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Global Stability for the Lotka-Volterra Mutualistic System with Time Delayy

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Abstract

In this paper, we are concerned with the dynamical behavior of a two-species Lotka-Volterra mutualistic system with time delay. First of all, we use three different methods to discuss the global stability of the unique positive equilibrium point of a two-species Lotka-Volterra mutualistic system without time delay. Secondly, we study the change of the global stability for a two-species Lotka-Volterra mutualistic system with time delay. Finally, we illustrative our results by some examples.

Keywords: Mutualistic system, delay, global stability, Lyapunov functional.

1 Introduction

One of the most important problems in the Lotka-Volterra mutualistic system is the global stability of mutualistic systems. The global stability analysis for mutualistic systems without time delay has been done by some authors [1–3]. In [1], Brauer and Soudack gave some hypotheses so that they can obtain complete information about the global behavior of solutions. In [2], Freedman and Rai derived conditions for a positive equilibrium point of mutualistic system to be globally asymptotically stable. In [3], Goh constructed a Lyapunov function to show that the unique positive equilibrium point is globally asymptotically stable in a nonlinear model of mutualism.

In recent years, the mutualistic systems were extended to include the time delays by some authors [5, 9]. In [5], the authors obtained the conditions for the global stability of facultative mutualism system with time delay by constructing a Lyapunov functional. In [9], Mukherjee found out the global stability condition of facultative mutualism system with different time delays.

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In this paper, we consider a two-species Lotka-Volterra mutualistic system with time delay,

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[r_1(1-\frac{x_1(t-\tau)}{K_1})+\alpha_{21}x_2(t)\right] \\ \dot{x}_2(t) &= x_2(t)\left[r_2(1-\frac{x_2(t-\tau)}{K_2})+\alpha_{12}x_1(t)\right] \end{aligned} \tag{1.1}$$

with the initial conditions

$$x_i(\theta) \;=\; \phi_i(\theta) > 0 \;\;,\;\; \theta \in [-\tau,0]\;,\; \phi_i \in C([-\tau,0],R)\;,\;\; i=1,2 \tag{1.2}$$

where $\cdot = d/dt,\ r_1,\ r_2,\ K_1,\ K_2,\ \alpha_{12},\ \alpha_{21}$ and τ are all positive constants. $\phi_i(t)(i=1,2)$ are continuous bounded functions on the interval $[-\tau,0]$. $x_1(t)$ and $x_2(t)$ denote the population densities of two mutualistic populations.

The main purpose of this paper is to establish global stability of a two-species Lotka-Volterra mutualistic system with time delay. In section 2, we introduce some useful definitions and theorems. In section 3, we analyze the global stability of the Lotka-Volterra mutualistic system without time delay by using Dulac's Criterion plus Poincaré−Bendixson Theorem, the construction of the Lyapunov function or stable limit cycle analysis. In section 4, we discuss the global stability of the Lotka-Volterra mutualistic system with a single delay by constructing a Lyapunov functional. In section 5, we illustrate our results by some examples.

2 THE MODEL WITHOUT TIME DELAY

Consider a two-species Lotka-Volterra mutualistic system without time delay modelled by

$$\begin{aligned} \dot{x}_1 &= x_1\left[r_1(1-\frac{x_1}{K_1})+\alpha_{21}x_2\right] \;\equiv\; x_1f_1(x_1,x_2) \;\equiv\; g_1(x_1,x_2) \\ \dot{x}_2 &= x_2\left[r_2(1-\frac{x_2}{K_2})+\alpha_{12}x_1\right] \;\equiv\; x_2f_2(x_1,x_2) \;\equiv\; g_2(x_1,x_2) \end{aligned} \tag{2.1}$$

with the constraints

$$x_1(t) \;>\; 0 \;\;,\;\; x_2(t) \;>\; 0 \text{ for all } t. \tag{2.2}$$

where $r_1,\ r_2,\ K_1,\ K_2,\ \alpha_{12}$ and α_{21} are all positive constants. x_1 and x_2 denote the population densities of two mutualistic populations. All we want to discuss is biological population, so we just consider the first quadrant in the x_1-x_2 plane.

Analysing the behavior of the system (2.1), firstly, we discuss the local stability of the equilibrium points of the system (2.1) by Hartman-Grobman Theorem.

Secondly, we want to analyse the global stability of the unique positive equilibrium point E^* of the system (2.1).

2.1 Local Stability

Clearly, $\overline{E} = (0,0)$, $\widetilde{E} = (K_1,0)$ and $\widehat{E} = (0,K_2)$ are the equilibrium points and $E^* = (x_1^*, x_2^*)$ is the unique positive equilibrium point in the first quadrant of the system (2.1) with the initial condition (2.2), where

$$x_1^* = \frac{-r_2K_1(K_2\alpha_{21}+r_1)}{K_1K_2\alpha_{12}\alpha_{21}-r_1r_2} \; , \; x_2^* = \frac{-r_1K_2(K_1\alpha_{12}+r_2)}{K_1K_2\alpha_{12}\alpha_{21}-r_1r_2} \tag{2.3}$$

Remark 2.1 If

$$K_1K_2\alpha_{12}\alpha_{21}-r_1r_2 < 0 \tag{2.4}$$

then E^* is the unique positive equilibrium point of the system (2.1).

Now let us analyse the local behavior of the system (2.1) at the equilibrium points $\overline{E} = (0,0)$, $\widetilde{E} = (K_1,0)$, $\widehat{E} = (0,K_2)$ and $E^* = (x_1^*, x_2^*)$. The Jacobian matrix of the system (2.1) takes the form

$$J \equiv \begin{bmatrix} r_1(1-\frac{2x_1}{K_1})+\alpha_{21}x_2 & \alpha_{21}x_1 \\ \alpha_{12}x_2 & r_2(1-\frac{2x_2}{K_2})+\alpha_{12}x_1 \end{bmatrix}$$

The Jacobian matrix of the system (2.1) at $\overline{E}$ is

$$\bar{J} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

The eigenvalues $\lambda_1 = r_1$, $\lambda_2 = r_2$ of $\bar{J}$ are positive. Thus, the equilibrium point $\overline{E}$ of the system (2.1) is unstable.

The Jacobian matrix of the system (2.1) at $\widetilde{E}$ is

$$\tilde{J} = \begin{bmatrix} -r_1 & \alpha_{21}K_1 \\ 0 & r_2+\alpha_{12}K_1 \end{bmatrix}$$

The eigenvalues of $\tilde{J}$ are $\lambda_1 = -r_1$, $\lambda_2 = r_2+\alpha_{12}K_1$. Since $\lambda_1 < 0$ and $\lambda_2 > 0$, the equilibrium point $\widetilde{E}$ of the system (2.1) is a saddle point. Furthermore, we know that

$$\Gamma_1 = \{(x_1,x_2) \mid x_1 > 0, x_2 = 0\}$$

is the stable manifold of the equilibrium point $\widetilde{E}$.

The Jacobian matrix of the system (2.1) at $\widehat{E}$ is

$$\hat{J} = \begin{bmatrix} r_1 + \alpha_{21} K_2 & 0 \\ \alpha_{12} K_2 & -r_2 \end{bmatrix}$$

The eigenvalues of $\hat{J}$ are $\lambda_1 = r_1 + \alpha_{21} K_2$ and $\lambda_2 = -r_2$. Since $\lambda_1 > 0$ and $\lambda_2 < 0$, the equilibrium point $\widehat{E}$ of the system (3.1) is a saddle point. Furthermore, we know that

$$\Gamma_2 \;=\; \{(x_1, x_2) \mid x_1 = 0, x_2 > 0\}$$

is the stable manifold of the equilibrium point $\widehat{E}$.

Lemma 2.2 *If*

$$K_1 K_2 \alpha_{12} \alpha_{21} - r_1 r_2 \;<\; 0$$

then the unique positive equilibrium point E^* of the system (2.1) is locally asymptotically stable.

