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Is the Inverse Square Law an Optimal Controller from Nature? What is the Action?

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Abstract

The usual way in developing the formulation for classical mechanics is to define the Lagrangian for the action first and then the equation of motion is obtained by using calculus of variation to minimize the action. If we recognize the inverse square law being an optimal controller presented by nature, what is the corresponding action (or Lagrangian) for it? First of all, a planar motion of two bodies is considered. An optimal control problem is then formulated with a presumed unknown Lagrangian. By using Pontryagin minimal principle to minimize the action, a partial differential equation for the Lagrangian is obtained and solved. For this action, it can be verified directly that the inverse square law is the corresponding optimal controller. Finally, the generalization of this mechanization is presented for more complicated dynamical systems. This type of problem is considered as an inverse problem from the optimal control theory point of view.

Keywords: inverse square law, Lagrangian, Pontryagin minimal principle, optimal control, inverse problem
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1 Introduction

"Everything happens as if matter attracts matter in direct proportion to the products of masses and in inverse proportion to the square of the distances".

This famous proposition of Newton's "Principia"(1687) was a decisive step in our understanding of the Universe. It describes the law of universal attraction which also known as the *inverse square law* in gravity field. Two point masses M and m are separated by the distance r and are attracted by each other, see Figure 1. The mass M experiences the force $F_{1,2}$ and the mass m the opposite force $F_{2,1}$:

$$\|F_{1,2}\| = \|F_{2,1}\| = \frac{GMm}{r^2}$$

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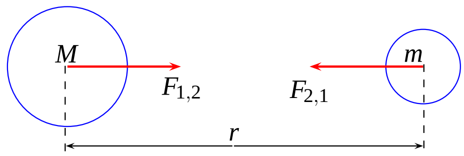

Figure 1: The law of universal attraction

where G is the Newtonian constant which is equal to $6.672 \times 10^{-11} m^2/s^2 kg$. Newton demonstrated that the forces $F_{1,2}$ and $F_{2,1}$ remain the same if the masses M and m have spherical symmetry instead of being point masses (r being the distance between centers). And the motion of point masses is derived based on the calculus of variation or the so-called *Hamilton's principle*.

Beside the gravity field, point sources of electric field, light, sound or radiation also obey the inverse square law. The inverse square law has long been recognized as the fundamental law in nature which can only be verified experimentally but can't be derived from other nature laws. Regarding to this point, a question has been asked in [6]: "Is there any special meanings inherent in inverse square law?" or in another way, "Is inverse square law the outcome of some optimal action taken by nature?" Motivated by the assumption of Maupertuis (1698-1759) who claimed that all the phenomena of nature can be derived by minimizing a quantity called "action", the present paper attempts to "derive" inverse square law from an optimal control sense. We also note that this Maupertuis's assumption is commonly associated with the principle of least action today.

The least action principle is the condition that the action is stationary under small variations around the optimal orbit when the initial and final positions are fixed and the Hamiltonian is constant along the optimal and varied orbits. Although the time integral of the Lagrangian in the "Hamilton's principle" also is very commonly called the action, these two principles do not coincide (for detail, see Goldstein [4]). In optimal control theory, the action is called cost function or performance index. Thus the least action principle is quite the same as the minimal principle in control literature.

As a result, our answer to the previous question is that inverse square law is truly a least action taken by nature to minimize some "action". This paper is organized as follows. In section 2, our main tool: the minimal principle of Pontryagin is reviewed. Then the previous question is formulated into an inverse optimal control problem in section 3. Section 4 presentes our main results. The Lagrangian in the action related to inverse square law will be derived first by solving analytically a second-order partial differential equation, then we show that the optimal control law minimizing the action is exactly the inverse square law. By treating inverse square law as a special example in section 5, we further generalize the formulation to give a necessary condition under which an arbitrary nature law can be considered the optimal control law manipulated by nature. Some concluding remarks are given in final section.

2 Preliminaries

In this section, the Pontryagin Minimal Principle is reviewed for later use. We use bold letter to denote vector or matrix and the symbol $\mathbb{R}$ for real number system.

