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# Global Stability for a Predator-Prey System with Predator Self-Limitation

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## Abstract

The aim of this paper is to study the dynamical behavior in a class of predator-prey system with predator self-limitation. We present some global stability results obtained from Dulac's criterion and Poincaré-Bendixson theorem, comparison method, and stable limit cycle analysis for the predator-prey systems with predator self-limitation in the first quadrant.

**Keywords:** global stability, self-limitation, predator-prey system, Dulac's criterion, Ponicaré-Bendixson theorem, comparison method, limit cycle

## 1 Introduction

Predator-prey models have been studied for a long time. Many biologists believe that if the unique positive equilibrium point of a predator-prey system is locally asymptotically stable, then it is globally asymptotically stable. However, this is not always true. In [14], Josef found that a unique positive locally asymptotically stable equilibrium point has at least one limit cycle surrounding the equilibrium point under suitable condition. Thus many mathematicians try to use some well-known methods to find the conditions for global stability for the equilibrium point of predator-prey systems. Firstly, they construct a Lyapunov function and establish the global stability by LaSalle's invariance principle (See [2,8,9,11]). Secondly, they employ the Dulac's criterion to eliminate the existence of the periodic solution and prove the global stability by Poincaré-Bendixson theorem (See [10,12,15,19]). Thirdly, comparison method is used to prove the global stability for some predator-prey systems (See [5,15,17]). Furthermore, there is a method which people can compares the trajectory of the system with an auxiliary system which is obtained by "mirror" reflection (See [6,15]). Finally, there still exists a method to analyze the global stability

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of the predator-prey systems. It's stable limit cycle analysis (See [10,12,13,22,23]). However, in these models which have been discussed, the assumptions for predator is that the populations will infinitely increase if the prey populations is large enough. From the viewpoint of the biology, this is false. Hence we are trying to find the conditions of predator which is more appropriating biology.

At first, in the Lotka-Volterra model, the predator zero isocline is vertical. That is, the same number of prey is assumed to be sufficient to maintain any number of predators (See Figure 1 curve A). This assumption can also be found in [7,8,20,21]. But this is most unlikely since larger populations of predator require larger populations of prey to maintain them (See Figure 1 curve B). This situation is discussed in [12] and [19]. Besides, mutual interference will also reduce individual consumption rate (See Figure 1 curve C). Recently, many biologists discover that predator populations might be influenced by the prey populations and predator's self-limitation. That is, even if predators have excess food, their populations are still limited by availability of some other resource: nesting sites perhaps, or safe refuges of their own. This will put an upper limit on the predator population irrespective of prey numbers (See Figure 1 curve D).

In order to explain the stabilizing effects of self-limitation in predator-prey interactions, we give an example to state this. In 1983, Batzli [3] discovered that in the Arctic, the ground squirrels have populations that remain remarkably constant from year to year. This is because the ground squirrels are strongly self-limitation by their aggressive territorial defence of burrows used for breeding and hibernating.

Next, we give the instance to describe the influence of the self-limitation. In 1972, Watson and Moss [18] discuss the red grouse (*Lagopus lagopus scoticus*) feeding on heather(*Calluna vulgaris*) on Scottish moorlands (See also Caughley and Lawton [4], 1981). They find that heather comprises at least 90% of red grouse's diet over most of the year, and it is the dominant higher plant on the moors where the grouse live. The grouse themselves are strongly territorial, with the size of the spring breeding population being determined by the number of territories established by cocks in the previous autumn. And the larger the self-limitation is, the less fluctuating the populations of the predator is (See Figure 2 curve (i) (ii) and (iii)).

The main purpose of this paper is to establish the global stability of the predator-prey model with self-limitation predator.

In section 2, we introduce the primary model (2.1) with some assumptions and prove that all solutions of the system (2.1) are positive and bounded.

In section 3, we use the four renowned methods, such as Dulac's criterion, Lyapunov function, comparison method, and stable limit cycle analysis, to discuss the global stability of the model (3.1).

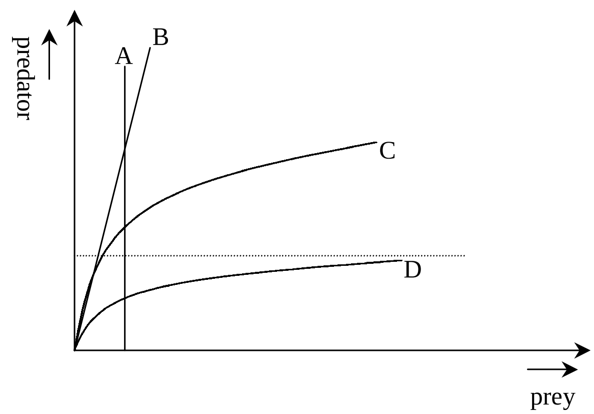


Figure 1: curves A to D are predator zero isoclines of increasing complexity.

## 2 The Models

Consider the following predator-prey system

$$\begin{aligned} \dot{x} &= xg(x) - yp(x) \\ \dot{y} &= y\left[\delta - \beta \frac{y}{q(x)}\right] \\ x(0) > 0 \quad &, \quad y(0) > 0 \end{aligned} \tag{2.1}$$

where $\cdot = d/dt$, $x$ and $y$ represent the prey population and the predator population, respectively. Because all we want to discuss is biological population, we only consider the first quadrant in the $x-y$plane. The following assumptions are consistent with the system (2.1).

