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The Influence of Time Delay on Local Stability for a Predator-Prey System

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Abstract

The aim of this paper is to study the influence of time delay on local stability for a predator-prey system. We first discuss the local stability of the system without delay. Then we study the change of the local stability of the predator-prey system with a single delay. Finally, we conclude with an example.

Keywords: global stability, predator-prey system, Dulac's criterion, Ponicaré-Bendixson theorem

1 Introduction

Predator-prey models have been studied for a long time. Many researchers either have no concern with time delays or tend to ignore delays in their models. Such a common practice generally works. However, it is not true without qualification. More realistic models should include some of the past states of the population systems; that is, ideally, a real system should be modeled with time delays.

Although concern for time delays in population models dates back to the 1920s, when Volterra [10,11] investigated the predator-prey model, the momentum did not pick up until the last three decades. Starting from Hutchinson's [6] delay logistic equation, May [9] has proposed the following system

$$\begin{aligned} \dot{x}_1(t) &= rx_1(t)\left[1-\frac{x_1(t-\tau)}{K}\right]-mx_1(t)x_2(t) \\ \dot{x}_2(t) &= -dx_2(t)+cx_1(t)x_2(t) \end{aligned} \tag{1.1}$$

where x_1 and x_2 are density of prey and predator, respectively, and r, τ, K, m, d, c are positive constants. If we think the gestation period of predator is τ, then the per capita growth rate function

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should carry a time delay τ, which results in (1.1). Here, we are trying to analyze the possible influence of delays on the stability of this system.

Most studies on time-delay systems start from the local stability analysis of some special solutions. For this purpose, the standard approach is to analyze the stability of the linearized systems about the special solution. By a similar technique as is utilized in [7], we can easily determine the delay margins for stability of the linearized form of the system (1.1).

We will use the characteristic equation of the linearized form of the system (1.1) to obtain the intervals of delay value for which the linearized system is stable. The stability of the trivial solution (i.e., the zero solution) of the linearized system depends on the location of the roots of the associated characteristic equation. As the length of delay changes, the stability of the trivial solution may also change. Such phenomena are often referred to as stability switches [1,3,8].

The main purpose of this paper is to analyze the possible influence of delays on the stability of the system (1.1). Firstly, we analyze the stability of the system (1.1) without time delay (i.e.,$\tau = 0$). In section 3, we use the above two methods to obtain the delay margins and intervals for stability with a single delay. In section 4, we illustrate our results by one example.

2 The Model without Delay

Consider the following predator-prey system

$$\begin{aligned} \dot{x}_1(t) &= rx_1(t)\left[1-\frac{x_1(t)}{K}\right]-mx_1(t)x_2(t) \\ \dot{x}_2(t) &= -dx_2(t)+cx_1(t)x_2(t) \end{aligned} \tag{2.1}$$

where $r,\ K,\ m,\ d,\ c$ are positive constants, and $Kc > d$, x_1 and x_2 represent the prey population and the predator population, respectively.

Clearly, $E_0 \equiv (0,0)$ and $E_K \equiv (K,0)$ are equilibrium points of the system (2.1). Since $Kc > d$, there is an unique positive equilibrium point

$$E^* \equiv (x_1^*, x_2^*) \equiv \left(\frac{d}{c}, \frac{r(Kc-d)}{mKc}\right)$$

in the first quadrant for the system (2.1). To understand the local behavior of the system (2.1), firstly we discuss the stability of the equilibrium points in the system (2.1) by the Hartman-Grobman theorem. Secondly, we use the Poincaré-Bendixson theorem to show that the global stability of E^*.

Now let us study the stability of the system (2.1) at equilibrium points E_0, E_K and E^*. The

Jacobian of (2.1) takes the form

$$A = \begin{bmatrix} r - \dfrac{2r}{K}x_1 - mx_2 & -mx_1 \\ & \\ cx_2 & -d + cx_1 \end{bmatrix}$$

At E_0

$$A_0 = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix}$$

Since $\det(A_0) = -rd < 0$, the equilibrium point E_0 is a saddle point with the x_2-axis as its stable manifold and the x_1-axis as its unstable manifold.

At E_K

$$A_K = \begin{bmatrix} -r & -mK \\ 0 & -d + cK \end{bmatrix}$$

Since $-r < 0$, $-d + cK > 0$ and $det(A_K) < 0$, the equilibrium point E_K is a saddle point with the x_1-axis as its stable manifold.

