$\mathcal{H}_\infty$ state feedback control of smart beam-plates via the descriptor system approach

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Abstract

Through the calculus of variation for establishing one-player game theory, an optimal control theory and bounded real lemma for descriptor systems are conducted to develop a design method for robust $\mathcal{H}_\infty$ control laws. Thereafter, the developed control methods are applied to treat vibration suppression for beam-plate structures. Numerical simulation is presented to illustrate the design procedure for state feedback controller.

1 Introduction

Recently, the numerical computation on solving Differential-Algebraic Equations (DAEs, also known as descriptor systems, singular systems, generalized state-space systems) is an active area of applied mathematical research [1]. This type of systems arises from circuit and network analysis [2,3], as well as distributed-parameter consideration in other engineering disciplines [3,4]. Examples may include a safe reentry profile for the space shuttle [5], a motion analysis of constrained robot systems [6], the dynamic Leontief model for economic production sectors [7], and the derivation of governing equations for chemical reactors [8] *etc.*. Matured numerical softwares [4,9–11] concerning DAE have been established to provide accurate and reliable solutions for descriptor systems. The direct application of DAE formulation to physical systems becomes feasible and saves great effort for preconditioning particular DAEs by eliminating as many redundant variables as possible.

Lewis [12] proved that the descriptor system is stable and impulse-free if and only if the system matrix pair satisfies the generalized Lyapunov equation (GLE). Cobb [13] also considered the duality

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between controllability and observability. At the same time, many researchers have moved forward to treat linear regulator problems for descriptor systems for minimizing a given quadratic functional. Cobb [14] first gave a necessary and sufficient condition for the existence of an optimal solution to this problem. But variational calculus for descriptor system was derived by Jonckheere [15] until 1988. Some other results on the robustness for linear regulator problems [16, 17] were also obtained. Recently, Kunkel and Mehrmann [18] studied linear regulator problems with time-variant coefficients.

The robust $\mathcal{H}_\infty$ control theory is the most active research field in control theory during the past three decades. Nowadays, this line of research is moving toward establishing control theories for descriptor systems. Solutions of the $\mathcal{H}_\infty$ control problem for descriptor system were given by Takaba *et al.* [19]. They proved the sufficient conditions for the existence of solutions by using the J-spectral factorization. More recently, Masubuchi *et al.* [20] have considered a similar problem by using a matrix inequalities approach. Although, their solutions may be obtained by using LMI numerical tool, but they rely on two generalized algebraic Riccati inequalities (GARI) involving two unknown parameters plus two to-be-determined variables. Katayama and Minamino [21] used the generalized Lyapunov theorem (see Takaba *et al.* [22]) to solve the linear regular problem for which the control law and optimal cost are computed based on the solution of the generalized Riccati differential equation (GRDE) and the generalized algebraic Riccati equation (GARE) for finite- and infinite- horizon cases, respectively. The corresponding GARE solutions for $\mathcal{H}_\infty$ control problems are presented by Wang *et al.* [23]. They gave the necessary and sufficient conditions for the existence of a solution to $\mathcal{H}_\infty$ control in terms of two GARE's. However, the construction of all solutions from center controller is still under development.

Attempts made in this paper are to develop the $\mathcal{H}_\infty$ control theory for descriptor systems through a game theoretical approach which has successfully used to develop the $\mathcal{H}_\infty$ control theory for nonlinear state-space systems [24]. This paper is organized as follows. Section 2 is concerned with the establishment of optimization theories for nonlinear descriptor systems through the use of the calculus of variation. Section 3 presents a unified approach to treat linear regulator control and robust $\mathcal{H}_\infty$ control problems. The bounded real lemma for linear descriptor system is established and the $\mathcal{H}_\infty$ control is then synthesized by using this lemma. In Section 4, the control theory developed here is applied to suppress unacceptable vibration behaviors of the smart beam-plate structures. Section 5 presents some conclusions based on the findings of the study.

2 Optimization Theory for Descriptor Systems

In this section, optimization theories for descriptor systems are established. The optimization process is performed by minimizing a cost functional which is selected to make the descriptor system exhibit a desired type of performances.

The calculus of variation is applied to establish the one-player game theory associated with descriptor systems which is the base for further developing the control theory. We consider the following dynamical system:

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad \mathbf{E}\mathbf{x}(t_0) = \mathbf{E}\mathbf{x}_0 \tag{2.1}$$

where $\mathbf{f}$ satisfies the required regularity conditions. The associated cost functional $J(\mathbf{x}_0, \mathbf{u}, t_0, t_f)$ is given by

$$J(\mathbf{x}_0, \mathbf{u}, t_0, t_f) = \phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} F(\mathbf{x}(t), \mathbf{u}(t), t)dt, \tag{2.2}$$

where $\phi(\mathbf{x}(t), t)$ and $F(\mathbf{x}(t), \mathbf{u}(t), t)$ are sufficiently differentiable real-value functions. And, the final state condition is represented as:

$$\psi(\mathbf{x}(t_f), t_f) = \mathbf{0} \tag{2.3}$$

for a given continuous function ψ. The admissible control set U is defined as

$$U = \left\{ \mathbf{u} \in \mathcal{L}^2(\mathbb{R}, \mathbb{R}^{m\times 1}) \mid \text{the solution of (2.1) satisfy (2.3)} \right\} \tag{2.4}$$

Our purpose is to select an optimal strategy $\mathbf{u}^*(t)$ from U within the interval $[t_0, t_f]$ that drives the plant (2.1) along a trajectory $\mathbf{x}^*(t)$ such that $J(\mathbf{x}_0, \mathbf{u}, t_0, t_f)$ is minimized, and (2.3) is also satisfied.

Since the state variable $\mathbf{x}$ during the minimization of (2.2) must satisfy the two constraints: (2.1) and (2.3), we cannot apply the calculus of variation directly. The common way to treat this problem is by using the method of Lagrange multipier. As point out by Luenberger [25, section 9], a sufficient condition for the Lagrange multiplier to exists is that the constraint have a property known as "regularity". Unfortunately, in the case of the constraint (2.1), regularity is equivalent to complete reachability [26]. Jonckheere [15] does not use Lagrange multiplier for the dynamic constraint; instead he applies the calculus of variations directly by taking a weak variation about the trajectory of $\mathbf{x}$, $\mathbf{E}\dot{\mathbf{x}}$, and $\mathbf{u}$ with additional requirement that this trajectory must be impulsive-free. Otherwise, if the descriptor system (2.1) has impulsive mode, the second term in (2.2) may become infinity. This explains why Jonckheere [15] assumes the trajectory must be impulsive free.

