\; T_2 < T_5 \le T_3,$$

and $x_1(t) > M_1^*$ for $T_4 < t < T_5$. Hence there is a $T_6 \in (T_4, T_5)$ such that $x_1(T_6)$ is a local maximum, and hence it follows from (3.6) that

$$0 = \dot{x}_1(T_6) \le x_1(T_6)[r_1 - b_1 x_1(T_6-\tau)]$$

and this implies

$$x_1(T_6-\tau) \le \frac{r_1}{b_1}.$$

Integrating both side of (3.6) on the interval $[T_6-\tau, T_6]$, we have

$$\ln\left[\frac{x_1(T_6)}{x_1(T_6-\tau)}\right] = \int_{T_6-\tau}^{T_6} \frac{\dot{x}_1(s)}{x_1(s)} ds \le \int_{T_6-\tau}^{T_6} [r_1 - b_1 x_1(s-\tau)] ds \le r_1\tau.$$

It follows that

$$x_1(T_6) \le x_1(T_6-\tau)e^{r_1\tau} \le \frac{r_1}{b_1} e^{r_1\tau} = M_1.$$

Similarly, if there exists other local maxima of $x_1(t)$ for $t > T_6$, then they are also less or equal to M_1 by above argument. So we can conclude that there exists a $T > 0$ such that

$$x_1(t) \le M_1 \quad for \quad t \ge T. \tag{3.9}$$

Suppose $x_1(t)$ is oscillatory about M_1^*, for this case, the proof is similarly to above one. Now we want to show that $x_2(t)$ is bounded above by M_2 eventually for all large t. By (3.9), it follows that for $t > T$

$$\begin{aligned}
\dot{x}_2(t) &= x_2(t)\left[r_2 - \frac{a_2 x_2(t)}{x_1(t)+k_2}\right] \\
&\le r_2 x_2(t)\left[1 - \frac{a_2 x_2(t)}{r_2(M_1+k_2)}\right] \\
&= r_2 x_2(t)\left[1 - \frac{x_2(t)}{\frac{r_2(M_1+k_2)}{a_2}}\right].
\end{aligned}$$

Case1 : If $x_2(0) < r_2(M_1+k_2)/a_2$, then $x_2(t) \le r_2(M_1+k_2)/a_2 \le M_2$ for $t > T$.
Case2 : If $x_2(0) > r_2(M_1+k_2)/a_2$, then

$$\limsup_{t\to\infty} x_2(t) \le \frac{r_2(M_1+k_2)}{a_2}.$$

So, for large $t,\ x_2(t) \le 3r_2(M_1+k_2)/2a_2 = M_2$. This completes the proof.

Lemma 3.3 *Suppose that the system (3.1) satisfies*

$$r_1 - \frac{a_1 M_2}{k_1} > 0. \tag{3.10}$$

where M_2 defined by (3.5). Then the system (3.1) is uniformly persistent. That is, there exists $m_1 > 0, m_2 > 0$ and $T^* > 0$ such that $m_i \le x_i(t) \le M_i$ for $i = 1,2$, $t \ge T^*$. Where

$$\begin{aligned}
m_1 &= \frac{r_1 - \frac{a_1 M_2}{k_1}}{b_1} e^{\left(r_1 - b_1 M_1 - \frac{a_1 M_2}{k_1}\right)\tau}, \\
m_2 &= \frac{r_2(m_1+k_2)}{a_2}.
\end{aligned}$$

Proof : By Lemma 3.2, equation (3.1) follows that for $t \ge T+\tau$

$$\dot{x}_1(t) \ge x_1(t)\left[r_1 - b_1 M_1 - \frac{a_1 M_2}{k_1}\right]. \tag{3.11}$$

Integrating both of sides of (3.11) on $[t-\tau, t]$, where $t \ge T+\tau$, then we have

$$x_1(t) \ge x_1(t-\tau) e^{\left(r_1 - b_1 M_1 - \frac{a_1 M_2}{k_1}\right)\tau},$$

that is,

$$x_1(t-\tau) \le e^{-\left(r_1 - b_1 M_1 - \frac{a_1 M_2}{k_1}\right)\tau} x_1(t). \tag{3.12}$$

By (3.1) and (3.12), for $t \geq T + \tau$,

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)\left[r_1 - b_1 x_1(t-\tau) - \frac{a_1 x_2(t)}{x_1(t)+k_1}\right] \\
&= x_1(t)\left[r_1 - \frac{a_1 x_2(t)}{x_1(t)+k_1} - b_1 x_1(t-\tau)\right] \\
&\geq x_1(t)\left[\left(r_1 - \frac{a_1 M_2}{k_1}\right) - b_1 e^{-\left(r_1 - b_1 M_1 - \frac{a_1 M_2}{k_1}\right)\tau} x_1(t)\right] \\
&= \left(r_1 - \frac{a_1 M_2}{k_1}\right) x_1(t)\left[1 - \frac{b_1 e^{-\left(r_1 - b_1 M_1 - \frac{a_1 M_2}{k_1}\right)\tau}}{r_1 - \frac{a_1 M_2}{k_1}} x_1(t)\right] \\
&= \left(r_1 - \frac{a_1 M_2}{k_1}\right) x_1(t)\left[1 - \frac{x_1(t)}{\frac{\left(r_1 - \frac{a_1 M_2}{k_1}\right)}{b_1} e^{\left(r_1 - b_1 M_1 - \frac{a_1 M_2}{k_1}\right)\tau}}\right].
\end{aligned}$$

It follows that

$$\liminf_{t\to\infty} x_1(t) \geq \frac{r_1 - \frac{a_1 M_2}{k_1}}{b_1} e^{\left(r_1 - b_1 M_1 - \frac{a_1 M_2}{k_1}\right)\tau} \equiv \overline{m}_1.$$

By (3.10), we have $\overline{m}_1 > 0$. So, for large t, $x_1 \geq \overline{m}_1/2 \equiv m_1 > 0$. It follows that

$$\begin{aligned}
\dot{x}_2(t) &= x_2(t)\left[r_2 - \frac{a_2 x_2(t)}{x_1(t)+k_2}\right] \\
&\geq x_2(t)\left[r_2 - \frac{a_2 x_2(t)}{m_1+k_2}\right] \\
&= r_2 x_2(t)\left[1 - \frac{a_2 x_2(t)}{r_2(m_1+k_2)}\right] \\
&= r_2 x_2(t)\left[1 - \frac{x_2(t)}{\frac{r_2(m_1+k_2)}{a_2}}\right].
\end{aligned}$$

Then

$$\liminf_{t\to\infty} x_2(t) \geq \frac{r_2(m_1+k_2)}{a_2} \equiv \overline{m}_2.$$

So, for large t, $x_2(t) \geq \overline{m}_2/2 \equiv m_2 > 0$. Let

$$D = \{(x_1, x_2) | m_1 \leq x_1 \leq M_1, m_2 \leq x_2 \leq M_2\}.$$

Then D is a bounded compact region in $\mathbb{R}^2_+$ that has positive distance from coordinate hyperplanes. Hence we obtain that there exists a $T^* > 0$ such that if $t \geq T^*$, then every positive solution of system (3.1) with the initial conditions (3.2) eventually enters and remains in the region D, that is, system (3.1) is uniformly persistent.