Consider the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{2.1}$$

with $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^m$, such that

$$\mathbf{u}(t) \in U \subset \mathbb{R}^m. \tag{2.2}$$

Let t_0 and $t_f > t_0$ be given, and assume

$$\mathbf{x}(t_0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f \tag{2.3}$$

Define the cost

$$J(\mathbf{x}, \mathbf{u}, t_f) = \int_{t_0}^{t_f} L(\mathbf{x}, \mathbf{u})\, dt \tag{2.4}$$

where L is a smooth function

$$L : \mathbb{R}^n \times U \to \mathbb{R}.$$

We call L the *Lagrangian*. A typical optimal control problem is to find (if possible) the value t_f^*, the control law $\mathbf{u}^*_{[t_0,t_f^*]}$, the state trajectory $\mathbf{x}^*_{[t_0,t_f^*]}$ which satisfy the differential constraint (2.1), the constraints (2.3) and (2.4) and minimize the cost (2.4). This type of problem can be solved by calculus of variation or Pontryagin minimal principle [1, 3].

Theorem 2.1 (**Pontryagin Minimal Principle**) *Let $(\mathbf{x}^*, \mathbf{u}^*, t_f^*)$ be an admissible solution of the considered optimal control problem. Let*

$$H(\mathbf{x}, \mathbf{u}, \lambda_0, \lambda) = \lambda_0 L(x, u) + \lambda^T \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{2.5}$$

Then $(\mathbf{x}^*, \mathbf{u}^*, t_f^*)$ is an optimal solution only if there exist a constant $\lambda_0^* \geq 0$ and a (vector) function $\lambda^*(t) \in \mathbb{R}^n$, not simultaneously zero on any time instant $t \in [t_0, t_f]$, such that, for all $t \in [t_0, t_f]$, one has

$$\dot{\lambda}^* = -\left.\frac{\partial H}{\partial \mathbf{x}}\right|_{(\mathbf{x}^*, \mathbf{u}^*, \lambda_0^*, \lambda^*)}, \tag{2.6}$$

$$H(\mathbf{x}^*, \mathbf{u}, \lambda_0^*, \lambda^*) \geq H(\mathbf{x}^*, \mathbf{u}^*, \lambda_0^*, \lambda^*), \quad \forall \mathbf{u} \in U, \tag{2.7}$$

$$H(\mathbf{x}^*, \mathbf{u}^*, \lambda_0^*, \lambda^*) = 0. \tag{2.8}$$

Moreover, discontinuities in $\dot{\lambda}^*$ occur only at the time instants $\bar{t}$ in which $\mathbf{u}^*$ is discontinuous.

Remark 2.2 Equation (2.6) is called the *adjoint equation* or *costate equation*. Thus, the Lagrange multiplier λ is also known as *costate*. The inequality (2.7) expresses the so-called *minimum principle*. Without loss of generality, we can assume $\lambda_0^* = 1$.

Remark 2.3 If the final time t_f is fixed, equation (2.8) has to be replaced by

$$H(\mathbf{x}^*, \mathbf{u}^*, \lambda_0^*, \lambda^*) = k, \qquad k \in \mathbb{R} \tag{2.9}$$

i.e., the function $H(\mathbf{x}^*, \mathbf{u}^*, \lambda_0^*, \lambda^*)$ is constant, but not necessarily zero, for all $t \in [t_0, t_f]$.

Remark 2.4 If the final state is not fixed and the cost includes a term depending on the final state, i.e.,

$$J(\mathbf{x}, \mathbf{u}, t_f) = \int_{t_0}^{t_f} L(\mathbf{x}, \mathbf{u})\; d\,t + G(t_f, \mathbf{x}(t_f))$$

then, as the boundary condition $\mathbf{x}(t_f) = \mathbf{x}_f$ is no more applicable, we need to specify other n boundary conditions. In this case, these are

$$\lambda^*(t_f) = \lambda_0^* \frac{\partial G(t, \mathbf{x})}{\partial \mathbf{x}}\bigg|_{(\mathbf{x}^*, \mathbf{u}^*, t_f^*)}.$$