Suppose the system (2.1) is impulsive free, we use the same method in [15] to treat this dynamic constraint. On the other hand, for the static constraint (2.3), we apply the Lagrange multipiler (sometimes called *costate vector*) $\nu \in \mathbb{R}^{\ell \times 1}$ to ajoin it into the cost functional (2.2). Define the *Hamiltonian function* as

$$H(\mathbf{x}(t),\mathbf{u}(t),\lambda(t),t) = F(\mathbf{x}(t),\mathbf{u}(t),t) + \lambda(t)^{\mathrm{T}}\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),t), \tag{2.5}$$

where $\lambda(t) \in \mathbb{R}^{n\times 1}$. The augmented cost functional is then defined by

$$\begin{aligned} J'(\mathbf{x}_0,\mathbf{u},\lambda,\nu,t_0,t_f) &= \phi(\mathbf{x}(t_f),t_f) + \nu^{\mathrm{T}}\psi(\mathbf{x}(t_f),t_f) \\ &\quad + \int_{t_0}^{t_f} \Big[H(\mathbf{x}(t),\mathbf{u}(t),\lambda(t),t) - \lambda^{\mathrm{T}}(t) f(\mathbf{x}(t),\mathbf{u}(t),t) \Big]\, dt. \end{aligned}$$

after using the relationship (2.5). Since the dynamic constraint (2.1) needs to be satisfied and perform the integration by part to eliminate $\dot{\mathbf{x}}$, one arrives at

$$\begin{aligned} J'(\mathbf{x}_0,\mathbf{u},\lambda,\nu,t_0,t_f) &= \Phi(\mathbf{x}(t_f),t_f) - \lambda^{\mathrm{T}}(t_f)\mathbf{E}\mathbf{x}(t_f) + \lambda^{\mathrm{T}}(t_0)\mathbf{E}\mathbf{x}(t_0) \\ &\quad + \int_{t_0}^{t_f} \Big[H(\mathbf{x},\mathbf{u},\lambda,t) + \dot{\lambda}^{\mathrm{T}}\mathbf{E}\mathbf{x} \Big]\, dt, \end{aligned} \tag{2.6}$$

where

$$\Phi(\mathbf{x}(t_f),t_f) = \phi(\mathbf{x}(t_f),t_f) + \nu^{\mathrm{T}}\psi(\mathbf{x}(t_f),t_f). \tag{2.7}$$

In order to solve this minimization problem, we consider the following small perturbations:

$$\mathbf{x}(t) \to \mathbf{x}(t) + \delta\mathbf{x}(t), \quad \mathbf{u}(t) \to \mathbf{u}(t) + \delta\mathbf{u}(t), \quad t_f \to t_f + \delta t_f.$$

The *first* and *second variations* of augmented cost functional be $\delta J'$ and $\delta^2 J'$, respectively, are equal to the linear and quadratic increments in J' due to the variations $\delta\mathbf{x}$, $\delta\mathbf{u}$, and δt_f, i.e., the difference between
$J'(\mathbf{x}_0,\mathbf{u}+\delta\mathbf{u},\lambda,\nu,t_0,t_f+\delta t_f)\big|_{\mathbf{x}=\mathbf{x}+\delta\mathbf{x}}$ and $J'(\mathbf{x}_0,\mathbf{u},\lambda,\nu,t_0,t_f)$.

Since the final state is governed by the function ψ, hence $\delta\mathbf{x}(t_f+\delta t_f)$ and δt_f must also satisfy the following relationship:

$$\delta\psi(\mathbf{x}(t_f+\delta t_f) + \delta\mathbf{x}(t_f+\delta t_f), t_f+\delta t_f) = 0 \tag{2.8}$$

where $\delta\mathbf{x}(t_f+\delta t_f)$ satisfies the following relation:

$$\delta[\mathbf{x}(t_f)] = \frac{d}{d\tau}[\mathbf{x}(t_f+\tau\delta t_f) + \tau\delta\mathbf{x}(t_f+\tau\delta t_f)]\Big|_{\tau=0} = \delta\mathbf{x}(t_f) + \dot{\mathbf{x}}(t_f)\delta t_f \tag{2.9}$$

with $\delta\mathbf{x}_f(t_f)$ denoting the variation of $\mathbf{x}(t)$ when the time t is held fixed and equal to t_f. Thus the variations between the final state, $\delta\mathbf{x}(t_f)$, and final time, δt_f, are linked to each other with the relationship.

After introducing the Lagrange multiplier ν to adjoin (2.3) into the augmented cost functional J', these two perturbations can be considered to variate independently. Using Leibniz's rule, $\delta J'$ and $\delta^2 J$ are computed and represented by

$$\begin{aligned}\delta J' &= \delta t_f \left[\frac{d\Phi}{dt} + F\right]_{t_f} + \delta\mathbf{x}(t_f)^{\mathrm{T}} \left[\frac{\partial\Phi}{\partial\mathbf{x}} - \mathbf{E}^{\mathrm{T}}\lambda\right]_{t_f} \\ &\quad + \int_{t_0}^{t_f} \left[\delta\mathbf{u}^{\mathrm{T}} \frac{\partial H}{\partial\mathbf{u}} + \delta\mathbf{x}^{\mathrm{T}} \left(\mathbf{E}^{\mathrm{T}}\dot{\lambda} + \frac{\partial H}{\partial\mathbf{x}}\right)\right] dt \end{aligned} \tag{2.10}$$

$$\begin{aligned}\delta^2 J' &= \begin{bmatrix} \delta\mathbf{x}(t_f)^{\mathrm{T}} & \delta t_f \end{bmatrix} \begin{bmatrix} \frac{\partial^2\Phi}{\partial\mathbf{x}^2} & \frac{\partial}{\partial\mathbf{x}}\left(\frac{d\Phi}{dt} + F\right) \\ \frac{\partial}{\partial\mathbf{x}}\left(\frac{d\Phi}{dt} + F\right)^{\mathrm{T}} & \frac{d}{dt}\left[\frac{d}{dt}\left(\Phi + \lambda^{\mathrm{T}}\mathbf{E}\mathbf{x}\right) + 2F\right] \end{bmatrix}_{t_f} \begin{bmatrix} \delta\mathbf{x}(t_f) \\ \delta t_f \end{bmatrix} \\ &\quad + 2\delta\mathbf{u}(t_f)^{\mathrm{T}} \left.\frac{\partial H}{\partial\mathbf{u}}\right|_{t_f} + \int_{t_0}^{t_f} \begin{bmatrix} \delta\mathbf{x}^{\mathrm{T}} & \delta\mathbf{u}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 H}{\partial\mathbf{x}^2} & \frac{\partial^2 H}{\partial\mathbf{x}\partial\mathbf{u}} \\ \frac{\partial^2 H}{\partial\mathbf{u}\partial\mathbf{x}} & \frac{\partial^2 H}{\partial\mathbf{u}^2} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x} \\ \delta\mathbf{u} \end{bmatrix} dt. \end{aligned} \tag{2.11}$$

Setting to zero the coefficients of the independent variations $\delta\mathbf{x}$ and $\delta\mathbf{u}$ in (2.10) with some algebraic operations yields the necessary condition for a minimum:

Minimum Principle

$$\begin{array}{ll} \min\limits_{u\in U} H(\mathbf{x},\mathbf{u},\lambda,t), \text{ i.e., } \dfrac{\partial H}{\partial\mathbf{u}} = 0 & \text{for all } t \in [t_0, t_f] \\ \dot{\mathbf{x}} = \dfrac{\partial H}{\partial\lambda}, \quad \mathbf{E}\mathbf{x}(0) = \mathbf{E}\mathbf{x}_0 & \text{equation of motion for } \mathbf{x} \\ \dot{\lambda} = -\dfrac{\partial H}{\partial\mathbf{x}}, \quad \mathbf{E}^{\mathrm{T}}\lambda(t_f) = \dfrac{\partial\Phi}{\partial\mathbf{x}} & \text{equation of motion for } \lambda \\ \left\{ \begin{array}{l} \left.\dfrac{\partial\Phi}{\partial t}\right|_{t_f} + H(\mathbf{x}(t_f),\mathbf{u}(t_f),\lambda(t_f),t_f) = 0 \\ \psi(\mathbf{x}(t_f),t_f) = 0 \end{array} \right. & \text{transversality condition} \end{array} \tag{2.12}$$