Proof : The Jacobian matrix of the system (2.1) at E^* is

$$J^* = \begin{bmatrix} -\dfrac{r_1}{K_1} x_1^* & \alpha_{21} x_1^* \\ \alpha_{12} x_2^* & -\dfrac{r_2}{K_2} x_2^* \end{bmatrix}$$

Since

$$\begin{aligned} det(J^*) &= \frac{r_1 r_2}{K_1 K_2} x_1^* x_2^* - \alpha_{12}\alpha_{21} x_1^* x_2^* \\ &= x_1^* x_2^* \left(\frac{r_1 r_2}{K_1 K_2} - \alpha_{12}\alpha_{21} \right) \\ &= x_1^* x_2^* \left(\frac{r_1 r_2 - K_1 K_2 \alpha_{12}\alpha_{21}}{K_1 K_2} \right) \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} trace(J^*) &= -\frac{r_1}{K_1} x_1^* - \frac{r_2}{K_2} x_2^* \\ &= -\left(\frac{r_1}{K_1} x_1^* + \frac{r_2}{K_2} x_2^* \right) \\ &< 0 \end{aligned}$$

the unique positive equilibrium point E^* of the system (2.1) is locally asymptotically stable.

Lemma 2.3 *All solutions* $(x_1(t), x_2(t))$ *of the system (2.1) with the initial condition (2.2) are positive and bounded.*

Proof : Firstly, we want to show that all solutions $(x_1(t), x_2(t))$ of the system (2.1) with the initial condition (2.2) are positive. In other words, if the initial point $(x_1(0), x_2(0))$ is in the first quadrant, so is the solution $(x_1(t), x_2(t))$ for all $t > 0$. Since x_1-axis and x_2-axis are the solutions of the system (2.1), the trajectory of the solutions $(x_1(t), x_2(t))$ with the initial point $(x_1(0), x_2(0))$ in the first quadrant cannot cross with x_1-axis and x_2-axis by the uniqueness of the solution. Therefore, we know that all solutions $(x_1(t), x_2(t))$ of the system (2.1) with the initial condition (2.2) are positive.

Secondly, we want to show that all solutions $(x_1(t), x_2(t))$ of the system (2.1) with the initial condition (2.2) are bounded. That is, $x_1(t)$ and $x_2(t)$ are both bounded for all $t \geq 0$. From the Figure 2.1, we find that all solutions $(x_1(t), x_2(t))$ of the system (2.1) with the initial condition (2.2) are bounded.

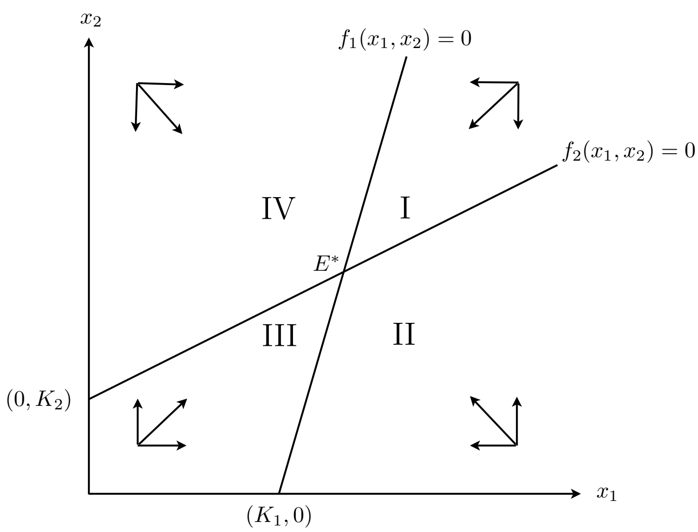

Figure (2.1). The slope of the trajectory of the system (2.1)

2.2 Global Stability

Now, we want to use the following three methods to analyse the global stability of the unique positive equilibrium point E^* of the system (2.1) :

(i) Dulac's criterion plus Poincaré- Bendixson theorem

(ii) Stable limit cycle analysis

(iii) Lyapunov function

Theorem 2.4 *If (2.4) holds, then the unique positive equilibrium point E^* of the system (2.1) is globally asymptotically stable.*

Proof : Firstly, we use the method (i) to analyse the system (2.1). Consider

$$H(x_1,x_2) = \frac{1}{x_1x_2}, \qquad x_1 > 0,\ x_2 > 0$$

Then

$$\begin{aligned}
\nabla\cdot(Hg) &= \frac{\partial}{\partial x_1}(Hg_1)+\frac{\partial}{\partial x_2}(Hg_2) \\
&= \frac{\partial}{\partial x_1}\left(\frac{x_1f_1}{x_1x_2}\right)+\frac{\partial}{\partial x_2}\left(\frac{x_2f_2}{x_1x_2}\right) \\
&= \frac{\partial}{\partial x_1}\left(\frac{f_1}{x_2}\right)+\frac{\partial}{\partial x_2}\left(\frac{f_2}{x_1}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{x_2}\cdot\frac{\partial}{\partial x_1}f_1+\frac{1}{x_1}\cdot\frac{\partial}{\partial x_2}f_2 \\
&= \frac{1}{x_2}\cdot\left(-\frac{r_1}{K_1}\right)+\frac{1}{x_1}\cdot\left(-\frac{r_2}{K_2}\right) \\
&= -\left(\frac{r_1}{K_1x_2}+\frac{r_2}{K_2x_1}\right) \\
&< 0
\end{aligned}$$

Hence by the Dulac's criterion, there is no closed orbit in the first quadrant. From Lemma 2.1, we know that the unique positive equilibrium point E^* is locally asymptotically stable. By the Lemma 2.2 and the Poincaré-Bendixson theorem, it suffices to show that the unique positive equilibrium point E^* is globally asymptotically stable in the first quadrant.

Secondly, we introduce the method (ii) to analyse the global stability of the system (2.1). Now, we want to show that the system (2.1) has no closed orbit in the first quadrant. Suppose on the contrary that there is a T-periodic orbit $\Gamma = \{(x_1(t),x_2(t)) \mid 0 \le t \le T\}$ in the first quadrant. Compute

$$\begin{aligned}
\triangle &= \int_{\Gamma}\left(\frac{\partial g_1}{\partial x_1}+\frac{\partial g_2}{\partial x_2}\right)ds \\
&= \int_0^T \left\{ \frac{\partial}{\partial x_1}\left[r_1 x_1(t)\left(1-\frac{x_1(t)}{K_1}\right)+\alpha_{21}x_1(t)x_2(t)\right]\right. \\
&\qquad \left. + \frac{\partial}{\partial x_2}\left[r_2 x_2(t)\left(1-\frac{x_2(t)}{K_2}\right)+\alpha_{12}x_1(t)x_2(t)\right]\right\} dt \\
&= \int_0^T \left\{\left[r_1\left(1-\frac{2x_1(t)}{K_1}\right)+\alpha_{21}x_2(t)\right]+\left[r_2\left(1-\frac{2x_2(t)}{K_2}\right)+\alpha_{12}x_1(t)\right]\right\}dt \\
&= \int_0^T \left[\frac{\dot{x}_1(t)}{x_1(t)}-\frac{r_1}{K_1}x_1(t)+\frac{\dot{x}_2(t)}{x_2(t)}-\frac{r_2}{K_2}x_2(t)\right]dt \\
&= \int_0^T \frac{\dot{x}_1(t)}{x_1(t)}dt-\frac{r_1}{K_1}\int_0^T x_1(t)dt+\int_0^T \frac{\dot{x}_2(t)}{x_2(t)}dt-\frac{r_2}{K_2}\int_0^T x_2(t)dt \\
&= \int_{x_1(0)}^{x_1(T)}\frac{1}{x_1}dx_1+\int_{x_2(0)}^{x_2(T)}\frac{1}{x_2}dx_2-\int_0^T\left[\frac{r_1}{K_1}x_1(t)+\frac{r_2}{K_2}x_2(t)\right]dt
\end{aligned}$$

Since Γ is a T-periodic,

$$\int_{x_1(0)}^{x_1(T)}\frac{1}{x_1}dx_1 = 0 \quad \text{and} \quad \int_{x_2(0)}^{x_2(T)}\frac{1}{x_2}dx_2 = 0$$

Hence we obtain that

$$\begin{aligned}
\triangle &= -\int_0^T\left[\frac{r_1}{K_1}x_1(t)+\frac{r_2}{K_2}x_2(t)\right]dt \\
&< 0
\end{aligned}$$

This implies that all closed orbits of the system (2.1) in the first quadrant are orbitally stable. Since every closed orbit is orbitally, there is a unique stable limit cycle in the first quadrant. That is, the unique positive equilibrium point E^* is unstable. However, by Lemma 2.1, E^* is locally asymptotically stable. Thus there is no periodic orbit in the first quadrant. By Lemma 2.2 and the Poincaré-Bendixson theorem, it suffices to show that the unique positive equilibrium point E^* is globally asymptotically stable in the first quadrant.