Remark 2.5 Condition (2.7) is a condition of minimum for the function

$$H(\mathbf{x}, \mathbf{u}, \lambda_0, \lambda) = \lambda_0 L(\mathbf{x}, \mathbf{u}) + \lambda^T \mathbf{f}(\mathbf{x}, \mathbf{u})$$

If no constraints on the control are present, this condition must be substituted by the (obvious) condition

$$-\frac{\partial H}{\partial \mathbf{u}}\bigg|_{(\mathbf{x}^*, \mathbf{u}^*, \lambda_0^*, \lambda^*)} = 0. \tag{2.10}$$

3 Problem Formulation

Consider the motion of a system consisting of two bodies affected by a force directed along the line connecting the centers of the two bodies. We restricted our attention to systems without friction losses and for which the potential energy is a function only of their distance. According to the standard technique in classical mechanics, this problem can be formally reduced to an equivalent one-body problem in which only the motion of a "particle" of mass m in the central field described by the potential function $U(r)$ where r is the distance to the original as shown in Figure 2.

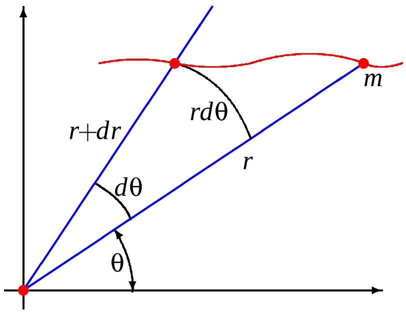

Figure 2: The trajectory $r(t)$ of a moving particle in the planar motion

The system satisfies the following assumption:

Assumption 1 ***(Conservation of Angular Momentum)*** *The total angular momentum of the system is constant.*

Based on this assumption, the trajectory of the particle stays on the same plane [2]. The angular momentum of this system is denoted by

$$mr^2\dot{\theta} = \ell,$$

or equivalently,

$$\dot{\theta} = \frac{\ell}{mr^2} \tag{3.1}$$

where ℓ denotes the total angular momentum of this system. The conservation of angular momentum requires ℓ be a constant or

$$\frac{d}{dt}mr^2\dot{\theta} = 0 \tag{3.2}$$

The trajectory of the particle m is governed by the differential equations in r and θ directions, respectively,

$$m\ddot{r} - mr\dot{\theta}^2 = u \tag{3.3a}$$

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \tag{3.3b}$$

where u denotes the control force. The gravity force u^* is equal to the gradient of the potential function $U(r)$ in the gravity filed, i.e.

$$u^*(r) = -\nabla U = -\frac{\partial}{\partial r}\left(-\frac{K}{r}\right) = -\frac{K}{r^2}$$

with a constant K. Our purpose is to show that this unique force u^* with its magnitude K/r^2 (satisfying the inverse square law) can be recognized as an optimal controller introduced by nature to minimize certain action.

Since the left hand side (LHS) of (3.3b) can be rewritten as

$$m(2\dot{r}\dot{\theta}+r\ddot{\theta}) = \frac{1}{r}\frac{d}{dt}\left(mr^2\dot{\theta}\right)$$

and by the conservation of angular momentum (3.3b) holds. From (3.1) the independent variable in (3.3a) can be changed from t to θ. Now, by using (3.1) it follows that

$$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = \frac{\ell}{mr^2}\frac{dr}{d\theta} = -\frac{\ell}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right) \tag{3.4a}$$

$$\frac{d^2r}{dt^2} = \frac{d}{d\theta}\left(\frac{dr}{dt}\right)\cdot\frac{d\theta}{dt} = -\frac{\ell^2}{m^2r^2}\frac{d}{d\theta}\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\ell^2}{m^2r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \tag{3.4b}$$

and the substitution of (3.1) and (3.4) into (3.3a) leads to

$$\frac{\ell^2}{mr^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right] = -u \tag{3.5}$$