Another necessary condition for an extremum of a functional to be a minimum was the nonnegativeness of the second variation, i.e.

$$\delta^2 J' \geq 0 \tag{2.13}$$

From (2.11), after using the fact that $\partial H^*/\partial\mathbf{u}|_{t_f} = 0$, the following two matrices should be nonnegative:

$$\left[\begin{array}{cc} \frac{\partial^2\Phi^*}{\partial\mathbf{x}^2} & \frac{\partial}{\partial\mathbf{x}}\left(\frac{d\Phi^*}{dt}+F^*\right) \\ \frac{\partial}{\partial\mathbf{x}}\left(\frac{d\Phi^*}{dt}+F^*\right)^{\mathrm{T}} & \frac{d}{dt}\left[\frac{d}{dt}\left(\Phi^*+(\lambda^*)^{\mathrm{T}}\mathbf{E}\mathbf{x}^*\right)+2F^*\right] \end{array}\right]_{t_f^*} \geq 0, \tag{2.14}$$

and

$$\left[\begin{array}{cc} \frac{\partial^2 H^*}{\partial\mathbf{x}^2} & \frac{\partial^2 H^*}{\partial\mathbf{x}\partial\mathbf{u}} \\ \frac{\partial^2 H^*}{\partial\mathbf{u}\partial\mathbf{x}} & \frac{\partial^2 H^*}{\partial\mathbf{u}^2} \end{array}\right] \geq 0 \quad \text{for all } t\in[t_0,t_f^*] \tag{2.15}$$

where $\Phi^*=\Phi(\mathbf{x}^*(t),t)$ and $F^*=F(\mathbf{x}^*(t),\mathbf{u}^*(t),t)$. The detail derivation can be found in [29].

3 Robust Control Theory for Descriptor Systems

In this section, we develop the necessary control design method on the basis of the optimization theory developed in section 2. Two types of control problems are considered herein: one is the linear quadratic optimal control problem and the other is the robust $\mathcal{H}_\infty$ state feedback control problem.

3.1 Control Concepts in Linear Descriptor Systems

Let us consider the descriptor system as given by

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}+\mathbf{B_1}\mathbf{w}+\mathbf{B_2}\mathbf{u}, \tag{3.1a}$$

$$\mathbf{z} = \mathbf{C_1}\mathbf{x}+\mathbf{D_{12}}\mathbf{u}, \tag{3.1b}$$

$$\mathbf{y} = \mathbf{C_2}\mathbf{x}, \tag{3.1c}$$

here $\mathbf{x}\in\mathbb{R}^n$ is the descriptor variable, $\mathbf{w}\in\mathbb{R}^q$ is the exogenous input, $\mathbf{u}\in\mathbb{R}^m$ is the control input, $\mathbf{z}\in\mathbb{R}^s$ is the controlled output and $\mathbf{y}\in\mathbb{R}^p$ is the measured output. The matrix $\mathbf{E}\in\mathbb{R}^{n\times n}$ has the rank $r\leq n$ and the other matrices have appropriate sizes. In real systems, the control input $\mathbf{u}(t)$ often subjects to a constraint π. We shall use U to denote the collection of $\mathbf{u}(t)$ subject to constraint π, which is called the admissible control set, and any $\mathbf{u}(t)\in U$ is called admissible control input. For descriptor systems, the admissible control input should be a piecewise sufficiently continuously differentiable function such that the system (3.1) has consistent solution. The constraint π often is a vector function of state, control input, and time t, or $\pi(\mathbf{x}(t),\mathbf{u}(t),t)$.

Definition 3.1

(I) A pencil $s\mathbf{E}-\mathbf{A}$ (or a pair $(\mathbf{E},\mathbf{A})$) is *regular* if $\det(s\mathbf{E}-\mathbf{A})$ is not identically zero.

(II) For regular pencil $s\mathbf{E}-\mathbf{A}$, the finite eigenvalues of $s\mathbf{E}-\mathbf{A}$ is said to be the *finite dynamic modes* of $(\mathbf{E},\mathbf{A})$. If all the finite dynamic modes are stable, we say that $(\mathbf{E},\mathbf{A})$ is *stable*.

(III) For regular pencil $s\mathbf{E}-\mathbf{A}$, we define two modes for the infinite eigenvalues of the $s\mathbf{E}-\mathbf{A}$, which are the zero eigenvalues of the pencil $\mathbf{E}-\lambda\mathbf{A}$. The infinite eigenvalues corresponding to the generalized eigenvectors v with $\mathbf{E}v=0$ are the *nondynamic modes* of $(\mathbf{E},\mathbf{A})$. Suppose that $\mathbf{E}v_1=0$, then the infinite eigenvalues associated with the generalized principal vectors v_k satisfying $\mathbf{E}v_k=\mathbf{A}v_{k-1}$ $(k\geq 2)$ are the *impulsive modes* of $(\mathbf{E},\mathbf{A})$.

(IV) A pencil $(\mathbf{E},\mathbf{A})$ is *admissible* if it is regular and has neither impulsive modes nor unstable finite modes.

Takaba *et. al.* [22] proposed a necessary and sufficient algebraic condition (generalized Lyapunov equation, GLE) for the admissibility of a pair $(\mathbf{E},\mathbf{A})$, which is assumed to be *regular*:

Lemma 3.2 [22] *Suppose that the pencil $s\mathbf{E}-\mathbf{A}$ is regular and that $(\mathbf{E},\mathbf{A},\mathbf{C})$ is impulse observable and finite dynamics detectable, then $(\mathbf{E},\mathbf{A})$ is admissible if and only if there exists a solution $\mathbf{X}$ to Generalized Lyapunov Equation (GLE):*

$$\mathbf{E}^{\mathrm{T}}\mathbf{X}=\mathbf{X}^{\mathrm{T}}\mathbf{E}\geq 0, \tag{3.2a}$$

$$\mathbf{A}\mathbf{X}+\mathbf{X}^{\mathrm{T}}\mathbf{A}+\mathbf{C}^{\mathrm{T}}\mathbf{C}=0. \tag{3.2b}$$

When $(\mathbf{E},\mathbf{A})$ is in SVD coordinate system of the form

$$\mathbf{E}=\begin{bmatrix}\mathbf{I}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\quad \mathbf{A}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}$$

then $(\mathbf{E},\mathbf{A})$ is regular and impulsive free if and only if $\mathbf{A}_{22}$ is invertible (see [26] for the proof).

Since the regularity of the plant happens to be destroyed by feedback input, Masubuchi [20] proposed a matrix inequality condition to state the admissibility of $(\mathbf{E},\mathbf{A})$ without assuming that $(\mathbf{E},\mathbf{A})$ is regular.

