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A New Approach on the Balanced Realization of Linear Time-Invariant Systems

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Abstract

This paper deals with the balanced realization of linear time invariant systems using an one parameter optimization technique through a variational approach . The Hankel norm is established first. As balanced coordinate vectors are parallel to Hankel singular vectors, a new algorithm for balanced realization by a newly defined balanced map is obtained without performing the Cholesky factorization to the controllability or observability gramians. A simple example is presented for illustrative purposes.

Keywords: Hankel norm, parameter optimization, variational approach, balancing, LTI system

1 Introduction

The design of a controller for a high-order plant through numerical procedures such as finite-element methods is usually difficult and expensive. Therefore, to find a suitable approximate design process for controlling high-order models become a crucial issue. Hankel norm approximation method is one of the popular approaches for this purpose. Hankel norm model reduction in state-space as well as frequency-domain approaches have been addressed in many papers, e.g. [2, 5, 6], [8-9]. The key point of the Hankel approximation is to establish the Hankel norm by dilating the system to be all-pass in order to obtain a balanced realization. The idea of the balancing was first proposed by Moore[7]. Then Glover[3] introduced a theory for the balanced realization and optimal Hankel approximation of multivariable system, in which the Cholesky factorization of the controllability gramian is used. In this paper, we define a balanced state map, and develop an algorithm accordingly for constructing a transformation matrix to perform the balanced realization without using the Cholesky factorization of controllability or observability functions.

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Firstly, we review the algorithm of Glover[3]. After defining the system's Hankel operator, and the controllability and observability functions, the Hankel norm of the system is computed by solving a minimization problem for the full time domain on the basis of a variational principle. The separation property at the present time to split the time domain into past and future time ranges is not involved. The continuity of the Lagrange multiplier at the current time gives the same relationship between the Hankel singular value and Hankel singular vector in the Hankel operator approach. Thereafter, we derive the transformation matrix and establish a new algorithm for the balanced realization. A simple example of performing the balanced realization is presented for illustrating the new computational algorithm.

2 Background

In this section, we review some basic properties related to the balancing of linear system. Let G be a stable system governed by a differential equation of the form:

$$G: \begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bu(t), \quad x(0)=x_0 \qquad (1.a) \\ y(t) & = & Cx(t) \qquad (1.b) \end{array}$$

with $t \in (-\infty,+\infty)$, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{m\times n}$ are matrices of continuous real value functions in which $Re\ \lambda_i(A) < 0$, where $\lambda_i(A)$ is the i-th eigenvalue of A, and $\mathbb{R}$ is the set of all real numbers.

The triple of matrices (A,B,C) is called a realization of the system. If this system is both controllable and observable, then the pair (A,B) is controllable and (C,A) is observable. The triple (A,B,C) is then called minimal realization. It is usually convenient to assume that the system G is relaxed in the infinitely remote pass, i.e., $\lim_{t\to-\infty} x(t) = 0$. Hence, there exists a certain control function $u \in L^2((-\infty,0])$ which can drive the system from rest to the current state x_0,

$$x_0 = \int_{-\infty}^{0} e^{-A\tau}Bu(\tau)\, d\tau \qquad (2)$$

2.1 Hankel operators, Hankel singular values and Hankel norm

Let L_+^2 and L_-^2 denote $L^2([0,+\infty))$ and $L^2((-\infty,0])$, respectively. The Hankel operator Γ_G of the system G is defined as

$$\Gamma_G: \ L^2((-\infty,0]) \to L^2([0,\infty)) \ : \ u \mapsto y$$

where

$$y(t) \stackrel{\triangle}{=} (\Gamma_G u)(t) = Ce^{At}\int_{-\infty}^{0} e^{-A\tau}Bu(\tau)\, d\tau \qquad (3)$$

Thus, the Hankel operator is an operator that maps the pass control input function on L^2_- space into the future output response function on L^2_+. The adjoint operator of the Hankel operator Γ^*_G is defined as

$$\Gamma^*_G: \ L^2([0,\infty)) \to L^2((-\infty,0]) \ \ : \ y \mapsto u$$

and

$$u(t) \overset{\triangle}{=} (\Gamma^*_G y)(t) = \int_0^\infty B^{\mathrm{T}} e^{A^{\mathrm{T}}(-t+\tau)} C^{\mathrm{T}} y(\tau)\ d\tau \tag{4}$$

in which Γ^*_G is related to Γ_G by

$$< \Gamma^*_G y, u >_{L^2_-} = < y, \Gamma_G u >_{L^2_+} \tag{5}$$

where $< \cdot,\cdot >_{\mathcal{H}}$ denotes the inner product on Hilbert space $\mathcal{H}$ which may be L^2_+ or L^2_-, and the superscript T denotes transpose of the matrix.