Let $a = m/\ell^2$, $x_1 = 1/r$ and $x_2 = dx_1/d\theta = -m\dot{r}/\ell$, then the equation (3.5) can be expressed as a system of first order nonlinear differential equations

$$\frac{dx_1}{d\theta} = x_2 \tag{3.6a}$$

$$\frac{dx_2}{d\theta} = -x_1 - \frac{a}{x_1^2}u \tag{3.6b}$$

For convenience, we use above prime to denote the derivative with respect to θ, i.e., $x_1' = dx_1/d\theta$ and $x_2' = dx_2/d\theta$. The corresponding optimal controller is given by

$$u^* = -\frac{K}{r^2} = -Kx_1^2$$

thus, we can recognize u as a function depending on the variables x_1 and x_2. If the planetary motion is limited for $r \in [r_0,\infty)$ for some positive constant r_0, the variables x_1 and x_2 can be considered belongs to a closed bounded domain D in $\mathbb{R}^2$. Thus u belongs to the admissible set

$$U = \{u \in \mathcal{C}^1(D,\mathbb{R}) : (3.6) \text{ is stable}\}$$

where $\mathcal{C}^1$ denotes the class of functions having continuous first order derivatives. And we may ask the following question:

Problem 1 *What is the corresponding decision principle made by nature to choose* $u^*(x_1,x_2) = -Kx_1^2$ *from all admissible functions* U*?*

From the optimal control theory point of view, we suggest that there is an cost function to be minimized for decision purpose. Thus we make the following assumption.

Assumption 2 ***(Action Functional)*** *There exists a Lagrangian* $L(x_1, x_2, u)$ *such that the mechanism of nature for gravity is to minimize the following action functional*

$$I(u) = \int_0^{\theta_f} L(x_1(\theta), x_2(\theta), u(x_1(\theta), x_2(\theta)))d\theta \tag{3.7}$$

This action functional $I(u)$ is also known as *the action* for simplicity. And then the following question needs to be answered:

Problem 2 *What does the Lagrangian* $L(x_1, x_2, u)$ *look like? What is the corresponding physical meaning of the action?*

In next section, Pontryagin Minimal Principle is adopted to answer these questions.

4 Main Results

The corresponding variational problem is to find a function u to minimize

$$I(u) = \int_0^{\theta_f} L(x_1(\theta), x_2(\theta), u(x_1(\theta), x_2(\theta)))\, d\theta \tag{4.1}$$

subjected to

$$\frac{dx_1}{d\theta} = x_2, \qquad x_1(0) = x_{10} \tag{4.2a}$$

$$\frac{dx_2}{d\theta} = -\left(x_1 + \frac{a}{x_1^2}u\right), \qquad x_2(0) = x_{20} \tag{4.2b}$$

Here θ_f is considered to be free. Let λ_1 and λ_2 be Lagrange multipliers (or costate) and let H be the Hamiltonian associated with the optimization problem, i.e.,

$$H(x_1, x_2, \lambda_1, \lambda_2, u) = L(x_1, x_2, u) + \lambda_1 x_2 - \lambda_2 \left(x_1 + \frac{a}{x_1^2}u\right) \tag{4.3}$$

Then the constraint optimization problem becomes as to minimize the augmented cost

$$J(x_1, x_2, u) = \int_0^{\theta_f} \left(H(x_1, x_2, \lambda_1, \lambda_2, u) - \lambda_1 x_1' - \lambda_2 x_2'\right) d\theta, \tag{4.4}$$

with $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$ from all possible u in the admissible set U. Since there is no other constraint on controller u, we can use Pontryagin Minimal Principle to solve this problem.