(A1) $g \in C^1([0,\infty),R)$; $g(0) > 0$, and there exists $K$ such that $(x-K)g(x) < 0$ for $x \neq K$. Firstly, in the assumption (A1), $K$ is defined as the prey environmental carrying capacity. Secondly, the specific growth rate, $g(x)$, governs the growth of the prey in the absence of predators. Several forms of $g(x)$ have been catalogued in [1] or [2]. For examples, $g(x) = r[1-(x/K)^c]$; $0 < c \leq 1$, and $g(x) = r(K-x)/(K+\varepsilon x)$. We remark that a new feature of such predator-prey models is incorporated into the case when $g'(0) > 0$. This represents a prey population which can exhibit an accelerated population specific growth rate for intermediate values of population as a strategy to avoid extinction.

(A2) $p \in C^1([0,\infty),R)$; $p(0) = 0$ and $p'(x) \geq 0$ for all $x \geq 0$.

The functional response of the predator, $p(x)$, has been discussed in the literature. Several forms of $p(x)$ can be found in [16] or [17]. In some models $p(x)$ is assumed unbounded, for example, $p(x) = mx$ in the Lotka-Volterra model or in the Holling-type I model. In others $p(x)$ is assumed bounded, for instance, $p(x) = mx/(a+x)$, $p(x) = mx^2/(a+x^2)$, $p(x) = mx^c$, $0 < c \leq 1$, or $p(x) = m(1-e^{-cx})$.

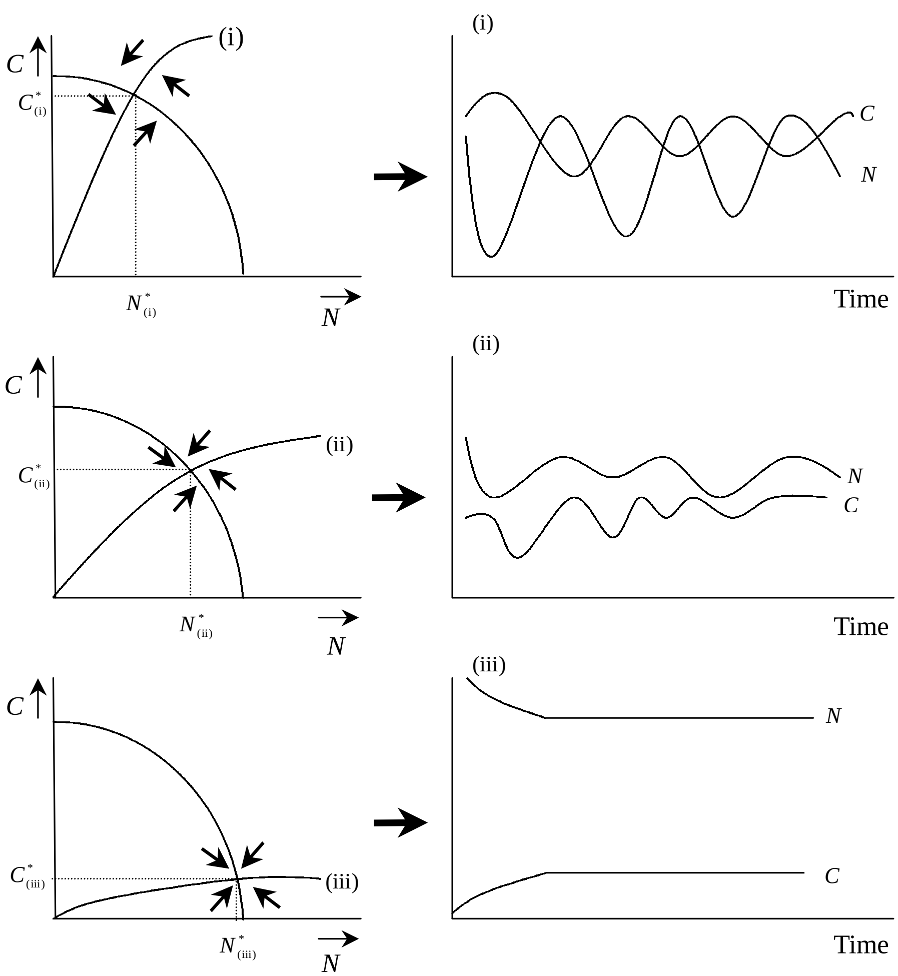


Figure 2: This prey isocline combined with predator zero isoclines with increasing levels of self-limitation. $(x^*,y^*)$ is the equilibrium point. (See [4] and [18])

(A3) $q \in C^1([0,\infty),R)$; $q(0)=0$, $q'(x)>0$, $\forall x \geq 0$.
The predator environmental carrying capacity is influenced by the density of the prey, say $q(x)$. In [9], $q(x)=x$ is a special case. Besides, by the Figure 1, the curve B is a sloping line with $q(x)=x$, the curve C is a curve with $q(x)$ increasing infinitely without any limitation, and the curve D is a curve with $\lim_{x\to\infty} q(x)=q_\infty>0$ where $q(x)$ can't increase infinitely.

Clearly, $E_K=(K,0)$ is an equilibrium point of the system (2.1). As we are interested in the stability of the unique positive equilibrium point in the first quadrant for the system (2.1), we need the following assumption:

(A4) There exists a unique positive equilibrium point $E^*=(x^*,y^*)$ such that $x^*g(x^*)-y^*p(x^*)=0$ and $\delta-\beta y^*/q(x^*)=0$.