At E^*

$$A^* = \begin{bmatrix} -\dfrac{rd}{Kc} & -\dfrac{md}{c} \\ & \\ \dfrac{r\,(Kc-d)}{mK} & 0 \end{bmatrix}$$

Since

$$\begin{aligned} \operatorname{trace}(A^*) &= -\frac{rd}{Kc} < 0 \\ \det(A^*) &= \frac{rd\,(Kc-d)}{Kc} \; > \; 0 \end{aligned}$$

therefore the equilibrium point E^* is locally asymptotically stable.

Before mention Lemma 2.1 we need the following notation(see Figure 1). Let

$$\begin{aligned} \Gamma &= \{(x_1(t), x_2(t)) : x_1(0) = K, x_2(0) = \hat{x}_2 > x_2^*, t \geq 0\} \\ \Gamma_1 &= \{(x_1(t), x_2(t)) : x_1(0) = K, x_2(0) = \alpha, t \geq 0\} \\ \Gamma_2 &= \{(x_1(t), x_2(t)) : x_1(0) = K, x_2(0) = \beta, t \geq 0\} \end{aligned}$$

where $\alpha < \hat{x}_2 < \beta$ and say $p_1 \equiv (K, \widehat{x}_2)$. Thus we know that $\dot{x}_2$ is positive, negative and zero as $x_1 > x_1^*$, $x_1 < x_1^*$ and $x_1 = x_1^*$, respectively. Therefore, there is a point $(x_1^*, \gamma) \equiv p_2 \in \Gamma$ such that the solution $(x_1(t), x_2(t)) \in \Gamma$ with $x_2(t) \leq \gamma$ for all $t \geq 0$.

Lemma 2.1 *All solutions $(x_1(t), x_2(t))$ of the system (2.1) are positive and bounded. That is, any solution $(x_1(t), x_2(t))$ of the system (2.1) will enter into a positive and bounded region Ω, where the region Ω is bounded by $x_1 = 0$, $x_1 = K$, $x_2 = 0$, $x_2 = \gamma$ and the curve $\widehat{p_1 p_2}$ (See Figure 1).*

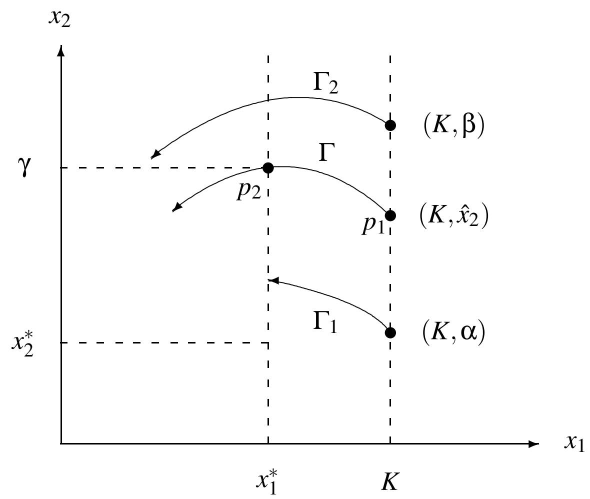

Figure 1: $p_1 = (K,\hat{x}_2)$, $p_2 = (x_1^*,\gamma)$

Proof. Firstly, we want to show that all solutions $(x_1(t),x_2(t))$ of the system (2.1) are positive. That is, if $(x_1(0),x_2(0))$ is in the first quadrant then $(x_1(t),x_2(t))$ is also in the first quadrant for all $t > 0$. If not, there is a $t_1 > 0$ such that $(x_1(t_1),x_2(t_1))$ is equal to the point $(x_1,0)$ or $(0,x_2)$ which belong to the x_1-axis or x_2-axis. But this contradicts the fact that the positive x_1-axis and the positive x_2-axis are the unique stable manifold of E_K and E_0, respectively.

Secondly, we want to show that all solutions $(x_1(t),x_2(t))$ of the system (2.1) are bounded. In the first quadrant, we know that $\dot{x_1}$ is negative for all $x_1 \geq K$; $x_2 > 0$. Hence, for all solutions $(x_1(t),x_2(t))$ of the system (2.1) with initial point $(x_1(0),x_2(0))$ and $x_1(0) \geq K$, there exists $s > 0$ such that $x_1(t) < K$ for $t \geq s$.

It is easy to show that the orbit Γ_1 is contained in the region Ω, because the orbits Γ_1 and Γ cannot cross at any point except the unique equilibrium point $E^* = (x_1^*,x_2^*)$. Let A be the region which is bounded by $x_1 = 0$, $x_1 = x_1^*$ and $x_2 > \frac{r}{m}(1-\frac{x_1}{K})$ (See Figure 2). The orbit Γ_2 starts at (K,β) and will enter into the region A since $\dot{x}_2$ is positive, negative and zero as $x_1 > x_1^*$, $x_1 < x_1^*$ and $x_1 = x_1^*$, respectively. By the slope of the trajectory in the region A (See Figure 2), the orbit Γ_2 finally will enter into the region Ω, too. If not, it will run into the x_2-axis. But this contradicts the fact that the positive x_2-axis is the unique stable manifold of E_0.