The corresponding optimal controller u^* must satisfy

$$\left.\frac{\partial H}{\partial u}\right|_{u^*} = \left.\frac{\partial L}{\partial u}\right|_{u^*} - \frac{a}{x_1^2}\lambda_2 = 0 \tag{4.5}$$

which leads to

$$\lambda_2 = \frac{x_1^2}{a}\left.\frac{\partial L}{\partial u}\right|_{u^*} \triangleq \frac{x_1^2}{a}\frac{\partial L}{\partial u^*} \tag{4.6}$$

On the other hand, H is not an explicit function of θ and by direction computation we obtain

$$\left.\frac{dH}{d\theta}\right|_{u^*} = \left.\left(\frac{\partial H}{\partial x_1}x_1' + \frac{\partial H}{\partial x_2}x_2' + \frac{\partial H}{\partial u}u' + \frac{\partial H}{\partial \lambda_1}\lambda_1' + \frac{\partial H}{\partial \lambda_2}\lambda_2'\right)\right|_{u^*} = 0$$

along the optimal trajectory. Therefore without loss of generality we arrive zero Hamiltonian function along the optimal trajectory

$$H^* = H(x_1, x_2, \lambda_1, \lambda_2, u^*) = L^* + \lambda_1 x_2 - \lambda_2(x_1 - aK) = 0 \tag{4.7}$$

where $L^* = L(x_1, x_2, u^*)$. The combination of (4.5), (4.6), and (4.7) gives us

$$\lambda_1 = \frac{1}{x_2}\left[-L^* + (x_1 - aK)\lambda_2\right] \tag{4.8}$$

Before doing further, we need to compute the following derivatives first along the optimal trajectory. The total derivative of the optimal control u^* is given by

$$\frac{d}{d\theta}u^*(x_1, x_2) = \frac{\partial u^*}{\partial x_1}x_1' = -2Kx_1x_2$$

and those of L^* and $\partial L/\partial u^*$ are

$$\begin{aligned} \frac{d}{d\theta}L^* &= \frac{\partial L^*}{\partial x_1}x_1' + \frac{\partial L^*}{\partial x_2}x_2' = x_2\frac{\partial L^*}{\partial x_1} - (x_1 - aK)\frac{\partial L^*}{\partial x_2} \\ \frac{d}{d\theta}\left(\frac{\partial L}{\partial u^*}\right) &= \frac{\partial^2 L}{\partial x_1 \partial u^*}x_1' + \frac{\partial^2 L}{\partial x_2 \partial u^*}x_2' = x_2\frac{\partial^2 L}{\partial x_1 \partial u^*} - (x_1 - aK)\frac{\partial^2 L}{\partial x_2 \partial u^*} \end{aligned}$$

At the meanwhile, λ_1 and λ_2 for optimal trajectory must satisfy the adjoint equations

$$\frac{d\lambda_1}{d\theta} = -\frac{\partial H^*}{\partial x_1} = -\frac{\partial L^*}{\partial x_1} + \lambda_2 \tag{4.9}$$

Since λ_1' could also be computed by

$$\frac{d\lambda_1}{d\theta} = -\left.\frac{\partial H}{\partial x_1}\right|_{u^*}$$

it follows that

$$\frac{\partial L^*}{\partial x_1} = \left.\frac{\partial L}{\partial x_1}\right|_{u^*} - 2\frac{aK}{x_1}\lambda_2$$

Similarly,

$$\frac{d\lambda_2}{d\theta} = -\frac{\partial H^*}{\partial x_2} = -\frac{\partial L^*}{\partial x_2} - \lambda_1 = -\frac{\partial L^*}{\partial x_2} + \frac{1}{x_2}\left[L^* - (x_1 - aK)\lambda_2\right] \tag{4.10}$$

and

$$\frac{\partial L^*}{\partial x_2} = \left.\frac{\partial L}{\partial x_2}\right|_{u^*}$$