Theorem 2.5 *If*

$$\frac{r_1}{\alpha_{21}K_1} > 1 \ , \qquad \frac{r_2}{\alpha_{12}K_2} > 1$$

holds, then the unique positive equilibrium point E^* of the system (2.1) is globally asymptotically stable.

Proof : Now, we use the method (iii) to analyse the system (2.1). Let's construct the following Lyapunov function

$$V(x_1,x_2) = V_1(x_1) + V_2(x_2)$$

where

$$V_1(x_1) = \frac{1}{\alpha_{21}}\left(x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*}\right)$$

$$V_2(x_2) = \frac{1}{\alpha_{12}}\left(x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*}\right)$$

on $G = \{(x_1,x_2)|x_1 > 0, x_2 > 0\}$. It is obvious that $V(x_1,x_2) \in C^1(G,R)$ and $V(x_1^*,x_2^*) = 0$. The function $V(x_1,x_2)$ satisfies

$$V(x_1,x_2) \ > \ V(x_1^*,x_2^*) \ = \ 0$$

which holds for all $(x_1,x_2) \in G - \{E^*\}$. Then the time derivatives of $V_i(x_1,x_2), i = 1,2$ computed along the solution of the system (2.1) are

$$
\begin{aligned}
\dot{V}_1(x_1) &= V_1'(x_1)\dot{x}_1 \\
&= \frac{1}{\alpha_{21}}(1-\frac{x_1^*}{x_1})\cdot\dot{x}_1 \\
&= \frac{1}{\alpha_{21}}\cdot\frac{\dot{x}_1}{x_1}(x_1-x_1^*) \\
&= \frac{1}{\alpha_{21}}\cdot\left[r_1\left(1-\frac{x_1}{K_1}\right)+\alpha_{21}x_2\right](x_1-x_1^*) \\
&= \frac{1}{\alpha_{21}}\cdot\left(r_1-\frac{r_1}{K_1}x_1+\alpha_{21}x_2\right)(x_1-x_1^*) \\
&= \frac{1}{\alpha_{21}}\cdot\left(\frac{r_1}{K_1}x_1^*-\alpha_{21}x_2^*-\frac{r_1}{K_1}x_1+\alpha_{21}x_2\right)(x_1-x_1^*) \\
&= \frac{1}{\alpha_{21}}\cdot\frac{r_1}{K_1}x_1^*x_1-\frac{1}{\alpha_{21}}\cdot\frac{r_1}{K_1}(x_1^*)^2-x_2^*x_1+x_1^*x_2^* \\
&\quad -\frac{1}{\alpha_{21}}\cdot\frac{r_1}{K_1}x_1^2+\frac{1}{\alpha_{21}}\frac{r_1}{K_1}x_1^*x_1+x_1x_2-x_1^*x_2 \\
&= -\frac{r_1}{\alpha_{21}K_1}(x_1-x_1^*)^2+(x_1-x_1^*)(x_2-x_2^*) \qquad (2.5)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{V}_2(x_2) &= V_2'(x_2)\dot{x}_2 \\
&= \frac{1}{\alpha_{12}}(1-\frac{x_2^*}{x_2})\cdot\dot{x}_2 \\
&= \frac{1}{\alpha_{12}}\cdot\frac{\dot{x}_2}{x_2}(x_2-x_2^*) \\
&= \frac{1}{\alpha_{12}}\cdot\left[r_2\left(1-\frac{x_2}{K_2}\right)+\alpha_{12}x_1\right](x_2-x_2^*)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\alpha_{12}} \cdot \left(r_2 - \frac{r_2}{K_2} x_2 + \alpha_{12} x_1 \right) (x_2 - x_2^*) \\
&= \frac{1}{\alpha_{12}} \cdot \left(\frac{r_2}{K_2} x_2^* - \alpha_{12} x_1^* - \frac{r_2}{K_2} x_2 + \alpha_{12} x_1 \right) (x_2 - x_2^*) \\
&= \frac{1}{\alpha_{12}} \cdot \frac{r_2}{K_2} x_2^* x_2 - \frac{1}{\alpha_{12}} \cdot \frac{r_2}{K_2} (x_2^*)^2 - x_1^* x_2 + x_1^* x_2^* \\
&\quad - \frac{1}{\alpha_{12}} \cdot \frac{r_2}{K_2} x_2^2 + \frac{1}{\alpha_{12}} \frac{r_2}{K_2} x_2^* x_2 + x_1 x_2 - x_1 x_2^* \\
&= -\frac{r_2}{\alpha_{12} K_2} (x_2 - x_2^*)^2 + (x_1 - x_1^*)(x_2 - x_2^*) \qquad (2.6)
\end{aligned}
$$

Therefore, the time derivative of $V(x_1, x_2)$ is given by

$$
\begin{aligned}
\dot{V}(x_1, x_2) &= \frac{d}{dt} \dot{V}(x_1, x_2) \\
&= \frac{d}{dt} V_1(x_1) + \frac{d}{dt} V_2(x_2) \\
&= -\frac{r_1}{\alpha_{21} K_1} (x_1 - x_1^*)^2 + (x_1 - x_1^*)(x_2 - x_2^*) \\
&\quad -\frac{r_2}{\alpha_{12} K_2} (x_2 - x_2^*)^2 + (x_1 - x_1^*)(x_2 - x_2^*) \\
&= -\frac{r_1}{\alpha_{21} K_1} (x_1 - x_1^*)^2 + 2(x_1 - x_1^*)(x_2 - x_2^*) - \frac{r_2}{\alpha_{12} K_2} (x_2 - x_2^*)^2 \\
&= -[(x_1 - x_1^*)^2 - 2(x_1 - x_1^*)(x_2 - x_2^*) + (x_2 - x_2^*)^2] \\
&\quad +(x_1 - x_1^*)^2 - \frac{r_1}{\alpha_{21} K_1} (x_1 - x_1^*)^2 \\
&\quad +(x_2 - x_2^*)^2 - \frac{r_2}{\alpha_{12} K_2} (x_2 - x_2^*)^2 \\
&= -[(x_1 - x_1^*) - (x_2 - x_2^*)]^2 - \left(\frac{r_1}{\alpha_{21} K_1} - 1 \right) (x_1 - x_1^*)^2
\end{aligned}
$$

$$
\begin{aligned}
& -\left(\frac{r_2}{\alpha_{12}K_2}-1\right)(x_2-x_2^*)^2 \\
< \;\; & 0
\end{aligned}
$$

This shows that $\dot{V}(x_1,x_2) \; < \; 0$ on G. Therefore, the unique positive equilibrium point E^* of the system (2.1) is globally asymptotically stable on G.

Remark 2.6 In [3], Goh constructed a Lyapunov function to show that the unique positive equilibrium point is globally asymptotically stable in a nonlinear model of mutualism.