To analyze the behavior of the system (2.1), firstly, we discuss the local behavior of the equilibrium points in the system (2.1) by the Hartman-Grobman Theorem. Now let us study the local behavior of the system (2.1) at equilibrium points $E_K$ and $E^*$. The Jacobian matrix of the system (2.1) takes the form

$$J=\begin{bmatrix} g(x)+xg'(x)-yp'(x) & -p(x) \\ \beta y^2\dfrac{q'(x)}{[q(x)]^2} & \delta-2\beta\dfrac{y}{q(x)} \end{bmatrix}$$

At $E_K$

$$\begin{aligned} J_K &= \begin{bmatrix} g(K)+Kg'(K) & -p(K) \\ 0 & \delta \end{bmatrix} \\ &= \begin{bmatrix} Kg'(K) & -p(K) \\ 0 & \delta \end{bmatrix} \end{aligned}$$

From (A1), $Kg'(K)<0$. Therefore, $\det(J_K)=\delta Kg'(K)<0$. Hence the equilibrium point $E_K$ is a saddle point. Let's find out the stable manifold. We consider one orbit of the system (2.1) along $x$-axis. As $0<x<K$, by the system (2.1) and (A1), $\dot{x}$ is positive. Thus the orbit along $x$-axis with $0<x<K$ goes to $E_K$. Similarly when $x>K$, the orbit also goes to $E_K$. Hence $x$-axis is the stable manifold. The variation matrix of the system (2.1) at $E^*$ is

$$\begin{aligned} J^* &= \begin{bmatrix} g(x^*)+x^*g'(x^*)-y^*p'(x^*) & -p(x^*) \\ \beta(y^*)^2\dfrac{q'(x^*)}{[q(x^*)]^2} & -\delta \end{bmatrix} \\ &= \begin{bmatrix} p(x^*)h'(x^*) & -p(x^*) \\ \dfrac{\delta^2}{\beta}q'(x^*) & -\delta \end{bmatrix} \end{aligned}$$

where $h(x) = xg(x)/p(x)$ is the prey isocline and

$$\begin{aligned} tr(J^*) &= p(x^*)h'(x^*) - \delta \\ \det(J^*) &= \delta p(x^*)\left[\frac{\delta}{\beta} q'(x^*) - h'(x^*)\right] \end{aligned}$$

We know that if $tr(J^*) < 0$ and $\det(J^*) > 0$, then the equilibrium point $E^*$ of the system (2.1) is locally asymptotically stable. In other words, $p(x^*)h'(x^*) < \delta$ and $[\delta q'(x^*)/\beta] > h'(x^*)$. That is,

$$h'(x^*) < \min\{\delta/p(x^*),\ \delta q'(x^*)/\beta\}.$$

Hence $E^*$ is locally asymptotically stable if the following inequality holds.

(A5) $h'(x^*) \leq 0$

Next, we state the following lemma to satisfy the meanings of the biology, so that the system (2.1) is as ”well behaved” as one intuits from the biological problem.

**Lemma 2.1** *Assume (A1), (A2) and (A3) hold. Then all solutions $(x(t), y(t))$ of the system (2.1) are positive and bounded.*

*Proof.* Firstly, we want to show that all the solutions $(x(t), y(t))$ of the system (2.1) are positive. In other words, if the initial point $(x_0, y_0) \equiv (x(0), y(0))$ is positive, then $(x(t), y(t))$ is in the first quadrant for all $t > 0$. Let's divide the first quadrant into four regions I-IV which are defined below:

$$\begin{aligned} \mathrm{I} &= \{(x,y)|h(x) < y,\ k(x) > y,\ x > 0,\ y > 0\} \\ \mathrm{II} &= \{(x,y)|h(x) > y,\ k(x) > y,\ x > 0,\ y > 0\} \\ \mathrm{III} &= \{(x,y)|h(x) > y,\ k(x) < y,\ x > 0,\ y > 0\} \\ \mathrm{IV} &= \{(x,y)|h(x) < y,\ k(x) < y,\ x > 0,\ y > 0\} \end{aligned}$$

where $h(x) = xg(x)/p(x)$ and $k(x) = \delta q(x)/\beta$; see Figure 3 or Figure 4.

Consider the following two cases:

(A) $(x_0, y_0)$ is near the positive $x$-axis;

(B) $(x_0, y_0)$ is near the positive $y$-axis.

In the case (A), the initial point $(x_0, y_0)$ will be in the region I or II. From Figure 3 or 4, since $\dot{y}$ is positive in the region I or II, the solution $(x(t), y(t))$ with the initial point $(x_0, y_0)$ will run away the positive $x$-axis. In the cases (B), the initial point $(x_0, y_0)$ will be in region III or IV. From Figure 3 or 4, since $\dot{x}$ is positive in the region III, the solution $(x(t), y(t))$ with the initial point $(x_0, y_0)$ in the region III will run away the positive $y$-axis. Next we want to show that if the initial point starts in IV, then the trajectory of the solution $(x(t), y(t))$ will go into the region III. That is, the trajectory