Let (u,v) be a Hilbert Schmidt pair corresponding to the Hankel singular value σ of Γ_G, then

$$\Gamma_G u = \sigma v, \qquad \Gamma^*_G v = \sigma u \tag{6}$$

where u is the left Hankel singular vector, and v is the right Hankel singular vector. The singular value and vector are computed by using the following lemma:

Lemma 2.1 *Suppose that there are two positive matrices P and Q for the system's controllability and observability gramians, respectively, which are the solutions of the following Lyapunov equations:*

$$AP + PA^{\mathrm{T}} + BB^{\mathrm{T}} = 0 \tag{7.a}$$

$$A^{\mathrm{T}}Q + QA + C^{\mathrm{T}}C = 0 \tag{7.b}$$

Then we have

$$\textit{(1)}\ PQx_0 = \sigma^2 x_0, \qquad \textit{(2)}\ u = \sigma^{-2} B^{\mathrm{T}} e^{-A^{\mathrm{T}}t} Q x_0, \qquad \textit{(3)}\ v = \sigma^{-2} C e^{At} x_0.$$

Proof: By using equations (2), (3) and (4), and knowing that $\Gamma^*_G \Gamma_G u = \sigma^2 u$, we have

$$\Gamma^*_G \Gamma_G u = B^{\mathrm{T}} e^{-A^{\mathrm{T}}t} Q x_0 \tag{8}$$

where

$$Q = \int_0^\infty e^{A^{\mathrm{T}}\tau} C^{\mathrm{T}} C e^{A\tau}\ d\tau \tag{9}$$

is the solution of (7.b). Thus, the left Hankel singular value is

$$u(t) = \sigma^{-2} B^{\mathrm{T}} e^{-A^{\mathrm{T}}t} Q x_0 \tag{10}$$

The substitution of u from (10) into (2) leads to

$$x_0 = \sigma^{-2} PQ x_0$$

where

$$P = \int_{-\infty}^{0} e^{-A\tau} BB^{\mathrm{T}} e^{-A^{\mathrm{T}}\tau}\, d\tau \tag{11}$$

is the solution of (7.a). Hence

$$PQx_0 = \sigma^2 x_0 \tag{12}$$

Similarly, since

$$\Gamma_G \Gamma_G^* v = \sigma^2 v = Ce^{At} x_0$$

so the right Hankel singular vector has the form

$$v(t) = \sigma^{-2} Ce^{At} x_0 \tag{13}$$

□

We remark that for a stable system G, the matrix PQ has n positive eigenvalues of σ^2 with each corresponding eigenvector x_0. Hence, the singular values of PQ are $\sigma_1, \sigma_2, \ldots, \sigma_n$ represented in the order of decreasing magnitude by a diagonal matrix $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$.

Definition 2.2 *The* Hankel norm *of the system G is defined as*

$$\|G\|_H \triangleq \sup_{\substack{u \in L_-^2 \\ u \neq 0}} \frac{\|y\|_{L_+^2}}{\|u\|_{L_-^2}} \tag{14}$$

where sup *denotes the supremum, and*

$$\|u\|_{L_-^2} \triangleq \left[\int_{-\infty}^{0} u(t)^{\mathrm{T}} u(t)\, dt \right]^{\frac{1}{2}}, \quad \|y\|_{L_+^2} \triangleq \left[\int_{0}^{\infty} y(t)^{\mathrm{T}} y(t)\, dt \right]^{\frac{1}{2}}$$

2.2 Glover Algorithm for balanced realization

Let T be an invertible matrix such that the new state $\bar{x}$ is related to the original state x by $\bar{x}(t) = T\,x(t)$. Then the system G becomes

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) \tag{15.a}$$

$$y(t) = \bar{C}\bar{x}(t) \tag{15.b}$$

where $\bar{A} = TAT^{-1}$, $\bar{B} = TB$ and $\bar{C} = CT^{-1}$.