3 THE MODEL WITH TIME DELAY

Consider a two-species Lotka-Volterra mutualistic system with time delay τ modelled by

$$
\begin{aligned}
\dot{x}_1(t) &= x_1(t)\left[r_1(1-\frac{x_1(t-\tau)}{K_1})+\alpha_{21}x_2(t)\right] \\
\dot{x}_2(t) &= x_2(t)\left[r_2(1-\frac{x_2(t-\tau)}{K_2})+\alpha_{12}x_1(t)\right] \qquad (3.1)
\end{aligned}
$$

with the initial conditions

$$
x_i(\theta) \; = \; \phi_i(\theta) > 0 \;\; , \;\; \theta \in [-\tau,0] \; , \; \phi_i \in C([-\tau,0],R) \; , \;\; i=1,2 \qquad (3.2)
$$

Lemma 3.1 *Every solution of the system (3.1) with the initial conditions (3.2) remains positive for all* $t \geq 0$.

Proof : It is true because

$$
x_1(t) = x_1(0)exp\left\{\int_0^t [r_1\left(1-\frac{x_1(s-\tau)}{K_1}\right)+\alpha_{21}x_2(s)]ds\right\}
$$

$$
x_2(t) = x_2(0)exp\left\{\int_0^t [r_2\left(1-\frac{x_2(s-\tau)}{K_2}\right)+\alpha_{12}x_1(s)]ds\right\}
$$

and $x_i(0) > 0$ for $i = 1,2$. Therefore, we obtain that all solutions $(x_1(t),x_2(t))$ of the system (4.1) with the initial conditions (3.2) are positive.

Lemma 3.2 *Let* $(x_1(t),x_2(t))$ *denote the solution of (3.1) with the initial conditions (3.2).*

(a) If

$$\limsup_{t\to\infty} x_2(t) < \infty \tag{3.3}$$

then

$$0 < \liminf_{t\to\infty} x_2(t) \quad \text{and} \quad 0 < \liminf_{t\to\infty} x_1(t) \le \limsup_{t\to\infty} x_1(t) < \infty \tag{3.4}$$

(b) If

$$\limsup_{t\to\infty} x_1(t) < \infty$$

then

$$0 < \liminf_{t\to\infty} x_1(t) \quad \text{and} \quad 0 < \liminf_{t\to\infty} x_2(t) \le \limsup_{t\to\infty} x_2(t) < \infty$$

(c) If there exists a $M_2 > 0$ such that for all the positive solutions $(x_1(t), x_2(t))$ of the system (3.1) with the initial conditions (3.2),

$$\limsup_{t\to\infty} x_2(t) \le M_2 \tag{3.5}$$

then there exists positive numbers m_1, m_2 and M_1 such that

$$m_2 \le \liminf_{t\to\infty} x_2(t) \quad \text{and} \quad m_1 \le \liminf_{t\to\infty} x_1(t) \le \limsup_{t\to\infty} x_1(t) \le M_1 \tag{3.6}$$

(d) If there exists a $M_1 > 0$ such that for all the positive solutions $(x_1(t), x_2(t))$ of the system (3.1) with the initial conditions (3.2),

$$\limsup_{t\to\infty} x_1(t) \le M_1$$

then there exists positive numbers m_1, m_2 and M_2 such that

$$m_1 \le \liminf_{t\to\infty} x_1(t) \quad \text{and} \quad m_2 \le \liminf_{t\to\infty} x_2(t) \le \limsup_{t\to\infty} x_2(t) \le M_2$$

Proof : We shall now prove the result using the technique developed in [5]. It is sufficient to prove (a) and (c). Suppose (3.3) holds, then there exist $M_2 > 0$ and $t_1 > 0$ such that

$$0 < x_2(t) \le M_2 \qquad \text{for} \quad t \ge t_1 \tag{3.7}$$

which together with (4.1) yield

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_1(t)[r_1 - \frac{r_1}{K_1} x_1(t-\tau) + \alpha_{21} x_2(t)] \\ &\le x_1(t)[r_1 - \frac{r_1}{K_1} x_1(t-\tau) + \alpha_{21} M_2] \quad \text{for} \quad t \ge t_1 \end{aligned} \tag{3.8}$$

By the positivity of the solution and (3.8), we have

$$\frac{dx_1(t)}{dt} \leq x_1(t)[r_1 + \alpha_{21}M_2] \qquad t \geq t_1 \tag{3.9}$$

Integrating both sides of (3.9) on $[t-\tau, t]$, where $t \geq t_1 + \tau$, we have

$$x_1(t) \leq x_1(t-\tau)e^{(r_1+\alpha_{21}M_2)\tau}$$

That is,

$$x_1(t-\tau) \geq x_1(t) \cdot e^{-(r_1+\alpha_{21}M_2)\tau} \tag{3.10}$$

It follows from (3.8) that for $t \geq t_1 + \tau$

$$\begin{aligned}
\frac{dx_1(t)}{dt} &\leq x_1(t)\left[(r_1 + \alpha_{21}M_2) - \frac{r_1}{K_1}e^{-(r_1+\alpha_{21}M_2)\tau} \cdot x_1(t)\right] \\
&= (r_1 + \alpha_{21}M_2)x_1(t)\left[1 - \frac{r_1}{K_1(r_1 + \alpha_{21}M_2) \cdot e^{(r_1+\alpha_{21}M_2)\tau}} \cdot x_1(t)\right] \\
&= (r_1 + \alpha_{21}M_2)x_1(t)\left[1 - \frac{x_1(t)}{\frac{K_1(r_1+\alpha_{21}M_2)}{r_1}\, e^{(r_1+\alpha_{21}M_2)\tau}}\right]
\end{aligned}$$

This implies that

$$x_1(t) \leq \frac{K_1(r_1 + \alpha_{21}M_2)}{r_1}\, e^{(r_1+\alpha_{21}M_2)\tau} \equiv M_1 \quad \text{for} \quad t \geq t_2 \tag{3.11}$$

for some $t_2 \geq t_1 + \tau$. Then

On the other hand, positivity of the solution and (3.1) give

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= x_1(t)[r_1 - \frac{r_1}{K_1}x_1(t-\tau) + \alpha_{21}x_2(t)] \\
&\geq x_1(t)\left[r_1 - \frac{r_1}{K_1}x_1(t-\tau)\right] \qquad (3.12) \\
&\geq x_1(t)\left[r_1 - \frac{r_1}{K_1} \cdot M_1\right] \quad \text{for} \quad t \geq t_2 + \tau \qquad (3.13)
\end{aligned}$$

Integrating both sides of (3.13) on $[t-\tau, t]$, where $t \geq t_2 + 2\tau$, we have

$$x_1(t) \geq x_1(t-\tau) \cdot e^{(r_1 - \frac{r_1}{K_1}M_1)\tau}$$

That is,

$$x_1(t-\tau) \leq x_1(t) \cdot e^{-(r_1 - \frac{r_1}{K_1}M_1)\tau}$$

for $t \geq t_2 + 2\tau$. This, together with (3.12), gives

$$\begin{aligned} \frac{dx_1(t)}{dt} &\geq x_1(t)\left[r_1 - \frac{r_1}{K_1}x_1(t-\tau)\right] \\ &\geq x_1(t)\left[r_1 - \frac{r_1}{K_1}\cdot e^{-(r_1-\frac{r_1}{K_1}M_1)\tau}x_1(t)\right] \\ &= r_1x_1(t)\left[1 - \frac{x_1(t)}{K_1\cdot e^{(r_1-\frac{r_1}{K_1}M_1)\tau}}\right] \end{aligned}$$

It follows that

$$\liminf_{t\to\infty} x_1(t) \geq K_1\cdot e^{(r_1-\frac{r_1}{K_1}M_1)\tau} \equiv m_1$$

and $m_1 > 0$. Similarly, it follows from (3.1) that

$$\begin{aligned} \frac{dx_2(t)}{dt} &= x_2(t)\left[r_2 - \frac{r_2}{K_2}x_2(t-\tau) + \alpha_{12}x_1(t)\right] \\ &\geq x_2(t)\left[r_2 - \frac{r_2}{K_2}x_2(t-\tau)\right] && (3.14) \\ &\geq x_2(t)\left(r_2 - \frac{r_2}{K_2}M_2\right) \quad \text{for} \quad t \geq t_1 + \tau && (3.15) \end{aligned}$$