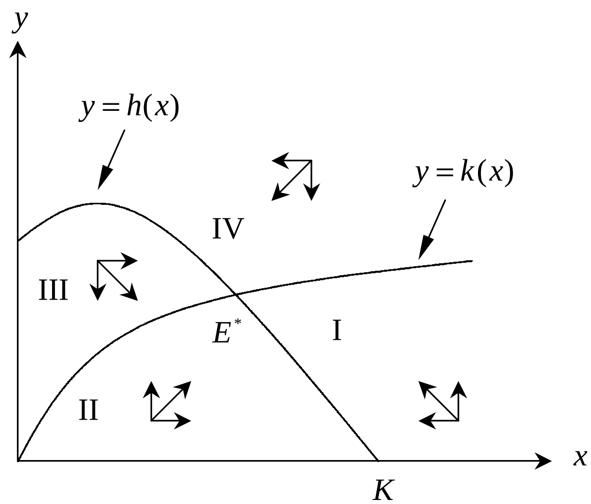


Figure 3: $p'(0) > 0$

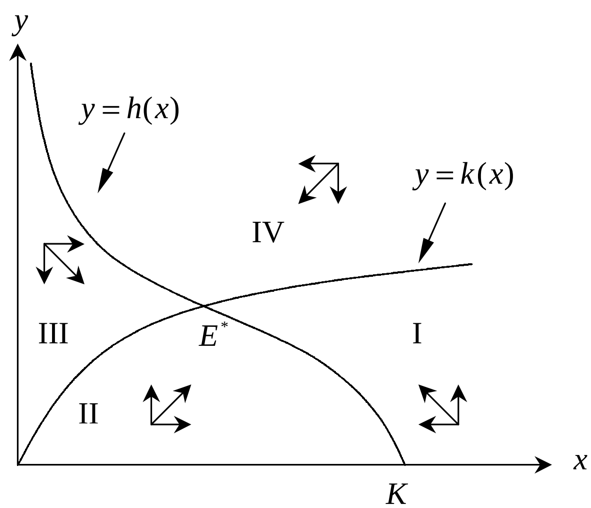


Figure 4: $p'(0) = 0$

of the solution $(x(t), y(t))$ will not stay in the region IV or not go to $y$-axis. Suppose the trajectory finally stays at some point $(x, y)$ in the region IV. Then the point $(x, y)$ will be an equilibrium point of the system (2.1). This is a contradiction to (A4). Thus any solution $(x(t), y(t))$ starts in the region IV won't stay in it. Besides, when the trajectory in the region IV approaches to $y$-axis, we find that $\dot{x} \to 0$ and $\dot{y} \to -\infty$ as $x \to 0$. Hence there exists some $t_1 > 0$ such that $(x(t), y(t))$ is in the region III whenever $t \geq t_1$. Therefore, all solutions $(x(t), y(t))$ are positive.

Secondly, we want to show that all solutions $(x(t), y(t))$ of the system (2.1) are bounded. That is, $x(t)$ and $y(t)$ are both bounded for all $t \geq 0$. At first, (a) we want to show that $x(t) < K$ for all $t \geq 0$ whenever $x(0) < K$. Suppose $x(t) \geq K$ for some $t > 0$. Then there exists $t_2 > 0$ such that $x(t_2) = K$ and $\dot{x}(t_2) \geq 0$. By (A1) and (A2) we know that $g(K) = 0$ and $p(K) > 0$. Thus

$$\begin{aligned}
\dot{x}(t_2) &= x(t_2)g(x(t_2)) - y(t_2)p(x(t_2)) \\
&= -y(t_2)p(K) \\
&< 0
\end{aligned}$$

This is a contradiction to $\dot{x}(t_2) \geq 0$. Hence $x(t) < K$ for all $t \geq 0$ whenever $x(0) < K$. Secondly, (b) we will show that $x(t) \leq x(0)$ for all $t \geq 0$ whenever $x(0) > K$. By (A1) and (A2) we know that $g(x) \leq 0$ and $p(x) > 0$ for $x \geq K$. Then, by the system (2.1),

$$\begin{aligned}
\dot{x} &= xg(x) - yp(x) \\
&\leq -yp(x) \\
&< 0
\end{aligned}$$

Thus $x$ is strictly decreasing for all $x \geq K$. Hence $x(t)$ decreases to some point $\hat{x} \geq K$ or there exists $\tau > 0$ such that $x(t) < K$ for all $t \geq \tau$. Hence from (a) and (b) we know that $x(t) \leq M \equiv \max\{K,\ x(0)\}$ for all $t \geq 0$. Next, (c) we are trying to show that $y(t) < [\delta q(M)/\beta]$ for all $t \geq 0$ whenever $y(0) < [\delta q(M)/\beta]$. Suppose $y(t) \geq [\delta q(M)/\beta]$ for some $t > 0$. Then there exists $t_3 > 0$ such that $y_3 = [\delta q(M)/\beta]$ and $\dot{y}(t_3) \geq 0$. Thus by (A3) and $x(t_3) \leq M$, we know that $q(x(t_3)) \leq q(M)$. Hence by the system (2.1)

$$\begin{aligned}
\dot{y}(t_3) &= y(t_3)\left[\delta - \beta\frac{y(t_3)}{q(x(t_3))}\right] \\
&= \frac{\delta q(M)}{\beta}\left[\delta - \beta\frac{\delta q(M)}{\beta q(x(t_3))}\right] \\
&< \frac{\delta q(M)}{\beta}[\delta - \delta] \\
&= 0
\end{aligned}$$

This is a contradiction to $\dot{y}(t_3) \geq 0$. Therefore, $y(t) < [\delta q(M)/\beta]$ for all $t \geq 0$ whenever $y(0) < [\delta q(M)/\beta]$. At last, (d) we will show that $y(t) \leq y(0)$ for all $t \geq 0$ whenever $y(0) \geq [\delta q(M)/\beta]$.