Then following theorem to assure the global stability of the equilibrium point E^* in the system (2.1).

Theorem 2.2 *If $Kc > d$, then the unique positive equilibrium point E^* of the system (2.1) is globally asymptotically stable.*

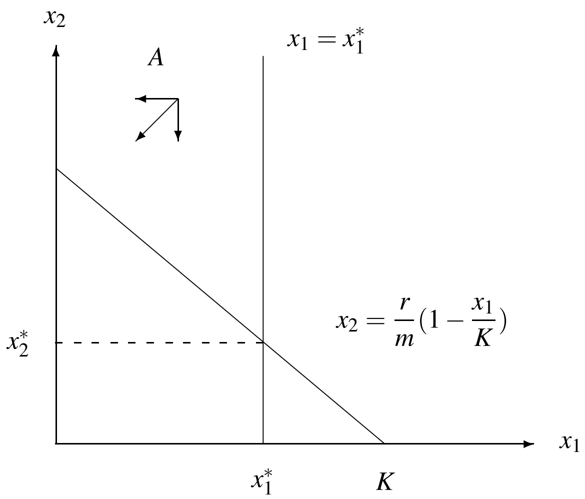

Figure 2: The slope of trajectory shown in the region A, where A is the region which is bounded by $x_1 = 0, x_1 = x_1^*$ and $x_2 > \frac{r}{m}(1 - \frac{x_1}{K})$

Proof. Let

$$H(x_1,x_2) = \frac{1}{x_1 x_2} \quad , \qquad x_1 > 0,\ x_2 > 0$$

Then

$$\begin{aligned}
& \frac{\partial}{\partial x_1} H \left\{ r x_1 \left[1 - \frac{x_1}{K} \right] - m x_1 x_2 \right\} + \frac{\partial}{\partial x_2} H \left\{ -d x_2 + c x_1 x_2 \right\} \\
= \quad & \frac{\partial}{\partial x_1} \left[\frac{r}{x_2} \left(1 - \frac{x_1}{K} \right) - m \right] + \frac{\partial}{\partial x_2} \left[-\frac{d}{x_1} + c \right] \\
= \quad & -\frac{r}{K x_2} \\
< \quad & 0
\end{aligned}$$

Hence by the Dulac's criterion, there is no closed orbit in the first quadrant. From above, we see that E^* is locally asymptotically stable. By Lemma 2.1 and the Poincaré-Bendixson theorem, it suffices to show that the unique positive equilibrium point E^* is globally asymptotically stable in the first quadrant.

3 The Model with a Single Delay

Consider the following predator-prey system with a single delay [9]

$$\begin{aligned} \dot{x}_1(t) &= rx_1(t)\left[1-\tfrac{x_1(t-\tau)}{K}\right]-mx_1(t)x_2(t) \\ \dot{x}_2(t) &= -dx_2(t)+cx_1(t)x_2(t) \end{aligned} \tag{3.1}$$

where x_1 and x_2 are density of prey and predator, respectively, and $r,\ \tau,\ K,\ m,\ d,\ c$ are positive constants. Clearly, the equilibrium points of the system (3.1) are the same as in the system (2.1).

We can use the techniques in [2,4,5,12] to check the boundedness and the point-wise positiveness of the solutions of (3.1) with the nonnegative initial value function $\phi \in C$ with positive components. Here our main purpose is to find the delay margins for local stability of the system (3.1) and analyze the stability switches of the linearized system of (3.1).

In order to understand the locally asymptotical stability of the equilibrium point E^* in (3.1), we analyze the associated linearized system with the perturbations $u=(u_1,u_2)$

$$\begin{aligned} \dot{u}_1(t) &= -\frac{rx_1{}^*}{K}u_1(t-\tau)-mx_1^*u_2(t) \\ \dot{u}_2(t) &= cx_2^*u_1(t) \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} u_1(t) &= x_1(t)+x_1^* \\ u_2(t) &= x_2(t)+x_2^* \end{aligned}$$

Hence, analyzing the local stability of E^* in the system (3.1) is equivalent to analyzing the stability of the zero solution in the system (3.2).