Integrating both sides of (3.15) on $[t-\tau, t]$, where $t \geq t_1 + 2\tau$, we have

$$x_2(t) \geq x_2(t-\tau)\cdot e^{(r_2-\frac{r_2}{K_2}M_2)\tau}$$

That is,

$$x_2(t-\tau) \leq x_2(t)\cdot e^{-(r_2-\frac{r_2}{K_2}M_2)\tau}$$

for $t \geq t_1 + 2\tau$. This, together with (3.14), gives

$$\begin{aligned} \frac{dx_2(t)}{dt} &\geq x_2(t)\left[r_2 - \frac{r_2}{K_2}x_2(t-\tau)\right] \\ &\geq x_2(t)\left[r_2 - \frac{r_2}{K_2}\cdot e^{-(r_2-\frac{r_2}{K_2}M_2)\tau}x_2(t)\right] \\ &= r_2x_1(t)\left[1 - \frac{x_2(t)}{K_2\cdot e^{(r_2-\frac{r_2}{K_2}M_2)\tau}}\right] \end{aligned}$$

It follows that

$$\liminf_{t\to\infty} x_2(t) \geq K_2\cdot e^{(r_2-\frac{r_2}{K_2}M_2)\tau} \equiv m_2$$

and $m_2 > 0$. Thus, (3.6) holds. This completes the proof.

Theorem 3.3 *Suppose $K_1K_2\alpha_{12}\alpha_{21} - r_1r_2 < 0$ and the system (3.1) is uniformly persistent. Assume that the delay τ in (3.1) satisfies*

$$2K_2r_1x_1^*(K_1 - r_1M_1\tau) > K_1\alpha_{21}x_2^*(2K_1K_2 + K_2r_1M_1\tau + K_1r_2M_2\tau) \tag{3.16}$$

$$2K_1r_2x_2^*(K_2 - r_2M_2\tau) > K_2\alpha_{12}x_1^*(2K_1K_2 + K_1r_2M_2\tau + K_2r_1M_1\tau) \tag{3.17}$$

where M_1 and M_2 are defined in the Lemma 3.2. Then the unique positive equilibrium point E^* of the system (3.1) is globally asymptotically stable.

Proof : Define $y(t) = (y_1(t), y_2(t))$ by

$$y_1(t) = \frac{x_1(t) - x_1^*}{x_1^*} \quad , \quad y_2(t) = \frac{x_2(t) - x_2^*}{x_2^*}$$

From (3.1), we have

$$\dot{y}_1(t) = [1 + y_1(t)]\left[-\frac{r_1x_1^*}{K_1}y_1(t-\tau) + \alpha_{21}x_2^*y_2(t)\right] \tag{3.18}$$

$$\dot{y}_2(t) = [1 + y_2(t)]\left[-\frac{r_2x_2^*}{K_2}y_2(t-\tau) + \alpha_{12}x_1^*y_1(t)\right] \tag{3.19}$$

Let

$$V_1(y(t)) = \frac{1}{\alpha_{21}x_2^*}\{y_1(t) - \ln[1 + y_1(t)]\} + \frac{1}{\alpha_{12}x_1^*}\{y_2(t) - \ln[1 + y_2(t)]\} \tag{3.20}$$

then we have, from (3.18) and (3.19), that

$$\begin{aligned}
\dot{V}_1(y(t)) &= \frac{1}{\alpha_{21}x_2^*} \cdot \frac{y_1(t)\dot{y}_1(t)}{1 + y_1(t)} + \frac{1}{\alpha_{12}x_1^*} \cdot \frac{y_2(t)\dot{y}_2(t)}{1 + y_2(t)} \\
&= -\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*} \cdot y_1(t)y_1(t-\tau) + 2y_1(t)y_2(t) \\
&\quad -\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*} \cdot y_2(t)y_2(t-\tau) \\
&= -\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*}\, y_1(t)\left[y_1(t) - \int_{t-\tau}^{t} \dot{y}_1(s)ds\right] \\
&\quad -\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*}\, y_2(t)\left[y_2(t) - \int_{t-\tau}^{t} \dot{y}_2(s)ds\right] \\
&\quad - [y_1^2(t) - 2y_1(t)y_2(t) + y_2^2(t)] + y_1^2(t) + y_2^2(t)
\end{aligned}$$

$$
\begin{aligned}
= & -\left(\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*}-1\right)\,y_1^2(t)-\left(\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*}-1\right)y_2^2(t) \\
& +\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*}\,y_1(t)\cdot\int_{t-\tau}^{t}[1+y_1(s)]\left\{-\frac{r_1x_1^*}{K_1}y_1(s-\tau)+\alpha_{21}x_2^*y_2(s)\right\}ds \\
& +\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*}\,y_2(t)\cdot\int_{t-\tau}^{t}[1+y_2(s)]\left\{-\frac{r_2x_2^*}{K_2}y_2(s-\tau)+\alpha_{12}x_1^*y_1(s)\right\}ds \\
& -(y_1(t)-y_2(t))^2 \\
\leq & -\left(\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*}-1\right)\,y_1^2(t)-\left(\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*}-1\right)y_2^2(t) \\
& +\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*}\,y_1(t)\cdot\int_{t-\tau}^{t}[1+y_1(s)]\left[-\frac{r_1x_1^*}{K_1}y_1(s-\tau)+\alpha_{21}x_2^*y_2(s)\right]ds \\
& +\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*}\,y_2(t)\cdot\int_{t-\tau}^{t}[1+y_2(s)]\left[-\frac{r_2x_2^*}{K_2}y_2(s-\tau)+\alpha_{12}x_1^*y_1(s)\right]ds \\
= & -\left(\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*}-1\right)\,y_1^2(t)-\left(\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*}-1\right)y_2^2(t) \\
& +\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*}\int_{t-\tau}^{t}[1+y_1(s)]\left[-\frac{r_1x_1^*}{K_1}y_1(t)y_1(s-\tau)+\alpha_{21}x_2^*y_1(t)y_2(s)\right]ds \\
& +\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*}\int_{t-\tau}^{t}[1+y_2(s)]\left[-\frac{r_2x_2^*}{K_2}y_2(t)y_2(s-\tau)+\alpha_{12}x_1^*y_2(t)y_1(s)\right]ds \\
\leq & -\left(\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*}-1\right)\,y_1^2(t)-\left(\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*}-1\right)y_2^2(t) \\
& +\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*}\int_{t-\tau}^{t}[1+y_1(s)]\left[\frac{r_1x_1^*}{K_1}|y_1(t)y_1(s-\tau)|+\alpha_{21}x_2^*|y_1(t)y_2(s)|\right]ds \\
& +\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*}\int_{t-\tau}^{t}[1+y_2(s)]\left[\frac{r_2x_2^*}{K_2}|y_2(t)y_2(s-\tau)|+\alpha_{12}x_1^*|y_2(t)y_1(s)|\right]ds \qquad (3.21)
\end{aligned}
$$