From (A3) and $0 < x(t) \leq M$ for all $t \geq 0$, we know that $g(x(t)) \leq q(M)$ for all $t \geq 0$. Hence we have $\dot{y} < 0$. Therefore, $y(t)$ strictly decreases to some $y \geq [\delta q(M)/\beta]$ or there exists $s > 0$ such that $y < [\delta q(M)/\beta]$ for $t \geq s$. From (c) and (d), $y(t) \leq N \equiv \max\{\delta q(M)/\beta,\ y(0)\}$ for all $t \geq 0$.

# 3 Global Stability

In this section, we want to introduce the following methods to analyze the global stability for the system (2.1):

**(i)** Dulac's criterion and Poincaré-Bendixson theorem

**(ii)** Comparison method

**(iii)** Lyapunov function

**(iv)** Stable limit cycle analysis

At first, we use the method (i) to analyze the system (2.1). We rewrite the system (2.1) to the following form:

$$\begin{aligned} \dot{x} &= xg(x) - yp(x) \equiv f(x,y) \\ \dot{y} &= y\left[\delta - \beta \frac{y}{q(x)}\right] \equiv g(x,y) \end{aligned}$$

**Theorem 3.1** *Let (A1)-(A4) hold. If*

$$h'(x)p(x) - \delta < 0, \tag{3.2}$$

*then the equilibrium point $E^*$ of the system (3.1) is globally asymptotically stable.*

**Proof:** Consider

$$H(x,y) = \frac{1}{p(x)y^2} \qquad x > 0,\ y > 0$$

Then

$$\begin{aligned} Hf &= \frac{h(x)}{y^2} - \frac{1}{y} \\ Hg &= \frac{1}{p(x)}\left[\frac{\delta}{y} - \beta\frac{1}{q(x)}\right] \end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial(Hf)}{\partial x} &= \frac{h'(x)}{y^2} \\
\frac{\partial(Hg)}{\partial y} &= -\frac{\delta}{y^2 p(x)} \\
\frac{\partial(Hf)}{\partial x} + \frac{\partial(Hg)}{\partial y} &= \frac{1}{y^2}\left[h'(x) - \frac{\delta}{p(x)}\right]
\end{aligned}$$

By the assumption $h'(x)p(x) - \delta < 0$,

$$\frac{\partial(Hf)}{\partial x} + \frac{\partial(Hg)}{\partial y} < 0$$

Hence by Dulac criterion, there exists no periodic orbit in the first quadrant. From (A4), the equilibrium point $E^*$ is locally asymptotically stable. By Lemma 2.1 and Poincaré-Bendixson theorem, it suffices to show that the unique positive equilibrium point $E^*$ is globally asymptotically stable in the first quadrant. □

**Remark 3.2** The global stability for the system (3.1) with $g(x) = (1-x)$, $q(x) = x$ and that $p(x) = mx$, $p(x) = mx/(a+x)$, or $p(x) = x^2/(a+x)(b+x)$ has been studied by Hsu and Huang in [12]. In that paper, they used the Dulac's criterion with $H(x,y) = [p(x)]^{-1}y^{-2}$ and Poincaré-Bendixson theorem to prove the global stability of the equilibrium point.

**Remark 3.3** The condition (3.2) $h'(x)p(x) - \delta < 0$ is weaker than the assumption $h'(x) \le 0$ which satisfies that the prey isocline is decreasing. That is, if the prey isocline is decreasing, then the Theorem 3.1 will be true.

Secondly, we use the method (ii) to analyze the system (2.1). Let's consider the following systems

$$\begin{aligned}
\dot{x} &= xg(x) - yp(x) \\
\dot{y} &= y\left[\delta - \beta\frac{y}{q(x)}\right]
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
\dot{x} &= p(x)(y - y^*) \\
\dot{y} &= y\left[\delta - \beta\frac{y^*}{q(x)}\right]
\end{aligned} \tag{3.3}$$

Where the system (2.1) has a unique positive equilibrium point $E^* = (x^*, y^*)$. And clearly, the point $E^*$ is also the unique positive equilibrium point of the system (3.3). We are trying to study the global stability of $E^*$ for the system (2.1) by using the comparison method. In order to present the Theorem 3.5, we need the following Lemma 3.4.

**Lemma 3.4** *If assumption (A1)-(A3) hold, then every orbit of the system (3.3) is a closed orbit which encloses around $E^*$.*

**Proof:** In the first quadrant, lines $y = y^*$ and $x = x^*$ cut it into four regions. Each orbit of the system (3.3) will be one of the following three cases: (i) a closed orbit encloses around $E^*$, or (ii) not a closed orbit and toward $E^*$, or (iii) not a closed orbit and away $E^*$ (See Figure 5, 6 and 7). For any solution $(x(t), y(t))$ of the system (3.3) with the initial point $(x(0), y(0)) = (x_0, y_0)$ which doesn't equal to $E^*$ where $x_0 = x^*$ and $y_0 > y^*$, there is $t_1 > 0$ such that $E_1 = (x(t_1), y(t_1))$ where $x(t_1) = x^*$ and $y(t_1) > y^*$. Equation (3.3) tells us that

$$\frac{dy}{dx} = \frac{y[\delta - \beta y^*/q(x)]}{p(x)[y^* - y]}$$

then, for any solution $(x(t), y(t))$ of the system (3.3) with initial point $(x_0, y_0)$

$$\int_{y_0}^{y} \frac{y^* - \eta}{\eta} d\eta = \int_{x_0}^{x} \frac{\delta - \beta y^*/q(\tau)}{p(\tau)} d\tau.$$