3.2 Local Stability

Sequentially, we will analyze the local stability of the system (3.1).

Theorem 3.4 *Let E^* be the unique positive equilibrium point of the system (3.1) and if the parameters satisfy*

$$\begin{aligned} \alpha_2+\alpha_3-\alpha_1 &> 0, & (3.13)\\ [2(\alpha_3-\alpha_1)-(\alpha_2+2\alpha_3-2\alpha_1)\alpha_3\tau] &> 0, & (3.14)\\ -\alpha_2\,(2\beta_2+\alpha_3\beta_1\tau) &> 0, & (3.15)\end{aligned}$$

where

$$\begin{aligned} &\alpha_1 = r_1 - b_1 x^* - \frac{a_1 k_1 x_2^*}{(x_1^*+k_1)^2},\ \ \alpha_2 = \frac{a_1 x_1^*}{x_1^*+k_1},\ \ \alpha_3 = b_1 x_1^*,\\ &\beta_1 = \frac{a_2 (x_2^*)^2}{(x_1^*+k_2)^2},\ \ \beta_2 = r_2 - \frac{2a_2 x_2^*}{x_1^*+k_2}. \end{aligned} \tag{3.16}$$

then the unique positive equilibrium point E^* of the system (3.1) is locally asymptotically stable.

Proof : Define $y(t) = (y_1(t), y_2(t))$ by

$$x_1(t) = y_1(t) + x_1^*,\ \ x_2(t) = y_2(t) + x_2^*.$$

Linearizing (3.1) at (x_1^*, x_2^*), we obtain

$$\begin{aligned} \dot{y}_1(t) &= \left[r_1 - b_1 x_1^* - \frac{a_1 k_1 x_2^*}{(x_1^*+k_1)^2}\right] y_1(t) - \frac{a_1 x_1^*}{x_1^*+k_1} y_2(t) - b_1 x_1^* y_1(t-\tau)\\ \dot{y}_2(t) &= \frac{a_2 (x_2^*)^2}{(x_1^*+k_2)^2} y_1(t) + \left(r_2 - \frac{2a_2 x_2^*}{x_1^*+k_2}\right) y_2(t). \end{aligned} \tag{3.17}$$

It is noticed that local asymptotical stability of E^* of (3.1) is determined by the asymptotic stability of the zero solution of the system (3.17) and which can be rewritten as

$$\begin{aligned} \dot{y}_1(t) &= \alpha_1 y_1(t) - \alpha_2 y_2(t) - \alpha_3 y_1(t-\tau),\\ \dot{y}_2(t) &= \beta_1 y_1(t) + \beta_2 y_2(t), \end{aligned} \tag{3.18}$$

where $\alpha_2 > 0$, $\alpha_3 > 0$ and $\beta_1 > 0$ from (3.16). Define

$$W_{11}(y(t)) = \left[y_1(t) - \alpha_3 \int_{t-\tau}^{t} y_1(s)ds\right]^2, \tag{3.19}$$

from (3.17) and (3.19), we have

$$\begin{aligned}
\dot{W}_{11}(y(t)) &\leq 2\left[y_1(t)-\alpha_3\int_{t-\tau}^{t} y_1(s)ds\right]\cdot\{\dot{y}_1(t)-\alpha_3[y_1(t)-y_1(t-\tau)]\}\\
&= 2\left[y_1(t)-\alpha_3\int_{t-\tau}^{t} y_1(s)ds\right]\cdot[\alpha_1 y_1(t)-\alpha_2 y_2(t)-\alpha_3 y_1(t-\tau)\\
&\quad -\alpha_3 y_1(t)+\alpha_3 y_1(t-\tau)]\\
&= 2\left[y_1(t)-\alpha_3\int_{t-\tau}^{t} y_1(s)ds\right]\cdot[(\alpha_1-\alpha_3)y_1(t)-\alpha_2 y_2(t)]\\
&\leq (\alpha_3-\alpha_1)\alpha_3\int_{t-\tau}^{t} 2|y_1(t)||y_1(s)|ds+\alpha_2\alpha_3\int_{t-\tau}^{t} 2|y_2(t)||y_1(s)|ds\\
&\quad +2(\alpha_1-\alpha_3)y_1^2(t)-2\alpha_2 y_1(t)y_2(t)\\
&\leq (\alpha_3-\alpha_1)\alpha_3\left[\tau y_1^2(t)+\int_{t-\tau}^{t} y_1^2(s)ds\right]+\alpha_2\alpha_3\left[\tau y_2^2(t)+\int_{t-\tau}^{t} y_1^2(s)ds\right]\\
&\quad +2(\alpha_1-\alpha_3)y_1^2(t)-2\alpha_2 y_1(t)y_2(t)\\
&= [2(\alpha_1-\alpha_3)+(\alpha_3-\alpha_1)\alpha_3\tau]y_1^2(t)+\alpha_2\alpha_3\tau y_2^2(t)-2\alpha_2 y_1(t)y_2(t)\\
&\quad +(\alpha_2+\alpha_3-\alpha_1)\alpha_3\int_{t-\tau}^{t} y_1^2(s)ds. \qquad (3.20)
\end{aligned}$$

Let

$$W_{12}(y(t))=(\alpha_2+\alpha_3-\alpha_1)\alpha_3\int_{t-\tau}^{t}\int_{s}^{t} y_1^2(\rho)d\rho ds. \qquad (3.21)$$

Then

$$\dot{W}_{12}=(\alpha_2+\alpha_3-\alpha_1)\alpha_3\left[\tau y_1^2(t)-\int_{t-\tau}^{t} y_1^2(s)ds\right]$$

Now, we let $W_1(y(t))$ be defined by

$$W_1(y(t))=\beta_1[W_{11}(y(t))+W_{12}(y(t))]. \qquad (3.22)$$

Then

$$\begin{aligned}
\dot{W}_1(y(t)) &\leq [2(\alpha_1-\alpha_3)+(\alpha_3-\alpha_1)\alpha_3\tau+(\alpha_2+\alpha_3-\alpha_1)\alpha_3\tau]\beta_1 y_1^2(t)\\
&\quad +\alpha_2\alpha_3\beta_1\tau y_2^2(t)-2\alpha_2\beta_1 y_1(t)y_2(t)\\
&\leq [2(\alpha_1-\alpha_3)+(\alpha_2+2\alpha_3-2\alpha_1)\alpha_3\tau]\beta_1 y_1^2(t)\\
&\quad +\alpha_2\alpha_3\beta_1\tau y_2^2(t)-2\alpha_2\beta_1 y_1(t)y_2(t). \qquad (3.23)
\end{aligned}$$