By Lemma 3.2, there exists a $T > 0$ such that $m_i \le x_i^*[1+y_i(t)] = x_i(t) \le M_i$ for $t > T,\ i = 1,2$. Then for $t > T+\tau \equiv \widehat{T}$, we have from (3.21) that

$$
\begin{aligned}
\dot{V}_1(y(t)) \quad \le \quad & -\left(\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*}-1\right) y_1^2(t) - \left(\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*}-1\right) y_2^2(t) \\
& + \frac{r_1M_1}{K_1\alpha_{21}x_2^*}\int_{t-\tau}^{t}\left[\frac{r_1x_1^*}{K_1}|y_1(t)||y_1(s-\tau)| + \alpha_{21}x_2^*|y_1(t)||y_2(s)|\right] ds \\
& + \frac{r_2M_2}{K_2\alpha_{12}x_1^*}\int_{t-\tau}^{t}\left[\frac{r_2x_2^*}{K_2}|y_2(t)||y_2(s-\tau)| + \alpha_{12}x_1^*|y_2(t)||y_1(s)|\right] ds \\
\le \quad & -\left(\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*}-1\right) y_1^2(t) - \left(\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*}-1\right) y_2^2(t) \\
& + \frac{r_1M_1}{K_1\alpha_{21}x_2^*}\left\{\frac{r_1x_1^*\tau}{2K_1}y_1^2(t) + \frac{r_1x_1^*}{2K_1}\int_{t-\tau}^{t} y_1^2(s-\tau)ds \right. \\
& \qquad \left. + \frac{\alpha_{21}x_2^*\tau}{2}y_1^2(t) + \frac{\alpha_{21}x_2^*}{2}\int_{t-\tau}^{t} y_2^2(s)ds\right\} \\
& + \frac{r_2M_2}{K_2\alpha_{12}x_1^*}\left\{\frac{r_2x_2^*\tau}{2K_2}y_2^2(t) + \frac{r_2x_2^*}{2K_2}\int_{t-\tau}^{t} y_2^2(s-\tau)ds \right. \\
& \qquad \left. + \frac{\alpha_{12}x_1^*\tau}{2}y_2^2(t) + \frac{\alpha_{12}x_1^*}{2}\int_{t-\tau}^{t} y_1^2(s)ds\right\} \\
= \quad & -\left(\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*} - 1 - \frac{r_1^2M_1x_1^*\tau}{2K_1^2\alpha_{21}x_2^*} - \frac{r_1M_1\tau}{2K_1}\right) y_1^2(t)
\end{aligned}
$$

$$-\left(\frac{r_2x_2^*}{K_2\alpha_{12}x_1^*}-1-\frac{r_2^2M_2x_2^*\tau}{2K_2^2\alpha_{12}x_1^*}-\frac{r_2M_2\tau}{2K_2}\right)y_2^2(t)$$

$$+\frac{r_1^2M_1x_1^*}{2K_1^2\alpha_{21}x_2^*}\int_{t-\tau}^{t}y_1^2(s-\tau)ds+\frac{r_1M_1}{2K_1}\int_{t-\tau}^{t}y_2^2(s)ds$$

$$+\frac{r_2^2M_2x_2^*}{2K_2^2\alpha_{12}x_1^*}\int_{t-\tau}^{t}y_2^2(s-\tau)ds+\frac{r_2M_2}{2K_2}\int_{t-\tau}^{t}y_1^2(s)ds \qquad (3.22)$$

Let

$$V_2(y(t)) = \frac{r_1^2M_1x_1^*}{2K_1^2\alpha_{21}x_2^*}\int_{t-\tau}^{t}\int_{s}^{t}y_1^2(z-\tau)dzds+\frac{r_1M_1}{2K_1}\int_{t-\tau}^{t}\int_{s}^{t}y_2^2(z)dzds$$

$$+\frac{r_2^2M_2x_2^*}{2K_2^2\alpha_{12}x_1^*}\int_{t-\tau}^{t}\int_{s}^{t}y_2^2(z-\tau)dzds$$

$$+\frac{r_2M_2}{2K_2}\int_{t-\tau}^{t}\int_{s}^{t}y_1^2(z)dzds \qquad (3.23)$$

then

$$\dot{V}_2(y(t)) = \frac{r_1^2M_1x_1^*\tau}{2K_1^2\alpha_{21}x_2^*}y_1^2(t-\tau)-\frac{r_1^2M_1x_1^*}{2K_1^2\alpha_{21}x_2^*}\int_{t-\tau}^{t}y_1^2(s-\tau)ds$$

$$+\frac{r_1M_1\tau}{2K_1}y_2^2(t)-\frac{r_1M_1}{2K_1}\int_{t-\tau}^{t}y_2^2(s)ds$$

$$+\frac{r_2^2M_2x_2^*\tau}{2K_2^2\alpha_{12}x_1^*}y_2^2(t-\tau)-\frac{r_2^2M_2x_2^*}{2K_2^2\alpha_{12}x_1^*}\int_{t-\tau}^{t}y_2^2(s-\tau)ds$$

$$+\frac{r_2M_2\tau}{2K_2}y_1^2(t)-\frac{r_2M_2}{2K_2}\int_{t-\tau}^{t}y_1^2(s)ds \qquad (3.24)$$

Furthermore, from (3.22) and (3.24), for $t>\widehat{T}$ we have

$$\dot{V}_1(y(t))+\dot{V}_2(y(t))\leq-\left(\frac{r_1x_1^*}{K_1\alpha_{21}x_2^*}-1-\frac{r_1^2M_1x_1^*\tau}{2K_1^2\alpha_{21}x_2^*}-\frac{r_1M_1\tau}{2K_1}-\frac{r_2M_2\tau}{2K_2}\right)y_1^2(t) \qquad (3.25)$$

$$V_3(y(t))=\frac{r_1^2M_1x_1^*\tau}{2K_1^2\alpha_{21}x_2^*}\int_{t-\tau}^{t}y_1^2(s)ds+\frac{r_2^2M_2x_2^*\tau}{2K_2^2\alpha_{12}x_1^*}\int_{t-\tau}^{t}y_2^2(s)ds \qquad (3.26)$$

then

$$\dot{V}_3(y(t)) = \frac{r_1^2 M_1 x_1^* \tau}{2K_1^2 \alpha_{21} x_2^*} y_1^2(t) - \frac{r_1^2 M_1 x_1^* \tau}{2K_1^2 \alpha_{21} x_2^*} y_1^2(t-\tau)$$

$$+ \frac{r_2^2 M_2 x_2^* \tau}{2K_2^2 \alpha_{12} x_1^*} y_2^2(t) - \frac{r_2^2 M_2 x_2^* \tau}{2K_2^2 \alpha_{12} x_1^*} y_2^2(t-\tau) \tag{3.27}$$

Now define a Lyapunov function candidate $V(y(t))$ as

$$V(y(t)) = V_1(y(t)) + V_2(y(t)) + V_3(y(t)) \tag{3.28}$$

Then, from (3.26) and (3.27), for $t > \widehat{T}$ we have

$$\dot{V}(y(t)) \leq -\left(\frac{r_1 x_1^*}{K_1 \alpha_{21} x_2^*} - 1 - \frac{r_1^2 M_1 x_1^* \tau}{2K_1^2 \alpha_{21} x_2^*} - \frac{r_1 M_1 \tau}{2K_1} - \frac{r_2 M_2 \tau}{2K_2} - \frac{r_1^2 M_1 x_1^* \tau}{2K_1^2 \alpha_{21} x_2^*} \right) y_1^2(t) \tag{3.29}$$

$$- \left(\frac{r_2 x_2^*}{K_2 \alpha_{12} x_1^*} - 1 - \frac{r_2^2 M_2 x_2^* \tau}{2K_2^2 \alpha_{12} x_1^*} - \frac{r_2 M_2 \tau}{2K_2} - \frac{r_1 M_1 \tau}{2K_1} - \frac{r_2^2 M_2 x_2^* \tau}{2K_2^2 \alpha_{12} x_1^*} \right) y_2^2(t)$$

$$= -\left(\frac{2K_1 K_2 r_1 x_1^* - 2K_1^2 K_2 \alpha_{21} x_2^* - 2K_2 r_1^2 M_1 x_1^* \tau - K_1 K_2 \alpha_{21} x_2^* r_1 M_1 \tau - K_1^2 \alpha_{21} x_2^* r_2 M_2 \tau}{2K_1^2 K_2 \alpha_{21} x_2^*} \right) y_1^2(t)$$

$$-\left(\frac{2K_1 K_2 r_2 x_2^* - 2K_2^2 K_1 \alpha_{12} x_1^* - 2K_1 r_2^2 M_2 x_2^* \tau - K_1 K_2 \alpha_{12} x_1^* r_2 M_2 \tau - K_2^2 \alpha_{12} x_1^* r_1 M_1 \tau}{2K_1^2 K_2 \alpha_{21} x_2^*} \right) y_2^2(t)$$

$$\equiv -\alpha y_1^2(t) - \beta y_2^2(t) \tag{3.30}$$

It follows from (3.16) and (3.17) that $\alpha > 0$ and $\beta > 0$. Let $w(s) = \widehat{N} s^2$ where $\widehat{N} = \min\{\alpha, \beta\}$, then w is nonnegative continuous on $[0, \infty)$, $w(0) = 0$ and $w(s) > 0$ for $s > 0$. It follows from (3.29) that for $t > \widehat{T}$

$$\dot{V}(y(t)) \leq -\widehat{N}[y_1^2(t) + y_2^2(t)] = -\widehat{N}|y(t)|^2 = -w(|y(t)|) \tag{3.31}$$