The triple $(\bar{A}, \bar{B}, \bar{C})$ is said to be a *balanced realization* if the solutions of the following two Lyapunov equations:

$$\bar{A}\bar{P} + \bar{P}\bar{A}^{\mathrm{T}} + \bar{B}\bar{B}^{\mathrm{T}} = 0, \qquad \bar{A}^{\mathrm{T}}\bar{Q} + \bar{Q}\bar{A} + \bar{C}^{\mathrm{T}}\bar{C} = 0 \tag{16}$$

satisfy the conditions $\bar{P} = \bar{Q} = \Sigma$.

The Glover algorithm[3] for balanced realization by using the Cholesky factorization of Q is summarized as follows:

(1) Compute P and Q of the system G.

(2) Perform Cholesky factorization of Q to obtain $Q^{\frac{1}{2}}$.

(3) Compute σ_i and v_i such that $\sigma_i^2 v_i = Q^{\frac{1}{2}} P Q^{\frac{1}{2}} v_i$.

(4) Form the matrix $V = [v_1 \quad v_2 \quad \cdots \quad v_n]$ which satisfies $V\Sigma^2 = Q^{\frac{1}{2}} P Q^{\frac{1}{2}} V$, calculate the transformation matrix $T = \Sigma^{-\frac{1}{2}} V^T Q^{\frac{1}{2}}$.

(5) Compute the balanced realization $(\overline{A}, \overline{B}, \overline{C})$ using $\overline{A} = TAT^{-1}, \quad \overline{B} = TB, \quad \overline{C} = CT^{-1}$.

Similar procedures using P instead of Q can be found in Green and Limebeer [4].

3 Main Results

We firstly compute the Hankel norm using an one parameter optimization technique via a variational approach. Then, we establish a new algorithm to perform the balancing for the system G.

3.1 Hankel norm computation via variational approach

Let the collection of all $u \in L^2_-$ satisfying (2) be denoted by U_0 such that

$$U_0 = \left\{ u \in L^2_- \,\middle|\, \int_{-\infty}^{0} e^{-At} B u(t) dt = x_0 \right\}$$

Since for a controllable and observable system G, the Hankel norm defined by Definition 2.2 can be re-expressed as

$$\|G\|_H = \sup_{x_0 \in \mathbb{R}^n} \mu(x_0) \tag{1}$$

where

$$\mu(x_0) = \max_{u \in U_0} \frac{\|y\|_{L^2_+}}{\|u\|_{L^2_-}} \tag{2}$$

The value of $\|G\|_H$ can be computed according to equations (1) and (2).

For a given x_0, suppose γ is an upper bound of $\mu(x_0)$, then

$$\mu^2(x_0) \le \gamma^2 \quad \text{if and only if} \quad \|y\|^2_{L^2_+} - \gamma^2 \|u\|^2_{L^2_-} \le 0 \;\; \forall\, u \in U_0$$

Let γ_* be the maximal value of $\mu(x_0)$, then there exists an optimal output function y_* and control input function u_* such that

$$\gamma_*^2 = \mu^2(x_0) = \frac{\|y_*\|^2_{L^2_+}}{\|u_*\|^2_{L^2_-}} \tag{3}$$

Therefore, the computation of $\mu(x_0)$ is equivalent to solve a minimal energy problem:

$$\min_{u\in U_0}\left\{\frac{\gamma_*^2}{2}\int_{-\infty}^{0}u^{\mathrm{T}}(t)u(t)dt-\frac{1}{2}\int_{0}^{\infty}y(t)^{\mathrm{T}}y(t)dt\right\}=0 \tag{4}$$

which can be solved by a variational method.