Next we define

$$W_2(y(t))=\alpha_2 y_2^2(t). \qquad (3.24)$$

Then

$$\begin{aligned}\dot{W}_2(y(t)) &= 2\alpha_2 y_2(t)\dot{y}_2(t)\\ &= 2\alpha_2 y_2(t)\left[\beta_1 y_1(t)+\beta_2 y_2(t)\right]\\ &= 2\alpha_2\beta_2 y_2^2(t)+2\alpha_2\beta_1 y_1(t)y_2(t).\end{aligned} \tag{3.25}$$

We take

$$W(t)=W(y(t))=W_1(y(t))+W_2(y(t)). \tag{3.26}$$

Combine (3.23) and (3.25). Then we obtain

$$\dot{W}(t)\le -\eta_1 y_1^2(t)-\eta_2 y_2^2(t), \tag{3.27}$$

where

$$\begin{aligned}\eta_1 &= \left[2(\alpha_3-\alpha_1)-(\alpha_2+2\alpha_3-2\alpha_1)\alpha_3\tau\right]\beta_1\\ \eta_2 &= -\alpha_2\left(2\beta_2+\alpha_3\beta_1\tau\right).\end{aligned}$$

Clearly, (3.13) and (3.14) imply that $\eta_1>0$ and $\eta_2>0$. Let $\eta=\min\{\eta_1,\eta_2\}$ and integrate both sides of (3.27) on the interval $[T,t]$, we have

$$W(t)+\eta\int_T^t\left[y_1^2(s)+y_2^2(s)\right]ds\le W(T)\qquad for\ \ t>T \tag{3.28}$$

and which implies that $y_1^2(t)+y_2^2(t)\in L_1[T,\infty)$. One can show $y_1^2(t)+y_2^2(t)$ is uniformly continuous by (3.17) and the boundedness of $y(t)=(y_1(t),y_2(t))$. Using Barbălat's Lemma (see [9]), we can conclude that

$$\lim_{t\to\infty}[y_1^2(t)+y_2^2(t)]=0.$$

Therefore the zero solution of (3.17) is asymptotically stable and this completes the proof.

Remark 3.5 Let E^* be the unique positive equilibrium point of the system (3.1). From Theorem 3.1 we know that if $\alpha_2+\alpha_3-\alpha_1>0$ and the delay τ satisfies $0<\tau<\bar{\tau}\equiv\min\{\bar{\tau_1},\bar{\tau_2}\}$, where

$$\begin{aligned}\bar{\tau_1} &\equiv \frac{2(\alpha_3-\alpha_1)}{[\alpha_2+2(\alpha_3-\alpha_1)]\alpha_3}>0,\\ \bar{\tau_2} &\equiv -\frac{2\beta_2}{\alpha_3\beta_1}>0,\end{aligned}$$

and those parameters are the same as (3.16), then the unique positive equilibrium point E^* of the system (3.1) is locally asymptotically stable.

3.3 Global Stability

Now we want to show that, under some assumptions, the unique positive equilibrium point E^* of the system (3.1) is globally asymptotically stable.

Theorem 3.6 *Let $E^* = (x_1^*, x_2^*)$ be the unique positive equilibrium point of the system (3.1) and if the parameters satisfy*

$$
\begin{aligned}
& r_1 - \frac{a_1 M_2}{k_1} > 0, && (3.29)\\
& r_2 x_1^*(m_1 + k_1) - a_1 x_2^*(M_1 + k_2) > 0, && (3.30)\\
& \frac{b_1 x_1^*(2x_1^* + k_1)}{M_1 + k_1} + \frac{a_1 x_2^* - 2r_1 x_1^* - 2b_1 x_1^*(x_1^* + M_1)}{2(m_1 + k_1)} - \frac{r_2 x_1^*}{2(m_1 + k_2)} && \\
& - \frac{M_1 b_1 (x_1^* + k_1)\tau}{2(m_1 + k_1)^2}\left[2r_1 x_1^* + a_1 x_2^* + 2b_1 x_1^*(3x_1^* + k_1 + M_1)\right] > 0, && (3.31)\\
& \frac{a_1 x_2^*}{2(m_1 + k_1)} + \frac{a_2 x_2^*}{M_1 + k_2} - \frac{r_2 x_1^*}{2(m_1 + k_2)} - \frac{M_1 a_1 b_1 x_2^*(x_1^* + k_1)\tau}{2(m_1 + k_1)^2} > 0, && (3.32)
\end{aligned}
$$

where m_1, M_1 and M_2 defined in Lemmas 3.2 and 3.3, then the unique positive equilibrium point E^* of the system (3.1) is globally asymptotically stable.

Proof : Define $z(t) = (z_1(t), z_2(t))$ by

$$
z_1(t) = \frac{x_1(t) - x_1^*}{x_1^*} \ , \ z_2(t) = \frac{x_2(t) - x_2^*}{x_2^*} \ .
$$

From (3.1), we have

$$
\begin{aligned}
\dot{z}_1(t) &= [1 + z_1(t)]\left\{\frac{(r_1 - b_1 x_1^*)x_1^* z_1(t)}{[1 + z_1(t)]x_1^* + k_1} - \frac{b_1 (x_1^*)^2 z_1(t) z_1(t-\tau)}{[1 + z_1(t)]x_1^* + k_1}\right.\\
&\qquad \left. - \frac{b_1 x_1^*(x_1^* + k_1) z_1(t-\tau)}{[1 + z_1(t)]x_1^* + k_1} - \frac{a_1 x_2^* z_2(t)}{[1 + z_1(t)]x_1^* + k_1}\right\}, && (3.33)\\
\dot{z}_2(t) &= [1 + z_2(t)]\left\{\frac{r_2 x_1^* z_1(t) - a_2 x_2^* z_2(t)}{[1 + z_1(t)]x_1^* + k_2}\right\}. && (3.34)
\end{aligned}
$$

Let

$$
V_1(z_t) = \{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\}. \qquad (3.35)
$$