Now, we want to find a function u such that $V(y(t)) \geq u(|y(t)|)$. It follows from (3.20), (3.23) and (3.26) that

$$V(y(t)) \geq \frac{1}{\alpha_{21} x_2^*} \{y_1(t) - \ln[1 + y_1(t)]\} + \frac{1}{\alpha_{12} x_1^*} \{y_2(t) - \ln[1 + y_2(t)]\} \tag{3.32}$$

By the Taylor Theorem, we have

$$y_i(t) - \ln[1 + y_i(t)] = \frac{y_i^2(t)}{2[1 + \theta_i(t)]^2} \tag{3.33}$$

where $\theta_i(t) \in (0, y_i(t))$ or $(y_i(t), 0)$ for $i = 1, 2$.

Case1 : If $0 < \theta_i(t) < y_i(t)$ for $i = 1, 2$, then

$$\frac{y_i^2(t)}{[1+y_i(t)]^2} < \frac{y_i^2(t)}{[1+\theta_i(t)]^2} < y_i^2(t) \tag{3.34}$$

By Lemma 3.2, it follows that for $t \geq T$

$$m_i \leq x_i^*[1+y_i(t)] = x_i(t) \leq M_i \quad \text{ for } \; i = 1, 2 \tag{3.35}$$

Then (3.33) implies that

$$\left(\frac{x_i^*}{M_i}\right)^2 y_i^2(t) \leq \frac{y_i^2(t)}{[1+\theta_i(t)]^2} < y_i^2(t) \quad , \; i = 1, 2 \tag{3.36}$$

It follows from (3.31), (3.32) and (3.35) that for $t \geq T$.

$$\begin{aligned}
V(y(t)) \quad &\geq \quad \frac{1}{2\alpha_{21}x_2^*} \cdot \frac{y_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2\alpha_{12}x_1^*} \cdot \frac{y_2^2(t)}{[1+\theta_2(t)]^2} \\
&\geq \quad \frac{1}{2\alpha_{21}x_2^*} \cdot \left(\frac{x_1^*}{M_1}\right)^2 y_1^2(t) + \frac{1}{2\alpha_{12}x_1^*} \cdot \left(\frac{x_2^*}{M_2}\right)^2 y_2^2(t) \\
&\geq \quad \min\left\{\frac{1}{2\alpha_{21}x_2^*}\left(\frac{x_1^*}{M_1}\right)^2 , \; \frac{1}{2\alpha_{12}x_1^*}\left(\frac{x_2^*}{M_2}\right)^2\right\}[y_1^2(t) + y_2^2(t)] \\
&\equiv \quad \widetilde{N}\,|y(t)|^2
\end{aligned}$$

Case2 : If $-1 < y_i(t) < \theta_i(t) < 0$ for $i = 1, 2$, then

$$y_i^2(t) < \frac{y_i^2(t)}{[1+\theta_i(t)]^2} < \frac{y_i^2(t)}{[1+y_i(t)]^2} \tag{3.37}$$

In view of (3.34), (3.36) implies that

$$y_i^2(t) < \frac{y_i^2(t)}{[1+\theta_i(t)]^2} \leq \left(\frac{x_i^*}{m_i}\right)^2 y_i^2(t) \quad , \; i = 1, 2 \tag{3.38}$$

It follows from (3.31), (3.32) and (3.37) that for $t \geq T$

$$
\begin{aligned}
V(y(t)) &\geq \frac{1}{2\alpha_{21}x_2^*}\cdot\frac{y_1^2(t)}{[1+\theta_1(t)]^2}+\frac{1}{2\alpha_{12}x_1^*}\cdot\frac{y_2^2(t)}{[1+\theta_2(t)]^2}\\
&> \frac{1}{2\alpha_{21}x_2^*}\cdot y_1^2(t)+\frac{1}{2\alpha_{12}x_1^*}\cdot y_2^2(t)\\
&\geq \frac{1}{2\alpha_{21}x_2^*}\left(\frac{x_1^*}{M_1}\right)^2 y_1^2(t)+\frac{1}{2\alpha_{12}x_1^*}\left(\frac{x_2^*}{M_2}\right)^2 y_2^2(t)\\
&\geq \widetilde{N}\,[y_1^2(t)+y_2^2(t)]\\
&= \widetilde{N}\,|y(t)|^2
\end{aligned}
$$

Case3 : If $0<\theta_1(t)<y_1(t)$ and $-1<y_2(t)<\theta_2(t)<0$, then it follows from (3.31),(3.32),(3.35) and (3.37) that for $t\geq T$

$$
\begin{aligned}
V(y(t)) &\geq \frac{1}{2\alpha_{21}x_2^*}\cdot\frac{y_1^2(t)}{[1+\theta_1(t)]^2}+\frac{1}{2\alpha_{12}x_1^*}\cdot\frac{y_2^2(t)}{[1+\theta_2(t)]^2}\\
&> \frac{1}{2\alpha_{21}x_2^*}\cdot\left(\frac{x_1^*}{M_1}\right)^2 y_1^2(t)+\frac{1}{2\alpha_{12}x_1^*}\cdot y_2^2(t)\\
&\geq \frac{1}{2\alpha_{21}x_2^*}\left(\frac{x_1^*}{M_1}\right)^2 y_1^2(t)+\frac{1}{2\alpha_{12}x_1^*}\left(\frac{x_2^*}{M_2}\right)^2 y_2^2(t)\\
&\geq \widetilde{N}\,[y_1^2(t)+y_2^2(t)]\\
&= \widetilde{N}\,|y(t)|^2
\end{aligned}
$$

Case4 : If $-1<y_1(t)<\theta_1(t)<0$ and $0<\theta_2(t)<y_2(t)$, then it follows from (3.31),(3.32),(3.35) and (3.37) that for $t\geq T$

$$
\begin{aligned}
V(y(t)) &\geq \frac{1}{2\alpha_{21}x_2^*}\cdot\frac{y_1^2(t)}{[1+\theta_1(t)]^2}+\frac{1}{2\alpha_{12}x_1^*}\cdot\frac{y_2^2(t)}{[1+\theta_2(t)]^2} \\
&> \frac{1}{2\alpha_{21}x_2^*}\cdot y_1^2(t)+\frac{1}{2\alpha_{12}x_1^*}\cdot\left(\frac{x_2^*}{M_2}\right)^2 y_2^2(t) \\
&\geq \frac{1}{2\alpha_{21}x_2^*}\left(\frac{x_1^*}{M_1}\right)^2 y_1^2(t)+\frac{1}{2\alpha_{12}x_1^*}\left(\frac{x_2^*}{M_2}\right)^2 y_2^2(t) \\
&\geq \widetilde{N}\,[y_1^2(t)+y_2^2(t)] \\
&= \widetilde{N}\,|y(t)|^2
\end{aligned}
$$

Let $u(s)=\widetilde{N}s^2$, then u is nonnegative continuous on $[0,\infty),\ u(0)=0,\ u(s)>0$ for $s>0$, and $\lim_{s\to\infty}u(s)=+\infty$. So, by Case1 $\sim$ Case4, we have

$$
V(y(t))\geq u(|y(t)|)\qquad \text{for}\quad t\geq T \tag{3.39}
$$

Thus, the equilibrium point E^* of the system (4.1) is globally asymptotically stable.

4 Examples

In this section, we present several simple examples to illustrate the procedures of applying our results.