Lemma 3.3 ([20], Lemma 2) *A pair* $(\mathbf{E},\mathbf{A})$ *is admissible if and only if there exists* $\mathbf{X}\in\mathbb{R}^{n\times n}$ *such that*

$$\mathbf{E}^{\mathrm{T}}\mathbf{X}=\mathbf{X}^{\mathrm{T}}\mathbf{E}\geq 0, \tag{3.3a}$$

$$\mathbf{A}\mathbf{X}+\mathbf{X}^{\mathrm{T}}\mathbf{A}<0. \tag{3.3b}$$

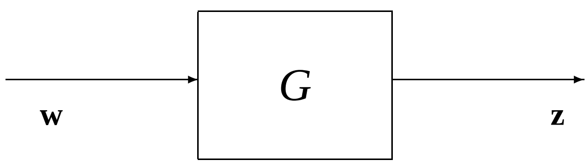
3.2 Bounded Real Lemma

Consider the descriptor system G from (3.1) without control input and measured output devices:

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{w}, \tag{3.4a}$$

$$\mathbf{z} = \mathbf{C}\mathbf{x} \tag{3.4b}$$

where the input-output block diagram of system G is shown as below:



We assume that the system is *regular* and *impulse free* and the $\mathcal{H}_\infty$-norm of system G is defined by

$$\|G\|_\infty=\sup_{\|\mathbf{w}\|_2\neq 0}\frac{\|\mathbf{z}\|_2}{\|\mathbf{w}\|_2}=\gamma_\infty, \quad \mathbf{w},\ \mathbf{z}\in\mathcal{L}_2[0,\infty) \tag{3.5}$$

where the infinite-horizon Lebesque 2-space $\mathcal{L}_2[0,\infty)$ is defined by

$$\mathcal{L}_2[0,\infty)=\left\{\mathbf{f}\,\middle|\,\mathbf{f}:[0,\infty)\rightarrow\mathbb{R}^n \text{ and } \|\mathbf{f}\|_2\triangleq\left(\int_0^\infty\mathbf{f}(t)^{\mathrm{T}}\mathbf{f}(t)dt\right)^{\frac{1}{2}}<\infty\right\}$$

whence all the functions in $\mathcal{L}_2[0,\infty)$ must be stable and converge to zero as $t\rightarrow\infty$. The linear descriptor system (3.4) is said to be γ-*dissipative* if its $\mathcal{H}$-norm is less than γ.

Theorem 3.4 (Bounded Real Lemma)
Consider the system (3.4). The pair $(\mathbf{E},\mathbf{A})$ *is admissible and* $\|G\|_\infty<\gamma$ *if and only if there exists* $\mathbf{X}\in\mathbb{R}^{n\times n}$ *such that*

$$\mathbf{E}^{\mathrm{T}}\mathbf{X}=\mathbf{X}^{\mathrm{T}}\mathbf{E}\geq 0, \tag{3.6a}$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{X}+\mathbf{X}^{\mathrm{T}}\mathbf{A}+\mathbf{C}^{\mathrm{T}}\mathbf{C}+\frac{1}{\gamma^2}\mathbf{X}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{X}=0. \tag{3.6b}$$

Proof: Suppose there exists a quadratic function $V(\xi) = \xi^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\mathbf{X}_T\xi$, $\mathbf{X}_T > 0$, and $\gamma > 0$ such that for all t [27],

$$\frac{d}{dt}V(x) + \mathbf{z}^{\mathrm{T}}\mathbf{z} - \gamma^2\mathbf{w}^{\mathrm{T}}\mathbf{w} \leq 0 \quad \text{for all } \mathbf{x} \text{ and } \mathbf{w} \text{ satisfying (3.4).} \tag{3.7}$$

Then, γ_∞ is less than γ. To show this, we integrate (3.7) from 0 to arbitrary T, with the initial state $\mathbf{x}(0) = \mathbf{0}$, to get

$$V(\mathbf{x}(T)) + \int_0^T \left(\mathbf{z}^{\mathrm{T}}\mathbf{z} - \gamma^2\mathbf{w}^{\mathrm{T}}\mathbf{w}\right) \leq 0. \tag{3.8}$$

since $V(\mathbf{x}(T)) \geq 0$, this implies

$$\frac{\|\mathbf{z}\|_2}{\|\mathbf{w}\|_2} \leq \gamma, \quad \text{for all } \mathbf{w}.$$

Hence,

$$\gamma_\infty = \inf\left\{\gamma \left| \begin{array}{l} \min\limits_{\mathbf{w}\in\mathcal{H}_2} \int_0^T \left(\gamma^2\mathbf{w}(t)^{\mathrm{T}}\mathbf{w}(t) - \mathbf{z}(t)^{\mathrm{T}}\mathbf{z}(t)\right) dt \geq 0, \\ \mathbf{x}, \mathbf{w}, \mathbf{z} \text{ satisfy (3.4) for all } T \end{array} \right.\right\}, \tag{3.9}$$

Thus, $\|G\|_\infty < \gamma_\infty$ if and only if for all T there exists a $V(\mathbf{x}(T) \geq 0$ such that $\gamma^2\|\mathbf{w}\|_2^2 - \|\mathbf{z}\|_2^2 \geq V(\mathbf{x}(T))$ for all $\mathbf{w} \in \mathcal{L}_2[0,\infty)$. This corresponds to a minimization problem with the associated cost function J defined by

$$J(\mathbf{x},\mathbf{w},T) = \frac{1}{2}\int_0^T \left(\gamma^2\mathbf{w}(t)^{\mathrm{T}}\mathbf{w}(t) - \mathbf{z}(t)^{\mathrm{T}}\mathbf{z}(t)\right) dt \tag{3.10}$$

subjected to the dynamic constraint (3.4).

The computation of the $\mathcal{H}_\infty$-norm γ_∞ can be divided into two steps:

1. For any given γ, compute the minimal value of J, i.e. $J^*(\mathbf{x}^*, \mathbf{w}*)$,
2. Determinate γ_∞ which is equal to the smallest γ such that the condition (3.9) holds.

Substituting (3.4b) into (3.9), it yields

$$J(\mathbf{x},\mathbf{w},T) = \frac{1}{2}\int_0^T \left(\gamma^2\mathbf{w}(t)^{\mathrm{T}}\mathbf{w}(t) - \mathbf{x}(t)^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{C}\mathbf{x}\right) dt \tag{3.11}$$

The condition (3.8) for all $\mathbf{w}$ leads to the non-negativity of the matrix

$$W_0 = \begin{bmatrix} -\mathbf{C}^{\mathrm{T}}\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \gamma^2\mathbf{I} \end{bmatrix}$$

Therefore, we can apply the **Minimum Principle** to solve this optimization problem. It gives

$$-\mathbf{E}^{\mathrm{T}}\dot{\mathbf{P}} = \mathbf{P}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{C}^{\mathrm{T}}\mathbf{C} - \frac{1}{\gamma^2}\mathbf{P}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{P}, \quad \mathbf{E}^{\mathrm{T}}\mathbf{P} = \mathbf{P}^{\mathrm{T}}\mathbf{E}, \quad \mathbf{P}(T) = -\mathbf{X}_T, \tag{3.12}$$

and $(\mathbf{E}, \mathbf{A} - \frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{P})$ is stable and impulsive free. Using the fact $\mathbf{x}(0) = 0$, the cost function in (3.11) can be rewritten as

$$\begin{aligned} J(\mathbf{x}, \mathbf{w}, T) &= \frac{1}{2}\int_0^T \left(\gamma^2\|\mathbf{w} - \mathbf{w}^*\|_2^2 - \frac{d}{dt}(\mathbf{x}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\mathbf{P}\mathbf{x})\right) dt \\ &= -\frac{1}{2}\mathbf{x}(T)^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\mathbf{P}(T)\mathbf{x} + \frac{1}{2}\int_0^T \gamma^2\|\mathbf{w} - \mathbf{w}^*\|_2^2 dt \geq \frac{1}{2}V(\mathbf{x}(T)), \quad \forall\, \mathbf{w} \end{aligned}$$

where

$$\mathbf{w}^*(t) = -\frac{1}{\gamma^2}\mathbf{B}^{\mathrm{T}}\mathbf{P}(t)\mathbf{x}(t).$$