Then by (3.33) and (3.34), we have

$$
\begin{aligned}
\dot{V}_1(z_t) &= \frac{\dot{z}_1(t)z_1(t)}{1+z_1(t)} + \frac{\dot{z}_2(t)z_2(t)}{1+z_1(t)} \\
&= \frac{(r_1-b_1x_1^*)x_1^*z_1^2(t)}{[1+z_1(t)]x_1^*+k_1} - \frac{b_1(x_1^*)^2z_1^2(t)z_1(t-\tau)}{[1+z_1(t)]x_1^*+k_1} \\
&\quad - \frac{b_1x_1^*(x_1^*+k_1)z_1(t)z_1(t-\tau)}{[1+z_1(t)]x_1^*+k_1} - \frac{a_1x_2^*z_1(t)z_2(t)}{[1+z_1(t)]x_1^*+k_1} \\
&\quad + \frac{r_2x_1^*z_1(t)z_2(t)}{[1+z_1(t)]x_1^*+k_2} - \frac{a_2x_2^*z_2^2(t)}{[1+z_1(t)]x_1^*+k_2} \\
&\leq \left\{ \frac{r_2x_1^*\{[1+z_1(t)]x_1^*+k_1\} - a_1x_2^*\{[1+z_1(t)]x_1^*+k_2\}}{\{[1+z_1(t)]x_1^*+k_1\}\{[1+z_1(t)]x_1^*+k_2\}} \right\} z_1(t)z_2(t) \\
&\quad + \frac{(r_1-b_1x_1^*)x_1^*z_1^2(t)}{[1+z_1(t)]x_1^*+k_1} - \frac{a_2x_2^*z_2^2(t)}{[1+z_1(t)]x_1^*+k_2} + \frac{b_1(x_1^*)^2z_1^2(t)|z_1(t-\tau)|}{[1+z_1(t)]x_1^*+k_1} \\
&\quad - \frac{b_1x_1^*(x_1^*+k_1)z_1(t)z_1(t-\tau)}{[1+z_1(t)]x_1^*+k_1}. \qquad (3.36)
\end{aligned}
$$

If $r_2x_1^*(m_1+k_1) - a_1x_2^*(M_1+k_2) > 0$, and by Lemma 3.3, there exists a $T^* > 0$ such that $m_1 \leq$

$[1+z_1(t)]x_1^* \leq M_1$, and $m_2 \leq [1+z_2(t)]x_2^* \leq M_2$ for $t > T^*$. Then (3.36) implies that

$$
\begin{aligned}
\dot{V}_1(z_t) \leq\ & \left[\frac{r_1x_1^*}{m_1+k_1} - \frac{b_1(x_1^*)^2}{M_1+k_1}\right] z_1^2 + \frac{b_1(x_1^*)^2}{m_1+k_1} z_1^2(t)|z_1(t-\tau)| \\
& + \left[\frac{r_2x_1^*}{2(m_1+k_2)} - \frac{a_1x_2^*}{2(m_1+k_1)}\right] z_1^2(t) + \left[\frac{r_2x_1^*}{2(m_1+k_2)} - \frac{a_1x_2^*}{2(m_1+k_1)}\right] z_2^2(t) \\
& - \frac{b_1x_1^*(x_1^*+k_1)}{[1+z_1(t)]x_1^*+k_1} z_1(t) \left[z_1(t) - \int_{t-\tau}^t \dot{z}_1(s)ds\right] - \frac{a_2x_2^*}{M_1+k_2} z_2^2(t) \\
\leq\ & \left[\frac{r_1x_1^*}{m_1+k_1} - \frac{b_1x_1^*(2x_1^*+k_1)}{M_1+k_1} + \frac{r_2x_1^*}{2(m_1+k_2)} - \frac{a_1x_2^*}{2(m_1+k_1)}\right] z_1^2(t) \\
& + \left[\frac{r_2x_1^*}{2(m_1+k_2)} - \frac{a_1x_2^*}{2(m_1+k_1)} - \frac{a_2x_2^*}{M_1+k_2}\right] z_2^2(t) + \frac{b_1(x_1^*)^2}{m_1+k_1} z_1^2(t)|z_1(t-\tau)| \\
& + \frac{b_1x_1^*(x_1^*+k_1)}{[1+z_1(t)]x_1^*+k_1} \int_{t-\tau}^t [1+z_1(s)] \left\{ \frac{(r_1-b_1x_1^*)x_1^*z_1(t)z_1(s)}{[1+z_1(s)]x_1^*+k_1} \right. \\
& - \frac{a_1x_2^*z_1(t)z_2(s)}{[1+z_1(s)]x_1^*+k_1} - \frac{b_1(x_1^*)^2 z_1(s)z_1(t)z_1(s-\tau)}{[1+z_1(s)]x_1^*+k_1} \\
& \left. - \frac{b_1x_1^*(x_1^*+k_1)z_1(t)z_1(s-\tau)}{[1+z_1(s)]x_1^*+k_1} \right\} ds \\
\leq\ & \left[\frac{r_1x_1^*}{m_1+k_1} - \frac{b_1x_1^*(2x_1^*+k_1)}{M_1+k_1} + \frac{r_2x_1^*}{2(m_1+k_2)} - \frac{a_1x_2^*}{2(m_1+k_1)}\right] z_1^2(t) \\
& + \left[\frac{r_2x_1^*}{2(m_1+k_2)} - \frac{a_1x_2^*}{2(m_1+k_1)} - \frac{a_2x_2^*}{M_1+k_2}\right] z_2^2(t) + \frac{b_1(x_1^*)^2}{m_1+k_1} z_1^2(t)|z_1(t-\tau)| \\
& - \frac{b_1x_1^*(x_1^*+k_1)}{[1+z_1(t)]x_1^*+k_1} \int_{t-\tau}^t [1+z_1(s)] \left\{ \frac{(r_1+b_1x_1^*)x_1^*|z_1(t)||z_1(s)|}{[1+z_1(s)]x_1^*+k_1} \right. \\
& \left. + \frac{a_1x_2^*|z_1(t)||z_2(s)|}{[1+z_1(s)]x_1^*+k_1} + \frac{b_1x_1^*(x_1^*+k_1)|z_1(t)||z_1(s-\tau)|}{[1+z_1(s)]x_1^*+k_1} \right\} ds \\
& - \frac{b_1x_1^*(x_1^*+k_1)}{[1+z_1(t)]x_1^*+k_1} \int_{t-\tau}^t \frac{b_1(x_1^*)^2 z_1(t)z_1(s-\tau)}{[1+z_1(s)]x_1^*+k_1} [1+z_1(s)]^2 ds \\
& + \frac{b_1x_1^*(x_1^*+k_1)}{[1+z_1(t)]x_1^*+k_1} \int_{t-\tau}^t \frac{b_1(x_1^*)^2 |z_1(t)||z_1(s-\tau)|}{[1+z_1(s)]x_1^*+k_1} [1+z_1(s)] ds.
\end{aligned}
\tag{3.37}
$$