Let the Hamiltonian function be

$$H(x,u,\lambda)=\begin{cases}\frac{\gamma_*^2}{2}u^{\mathrm{T}}u+\lambda^{\mathrm{T}}(Ax+Bu) & \text{if} \quad t\in(-\infty,0]\\ \lambda^{\mathrm{T}}(Ax)-\frac{1}{2}x^{\mathrm{T}}C^{\mathrm{T}}Cx & \text{if} \quad t\in[0,\infty)\end{cases} \tag{5}$$

where $\lambda(t)$ is the vectors of the Lagrange multipliers which must be continuous and satisfy the conditions $\lambda(-\infty)=\lambda(\infty)=0$. The sufficient condition for this minimal energy problem given in [1] requires that the following matrix

$$\begin{bmatrix}\frac{\partial^2 H}{\partial u^2} & \frac{\partial^2 H}{\partial u\partial x}\\ \frac{\partial^2 H}{\partial x\partial u} & \frac{\partial^2 H}{\partial x^2}\end{bmatrix}$$

be semipositive along the optimal trajectory $(x_*(t),u_*(t),\lambda_*(t))$, which is automatically satisfied here. Since

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}+\frac{\partial H}{\partial x}\dot{x}+\frac{\partial H}{\partial u}\dot{u}+\frac{\partial H}{\partial\lambda}\dot{\lambda}=0 \tag{6}$$

along the optimal trajectory, hence

$$H(x_*,u_*,\lambda_*)=H(x_*(-\infty),u_*(-\infty),\lambda_*(-\infty))=H(x_*(\infty),u_*(\infty),\lambda_*(\infty))=0. \tag{7}$$

The necessary condition for this extremal problem is as follows:

(1) $t\in(-\infty,0]$: Since $\frac{\partial H}{\partial u}=0$ leads to $\gamma_*^2u+B^{\mathrm{T}}\lambda=0$. The optimal control input u_* becomes

$$u_*=-B^{\mathrm{T}}\lambda_*/\gamma_*^2. \tag{8}$$

The corresponding adjoint equation is $\dot{\lambda}_*=-\frac{\partial H}{\partial x}=-A^{\mathrm{T}}\lambda_*$. Hence

$$\lambda_*(t)=e^{-A^{\mathrm{T}}t}\lambda_*(0) \tag{9}$$

which satisfies the requirement $\lambda_*(-\infty)=0$ and $\lambda_*(0)$ is a constant to be determined. Substitution of equation (9) into equation (8) yields $u_*(t)=-\frac{1}{\gamma_*^2}B^{\mathrm{T}}\mathrm{e}^{-A^{\mathrm{T}}t}\lambda_*(0)$. By equation (2), it finally follows that

$$\begin{aligned}x_0 &= \int_{-\infty}^{0}e^{-A\tau}B\Big(-\frac{1}{\gamma_*^2}B^{\mathrm{T}}e^{-A^{\mathrm{T}}\tau}\lambda_*(0)\Big)d\tau\\ &= -\frac{1}{\gamma_*^2}\left[\int_{-\infty}^{0}e^{-A\tau}BB^{\mathrm{T}}e^{-A^{\mathrm{T}}\tau}d\tau\right]\lambda_*(0) \quad = \quad -\frac{1}{\gamma_*^2}P\lambda_*(0)\end{aligned} \tag{10}$$

where P is given in equation (11). Thus the costate vector at current time is $\lambda_*(0)=-\gamma_*^2P^{-1}x_0$ and the minimal control law becomes

$$u_*(t)=B^{\mathrm{T}}e^{-A^{\mathrm{T}}t}P^{-1}x_0 \tag{11}$$

(2) $t \in ([0,\infty)$: Since there is no control on the input of the system, the optimal control law is $u_*(t) = 0$. The solution of the state equation is $x_*(t) = e^{At}x_0$, and the output is

$$y_*(t) = Ce^{At}x_0 \tag{12}$$

The corresponding adjoint equation is

$$\dot{\lambda}_* = -\frac{\partial H}{\partial x} = -A^{\mathrm{T}}\lambda_* + C^{\mathrm{T}}Cx_* \tag{13}$$