Setting $\mathbf{P}(t) = -\mathbf{X}(t)$ where $\mathbf{X}(t)$ is the solution of following GRDE:

$$\begin{aligned} -\mathbf{E}^{\mathrm{T}}\dot{\mathbf{X}} &= \mathbf{X}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{X} + \mathbf{C}^{\mathrm{T}}\mathbf{C} + \tfrac{1}{\gamma^2}\mathbf{X}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{X}, \\ \mathbf{E}^{\mathrm{T}}\mathbf{X} &= \mathbf{X}^{\mathrm{T}}\mathbf{E} \geq 0, \quad \mathbf{X}(T) = \mathbf{X}_T \end{aligned} \tag{3.13}$$

with stable and impulsive free $(\mathbf{E}, \mathbf{A} + \frac{1}{\gamma^2}\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{X})$, we have the minimum of J as

$$J^*(\mathbf{x}^*, \mathbf{w}^*) = \frac{1}{2}\mathbf{x}(T)^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\mathbf{X}_T\mathbf{x}(T) \geq 0 \tag{3.14}$$

or, equivalently,

$$\min_{\mathbf{w}\in\mathcal{H}_2}\int_0^T \left(\gamma^2\mathbf{w}(t)^{\mathrm{T}}\mathbf{w}(t) - \mathbf{z}(t)^{\mathrm{T}}\mathbf{z}(t)\right) dt = V(\mathbf{x}(T)), \tag{3.15}$$

and the worst case perturbation is then given by

$$\mathbf{w}^*(t) = \frac{1}{\gamma^2}\mathbf{B}^{\mathrm{T}}\mathbf{X}(t)\mathbf{x}^*(t). \tag{3.16}$$

For steady-state solution, we set $T \to \infty$, i.e. $\mathbf{X}(t) = \mathbf{X}$, and rewrite (3.13) and (3.16) in terms of $\mathbf{X}$ to get

$$\mathbf{X}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{X} + \mathbf{C}^{\mathrm{T}}\mathbf{C} + \frac{1}{\gamma^2}\mathbf{X}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{X} = 0, \quad \mathbf{E}^{\mathrm{T}}\mathbf{X} = \mathbf{X}^{\mathrm{T}}\mathbf{E} \geq 0, \tag{3.17}$$

and

$$\mathbf{w}^*(t) = \frac{1}{\gamma^2}\mathbf{B}^{\mathrm{T}}\mathbf{X}\mathbf{x}^*(t), \tag{3.18}$$

where $\mathbf{x}^*$ is the trajectory of the following stable and impulsive-free closed-loop system

$$\mathbf{E}\dot{\mathbf{x}}(t) = (\mathbf{A} + \frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{X})\mathbf{x}(t). \tag{3.19}$$

The solution of GARE (3.17) can then be expressed in terms of the Hamiltonian matrix $\mathbf{H}$:

$$\mathbf{P} = Ric(\mathbf{E}, \mathbf{H}), \quad \mathbf{H} = \begin{bmatrix} \mathbf{A} & \frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^{\mathrm{T}} \\ -\mathbf{C}^{\mathrm{T}}\mathbf{C} & -\mathbf{A}^{\mathrm{T}} \end{bmatrix}. \tag{3.20}$$

□

We need the following lemma to compute the $\mathcal{H}_\infty$-norm:

Lemma 3.5 *[23] Suppose that* $(\mathbf{E}, \mathbf{A})$ *is impulse-free,* $\mathbf{Q} = \mathbf{Q}^{\mathrm{T}}$, $\mathbf{R} \geq 0$*, and the pencil* $s[\mathbf{E}\ \ \mathbf{0}] + [\mathbf{A}\ \ \mathbf{R}]$ *has full row rank on the imaginary axis. Furthermore, suppose that GARI:*

$$\begin{cases} \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}^{\mathrm{T}}\mathbf{A} + \mathbf{Q} + \mathbf{P}^{\mathrm{T}}\mathbf{R}\mathbf{P} < 0 \\ \mathbf{E}^{\mathrm{T}}\mathbf{P} = \mathbf{P}^{\mathrm{T}}\mathbf{E} \end{cases} \tag{3.21}$$

has a solution $\mathbf{P}$ *with* $\mathbf{E}^{\mathrm{T}}\mathbf{P} = \mathbf{P}^{\mathrm{T}}\mathbf{E} \geq 0$*. Under these condition, the Hamiltonian pencil:*

$$s\begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} \mathbf{A} & \mathbf{R} \\ -\mathbf{Q} & -\mathbf{A}^{\mathrm{T}} \end{bmatrix} \triangleq s\bar{\mathbf{E}} - \bar{\mathbf{H}} \tag{3.22}$$

has no pure imaginary zeros and $(\bar{\mathbf{E}}, \bar{\mathbf{H}})$ *is impulse-free.*

For our problem, the Hamiltonian function $\mathbf{H}$ is defined by (3.20). The computation of γ_∞ is determined by the smallest γ such that the corresponding $s\bar{\mathbf{E}} - \mathbf{H}$ has no pure imaginary zeros.

3.3 $\mathcal{H}_\infty$ State Feedback Controller Synthesis

In the following analysis, the measure output equation is ignored. Setting $\mathbf{C_1} = \mathbf{C}$ and $\mathbf{D_{12}} = \mathbf{D}$ for simplicity, the system from (3.1) is then described by

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B_1}\mathbf{w} + \mathbf{B_2}\mathbf{u} \tag{3.23a}$$

$$\mathbf{z} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{3.23b}$$

When no control input is introduced, the system (3.23) becomes

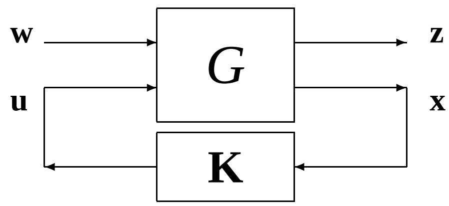
$$G: \begin{array}{c} \mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B_1}\mathbf{w} \\ \mathbf{z} = \mathbf{C}\mathbf{x} \end{array} \tag{3.24}$$

here G represents the uncontrolled system with $\|\mathbf{z}\|_2 \geq \gamma\|\mathbf{w}\|_2$. We can test the dissipativity of this system by using bounded real lemma (Theorem 3.4).