Then by (3.37), we have, for $t \geq T^* + \tau \equiv \widehat{T}$,

$$
\begin{aligned}
\dot{V}_1(z_t) \;\leq\; & -\left[\frac{b_1x_1^*(2x_1^*+k_1)}{M_1+k_1} + \frac{a_1x_2^*}{2(m_1+k_1)} - \frac{r_1x_1^*}{m_1+k_1} - \frac{r_2x_1^*}{2(m_1+k_2)}\right] z_1^2(t) \\
& -\left[\frac{a_1x_2^*}{2(m_1+k_1)} + \frac{a_2x_2^*}{M_1+k_2} - \frac{r_2x_1^*}{2(m_1+k_2)}\right] z_2^2(t) + \frac{b_1(x_1^*)^2}{m_1+k_1}\left(1+\frac{M_1}{x_1^*}\right) z_1^2(t) \\
& + \frac{M_1b_1(x_1^*+k_1)}{m_1+k_1}\int_{t-\tau}^{t}\left[\frac{(r_1+b_1x_1^*)x_1^*|z_1(t)||z_1(s)|}{m_1+k_1} + \frac{a_1x_2^*|z_1(t)||z_2(s)|}{m_1+k_1}\right. \\
& \left. + \frac{b_1x_1^*(x_1^*+k_1)|z_1(t)||z_1(s-\tau)|}{m_1+k_1}\right] ds \\
& + \frac{M_1^2b_1(x_1^*+k_1)}{(m_1+k_1)x_1^*}\int_{t-\tau}^{t}\frac{b_1(x_1^*)^2|z_1(t)||z_1(s-\tau)|}{m_1+k_1}ds \\
& + \frac{M_1b_1(x_1^*+k_1)}{m_1+k_1}\int_{t-\tau}^{t}\frac{b_1(x_1^*)^2|z_1(t)||z_1(s-\tau)|}{m_1+k_1}ds \\
\leq\; & -\left\{\frac{b_1x_1^*(2x_1^*+k_1)}{M_1+k_1} + \frac{a_1x_2^*}{2(m_1+k_1)} - \frac{r_1x_1^*}{m_1+k_1} - \frac{r_2x_1^*}{2(m_1+k_2)} - \frac{b_1x_1^*(x_1^*+M_1)}{m_1+k_1}\right. \\
& - \frac{M_1b_1(x_1^*+k_1)\tau}{(m_1+k_1)^2}\left[\frac{(r_1+b_1x_1^*)x_1^* + a_1x_2^* + b_1x_1^*(2x_1^*+k_1)}{2}\right] \\
& \left. + \frac{M_1^2b_1^2x_1^*(x_1^*+k_1)\tau}{2(m_1+k_1)^2}\right\} z_1^2(t) - \left[\frac{a_1x_2^*}{2(m_1+k_1)} + \frac{a_2x_2^*}{M_1+k_2} - \frac{r_2x_1^*}{2(m_1+k_2)}\right] z_2^2(t) \\
& + \frac{M_1b_1(x_1^*+k_1)(r_1+b_1x_1^*)x_1^*}{2(m_1+k_1)^2}\int_{t-\tau}^{t} z_1^2(s)ds + \frac{M_1b_1(x_1^*+k_1)a_1x_2^*}{2(m_1+k_1)^2}\int_{t-\tau}^{t} z_2^2(s)ds \\
& + \left[\frac{M_1b_1^2(x_1^*+k_1)^2x_1^* + M_1^2b_1^2x_1^*(x_1^*+k_1) + M_1b_1^2(x_1^*)^2(x_1^*+k_1)}{2(m_1+k_1)^2}\right]\int_{t-\tau}^{t} z_1^2(s-\tau)ds \\
=\; & -\left\{\frac{b_1x_1^*(2x_1^*+k_1)}{M_1+k_1} + \frac{a_1x_2^* - 2r_1x_1^* - 2b_1x_1^*(x_1^*+M_1)}{2(m_1+k_1)} - \frac{r_2x_1^*}{2(m_1+k_2)}\right. \\
& \left. - \frac{M_1b_1(x_1^*+k_1)\tau}{2(m_1+k_1)^2}\left[r_1x_1^* + a_1x_2^* + b_1x_1^*(3x_1^*+k_1+M_1)\right]\right\} z_1^2(t) \\
& - \left[\frac{a_1x_2^*}{2(m_1+k_1)} + \frac{a_2x_2^*}{M_1+k_2} - \frac{r_2x_1^*}{2(m_1+k_2)}\right] z_2^2(t) \\
& + \frac{M_1b_1x_1^*(x_1^*+k_1)(r_1+b_1x_1^*)}{2(m_1+k_1)^2}\int_{t-\tau}^{t} z_1^2(s)ds + \frac{M_1a_1b_1x_2^*(x_1^*+k_1)}{2(m_1+k_1)^2}\int_{t-\tau}^{t} z_2^2(s)ds \\
& + \frac{M_1b_1^2x_1^*(x_1^*+k_1)}{2(m_1+k_1)^2}(2x_1^*+k_1+M_1)\int_{t-\tau}^{t} z_1^2(s-\tau)ds. \qquad (3.38)
\end{aligned}
$$

Let

$$
\begin{aligned}
V_2(z_t) \;=\; & \frac{M_1b_1x_1^*(x_1^*+k_1)(r_1+b_1x_1^*)}{2(m_1+k_1)^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma)d\gamma ds \\
& + \frac{M_1a_1b_1x_2^*(x_1^*+k_1)}{2(m_1+k_1)^2}\int_{t-\tau}^{t}\int_{s}^{t} z_2^2(\gamma)d\gamma ds \\
& + \frac{M_1b_1^2x_1^*(x_1^*+k_1)(2x_1^*+k_1+M_1)}{2(m_1+k_1)^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma-\tau)d\gamma ds. \qquad (3.39)
\end{aligned}
$$

Then

$$\begin{aligned}\dot{V}_2(z_t) &= \frac{M_1 b_1 x_1^*(x_1^*+k_1)(r_1+b_1x_1^*)\tau}{2(m_1+k_1)^2} z_1^2(t) - \frac{M_1 b_1 x_1^*(x_1^*+k_1)(r_1+b_1x_1^*)}{2(m_1+k_1)^2}\int_{t-\tau}^{t} z_1^2(s)ds \\ &\quad + \frac{M_1 a_1 b_1 x_2^*(x_1^*+k_1)\tau}{2(m_1+k_1)^2} z_2^2(t) - \frac{M_1 a_1 b_1 x_2^*(x_1^*+k_1)}{2(m_1+k_1)^2}\int_{t-\tau}^{t} z_2^2(s)ds \\ &\quad + \frac{M_1 b_1^2 x_1^*(x_1^*+k_1)(2x_1^*+k_1+M_1)\tau}{2(m_1+k_1)^2} z_1^2(t-\tau) \\ &\quad - \frac{M_1 b_1^2 x_1^*(x_1^*+k_1)(2x_1^*+k_1+M_1)}{2(m_1+k_1)^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds, \end{aligned} \tag{3.40}$$