Hence we have

$$\int_{y_0}^{y(t_1)} \frac{y^* - \eta}{\eta} d\eta = \int_{x_0}^{x(t_1)} \frac{\delta - \beta y^*/q(\tau)}{p(\tau)} d\tau.$$

Since $x_0 = x(t_1) = x^*$,

$$\int_{x_0}^{x(t_1)} \frac{\delta - \beta y^*/q(\tau)}{p(\tau)} d\tau = 0$$

This implies

$$\int_{y_0}^{y(t_1)} \frac{y^* - \eta}{\eta} d\eta = 0$$

Hence we know that $y_0 = y(t_1)$. Therefore, cases (ii) and (iii) are not true. So, any orbit that is closed orbit encloses around $E^*$. □

**Theorem 3.5** *Let (A1)-(A4) hold. If*

$$(3.4) \qquad (x - x^*)\left[\frac{xg(x)}{p(x)} - y^*\right] < 0 \qquad for\ all\ 0 < x < K\ and\ x \neq x^*$$

*then $E^*$ is globally asymptotically stable for the system (2.1).*

**Proof:** Firstly, we know that systems (2.1) and (3.3) have the same equilibrium point in the first quadrant, say $E^*$. From Lemma 3.4., we know that the equilibrium point $E^*$ of the system (3.3) is a center. Let

$$\begin{aligned}
\vec{S_1} &= (M_1, N_1, 0) \\
\vec{S_2} &= (M_2, N_2, 0)
\end{aligned}$$

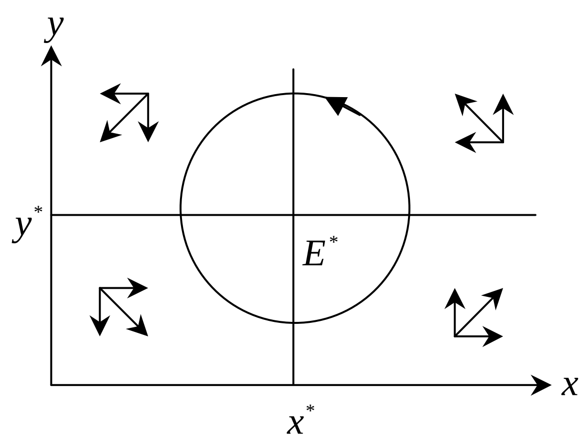


Figure 5: A closed orbit around $E^*$.

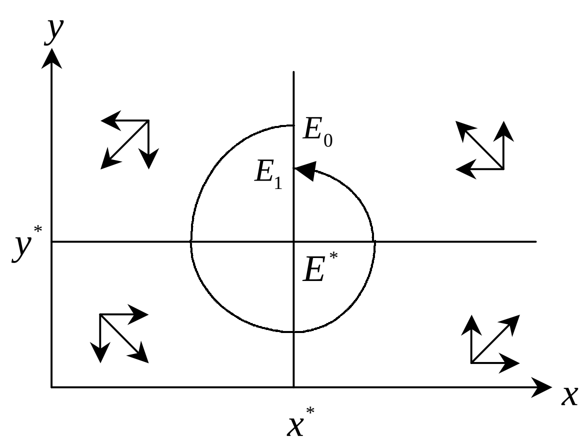


Figure 6: $E^*$ is a sink.

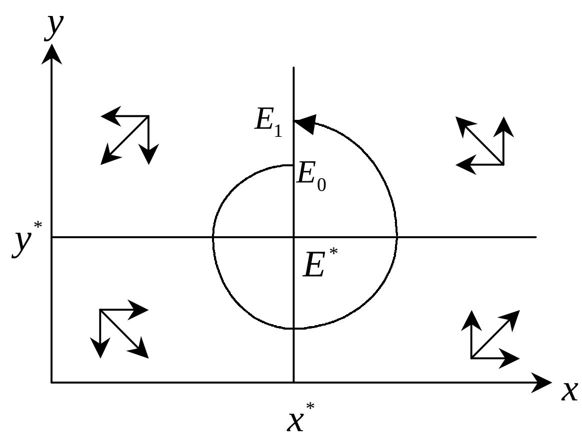


Figure 7: $E^*$ is a source.

and

$$\vec{S}_1 \times \vec{S}_2 \equiv (0, 0, \theta)$$

where

$$M_1 = xg(x) - yp(x) \quad , \quad N_1 = y\left[\delta - \beta \frac{y}{q(x)}\right]$$
$$M_2 = p(x)(y^* - y) \quad , \quad N_2 = y\left[\delta - \beta \frac{y^*}{q(x)}\right]$$

then by computation we have

$$\begin{aligned} \theta &= M_1 N_2 - M_2 N_1 \\ &= y\left\{[xg(x) - yp(x)]\left[\delta - \beta \frac{y^*}{q(x)}\right] - p(x)(y^* - y)\left[\delta - \beta \frac{y}{q(x)}\right]\right\} \\ &= y\left\{[xg(x)\left[\delta - \beta \frac{y^*}{q(x)}\right] - yp(x)\left[\delta - \beta \frac{y^*}{q(x)}\right] - p(x)(y^* - y)\left[\delta - \beta \frac{y}{q(x)}\right]\right]\right\} \\ &= y\left\{[xg(x) - y^* p(x)]\left[\delta - \beta \frac{y^*}{q(x)}\right] - \beta \frac{p(x)}{q(x)}(y^* - y)^2\right\} \end{aligned}$$