If it is not γ-dissipative, we may adjust the system properties via a designed controller **u** of the form

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \tag{3.25}$$

such that the closed loop system is γ-dissipative, and this controlled system with $\|\mathbf{z}\|_2 \leq \gamma\|\mathbf{w}\|_2$ is shown as the following block diagram:



The purpose of this subsection is devoted to design the control law **K** in (3.25) such that the closed loop system is γ-dissipative, i.e. the exogenous input **w** and the system output **z** satisfy the following relationship:

$$\int_0^T \mathbf{z}(t)^{\mathrm{T}}\mathbf{z}(t)dt \leq \gamma^2 \int_0^T \mathbf{w}(t)^{\mathrm{T}}\mathbf{w}(t)dt \tag{3.26}$$

which implies the following inequality:

$$\min_{\mathbf{u}}\max_{\mathbf{w}} \int_0^T (\mathbf{z}(t)^{\mathrm{T}}\mathbf{z} - \gamma^2\mathbf{w}(t)^{\mathrm{T}}\mathbf{w}(t))dt \leq 0$$

or, equivalently,

$$\max_{\mathbf{u}}\min_{\mathbf{w}} \int_0^T (\mathbf{z}(t)^{\mathrm{T}}\mathbf{z} - \gamma^2\mathbf{w}(t)^{\mathrm{T}}\mathbf{w}(t))dt \geq 0 \tag{3.27}$$

Thus, the $\mathcal{H}_\infty$ control problem is a two-player game theory. One player is **w** standing for the effect of the environment; the other player is **u** standing for the wisdom of the designer to overcome the defects or drawbacks.

Theorem 3.6 *Given the system (3.23), suppose that Assumption 2 holds and* **X** *be the solution of the GARE*

$$\begin{aligned}
-\mathbf{E}^{\mathrm{T}}\dot{\mathbf{X}} &= \mathbf{X}^{\mathrm{T}}[\mathbf{A}+\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}] + [\mathbf{A}+\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}]^{\mathrm{T}}\mathbf{X} \\
&\quad +\mathbf{C}^{\mathrm{T}}(\mathbf{I}-\mathbf{D}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}})\mathbf{C} + \mathbf{X}^{\mathrm{T}}(\tfrac{1}{\gamma^2}\mathbf{B_1}\mathbf{B_1}^{\mathrm{T}} - \mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{B_2}^{\mathrm{T}})\mathbf{X} \\
\mathbf{E}^{\mathrm{T}}\mathbf{X} &= \mathbf{X}^{\mathrm{T}}\mathbf{E} \geq 0,
\end{aligned} \tag{3.28}$$

such that $(\mathbf{E}, \mathbf{A} - \mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}(\mathbf{B_2}^{\mathrm{T}}\mathbf{X} + \mathbf{D}^{\mathrm{T}}\mathbf{C}))$ *are stable and impulse free. Then an state feedback control law is given by*

$$\mathbf{u} = -\mathbf{K}_{\infty}\mathbf{x}, \quad \mathbf{K}_{\infty} = (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}(\mathbf{D}^{\mathrm{T}}\mathbf{C} + \mathbf{B_2}^{\mathrm{T}}\mathbf{X}) \tag{3.29}$$

such that the closed-loop system

$$\mathbf{E}\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B_2}\mathbf{K}_{\infty})\mathbf{x} + \mathbf{B_1}\mathbf{w} \tag{3.30a}$$

$$\mathbf{z} = (\mathbf{C} - \mathbf{D}\mathbf{K}_{\infty})\mathbf{x} \tag{3.30b}$$

is γ*-dissipative.*

Proof: Define the cost function for this optimization problem as

$$\begin{aligned} J(\mathbf{u}, \mathbf{w}) &= \frac{1}{2}\int_0^T (\gamma^2 \mathbf{w}(t)^{\mathrm{T}}\mathbf{w}(t) - \mathbf{z}(t)^{\mathrm{T}}\mathbf{z}(t))dt \\ &= \frac{1}{2}\int_0^T \left(\gamma^2 \mathbf{w}^{\mathrm{T}}\mathbf{w} - (\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u})^{\mathrm{T}}(\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u})\right) dt \end{aligned} \tag{3.31}$$

after using the relation (3.23b) and the corresponding Hamiltonian function H is given by

$$\begin{aligned} H &= \frac{1}{2}\gamma^2 \mathbf{w}^{\mathrm{T}}\mathbf{w} - \frac{1}{2}(\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u})^{\mathrm{T}}(\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}) + \lambda^{\mathrm{T}}(\mathbf{A}\mathbf{x} + \mathbf{B_1}\mathbf{w} + \mathbf{B_2}\mathbf{u}) \\ &= \frac{1}{2}\gamma^2 \mathbf{w}^{\mathrm{T}}\mathbf{w} - \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{C}\mathbf{x} - \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{D}\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{C}\mathbf{x} + \lambda^{\mathrm{T}}(\mathbf{A}\mathbf{x} + \mathbf{B_1}\mathbf{w} + \mathbf{B_2}\mathbf{u}) \end{aligned} \tag{3.32}$$

The optimal control strategy $\mathbf{u}^*$ and the worst case disturbance $\mathbf{w}^*$ are computed according to $\frac{\partial H}{\partial \mathbf{u}^*} = 0$ and $\frac{\partial H}{\partial \mathbf{w}^*} = 0$, respectively, in which

$$\frac{\partial H}{\partial \mathbf{u}} = -\mathbf{D}^{\mathrm{T}}\mathbf{D}\mathbf{u} - \mathbf{D}^{\mathrm{T}}\mathbf{C}\mathbf{x} + \mathbf{B_2}^{\mathrm{T}}\lambda = \mathbf{0}$$

implies

$$\mathbf{u}^* = (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\left(\mathbf{B_2}^{\mathrm{T}}\lambda - \mathbf{D}^{\mathrm{T}}\mathbf{C}\mathbf{x}\right) \tag{3.33}$$

and

$$\frac{\partial H}{\partial \mathbf{w}} = \gamma^2\mathbf{w} + \mathbf{B_1}^{\mathrm{T}}\lambda = 0$$

implies

$$\mathbf{w}^* = -\frac{1}{\gamma^2}\mathbf{B_1}^{\mathrm{T}}\lambda \tag{3.34}$$

Putting (3.33) and (3.34) back into (3.32) yields

$$
\begin{aligned}
H(\mathbf{x},\lambda,\mathbf{u}^*,\mathbf{w}^*,t) &= \frac{1}{2}\frac{1}{\gamma^2}\lambda^{\mathrm{T}}\mathbf{B_1}\mathbf{B_1^{\mathrm{T}}}\lambda-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{C}\mathbf{x} \\
&\quad -(\lambda^{\mathrm{T}}\mathbf{B_2}-\mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{D})(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}\mathbf{x} \\
&\quad -\frac{1}{2}(\lambda^{\mathrm{T}}\mathbf{B_2}-\mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{D})(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{D}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}(\mathbf{B_2^{\mathrm{T}}}\lambda-\mathbf{D}^{\mathrm{T}}\mathbf{C}\mathbf{x}) \\
&\quad +\lambda^{\mathrm{T}}\mathbf{A}\mathbf{x}-\frac{1}{\gamma^2}\lambda^{\mathrm{T}}\mathbf{B_1}\mathbf{B_1^{\mathrm{T}}}\lambda+\lambda^{\mathrm{T}}\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}(\mathbf{B_2^{\mathrm{T}}}\lambda-\mathbf{D}^{\mathrm{T}}\mathbf{C}\mathbf{x}) \\
&= -\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{C}[\mathbf{I}-\mathbf{D}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}]\mathbf{C}\mathbf{x} \\
&\quad +\lambda^{\mathrm{T}}[\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{B_2^{\mathrm{T}}}-\frac{1}{\gamma^2}\mathbf{B_1}\mathbf{B_1^{\mathrm{T}}})\lambda \\
&\quad +\lambda^{\mathrm{T}}[\mathbf{A}+\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}]\mathbf{x} \qquad (3.35)
\end{aligned}
$$