and then we have from (3.38) and (3.40) that for $t > \widehat{T}$,

$$\begin{aligned} &\dot{V}_1(z_t) + \dot{V}_2(z_t) \\ \leq\ & -\Bigg\{ \frac{b_1 x_1^*(2x_1^*+k_1)}{M_1+k_1} + \frac{a_1 x_2^* - 2r_1 x_1^* - 2b_1 x_1^*(x_1^*+M_1)}{2(m_1+k_1)} \\ & - \frac{r_2 x_1^*}{2(m_1+k_2)} - \frac{M_1 b_1 (x_1^*+k_1)\tau}{2(m_1+k_1)^2}\left[2r_1x_1^* + a_1x_2^* + b_1x_1^*(4x_1^*+k_1+M_1)\right]\Bigg\} z_1^2(t) \\ & - \left[\frac{a_1x_2^*}{2(m_1+k_1)} + \frac{a_2x_2^*}{M_1+k_2} - \frac{r_2x_1^*}{2(m_1+k_2)} - \frac{M_1a_1b_1x_2^*(x_1^*+k_1)\tau}{2(m_1+k_1)^2}\right] z_2^2(t) \\ & + \frac{M_1 b_1^2 x_1^*(x_1^*+k_1)(2x_1^*+k_1+M_1)\tau}{2(m_1+k_1)^2} z_1^2(t-\tau). \end{aligned} \tag{3.41}$$

Let

$$V_3(z_t) = \frac{M_1 b_1^2 x_1^*(x_1^*+k_1)(2x_1^*+k_1+M_1)\tau}{2(m_1+k_1)^2}\int_{t-\tau}^{t} z_1^2(s)ds. \tag{3.42}$$

Then

$$\begin{aligned}\dot{V}_3(z_t) &= \frac{M_1 b_1^2 x_1^*(x_1^*+k_1)(2x_1^*+k_1+M_1)\tau}{2(m_1+k_1)^2} z_1^2(t) \\ &\quad - \frac{M_1 b_1^2 x_1^*(x_1^*+k_1)(2x_1^*+k_1+M_1)\tau}{2(m_1+k_1)^2} z_1^2(t-\tau). \end{aligned} \tag{3.43}$$

Now define a Lyapunov function $V(z(t))$ as

$$V(z_t)) = V_1(z_t) + V_2(z_t) + V_3(z_t). \tag{3.44}$$

Then by (3.44) and (3.45), we have that for $t > \widehat{T}$,

$$\begin{aligned}\dot{V}(z_t) &\leq -\Bigg\{ \frac{b_1 x_1^*(2x_1^*+k_1)}{M_1+k_1} + \frac{a_1 x_2^* - 2r_1 x_1^* - 2b_1 x_1^*(x_1^*+M_1)}{2(m_1+k_1)} - \frac{r_2 x_1^*}{2(m_1+k_2)} \\ &\quad - \frac{M_1 b_1 (x_1^*+k_1)\tau}{2(m_1+k_1)^2}\left[2r_1x_1^* + a_1x_2^* + 2b_1x_1^*(3x_1^*+k_1+M_1)\right]\Bigg\} z_1^2(t) \\ &\quad - \left[\frac{a_1x_2^*}{2(m_1+k_1)} + \frac{a_2x_2^*}{M_1+k_2} - \frac{r_2x_1^*}{2(m_1+k_2)} - \frac{M_1a_1b_1x_2^*(x_1^*+k_1)\tau}{2(m_1+k_1)^2}\right] z_2^2(t) \\ &\equiv -\xi_1 z_1^2(t) - \xi_2 z_2^2(t). \end{aligned} \tag{3.45}$$

Then it follows from (3.31) and (3.32) that $\xi_1 > 0$ and $\xi_2 > 0$. Let $w(s) = \xi s^2$ where $\xi = \min\{\xi_1,\ \xi_2\}$, then w is nonnegative continuous on $[0,\infty)$, $w(0) = 0$ and $w(s) > 0$ for $s > 0$. It follows from (3.45) that for $t > \widehat{T}$

$$\dot{V}(z_t) \leq -\xi[z_1^2(t) + z_2^2(t)] = -\xi|z(t)|^2 = -w(|z(t)|). \tag{3.46}$$

Now, we want to find a function u such that $V(z_t) \geq u(|z(t)|)$. It follows from (3.35), (3.39) and (3.42) that

$$V(z_t) \geq \{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\}. \tag{3.47}$$

By the Taylor Theorem, we have that

$$z_i(t) - \ln[1 + z_i(t)] = \frac{z_i^2(t)}{2[1 + \theta_i(t)]^2}, \tag{3.48}$$

where $\theta_i(t) \in (0, z_i(t))$ or $(z_i(t), 0)$ for $i = 1, 2$.
Case1 : If $0 < \theta_i(t) < z_i(t)$ for $i = 1, 2$, then

$$\frac{z_i^2(t)}{[1 + z_i(t)]^2} < \frac{z_i^2(t)}{[1 + \theta_i(t)]^2} < z_i^2(t). \tag{3.49}$$

By Lemma 3.3, it follows that for $t \geq T^*$,

$$m_i \leq x_i^*[1 + z_i(t)] = x_i(t) \leq M_i \quad \text{for } i = 1, 2. \tag{3.50}$$

Then (3.49) implies that

$$\left(\frac{x_i^*}{M_i}\right)^2 z_i^2(t) \leq \frac{z_i^2(t)}{[1 + \theta_i(t)]^2} < z_i^2(t) \quad , \ i = 1, 2. \tag{3.51}$$

It follows that (3.47), (3.48) and (3.51) that for $t \geq T^*$,

$$\begin{aligned} V(z_t) &\geq \frac{1}{2} \cdot \frac{z_1^2(t)}{[1 + \theta_1(t)]^2} + \frac{1}{2} \cdot \frac{z_2^2(t)}{[1 + \theta_2(t)]^2} \\ &\geq \frac{1}{2} \cdot \left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2} \cdot \left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq \min\left\{\frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2, \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2\right\}[z_1^2(t) + z_2^2(t)] \\ &\equiv \zeta\, |z(t)|^2. \end{aligned}$$

Case2 : If $-1 < z_i(t) < \theta_i(t) < 0$ for $i = 1, 2$, then

$$z_i^2(t) < \frac{z_i^2(t)}{[1 + \theta_i(t)]^2} < \frac{z_i^2(t)}{[1 + z_i(t)]^2}. \tag{3.52}$$

By (3.50) and (3.52) implies that

$$z_i^2(t) < \frac{z_i^2(t)}{[1 + \theta_i(t)]^2} \leq \left(\frac{x_i^*}{m_i}\right)^2 z_i^2(t) \quad , \ i = 1, 2. \tag{3.53}$$

It follows that (3.47), (3.48) and (3.51) that for $t \geq T^*$,

$$\begin{aligned} V(z_t) &\geq \frac{1}{2} \cdot \frac{z_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2} \cdot \frac{z_2^2(t)}{[1+\theta_2(t)]^2} \\ &> \frac{1}{2} \cdot z_1^2(t) + \frac{1}{2} \cdot z_2^2(t) \\ &\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq \zeta\,[z_1^2(t) + z_2^2(t)] \\ &= \zeta\,|z(t)|^2. \end{aligned}$$