Integration of equation (13) leads to

$$\lambda_*(t) = e^{-A^{\mathrm{T}}t}\left[\lambda_*(0) + \int_0^t e^{A^{\mathrm{T}}\tau}C^{\mathrm{T}}Cx_*(\tau)\,d\tau\right] = e^{-A^{\mathrm{T}}t}\left[\lambda_*(0) + \int_0^t e^{A^{\mathrm{T}}\tau}C^{\mathrm{T}}Ce^{A\tau}\,d\tau x_0\right]$$

Since $\lambda_*(\infty) = 0$, we must have

$$\lambda_*(0) + \int_0^\infty e^{A^{\mathrm{T}}\tau}C^{\mathrm{T}}Ce^{A\tau}\,d\tau x_0 \equiv 0$$

or equivalently

$$\lambda_*(0) = -Qx_0 \tag{14}$$

By equations (10) and (14), we have

$$\gamma_*^2 P^{-1}x_0 = Qx_0 \tag{15}$$

which is the continuity condition of the costate vector $\lambda_*(t)$ at $t = 0$. Furthermore, equation (15) is equivalent to $PQx_0 = \gamma_*^2 x_0$. By the *Lemma 2.1*, the γ_* is the Hankel singular value of the system. And, the vectors u_* and y_* given in (11) and (12) are parallel to the Hankel singular vectors. Therefore, we can compute the Hankel singular values and the associated Hankel singular vectors. The present idea provides a new way to compute the Hankel norm for more complicated nonlinear systems or certain type of linear systems with input/feedthrough delays.

The minimum energy required for $u \in U_0$ to transfer the state $x(t)$ from $x(-\infty) = 0$ to $x(0) = x_0$ is given by

$$\|u\|_{L_-^2}^2 = \int_{-\infty}^0 u_*(t)^{\mathrm{T}}u_*(t)dt = \int_{-\infty}^0 x_0^{\mathrm{T}}P^{-1}e^{-At}BB^{\mathrm{T}}e^{-A^{\mathrm{T}}t}P^{-1}x_0\,dt = x_0^{\mathrm{T}}P^{-1}x_0 \tag{16}$$

And, the free output response energy is

$$\|y\|_{L_+^2}^2 = \int_0^\infty y_*(t)^{\mathrm{T}}y_*(t)dt = \int_0^\infty x_0^{\mathrm{T}}e^{A^{\mathrm{T}}t}C^{\mathrm{T}}Ce^{At}x_0\,dt = x_0^{\mathrm{T}}Qx_0 \tag{17}$$

Substitution of equations (16) and (17) into equation (3) gives $\mu^2(x_0) = x_0^{\mathrm{T}}Qx_0/x_0^{\mathrm{T}}P^{-1}x_0$. Therefore, from equation (1) the Hankel norm $\|G\|_H$ is computed as

$$\|G\|_H^2 = \sup_{x_0\in\mathbb{R}^n}\mu^2(x_0) = \sup_{x_0\in\mathbb{R}^n}\frac{x_0^{\mathrm{T}}\sigma_i^2P^{-1}x_0}{x_0^{\mathrm{T}}P^{-1}x_0} = \sup_{x_0\in\mathbb{R}^n}\sigma_i^2\frac{x_0^{\mathrm{T}}P^{-1}x_0}{x_0^{\mathrm{T}}P^{-1}x_0} = \max_i \sigma_i^2 = \sigma_1^2$$

where x_0 is chosen to be the i-th eigenvector of the matrix PQ. Since σ_1 is the maximal singular value of the matrix PQ, thus the Hankel norm of linear system G is equal to the largest Hankel singular value, i.e. $\|G\|_H = \sigma_{\max}(PQ)$.

3.2 Balanced realization

Let $\ell_i \in \mathbb{R}^n$ be the eigenvector of PQ corresponding to σ_i, i.e. $\sigma_i^2 \ell_i = PQ\ell_i$ and $\ell_i^{\mathrm{T}} P^{-1} \ell_i = 1$. Defining $L = [\ \ell_1 \quad \ell_2 \quad \cdots \quad \ell_n\ ] \in \mathbb{R}^{n \times n}$, we have the following lemma:

Lemma 3.1 *The following identities hold:*

$$\text{(1) } L^{\mathrm{T}} P^{-1} L = I, \quad \text{(2) } L^{-1} P L^{-T} = I, \quad \text{(3) } L\Sigma^2 = PQL, \quad \text{(4) } L^{\mathrm{T}} Q L = \Sigma^2$$

where I denotes an $n \times n$ identity matrix.