Let $\mathbf{P}(t)$ be the solution of the following GRDE:

$$
\begin{aligned}
-\mathbf{E}^{\mathrm{T}}\dot{\mathbf{P}} &= \mathbf{P}^{\mathrm{T}}[\mathbf{A}+\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}]+[\mathbf{A}+\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}]^{\mathrm{T}}\mathbf{P} \\
&\quad -\mathbf{C}^{\mathrm{T}}(\mathbf{I}-\mathbf{D}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}})\mathbf{C}+\mathbf{P}^{\mathrm{T}}(\tfrac{1}{\gamma^2}\mathbf{B_1}\mathbf{B_1^{\mathrm{T}}}-\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{B_2^{\mathrm{T}}})\mathbf{P} \qquad (3.36) \\
\mathbf{E}^{\mathrm{T}}\mathbf{P} &= \mathbf{P}^{\mathrm{T}}\mathbf{E}
\end{aligned}
$$

such that $(\mathbf{E},\mathbf{A}-\frac{1}{\gamma^2}\mathbf{B_1}\mathbf{B_1^{\mathrm{T}}}\mathbf{P}+\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}(\mathbf{B_2^{\mathrm{T}}}\mathbf{P}-\mathbf{D}^{\mathrm{T}}\mathbf{C}))$ is stable and impulsive free. From (3.23a) and (3.36), we have

$$
\begin{aligned}
\frac{d}{dt}\mathbf{x}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\mathbf{P}\mathbf{x} &= \dot{\mathbf{x}}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\mathbf{P}\mathbf{x}+\mathbf{x}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\dot{\mathbf{P}}+\mathbf{x}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}} \\
&= \mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{C}\mathbf{x}+\mathbf{x}^{\mathrm{T}}[2\mathbf{P}^{\mathrm{T}}\mathbf{B_1}\mathbf{w}+\frac{1}{\gamma^2}\mathbf{B_1}\mathbf{B_1^{\mathrm{T}}}\mathbf{P}\mathbf{x}] \\
&\quad +\mathbf{x}^{\mathrm{T}}[2\mathbf{P}^{\mathrm{T}}\mathbf{B_2}\mathbf{u}-2\mathbf{C}^{\mathrm{T}}\mathbf{D}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{B_2^{\mathrm{T}}}\mathbf{P}-\mathbf{C}^{\mathrm{T}}\mathbf{D}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}]\mathbf{x}
\end{aligned}
\qquad (3.37)
$$

Substituting (3.37) into (3.31), it yields

$$
\begin{aligned}
J(\mathbf{x},\mathbf{w}) &= \frac{1}{2}\int_0^T\left(\gamma^2\|\mathbf{w}-\mathbf{w}^*\|_2^2-(\mathbf{u}-\mathbf{u}^*)^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{D}(\mathbf{u}-\mathbf{u}^*)-\frac{d}{dt}(\mathbf{x}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\mathbf{P}\mathbf{x})\right)dt \\
&= -\frac{1}{2}\mathbf{x}(T)^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\mathbf{P}(T)\mathbf{x}+\frac{1}{2}\int_0^T\left(\gamma^2\|\mathbf{w}-\mathbf{w}^*\|_2^2-\|\mathbf{D}(\mathbf{u}-\mathbf{u}^*)\|_2^2\right)dt,
\end{aligned}
\qquad (3.38)
$$

where $\mathbf{u}^*$ and $\mathbf{w}^*$ are

$$
\mathbf{u}^*(t)=(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\left(\mathbf{B_2^{\mathrm{T}}}\mathbf{P}-\mathbf{D}^{\mathrm{T}}\mathbf{C}\right)\mathbf{x}(t) \qquad (3.39)
$$

and

$$\mathbf{w}^*(t) = -\frac{1}{\gamma^2}\mathbf{B_1}^{\mathrm{T}}\mathbf{P}\mathbf{x}(t), \tag{3.40}$$

respectively. Comparing (3.39) and (3.40) with (3.33) and (3.34), respectively, the following relationship holds for optimal values of λ^* and $\mathbf{x}^*$, i.e.,

$$\lambda^*(t) = \mathbf{P}\mathbf{x}^*(t) \tag{3.41}$$

Thus, the optimal value of the cost function is given by

$$J(\mathbf{u}^*,\mathbf{w}^*) = \max_{\mathbf{u}}\min_{\mathbf{w}} J(\mathbf{u},\mathbf{w}) = -\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\mathbf{P}(t)\mathbf{x} \geq 0 \tag{3.42}$$

which implies $\mathbf{P}(t) \leq 0$. Setting $\mathbf{X}(t) = -\mathbf{P}(t)$, it follows

$$\begin{aligned}
-\mathbf{E}^{\mathrm{T}}\dot{\mathbf{X}} &= \mathbf{X}^{\mathrm{T}}[\mathbf{A}+\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}] + [\mathbf{A}+\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}]^{\mathrm{T}}\mathbf{X} \\
&\quad +\mathbf{C}^{\mathrm{T}}(\mathbf{I}-\mathbf{D}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}})\mathbf{C} + \mathbf{X}^{\mathrm{T}}(\tfrac{1}{\gamma^2}\mathbf{B_1}\mathbf{B_1}^{\mathrm{T}} - \mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{B_2}^{\mathrm{T}})\mathbf{X} \\
\mathbf{E}^{\mathrm{T}}\mathbf{X} &= \mathbf{X}^{\mathrm{T}}\mathbf{E} \geq 0,
\end{aligned} \tag{3.43}$$

such that $(\mathbf{E}, \mathbf{A}+\frac{1}{\gamma^2}\mathbf{B_1}\mathbf{B_1}^{\mathrm{T}}\mathbf{X}-\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}(\mathbf{B_2}^{\mathrm{T}}\mathbf{X}+\mathbf{D}^{\mathrm{T}}\mathbf{C}))$ is stable and impulsive free. And the optimal cost function becomes

$$J^*(\mathbf{u}^*,\mathbf{w}*) = \frac{1}{2}\mathbf{x}(T)^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\mathbf{X}(T)\mathbf{x}(T) \geq 0 \tag{3.44}$$

i.e. the closed loop system (3.24) is γ-dissipative.

Next, let $T \to \infty$, we have $\mathbf{X}(t) \to \mathbf{X}$ and $\dot{\mathbf{X}} = 0$ Thus $\mathbf{X}$ is the steady state solution of (3.43) which yields

$$\mathbf{X} = Ric(\mathbf{E},\mathbf{H}), \quad \mathbf{H} = \begin{bmatrix} \mathbf{A}+\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C} & \frac{1}{\gamma^2}(\mathbf{B_1}\mathbf{B_1}^{\mathrm{T}} - \mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{B_2}^{\mathrm{T}}) \\ -\mathbf{C}^{\mathrm{T}}(\mathbf{I}-\mathbf{D}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}})\mathbf{C} & -(\mathbf{A}+\mathbf{B_2}(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C})^{\mathrm{T}} \end{bmatrix} \tag{3.45}$$