Before we mention our results we consider the n-th order real scalar linear neutral delay differential equation:

$$\sum_{k=0}^{n} a_k y^{(k)}(t)+\sum_{k=0}^{n} b_k y^{(k)}(t-\tau)=0 \tag{3.3}$$

where $y^{(0)} \equiv y(t)$. Thus the stability analysis of (3.3) is equivalent to the problem of determining conditions under which all roots of its characteristic equation

$$\sum_{k=0}^{n} a_k\lambda^k+\left(\sum_{k=0}^{n} b_k\lambda^k\right)e^{-\lambda\tau}=0 \tag{3.4}$$

lie in the left of the complex plane and are uniformly bounded away from the imaginary axis. Without loss of generality, we assume $a_n=1$.

Theorem 3.1 [8] *If $|b_n|>1$, then, for all $\tau>0$, there is an infinite number of roots of (3.4) whose real parts are positive.*

An immediate consequence of this theorem is the following:

Theorem 3.2 [8] *If $|b_n| > 1$, then the trivial solution of (3.3) is unstable for all $\tau > 0$.*

Finally, we will quote a theorem in [7] to compute delay margins for stability of linear delay systems. Consider

$$y^{(n)}(t) + \sum_{j=0}^{n-1}\sum_{k=0}^{q} a_{kj} y^{(j)}(t - k\tau) = 0 \tag{3.5}$$

where a_{kj}, $k = 0, 1, ..., q$, $j = 0, 1, ..., n-1$ are given real constants, and $\tau \geq 0$ is a delay constant.

Theorem 3.3 [7] *Suppose that the system (3.5) is stable for $\tau = 0$. Let $H_n \equiv 0$, $T_n \equiv 0$, and*

$$H_j \equiv \begin{bmatrix} a_{qj} & a_{q-1\ 1} & \cdots & a_{1j} \\ 0 & a_{qj} & \cdots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{qj} \end{bmatrix}, \quad T_j \equiv \begin{bmatrix} a_{0j} & 0 & \cdots & 0 \\ a_{1j} & a_{0j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{q-1\ j} & a_{q-2\ j} & \cdots & a_{0j} \end{bmatrix}, \quad j = 0, 1, ..., n-1,$$

$$P_j \equiv \begin{bmatrix} (i)^j T_j & (i)^j H_j \\ (-i)^j H_j^T & (-i)^j T_j^T \end{bmatrix}, \quad j = 0, 1, ..., n.$$

Furthermore, define

$$P \equiv \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -P_n^{-1}P_0 & -P_n^{-1}P_1 & \cdots & -P_n^{-1}P_{n-1} \end{bmatrix}$$

Then the delay margin $\tau^* = \infty$ if $\sigma(P) \cap R_- = \emptyset$ or $\sigma(P) \cap R_+ = \{0\}$. Additionally, let

$$F(\lambda) \equiv \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0(\lambda) & -a_1(\lambda) & \cdots & -a_{q-1}(\lambda) \end{bmatrix},$$

$$G(\lambda) \equiv \mathrm{diag}(1 \ \cdots \ 1 \ a_q(\lambda))$$

Then the delay margin $\tau^* = \infty$ if $\sigma(F(i\omega_k), G(i\omega_k)) \cap \partial D = \emptyset$ for all $0 \neq \omega_k \in \sigma(P) \cap R_+$. In these cases, the system (3.5) is stable for all $\tau \in [0, \infty)$. Otherwise,

$$\tau^* \equiv \min_{1 \leq k \leq 2nq} \frac{\alpha_k}{\omega_k}$$

where $0 \neq \omega_k \in \sigma(P) \cap R_+$ and $\alpha_k \in [0, 2\pi]$ satisfy the relation $e^{-i\alpha_k} \in \sigma(F(i\omega_k), G(i\omega_k))$. The system (3.5) is stable for all $\tau \in [0, \tau^*)$ and is unstable at $\tau = \tau^*$.

Now, we will mention our results.

Theorem 3.4 *If $Kc > d$, then E^* is stable for all $\tau \in [0, \tau^*)$, and becomes unstable at $\tau = \tau^*$, where*

$$\tau^* = \frac{\pi/2}{\omega_+},$$

and $\omega_+ > 0$ satisfies

$$\begin{aligned}\omega_\pm^2 &= \frac{1}{2}\Bigg\{\left[\left(\frac{rx_1^*}{K}\right)^2 + 2mcx_1^*x_2^*\right] \\ &\quad \pm \left[\left(\left(\frac{rx_1^*}{K}\right)^2 + 2mcx_1^*x_2^*\right)^2 - 4\left(mcx_1^*x_2^*\right)^2\right]^{\frac{1}{2}}\Bigg\}. \qquad (3.6)\end{aligned}$$