Case3 : If $0 < \theta_1(t) < z_1(t)$ and $-1 < z_2(t) < \theta_2(t) < 0$, then it follows from (3.47), (3.48), (3.51) and (3.53) that for $t \geq T^*$,

$$\begin{aligned} V(z_t) &\geq \frac{1}{2} \cdot \frac{z_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2} \cdot \frac{z_2^2(t)}{[1+\theta_2(t)]^2} \\ &> \frac{1}{2} \cdot \left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2} \cdot z_2^2(t) \\ &\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq \zeta\,[z_1^2(t) + z_2^2(t)] \\ &= \zeta\,|z(t)|^2. \end{aligned}$$

Case4 : If $-1 < z_1(t) < \theta_1(t) < 0$ and $0 < \theta_2(t) < z_2(t)$, then it follows from (3.47), (3.48), (3.51) and (3.53) that for all $t \geq T^*$,

$$\begin{aligned} V(z_t) &\geq \frac{1}{2} \cdot \frac{z_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2} \cdot \frac{z_2^2(t)}{[1+\theta_2(t)]^2} \\ &> \frac{1}{2} \cdot z_1^2(t) + \frac{1}{2} \cdot \left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq \zeta\,[z_1^2(t) + z_2^2(t)] \\ &= \zeta\,|z(t)|^2. \end{aligned}$$

Let $u(s) = \zeta s^2$, then u is nonnegative continuous on $[0,\infty)$, $u(0) = 0$, $u(s) > 0$ for $s > 0$, and $\lim\limits_{s\to\infty} u(s) = +\infty$. So, by Case1 $\sim$ Case4, we have

$$V(z_t) \geq u(|z(t)|), \qquad \text{for all} \quad t \geq T^* \tag{3.54}$$

Thus, the unique positive equilibrium point E^* of the system (3.1) is globally asymptotically stable.

4 Examples

We present some simple examples to illustrate the procedures of applying our results.

Theorem 4.1

Consider the follow system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left[1.5 - 2.9x_1(t) - \frac{0.4x_2(t)}{x_1(t)+16}\right],\\ \dot{x}_2(t) &= x_2(t)\left[1.4 - \frac{4x_2(t)}{x_1(t)+3}\right],\end{aligned}$$

where $r_1 = 1.5$, $r_2 = 1.4$, $a_1 = 0.4$, $a_2 = 4$, $k_1 = 16$, $k_2 = 3$, and $b_1 = 2.9$. Since

$$\frac{r_1 k_1}{a_1} - \frac{r_2 k_2}{a_2} = 58.9500 > 0,$$

by (2.3), (2.4), and (2.5), the system (4.1) has the unique positive equilibrium point E^* and $E^* = (0.5070, 1.2274)$. Then

$$\begin{aligned}& b_1 - \frac{a_1 x_2^*}{(x_1^* + k_1)^2} = 2.8982 > 0\\ & a_1 = 0.4 < a_2 = 4\\ & k_2 = 3 < k_1 = 16.\end{aligned}$$

By theorem 2.1, the unique positive equilibrium point E^* of the system (4.1) is globally asymptotically stabl

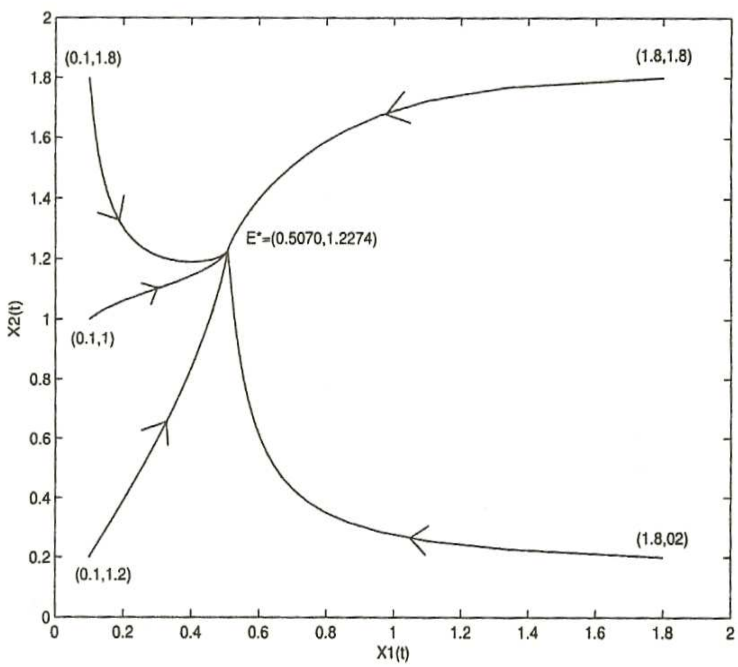

Figure 4.1: The global trajectory of the system (4.1).

Example 4.2

Sequentially, we consider the system (4.1) with time delay τ=0.3

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)\left[1.5 - 2.9x_1(t-0.3) - \frac{0.4x_2(t)}{x_1(t)+16}\right],\\
\dot{x}_2(t) &= x_2(t)\left[1.4 - \frac{4x_2(t)}{x_1(t)+3}\right].
\end{aligned}$$

Alike, the parameters $r_1 = 1.5$, $r_2 = 1.4$, $a_1 = 0.4$, $a_2 = 4$, $k_1 = 16$, $k_2 = 3$, $b_1 = 2.9$ and the system (4.2) has the same unique positive equilibrium point $E^* = (x_1^*, x_2^*) = (0.5070, 1.2274)$. Then

$$\begin{aligned}
\alpha_2 + \alpha_3 - \alpha_1 &= 1.4816 > 0,\\
[2(\alpha_3 - \alpha_1) - (\alpha_2 + 2\alpha_3 - 2\alpha_1)\alpha_3\tau] &= 1.6371 > 0,\\
-\alpha_2\,(2\beta_2 + \alpha_3\beta_1\tau) &= 0.0317 > 0,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1 &= r_1 - b_1x^* - \frac{a_1k_1x_2^*}{(x_1^*+k_1)^2}\ ,\ \alpha_2 = \frac{a_1x_1^*}{x_1^*+k_1}\ ,\ \alpha_3 = b_1x_1^*\ ,\\
\beta_1 &= \frac{a_2(x_2^*)^2}{(x_1^*+k_2)^2}\ ,\ \beta_2 = r_2 - \frac{2a_2x_2^*}{x_1^*+k_2}\ .
\end{aligned}$$

By theorem 3.1, the unique positive equilibrium point E^* of the system (4.2) is locally asymptotically stable. The local trajectory of the system (4.2) with $\tau = 0.3$ is depicted in Figure 4.2.