Proof: By using the definition of L, these identities can be verified directly. □

Since $\{\ell_i\}_1^n$ forms a basis for $\mathbb{R}^n$, we can express the state x_0 in terms of this basis as

$$x_0 = [x_{01}\ x_{02} \cdots x_{0n}]^{\mathrm{T}} = \alpha_1 \ell_1 + \alpha_2 \ell_2 + \cdots + \alpha_n \ell_n = L\alpha$$

with scalar α_i for i ranging from 1 to n. After some algebraic operations, we obtain $x_0^{\mathrm{T}} P^{-1} x_0 = \alpha^{\mathrm{T}} L^{\mathrm{T}} P^{-1} L\alpha = \alpha^{\mathrm{T}}\alpha$ and $x_0^{\mathrm{T}} Q x_0 = \alpha^{\mathrm{T}} L^{\mathrm{T}} Q L \alpha = \alpha^{\mathrm{T}} \Sigma^2 \alpha$. Suppose that the triple $(\bar{A}, \bar{B}, \bar{C})$ is the balanced realization of the system G, and $\bar{x}(t)$ is the corresponding balanced state which is partitioned as

$$\bar{x} = [\ \bar{x}_1 \quad \bar{x}_2 \quad \cdots \quad \bar{x}_n\ ]^{\mathrm{T}}$$

then by satisfying equation (16), we must have $\bar{x}(0)^{\mathrm{T}} \bar{P}^{-1} \bar{x}(0) = \bar{x}(0)^{\mathrm{T}} \Sigma^{-1} \bar{x}(0)$ and $\bar{x}(0)^{\mathrm{T}} \bar{Q} \bar{x}(0) = \bar{x}(0)^{\mathrm{T}} \Sigma \bar{x}(0)$. It follows that

$$\bar{x}(0) = \Sigma^{\frac{1}{2}} \alpha = \Sigma^{\frac{1}{2}} L^{-1} x_0 \tag{18}$$

with corresponding components $\bar{x}_i(0) = \sqrt{\sigma_i}\alpha_i$. And, we obtain

$$x_0 = \bar{x}(0)_1 \frac{\ell_1}{\sqrt{\sigma_1}} + \bar{x}(0)_2 \frac{\ell_2}{\sqrt{\sigma_2}} + \cdots + \bar{x}(0)_n \frac{\ell_n}{\sqrt{\sigma_n}}$$

Therefore, the basis of the balanced representation is given by $\{\ell_i/\sqrt{\sigma_i}\}_1^n$, which is parallel to the basis $\{\ell_i\}_1^n$. This shows that the unit base vector of the balanced realization is parallel to the Hankel singular vector. Hence, we can define the balanced map for the system G as

Definition 3.2 *The state transformation matrix T is a* balanced map *if it satisfies $T\ell_i/\sqrt{\sigma_i} = e_i$, $i = 1, 2, \ldots, n$ where $e_i = [\delta_{i1}, \cdots, \delta_{in}]^{\mathrm{T}}$ is the i-th vector of the canonical basis of $\mathbb{R}^n$ and δ_{ij} is the Kronecker delta. The state Tx is the* balanced state *and its corresponding realization $(TAT^{-1},\ TB,\ CT^{-1})$ is the* balanced realization.

We can perform the balanced realization of the system G by using the following theorem:

Theorem 3.3 *Suppose the system G is stable, controllable and observable, and* (A,B,C) *is minimal, but is not a balanced realization, then* $(\bar{A},\bar{B},\bar{C}) = (TAT^{-1},\ TB,\ CT^{-1})$ *is a balanced realization with*

$$T = \Sigma^{\frac{1}{2}} L^{-1} \tag{19}$$

And, the corresponding controllability and observability gramians are diagonal i.e. $\bar{P} = \bar{Q} = \Sigma$.