Proof. We can rewrite the system (3.2) as a delay differential equation

$$\ddot{u}_2(t) + \frac{rx_1^*}{K}\dot{u}_2(t-\tau) + mcx_1^*x_2^*u_2(t) = 0 \qquad (3.7)$$

Thus the characteristic equation is obtained as

$$\lambda^2 + \frac{rx_1^*}{K}\lambda e^{-\lambda\tau} + mcx_1^*x_2^* = 0 \qquad (3.8)$$

First, we want to determine the stability of the zero solution of (3.7) with $\tau = 0$. When $\tau = 0$ then the characteristic equation (3.8) becomes

$$\lambda^2 + \frac{rx_1^*}{K}\lambda + mcx_1^*x_2^* = 0$$

and it is easy to show that the zero solution of the system (3.7) is stable with $\tau = 0$ by checking the roots in the open left half plane.

Compare equation (3.7) with equation (3.5), now $q = 1$, $n = 2$. Hence using Theorem 3.3 we obtain readily

$$a_0(\lambda) = \lambda^2 + mcx_1^*x_2^* \qquad \text{and} \qquad a_1(\lambda) = \frac{rx_1^*}{K}\lambda$$

Then

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ mcx_1^*x_2^* & 0 & 0 & i\frac{rx_1^*}{K} \\ 0 & mcx_1^*x_2^* & i\frac{rx_1^*}{K} & 0 \end{bmatrix}.$$

So $\sigma(P) = \{\pm\omega_+, \pm\omega_-\}$, where $\omega_\pm > 0$ satisfy (3.6). Then

$$\sigma(P) \cap R_- = \{-\omega_\pm\} \qquad \text{and} \qquad \sigma(P) \cap R_+ = \{\omega_\pm\} \neq \{0\}.$$

Furthermore, we have

$$F(\lambda) = -a_0(\lambda) \qquad \text{and} \qquad G(\lambda) = a_1(\lambda).$$

Thus

$$\sigma(F(i\omega_\pm), G(i\omega_\pm)) = \{-i, i\} = \{e^{-i\pi/2}, e^{-i3\pi/2}\}$$

Hence

$$\tau^* = \frac{\pi/2}{\omega_+}.$$

Therefore, by Theorem 3.3, we know that the zero solution of the system (3.2) is stable for all $\tau \in [0, \tau^*)$, and that it becomes unstable at $\tau = \tau^*$.

Finally, we want to discuss the stability of the zero solution of the system (3.2) whenever $\tau > \tau^*$.

Theorem 3.5 *If $Kc > d$, then the stability of E^* of the system (3.1) changes a finite number of times (Suppose k times.) as τ is increased, and eventually it becomes unstable. And k switches from stability to instability to stability occur when the parameters are such that*

$$\tau_{0,1} < \tau_{0,2} < \tau_{1,1} < \tau_{1,2} < \cdots < \tau_{k-1,1} < \tau_{k-1,2} < \tau_{k,1} < \tau_{k+1,1} < \cdots$$

Here

$$\tau_{n,1} = \frac{\pi/2}{\omega_+} + \frac{2n\pi}{\omega_+}, \qquad \text{and} \qquad \tau_{n,2} = \frac{3\pi/2}{\omega_-} + \frac{2n\pi}{\omega_-},$$

where $n = 0, 1, 2, \cdots$; and $\omega_\pm > 0$ satisfy the equation (3.6).

Proof. The characteristic equation of (3.2) is the same as (3.8). Suppose $\lambda = i\omega$, $\omega > 0$, is a root of (3.8) for some τ (Since $mcx_1^*x_2^* \neq 0$, $\omega \neq 0$). So we have

$$\begin{aligned} \frac{rx_1^*}{K}\omega \sin\omega\tau &= \omega^2 - mcx_1^*x_2^* \\ \frac{rx_1^*}{K}\omega \cos\omega\tau &= 0 \end{aligned} \tag{3.9}$$

Then

$$\omega^4 - \left[\left(\frac{rx_1^*}{K}\right)^2 + 2mcx_1^*x_2^*\right]\omega^2 + (mcx_1^*x_2^*)^2 = 0 \tag{3.10}$$

Thus its roots satisfy (3.6), that is,

$$\begin{aligned} \omega_\pm^2 &= \frac{1}{2}\left\{\left[\left(\frac{rx_1^*}{K}\right)^2 + 2mcx_1^*x_2^*\right]\right. \\ &\qquad \left.\pm \left[\left(\left(\frac{rx_1^*}{K}\right)^2 + 2mcx_1^*x_2^*\right)^2 - 4\,(mcx_1^*x_2^*)^2\right]^{\frac{1}{2}}\right\}. \end{aligned}$$