Example 4.1 Consider the following system:

$$
\begin{aligned}
\dot{x}_1(t) &= x_1(t)\,[3(1-x_1(t))+x_2(t)] \\
\dot{x}_2(t) &= x_2(t)\left[\frac{3}{2}(1-\frac{x_2(t)}{3})+\frac{x_1(t)}{2}\right]
\end{aligned}
\tag{4.1}
$$

Comparing the system (4.1) with the system (2.1), we get $r_1=3$, $r_2=\frac{3}{2}$, $K_1=1$, $K_2=3$, $\alpha_{21}=1$, $\alpha_{12}=\frac{1}{2}$ and $E^*=(3,6)$. Then we conclude that the unique positive equilibrium point E^* of the system (4.1) is globally asymptotically stable by Theorem 4.1. The trajectory of the system (4.1) is depicted in Figure (4.1).

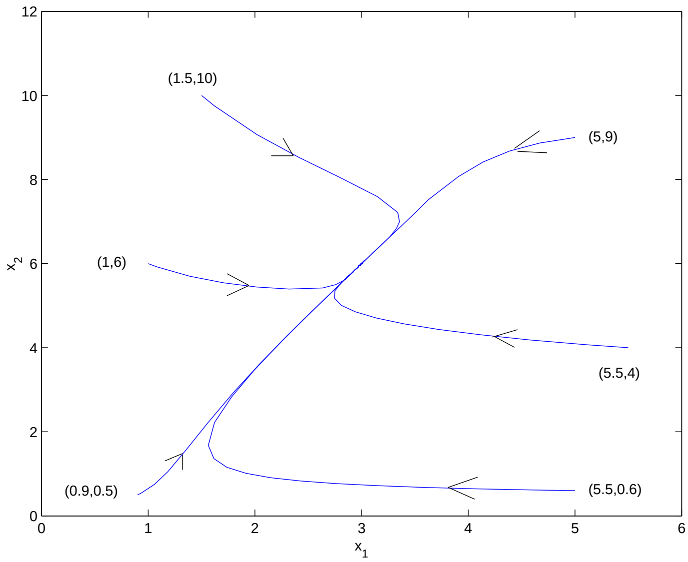

Figure (4.1). The trajectory of the system (4.1).

Example 4.2 Consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[3(1 - x_1(t-\tau)) + x_2(t)\right] \\ \dot{x}_2(t) &= x_2(t)\left[\frac{3}{2}(1 - \frac{x_2(t-\tau)}{3}) + \frac{x_1(t)}{2}\right] \end{aligned} \tag{4.2}$$

Comparing the system (4.2) with the system (3.1), we get $r_1 = 3$, $r_2 = \frac{3}{2}$, $K_1 = 1$, $K_2 = 3$, $\alpha_{21} = 1$ and $\alpha_{12} = \frac{1}{2}$. Moreover, the system (4.2) has the unique positive equilibrium point $E^* = (3,6)$. We find that if the time delay τ is small enough, then the unique positive equilibrium point E^* of the system (4.2) is globally asymptotically stable. The trajectory of the system (4.2) with $\tau = 0.01$ is depicted in Figure 4.2.

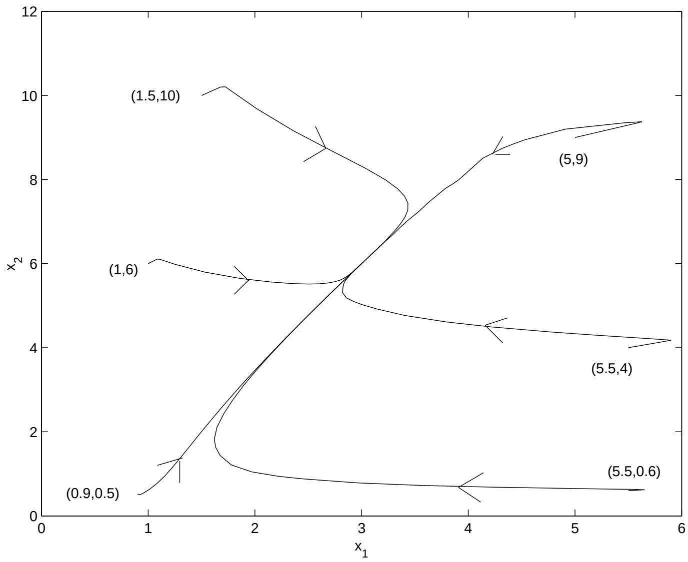

Figure (4.2). The trajectory of the system (4.2) with $\tau = 0.01$.

Example 4.3 Consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[3(1-x_1(t-\tau))+x_2(t)\right] \\ \dot{x}_2(t) &= x_2(t)\left[\frac{3}{2}(1-\frac{x_2(t-\tau)}{3})+\frac{x_1(t)}{2}\right] \end{aligned} \tag{4.3}$$

Comparing the system (4.3) with the system (3.1), we get $r_1 = 3$, $r_2 = \frac{3}{2}$, $K_1 = 1$, $K_2 = 3$, $\alpha_{21} = 1$ and $\alpha_{12} = \frac{1}{2}$. Moreover, the system (5.3) has the unique positive equilibrium point $E^* = (3,6)$. We find that if the time delay τ is not small enough, then the unique positive equilibrium point E^* of the system (4.3) may not be globally asymptotically stable. The trajectory of the system (4.3) with $\tau = 1$ is depicted in Figure 4.3.

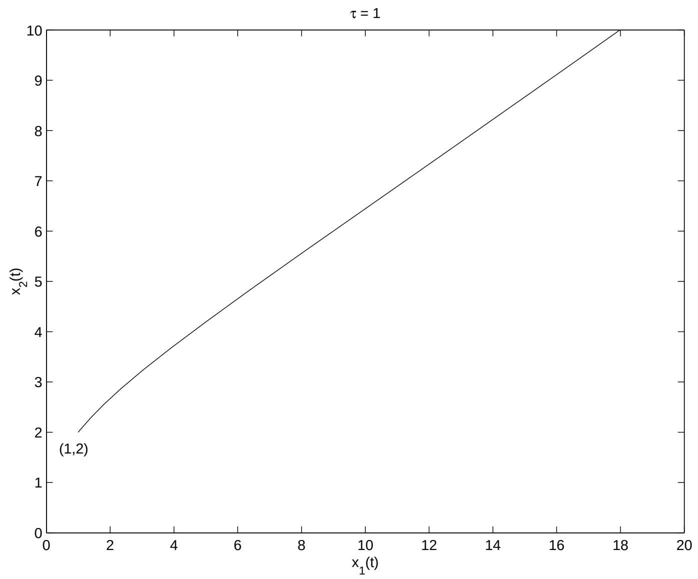

Figure (4.3). The trajectory of the system (4.3) with $\tau = 1$.

5 CONCLUSIONS

In this paper, we have shown that if at least one of the populations have a uniformly bounded, then a Lotka-Volterra mutualistic system will be uniformly persistent. If the corresponding delayed system is uniformly persistent, then the delayed system is globally asymptotically stable. We believe that a Lotka-Volterra mutualistic system with multiple delay described as follows will be an interested topic for future study.

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[r_1(1-\frac{x_1(t-\tau_1)}{K_1})+\alpha_{21}x_2(t-\tau_2)\right] \\ \dot{x}_2(t) &= x_2(t)\left[r_2(1-\frac{x_2(t-\tau_3)}{K_2})+\alpha_{12}x_1(t-\tau_4)\right] \end{aligned} \quad (5.1)$$

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具時滯參數之 Lotka-Volterra 互利共生系統之整體穩定性

徐楓淳　　何肇寶

摘　要

我們分析具時滯參數之 Lotka-Volterra 互利共生系統之整體穩定性。首先，我們利用三種不同的方法分析不具時滯參數之 Lotka-Volterra 互利共生系統之整體穩定性。緊接著，我們分析具時滯參數之 Lotka-Volterra 互利共生系統之整體穩定性。最後，我們用實例及電腦軌跡圖說明之。

關鍵詞：共生系統,時滯參數,整體穩定性,Lyapunov 涵數。