Proof: Since

$$\begin{aligned} \left[T\frac{\ell_1}{\sqrt{\sigma_1}}\ T\frac{\ell_2}{\sqrt{\sigma_2}}\ \cdots\ T\frac{\ell_n}{\sqrt{\sigma_n}} \right] &= T\left[\frac{\ell_1}{\sqrt{\sigma_1}}\ \frac{\ell_2}{\sqrt{\sigma_2}}\ \cdots\ \frac{\ell_n}{\sqrt{\sigma_n}} \right] \\ &= \Sigma^{\frac{1}{2}} L^{-1} [\ell_1\ \ell_2\ \cdots\ \ell_n] \Sigma^{-\frac{1}{2}} = I \end{aligned} \tag{20}$$

it follows that $T\ell_i/\sqrt{\sigma_i} = e_i$. Hence, by *Definition 3.2*, the matrix T is a balanced map, $\bar{x}$ is a balanced state and $(\bar{A},\bar{B},\bar{C})$ is the balanced representation. Substituting $\bar{A} = TAT^{-1}$, $\bar{B} = TB$ and $\bar{C} = CT^{-1}$ into equation (16), we have

$$TAT^{-1}\bar{P} + \bar{P}T^{-T}A^{\mathrm{T}}T^{\mathrm{T}} + TBB^{\mathrm{T}}T^{\mathrm{T}} = 0 \tag{21.a}$$

$$T^{-T}A^{\mathrm{T}}T^{\mathrm{T}}\bar{Q} + \bar{Q}TAT^{-1} + T^{-T}C^{\mathrm{T}}CT^{-1} = 0. \tag{21.b}$$

The multiplication of equation (6) by certain functions of T gives

$$TAPT^{\mathrm{T}} + TPA^{\mathrm{T}}T^{\mathrm{T}} + TBB^{\mathrm{T}}T^{\mathrm{T}} = 0 \tag{22.a}$$

$$T^{-T}A^{\mathrm{T}}QT^{-1} + T^{-T}QAT^{-1} + T^{-T}C^{\mathrm{T}}CT^{-1} = 0. \tag{22.b}$$

from which we obtain $\bar{P} = TPT^{\mathrm{T}}$ and $\bar{Q} = T^{-T}QT^{-1}$. Using the identities given in *Lemma 3.1*, we have

$$\bar{P} = \Sigma^{\frac{1}{2}}L^{-1}PL^{-\mathrm{T}}\Sigma^{\frac{1}{2}} = \Sigma \quad \text{and} \quad \bar{Q} = \Sigma^{-\frac{1}{2}}L^{\mathrm{T}}QL\Sigma^{-\frac{1}{2}} = \Sigma. \tag{23}$$

□

It is noted that the idea presented in *Definition 3.2* and *Theorem 3.3* can be extended to qualitatively discuss the balancing of the linear time-invariant system in behavioral approach.

The new computation algorithm developed in the present study for a balanced realization is summarized as follows:

The New Algorithm:

(1) Compute P and Q of the system G.

(2) Compute σ_i and v_i such that $\sigma_i^2 v_i = PQv_i$.

(3) Determine the eigenvectors ℓ_i which satisfy the requirement $\ell_i^{\mathrm{T}}P^{-1}\ell_i = 1$ from

$$\ell_i = \frac{v_i}{\sqrt{v_i^{\mathrm{T}}P^{-1}v_i}}$$

.

(4) Form the matrices Σ and L, calculate the transformation matrix T as $T = \Sigma^{\frac{1}{2}} L^{-1}$.

(5) The balanced realization $(\overline{A}, \overline{B}, \overline{C})$ is determined with $\overline{A} = TAT^{-1}$, $\overline{B} = TB$ and $\overline{C} = CT^{-1}$.

The advantage of the present new algorithm over Glover Algorithm [3] is that there is no need to perform the Cholesky factorization for the matrix Q in order to obtain the balanced realization. The matrix L here is equal to $Q^{\frac{1}{2}} V\Sigma$ as in the Glover Algorithm.