Now, we need to determine the sign of the derivative of $Re\ \lambda(\tau)$ at the points where $\lambda(\tau)$ is purely imaginary. From (3.8), we have

$$\left\{2\lambda + (1-\tau\lambda)\frac{rx_1^*}{K}e^{-\lambda\tau}\right\}\frac{d\lambda}{d\tau} = \lambda^2\frac{rx_1^*}{K}e^{-\lambda\tau}$$

For convenience, we study $\left(\frac{d\lambda}{d\tau}\right)^{-1}$ instead of $\frac{d\lambda}{d\tau}$. Then

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{2\lambda e^{\lambda\tau} + \frac{rx_1^*}{K}(1-\lambda\tau)}{\frac{rx_1^*}{K}\lambda^2} \\ &= \frac{2\lambda e^{\lambda\tau} + \frac{rx_1^*}{K}}{\frac{rx_1^*}{K}\lambda^2} - \frac{\tau}{\lambda}\end{aligned}$$

and from (3.8) we know

$$e^{\lambda\tau} = \frac{-\frac{rx_1^*}{K}\lambda}{\lambda^2 + mcx_1^*x_2^*}$$

So

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{-\lambda^2 + mcx_1^*x_2^*}{\lambda^2(\lambda^2 + mcx_1^*x_2^*)} - \frac{\tau}{\lambda}$$

Therefore,

$$\begin{aligned}&\text{sign}\left\{\frac{d(Re\lambda)}{d\tau}\right\}_{\lambda=i\omega} = \text{sign}\left\{Re(\frac{d\lambda}{d\tau})^{-1}\right\}_{\lambda=i\omega} \\ &= \text{sign}\left\{Re\left[\frac{-1}{\lambda^2 + mcx_1^*x_2^*}\right]_{\lambda=i\omega} + Re\left[\frac{mcx_1^*x_2^*}{\lambda^2(\lambda^2 + mcx_1^*x_2^*)}\right]_{\lambda=i\omega}\right\} \\ &= \text{sign}\left\{\frac{-1}{-\omega^2 + mcx_1^*x_2^*} + \frac{mcx_1^*x_2^*}{\omega^4 - mcx_1^*x_2^*\omega^2}\right\} \\ &= \text{sign}\left\{\omega^4 - (mcx_1^*x_2^*)^2\right\} \\ &= \text{sign}\left\{\omega^4 + \omega^4 - \left(\frac{rx_1^*}{K}\right)^2\omega^2 - 2mcx_1^*x_2^*\right\} \\ &= \text{sign}\left\{2\omega^2 - \left[\left(\frac{rx_1^*}{K}\right)^2 + 2mcx_1^*x_2^*\right]\right\} \qquad (3.11)\end{aligned}$$

By inserting the expression for $\omega_{\pm}^2$, it is seen that the sign is positive for ω_+^2 and negative for ω_-^2. Therefore, crossing from left to right with increasing τ occurs for values of τ corresponding to ω_+ and crossing from right to left occurs for values of τ corresponding to ω_-. From equations (3.6) and (3.9), we obtain the following two sets of values of τ for which there are imaginary roots:

$$\tau_{n,1} = \frac{\pi/2}{\omega_+} + \frac{2n\pi}{\omega_+}\,, \qquad \text{and} \qquad \tau_{n,2} = \frac{3\pi/2}{\omega_-} + \frac{2n\pi}{\omega_-}\,,$$

where $n = 0, 1, 2, \cdots$.

We observe that

$$\tau_{n+1,1} - \tau_{n,1} = \frac{2\pi}{\omega_+} < \frac{2\pi}{\omega_-} = \tau_{n+1,2} - \tau_{n,2}$$

where $n = 0, 1, 2, \cdots$.

Therefore, there can be only a finite number of switches between stability and instability. Moreover, it is easy to see that there exist values of the parameters that realize any number of such stability switches. However, there exists a value of τ, $\tau = \hat{\tau}$, such that at $\tau = \hat{\tau}$ a stability switch occurs from stable to unstable, and for $\tau > \hat{\tau}$ the solution remain unstable.