We know that $\theta < 0$ if

$$\left\{[xg(x) - y^* p(x)]\left[\delta - \beta \frac{y^*}{q(x)}\right] - \beta \frac{p(x)}{q(x)}(y^* - y)^2\right\} < 0$$

or

$$\left[\frac{xg(x)}{p(x)} - y^*\right]\left[\delta - \beta \frac{y^*}{q(x)}\right] - \frac{\beta}{q(x)}(y^* - y)^2 < 0$$

That is,

$$\frac{\beta y^*}{q(x)q(x^*)}[q(x) - q(x^*)]\left[\frac{xg(x)}{p(x)} - y^*\right] - \frac{\beta}{q(x)}(y^* - y)^2 < 0$$

Hence $\theta < 0$ whenever $[q(x)-q(x^*)][xg(x)/p(x)-y^*] < 0$ for all $x \neq x^*$. Since by (A3) $q'(x) > 0$ for all $x \geq 0$, $(x-x^*)[q(x)-q(x^*)] > 0$. That is, $[q(x)-q(x^*)][xg(x)/p(x)-y^*] < 0$ implies $(x-x^*)[xg(x)/p(x)-y^*] < 0$ for all $x \neq x^*$. Thus by Lemma 3.4, $E^*$ is globally asymptotically stable for the system (2.1) whenever the inequality (3.4) holds. $\square$

**Remark 3.6** We know the assumption

$$(x-x^*)\left[\frac{xg(x)}{p(x)}-y^*\right] < 0 \qquad \text{for all } x \neq x^*$$

means that

$$[k(x)-y^*][h(x)-y^*] < 0 \qquad \text{for all } x \neq x^*$$

That is, the prey isocline $y = h(x)$ lies above the horizontal $y = y^*$ whenever the predator isocline $y = k(x)$ lies below $y = y^*$.

Thirdly, we want to prove Theorem 3.5 by using the method (iii).

**Proof:** Let's consider the following Lyapunov function:

$$V(x,y) = \int_{x^*}^{x} \frac{q(\tau)-q(x^*)}{q(\tau)p(\tau)}d\tau + \frac{q(x^*)}{\beta y^*}\int_{y^*}^{y}\frac{\eta - y^*}{\eta}d\eta$$

on $G = \{(x,y) : x > 0,\ y > 0\}$. Then the time derivative of $V(x,y)$ computed along solution of the system (2.1) is

$$\begin{aligned}
\dot{V}(x,y) &= \frac{q(x)-q(x^*)}{q(x)p(x)}\dot{x} + \frac{q(x^*)}{\beta y^*}\left(\frac{y-y^*}{y}\right)\dot{y} \\
&= \frac{q(x)-q(x^*)}{q(x)}\left[\frac{xg(x)}{p(x)}-y\right] + \frac{q(x^*)}{\beta y^*}(y-y^*)\left[\beta\frac{y^*}{q(x^*)} - \beta\frac{y}{q(x)}\right] \\
&= \frac{q(x)-q(x^*)}{q(x)}\left[\frac{xg(x)}{p(x)}-y^*\right] - \frac{q(x)-q(x^*)}{q(x)}y + \frac{q(x)-q(x^*)}{q(x)}y^* \\
&\quad + \frac{q(x^*)}{y^*}(y-y^*)\left[\frac{y^*}{q(x^*)} - \frac{y}{q(x)}\right] \\
&= \frac{q(x)-q(x^*)}{q(x)}\left[\frac{xg(x)}{p(x)}-y^*\right] - \frac{q(x)-q(x^*)}{q(x)}(y-y^*) + (y-y^*) \\
&\quad - \frac{yq(x^*)}{y^*q(x)}(y-y^*) \\
&= \frac{q(x)-q(x^*)}{q(x)}\left[\frac{xg(x)}{p(x)}-y^*\right] + \frac{q(x^*)}{q(x)}(y-y^*) - \frac{yq(x^*)}{y^*q(x)}(y-y^*) \\
&= \frac{q(x)-q(x^*)}{q(x)}\left[\frac{xg(x)}{p(x)}-y^*\right] - \frac{q(x^*)}{y^*q(x)}(y-y^*)^2
\end{aligned}$$

From the assumption

$$(x-x^*)\left[\frac{xg(x)}{p(x)}-y^*\right] < 0 \qquad for\ all\ x \neq x^*$$

and by (A3) $q'(x) > 0$ for all $x \geq 0$,

$$[q(x) - q(x^*)]\left[\frac{xg(x)}{p(x)} - y^*\right] < 0 \qquad for\ x \neq x^*.$$