4 A Numerical Example

An example for the following simple dynamical system is considered:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{1.a}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \tag{1.b}$$

Comparing with the standard form of equation (1), the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{2}$$

Substituting equation (2) into equations (7.a) and (7.b), the controllability gramian P and observability gramian Q are obtained as follows:

$$P = \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{11}{12} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{12} \end{bmatrix}$$

By *Lemma 2.1*, the Hankel singular values, σ_i, and the corresponding Hankel singular vectors, ℓ_i satisfying the conditions $\sigma_i^2 \ell_i = PQ\ell_i$, and $\ell_i^T P^{-1} \ell_i = 1$ are found to be

$$\sigma_1 = 0.296796, \quad \sigma_2 = 0.0467961, \quad \ell_1 = \begin{bmatrix} 0.268298 \\ 0.150663 \end{bmatrix}, \quad \ell_2 = \begin{bmatrix} -0.106535 \\ 0.379430 \end{bmatrix}$$

Hence, by the present new Algorithm

$$L = \begin{bmatrix} 0.268298 & -0.106535 \\ 0.1507663 & 0.379430 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.296796 & 0 \\ 0 & 0.0467961 \end{bmatrix}$$

The transformation matrix T given by (19) becomes

$$T = \Sigma^{\frac{1}{2}} L^{-1} = \begin{bmatrix} -1.75399 & 0.492479 \\ -0.276553 & 0.492479 \end{bmatrix}$$

Since $\bar{x} = Tx$, the new realization of the triple $(\bar{A}, \bar{B}, \bar{C})$ with $\bar{A} = TAT^{-1}$,$\bar{B} = TB$, and $\bar{C} = CT^{-1}$ are found to be

$$\bar{A} = \begin{bmatrix} -0.40859 & 0.97142 \\ -0.97143 & -2.59141 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 0.492479 \\ 0.492479 \end{bmatrix}, \ \bar{C} = \begin{bmatrix} 0.492479 & -0.492479 \end{bmatrix}$$

After substituting the matrices $\bar{A}$, $\bar{B}$, and $\bar{C}$ into equations (7.a) and (7.b), we obtain

$$\bar{P} = \bar{Q} = \Sigma \begin{bmatrix} 0.296796 & 0 \\ 0 & 0.0467961 \end{bmatrix}$$

Thus, the triple $(\bar{A}, \bar{B}, \bar{C})$ is a balanced representation. As a result, the balanced state equations become

$$\begin{aligned} \dot{\bar{x}} &= \begin{bmatrix} -0.40859 & 0.97142 \\ -0.97143 & -2.59141 \end{bmatrix} \bar{x} + \begin{bmatrix} 0.492479 \\ 0.492479 \end{bmatrix} u \\ y &= \begin{bmatrix} 0.492479 & -0.492479 \end{bmatrix} \bar{x} \end{aligned}$$

5 Conclusions

A new algorithm for performing balanced realization of linear time-invariant systems has been established. Firstly, we have developed the Hankel norm for computational purpose as a minimal energy problem. A Hamiltonian function defined by the past control input function and future output response is used for solving the problem. The continuity condition of the Lagrange multiplier at the current time leads to the same relationship between the Hankel singular values and Hankel singular vectors as the result arrived on the basis of the Hankel operator. Consequently, the Hankel norm of the system is solved as an one parameter optimization problem. Thereafter, the balancing process is discussed previously. A new algorithm for performing balanced realization without using the Cholesky factorization of the controllability or observability gramians has been established. The simplification of the present algorithm over the existing Glover algorithm is made because of the fact that the bases of the balanced coordinate are found to be parallel to the Hankel singular vectors. The present approach can be extended without much difficulty to nonlinear and delay systems while it is difficult to derive an explicit form using the Hankel operator.

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線性非時變系統之平衡化處理演算法

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摘　要

本文提出以對局理論與參數最佳化技巧以實現線性系統之平衡化處理。首先，吾人推導Hankel範數之計算。因為平衡實現的座標向量係平行Hankel奇異向量，如此一來吾人定義了平衡對映，並據以推導出新的平衡化處理演算法，以簡化以往須先對可控制矩陣或可觀測矩陣進行Cholesky 分解之作法。文中並舉例說明此演算法之計算過程。

關鍵詞：Hankel範數，線性非時變系統，對局理論，參數最佳化，平衡化處理

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