The optimal control $\mathbf{u}^*$ and the worst case disturbance $\mathbf{w}^*$ are given by

$$\mathbf{u}^* = (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}(\mathbf{D}^{\mathrm{T}}\mathbf{C}+\mathbf{B_2}^{\mathrm{T}}\mathbf{X})\mathbf{x}^*$$

and

$$\mathbf{w}^* = \frac{1}{\gamma^2}\mathbf{B_1}^{\mathrm{T}}\mathbf{X}\mathbf{x}^*$$

respectively, where $\mathbf{x}^*$ is the optimal trajectory of the closed-loop system from (3.23) and (3.39) after substituting $\mathbf{u} = \mathbf{u}^*$ and $\mathbf{w} = \mathbf{w}^*$. When there exists a γ, say γ_∞, such that $J(\mathbf{u}^*, \mathbf{w}^*) = 0$, then the saddle point condition

$$J(\mathbf{u}, \mathbf{w}^*) \leq J(\mathbf{u}^*, \mathbf{w}^*) = 0 \leq J(\mathbf{u}^*, \mathbf{w})$$

is satisfied. This type of game is called zero-sum differential game which asserts the best strategies are used by both players. The corresponding controller is called the optimal controller. On the other hand, the controller is called suboptimal controller. □

4 Example: Vibration Suppression of Smart Beam-Plates

In order to study the effect on controlling the undesirable vibration phenomena, we consider a laminated beam-plate structure bonded with a pair of sensor and actuator made of PVDF materials under the influence of a impact force $P(t)$ acting on $x = L/2$ at $t = 0$. In addition, both ends of this structure are hinged with rotational restraints with spring constants $\bar{K_0} = \bar{K_1} = 1000$. The schematic diagram for the structure is shown in Fig. 1. The material properties are listed in Table 4 [28]. The mass per unit length r and effective bending stiffness D are 0.0773998 and 2.0122815, respectively.

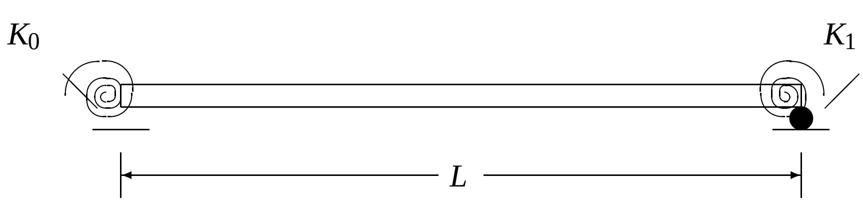


Figure 1: The configuration of a hinged beam-plate with rotational restraints.

After applying the technique from [29, Chapter 4], the descriptor model of this beam-plate structure is then given by

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B_1}P(t) + \mathbf{B_2}V_a(t) \quad (4.1a)$$

$$\mathbf{z}(t) = \mathbf{C}\mathbf{x}(t) \quad (4.1b)$$

$$(4.1c)$$

Table 1: Material properties for a typical smart beam-plate

I. PVDF material

piezoelectric strain constant	d_{31}	$23.0 \times 10^{-12}\ m/Volt$
piezoelectric stress constant	g_{31}	$216.0 \times 10^{-3}\ Volt \cdot m/N$
electro-mechanical coupling factor	k_{31}	0.12
dielectric constant	k_{3t}	12.0
Young's Modulus	E_{13}	$64.5 \times 10^{9}\ N/m^2$
density	ρ_s,ρ_a	$1800.0\ Kg/m^3$
Kelvin-Voigt damping	C_{Ds},C_{Da}	$2.010 \times 10^{-5}\ sec \cdot N/m^2$
thickness	t_s,t_a	$0.0254\ cm$
width	W_s,W_a	$2.03\ cm$

II. Laminated material

Young's modulus	E_b	$73.0 \times 10^{9}\ N/m^2$
density	ρ_b	$1240.0\ Kg/m^3$
Kelvin-Voigt damping	C_{Db}	$1.378 \times 10^{-5}\ sec \cdot N/m^2$
viscous-damping	$c(t)$	$0.0\ sec \cdot N/m^2$
thickness	t_b	$0.16\ cm$
width	W_b	$2.03\ cm$
length	L	$28.4\ cm$

where $P(t)$ and V_a denote the impact force and input voltage of piezo-actuator. We don't include the sensor equation here. Some typical values of the corresponding system matrices $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B_1}$, $\mathbf{B_2}$, and $\mathbf{C}$ with uniform sensor and actuator are shown in Table 2 for when $N = 5$ for illustration.

The state feedback controller is designed by using Theorem 3.6. Firstly, we choose the value for γ to be 1.0. If there is no solution $\mathbf{X}$ exist for the GARE (3.43), then we may increase the value of γ; otherwise, we decrease this value until a solution is found. Fig. 2 shows a satisfactory control effect to suppress unacceptable vibration for $N = 6$.

(a) before control (b) after control

($\gamma_\infty = 7.4815$, $\omega_\infty = 729$) ($\gamma_\infty = 0.1804$, $\omega_\infty = 984$)

Figure 2: Vibration suppression effect on a hinged beam-plate under the influence of concentrated load 5 $\delta(t)\delta(x - L/2)$

The corresponding control gain matrix is given by

$$\mathbf{K} = \begin{bmatrix} 10001 & 7560 & 9998 & 3303 & 6654 & 6283 & 8497 & 6857 & 8782 & 684 & 5608 & 6624 & 7264 & 1985 \end{bmatrix}$$

Simultaneously, the system's $\mathcal{H}_\infty$ norm is reduced from 7.8415 to 0.1804 after introducing control and the corresponding frequency ω_∞ for the occurrence of the $\mathcal{H}_\infty$ norm is also moved from 729 to a higher value 984. The result shows that the state feedback control theory developed in this study is effective and adequate.

5 Conclusions

This paper has presented the $\mathcal{H}_\infty$ state feedback control theory to treat the dynamic response and control of descriptor systems. The one-player and two-player game theories for descriptor systems have been derived. Linear state-feedback $\mathcal{H}_\infty$ control theory has been conducted by using bounded real lemma. On the basis of theoretical analysis without experiment, simulation results presented in the paper suggest that this approach for smart structure control is effective. And we also recognized that

Table 2: Typical values of system matrices $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B_1}$, $\mathbf{B_2}$, and $\mathbf{C}$ when $N = 5$

$$\mathbf{E} = \begin{bmatrix} 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ -97.41 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 62. & -6.3 \\ 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & -1558.55 & 0. & 0. & 0. & 0. & 0. & -496.1 & -12.6 \\ 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & -7890.14 & 0. & 0. & 0. & 1647.3 & -18.8 \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & -24936.73 & 0. & -3968.8 & -25.1 \\ -31.0 & 0. & 248.1 & 0. & -837.2 & 0. & 1984.4 & 0. & 592.2 & 1.0 \\ -3.1 & 0. & -6.3 & 0. & -9.4 & 0. & -12.6 & 0. & -1.0 & 0. \end{bmatrix}$$

$$\mathbf{B_1} = \begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 1. & 0. & 0. & 0. & 0. & 2. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 1. & 0. & 0. & -2. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 1. & 0. & 0. & -1. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. \end{bmatrix}$$

$$\mathbf{B_2} = \begin{bmatrix} 0. & 0. & 0. & 0.0047289 & 0. & 0.0072934 & 0. & 0.0189158 & 0. & 0.000094 \end{bmatrix}$$

$$\mathbf{C_1} = \begin{bmatrix} 14.70 & 0. & 29.41 & 0. & 44.11 & 0. & 58.82 & 0. & 0. & 0. \end{bmatrix}$$

the development of a good and reliable software to solve the generalized Riccati differential equation and algebraic Riccati equation is a key issue to the successful applications of control theories for descriptor systems to practical problems.

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