As τ is increased, the multiplicity of roots of (3.8) for which $Re\lambda > 0$ is increased by two whenever τ passes through a value of $\tau_{n,1}$, and it is decreased by two whenever τ passes through a value of $\tau_{n,2}$. The k switches from stability to instability to stability occur when the parameters are such that

$$\tau_{0,1} < \tau_{0,2} < \tau_{1,1} < \tau_{1,2} < \cdots < \tau_{k-1,1} < \tau_{k-1,2} < \tau_{k,1} < \tau_{k+1,1} < \cdots$$

Here $\hat{\tau} = \tau_{k,1}$.

Hence, we can see that the the value τ^* in Theorem 3.4 is the same as $\tau_{0,1}$ in Theorem 3.5.

4 Example

In this section, we illustrate our result by one example.

Example 4.1

$$\begin{aligned} \dot{x}_1(t) &= 0.8x_1(t)\left[1 - \frac{x_1(t-\tau)}{100}\right] - 0.4x_1(t)x_2(t) \\ \dot{x}_2(t) &= -0.9x_2(t) + 0.1x_1(t)x_2(t) \end{aligned} \tag{4.1}$$

Comparing the system (4.1) with the system (3.1), we get $r = 0.8$, $K = 100$, $m = 0.4$, $d = 0.9$ and $c = 0.1$. So

$$E^* = (x_1^*, x_2^*) = (9, 1.82) \ ,$$

and the linearized system of (4.1) is

$$\begin{aligned} \dot{u}_1(t) &= -0.072u_1(t-\tau) - 3.6u_2(t) \\ \dot{u}_2(t) &= 0.182x_1(t) \end{aligned} \tag{4.2}$$

Then using the same technique as in section 3, we have 5 stability switches, and for $\tau > \tau_{5,1}$ the solution remain unstable. Here

$$
\begin{array}{ll}
\tau_{0,1} \approx 0.5908\pi, & \tau_{0,2} \approx 1.9374\pi \\
\tau_{1,1} \approx 2.9542\pi, & \tau_{1,2} \approx 4.5205\pi \\
\tau_{2,1} \approx 5.3176\pi, & \tau_{2,2} \approx 7.1037\pi \\
\tau_{3,1} \approx 7.6810\pi, & \tau_{3,2} \approx 9.6869\pi \\
\tau_{4,1} \approx 10.0444\pi, & \tau_{4,2} \approx 12.2700\pi \\
\tau_{5,1} \approx 12.4078\pi, & \tau_{5,2} \approx 14.8532\pi \\
\tau_{6,1} \approx 14.7711\pi, &
\end{array}
$$

Finally, we show some figures of its trajectory with some values of τ by MATLAB.

(a)

(b)

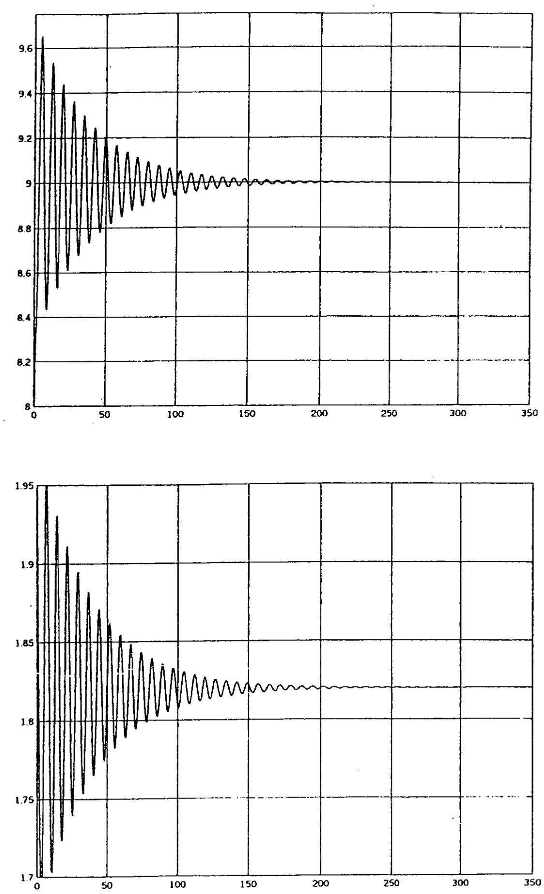

Figure 3: $0 < \tau = 1 < \tau_{0,1}$,$x_1(0) = 8$ and $x_2(0) = 2$. Figures (a) and (b) show that the trajectory of x_1 and x_2, respectively. It means that E^* is stable.