Hence $\dot{V}(x,y) \leq 0$ on $G$. Thus the equilibrium point $E^*$ is globally stable. At last, we want to show that $E^*$ is globally asymptotically stable. In other words, we are trying to show that only when $(x,y) = (x^*,y^*)$, $\dot{V}(x,y) = 0$. Since

$$(x - x^*)\left[\frac{xg(x)}{p(x)} - y^*\right] < 0 \qquad for\ x \neq x^*$$

we know that $(x - x^*)[xg(x)/p(x) - y^*] = 0$ only when $x = x^*$. This implies that $[q(x) - q(x^*)][xg(x)/p(x) - y^*] = 0$ only when $x = x^*$. Hence $\dot{V}(x,y) = 0$ if and only if $(y - y^*)^2 = 0$ and $[q(x) - q(x^*)][xg(x)/p(x) - y^*] = 0$. That is, $y = y^*$, $x = x^*$. Hence we know that only when $(x,y) = (x^*,y^*)$, $\dot{V}(x,y) = 0$. Therefore, the equilibrium point $E^*$ of the system (2.1) is globally asymptotically stable. □

**Remark 3.7** In Hsu and Huang [12], they use the same method to the system (2.1) with $q(x) = x$ where the Lyapunov function is

$$V(x,y) = \int_{x^*}^{x} \frac{\tau - x^*}{\tau p(\tau)} d\tau + \beta \frac{x^*}{y^*} \int_{y^*}^{y} \frac{\eta - y^*}{\eta} d\eta.$$

Finally, we introduce the method (iv) to prove the global stability of the system (2.1).

**Theorem 3.8** *Let (A1)-(A4) hold. If*

$$h'(x) < 0, \quad \forall x > 0 \tag{3.5}$$

*then the equilibrium point $E^*$ of the system (2.1) is globally asymptotically stable in the first quadrant.*

**Proof:** It suffices to show that the system (2.1) has no closed orbit in the first quadrant. Suppose on the contrary that there is a $T$-periodic orbit $\Gamma = \{(x(t),y(t))|0 \leq t \leq T\}$ in the first quadrant.

Compute

$$
\begin{aligned}
\Delta &= \int_\Gamma \left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right) ds \\
&= \int_0^T \left[\frac{\partial}{\partial x} f(x,y)+\frac{\partial}{\partial y} g(x,y)\right]\Big|_{\substack{x=x(t)\\y=y(t)}} dt \\
&= \int_0^T \left\{\frac{\partial}{\partial x}[xg(x)-yp(x)]+\frac{\partial}{\partial y}\left[y\left(\delta-\beta\frac{y}{q(x)}\right)\right]\right\}\Big|_{\substack{x=x(t)\\y=y(t)}} dt \\
&= \int_0^T \frac{\partial}{\partial x}\left[p(x)\left(\frac{xg(x)}{p(x)}-y\right)\right]\Big|_{\substack{x=x(t)\\y=y(t)}} dt+\int_0^T \frac{\partial}{\partial y}\left[y\left(\delta-\beta\frac{y}{q(x)}\right)\right]\Big|_{\substack{x=x(t)\\y=y(t)}} dt \\
&= \int_0^T \left[p'(x)\left(\frac{xg(x)}{p(x)}-y\right)+p(x)\left(\frac{xg(x)}{p(x)}\right)'\right]\Big|_{\substack{x=x(t)\\y=y(t)}} dt+\int_0^T \left[\delta-2\beta\frac{y}{q(x)}\right]\Big|_{\substack{x=x(t)\\y=y(t)}} dt \\
&= \int_0^T \frac{p'(x(t))}{p(x(t))}\dot{x}(t)dt+\int_0^T [p(x)h'(x)]\big|_{x=x(t)} dt+2\int_0^T \frac{1}{y(t)}\dot{y}(t)dt-\int_0^T \delta dt \\
&= \int_{p(x(0))}^{p(x(T))} \frac{1}{p}dp+\int_0^T [p(x)h'(x)]\big|_{x=x(t)} dt+2\int_{y(0)}^{y(T)} \frac{1}{y}dy-\int_0^T \delta dt
\end{aligned}
$$

Since $\Gamma$ is a $T$-periodic,

$$\int_{p(x(0))}^{p(x(T))} \frac{1}{p}dp=0 \qquad \text{and} \qquad \int_{y(0)}^{y(T)} \frac{1}{y}dy=0$$

Hence we have

$$\Delta=\int_0^T [p(x)h'(x)-\delta]\big|_{x=x(t)} dt$$

From (3.5), it follows that

$$\Delta<0.$$

This indicates that all periodic orbits of the system (2.1) in the first quadrant are orbitally stable. Since every periodic orbit is orbitally stable and then there is a unique stable limit cycle in the first quadrant. That is, $E^*$ is unstable. However, by (A4), $E^*$ is locally asymptotically stable. Thus there is no periodic orbit in the first quadrant. By Lemma 2.1 and Poincaré-Bendixson theorem, it suffices to show that the unique positive equilibrium point $E^*$ is globally asymptotically stable in the first quadrant. □

**Remark 3.9** From the Theorem 3.1, Theorem 3.5 and Theorem 3.7, we know that the assumption (3.5) is a special case of the assumptions (3.2) and (3.4).

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# 具有自我設限捕食者之捕食系統的整穩定性

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## 摘　　要

本篇論文主要在研究具有自我設限捕食者之捕食系統的整體穩定性。我們利用 Dulac 法則及 Poinecaré-Bendixson 定理、比較法則、Lyapunov 函數及極限環穩定分析法等四種方法得到上述系統之整體穩定性的充分條件。

關鍵詞：整體穩定性、自我設限、捕食系統、Dulac 法則、Poinecaré-Bendixson 定理、比較法則、極限環。

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