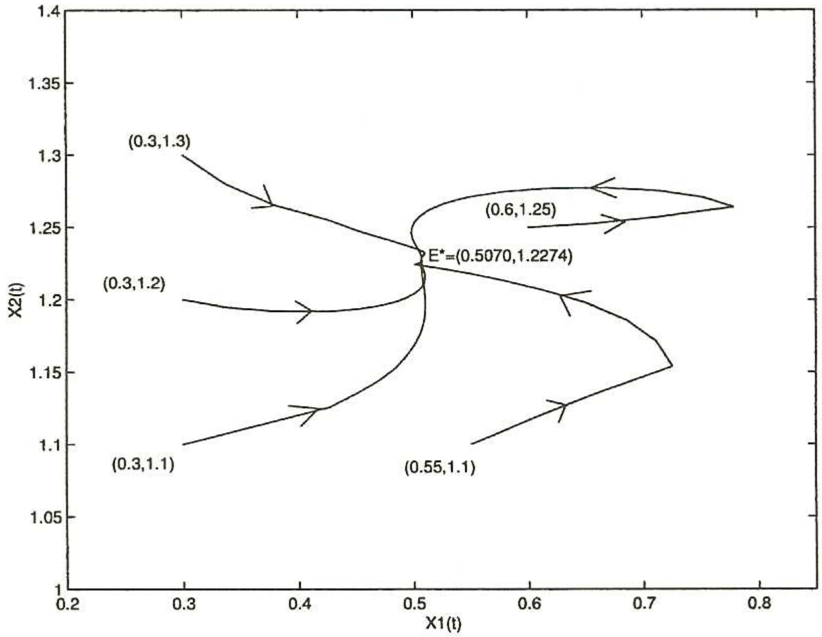

Figure 4.2: The local trajectory of the system (4.2) with $\tau = 0.3$.

Now, we illustrate the global stability of the system (4.2). Alike, the parameters $r_1 = 1.5,\ r_2 = 1.4,\ a_1 = 0.4,\ a_2 = 4,\ k_1 = 16,\ k_2 = 3,\ b_1 = 2.9$ and the system (4.2) has the same unique positive equilibrium point $E^* = (x_1^*, x_2^*) = (0.5070, 1.2274)$. Then

$$
\begin{aligned}
& r_1 - \frac{a_1 M_2}{k_1} = 1.4500 > 0, \\
& r_2 x_1^*(m_1 + k_1) - a_1 x_2^*(M_1 + k_2) = 9.7389 > 0, \\
& \frac{b_1 x_1^*(2x_1^* + k_1)}{M_1 + k_1} + \frac{a_1 x_2^* - 2r_1 x_1^* - 2b_1 x_1^*(x_1^* + M_1)}{2(m_1 + k_1)} - \frac{r_2 x_1^*}{2(m_1 + k_2)} \\
& -\frac{M_1 b_1 (x_1^* + k_1)\tau}{2(m_1 + k_1)^2}\left[2r_1 x_1^* + a_1 x_2^* + 2b_1 x_1^*(3x_1^* + k_1 + M_1)\right] = 0.0150 > 0, \\
& \frac{a_1 x_2^*}{2(m_1 + k_1)} + \frac{a_2 x_2^*}{M_1 + k_2} - \frac{r_2 x_1^*}{2(m_1 + k_2)} - \frac{M_1 a_1 b_1 x_2^*(x_1^* + k_1)\tau}{2(m_1 + k_1)^2} = 1.1869 > 0,
\end{aligned}
$$

whenever $\tau = 0.3$. By theorem 3.2, we conclude that unique positive equilibrium point $E^* = (x_1^*, x_2^*)$ of the system (4.2) is globally asymptotically stable. The global trajectory of the system (4.2) with $\tau = 0.3$ is depicted in Figure 4.3.

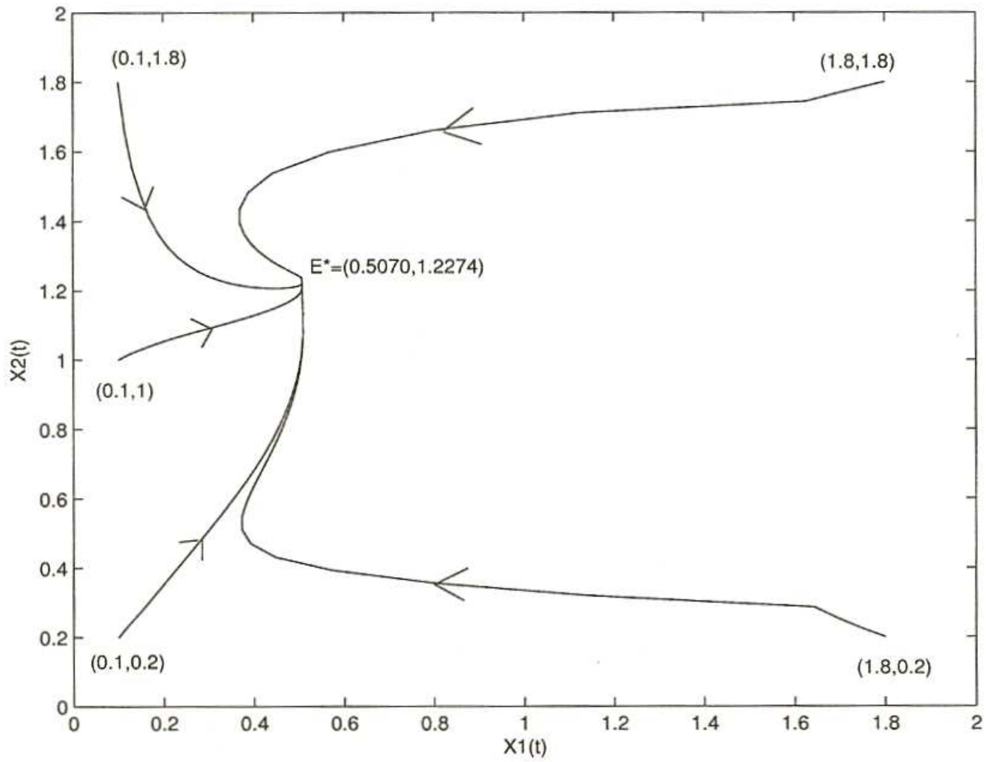

Figure 4.3: The global trajectory of the system (4.2) with $\tau = 0.3$.

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具時滯參數之 Leslie-Gower Holling-Type II 型捕食系統的局部及整體穩定性

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摘　　要

我們分析具時滯參數之 Leslie-Gower Holling-Type II 型捕食系統的局部及整體穩定性。首先我們分別探討不具時滯參數或具時滯參數之 Leslie-Gower Holling-Type II 型捕食系統的局部及整體穩定性。最後，用實例及電腦軌跡圖說明之。

關鍵詞：時滯參數，局部穩定性，整體穩定性，Lyapunov 涵數，均勻持久。