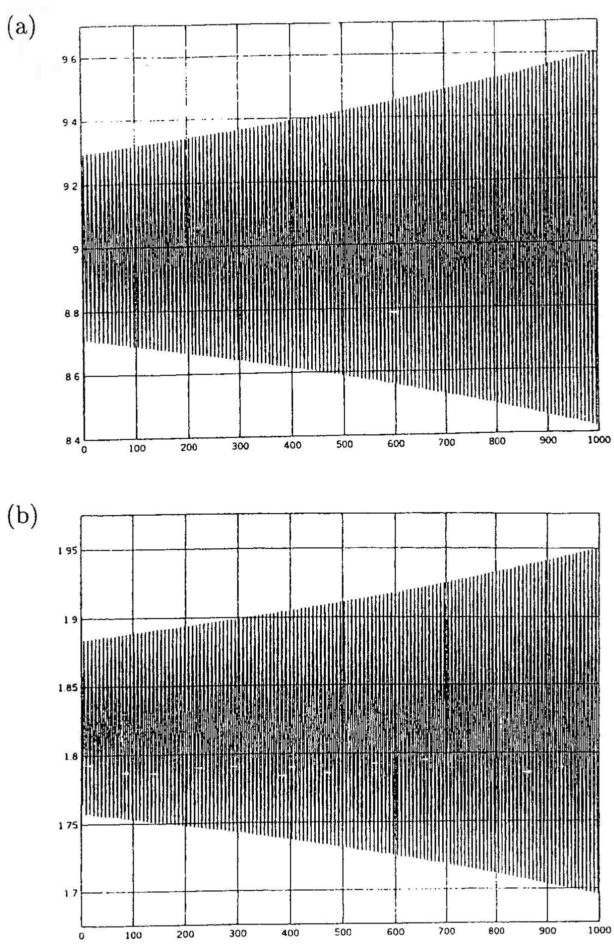

Figure 4: $\tau \approx \tau_{0,1}$,$x_1(0) = 8$ and $x_2(0) = 2$. Figures (a) and (b) show that the trajectory of x_1 and x_2, respectively. It means that E^* is unstable.

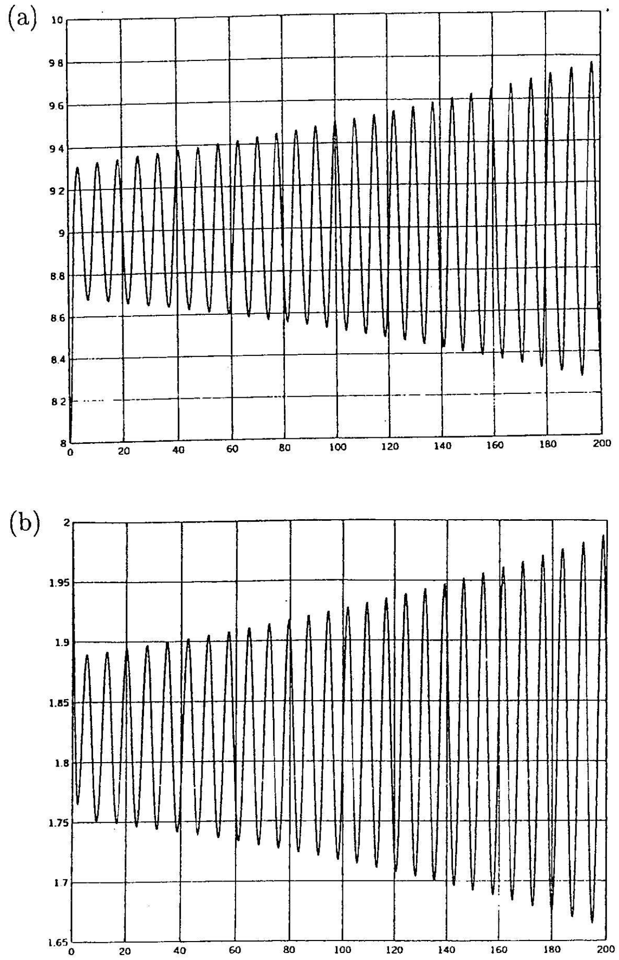

Figure 5: $\tau_{0,1} < \tau = 2 < \tau_{0,2}$,$x_1(0) = 8$ and $x_2(0) = 2$. Figures (a) and (b) show that the trajectory of x_1 and x_2, respectively. It means that E^* is unstable.

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時滯參數對捕食系統之局部穩定性的影響

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摘　　要

本篇文主要在研究時滯參數 τ 的大小是否會改變捕食系統

$$\dot{x}_1(t) = rx_1(t)\left[1 - \frac{x_1(t-\tau)}{K}\right] - mx_1(t)x_2(t)$$

$$\dot{x}_2(t) = -dx_2(t) + cx_1(t)x_2(t)$$

的局部穩定性。

關鍵詞：局部穩定性，穩定性切換。

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