But the terms $d\lambda_1/d\theta$ and $d\lambda_2/d\theta$ can also be obtained by differentiating (4.8) and (4.6), respectively. Thus we arrive at

$$\begin{aligned} \frac{d\lambda_1}{d\theta} &= -\frac{x_2'}{x_2^2}[-L^* + (x_1 - aK)\lambda_2] + \frac{1}{x_2}\left[-\frac{dL^*}{d\theta} + x_1'\lambda_2 + (x_1 - aK)\frac{d\lambda_2}{d\theta}\right] \\ &= -\frac{x_1 - aK}{x_2^2}[L^* - (x_1 - aK)\lambda_2] + \frac{1}{x_2}\left[-x_2\frac{\partial L^*}{\partial x_1} + (x_1 - aK)\frac{\partial L^*}{\partial x_2}\right] \\ &\quad + \lambda_2 + \frac{1}{x_2}(x_1 - aK)\frac{d\lambda_2}{d\theta} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \frac{d\lambda_2}{d\theta} &= \frac{2x_1x_1'}{a}\frac{\partial L}{\partial u^*} + \frac{x_1^2}{a}\frac{d}{d\theta}\left(\frac{\partial L}{\partial u^*}\right) \\ &= \frac{2x_1x_2}{a}\frac{\partial L}{\partial u^*} + \frac{x_1^2x_2}{a}\frac{\partial^2 L}{\partial x_1 \partial u^*} - \left(\frac{x_1^3}{a} + u^*\right)\frac{\partial^2 L}{\partial x_2 \partial u^*} \end{aligned} \tag{4.12}$$

Since $d\lambda_1/d\theta$ in (4.11) can be rewritten as

$$\frac{d\lambda_1}{d\theta} = -\frac{\partial L^*}{\partial x_1} + \lambda_2 + \frac{x_1 - aK}{x_2}\left[\frac{d\lambda_2}{d\theta} + \frac{\partial L^*}{\partial x_2} - \frac{1}{x_2}\left(L^* - (x_1 - aK)\lambda_2\right)\right]$$

which is the same as (4.9) after using the equation (4.10). For the term $d\lambda_1/d\theta$, the right hand sides (RHS) (4.10) and (4.12) must equal each other which leads to the relationship

$$\begin{aligned} 0 &= \frac{2x_1x_2}{a}\frac{\partial L}{\partial u^*} + \frac{x_1^2x_2}{a}\frac{\partial^2 L}{\partial x_1 \partial u^*} - \left(\frac{x_1^3}{a} + u^*\right)\frac{\partial^2 L}{\partial x_2 \partial u^*} \\ &\quad + \frac{\partial L^*}{\partial x_2} - \frac{1}{x_2}\left[L^* - (x_1 - aK)\lambda_2\right] \end{aligned}$$

or equivalently,

$$\begin{aligned} &-L^* + x_2\frac{\partial L^*}{\partial x_2} + \frac{1}{a}\left(x_1^3 + 2x_1x_2^2 + au^*\right)\frac{\partial L}{\partial u^*} \\ &+ \frac{x_1^2x_2^2}{a}\frac{\partial^2 L}{\partial x_1 \partial u^*} - \left(\frac{x_1^3}{a} + u^*\right)x_2\frac{\partial^2 L}{\partial x_2 \partial u^*} = 0 \end{aligned}$$

Thus the action function is the solution of this partial differential equation.

We can recapitulate as follows:

Lemma 4.1 *The optimal Lagrangian L^* for the least action of the system (4.2) imposed by the controller $u^* = -Kx_1^2$ is the solution of the following partial differential equation:*

$$\begin{aligned} -L^* + x_2\frac{\partial L^*}{\partial x_2} + \frac{1}{a}\left(x_1^3 + 2x_1x_2^2 + au^*\right)\frac{\partial L}{\partial u^*} & \\ + \frac{x_1^2x_2^2}{a}\frac{\partial^2 L}{\partial x_1 \partial u^*} - \left(\frac{x_1^3}{a} + u^*\right)x_2\frac{\partial^2 L}{\partial x_2 \partial u^*} &= 0 \end{aligned} \tag{4.13}$$

Although the existence and uniqueness of the solution for (4.13) is not clear, we could still find its solution in the following way. Rewrite (4.13) into the following form:

$$0 = -L^* + x_2\frac{\partial L^*}{\partial x_2} \tag{4.14a}$$

$$+\frac{x_1x_2^2}{a}\left(2\frac{\partial L}{\partial u^*} + x_1\frac{\partial^2 L}{\partial x_1 \partial u^*}\right) \tag{4.14b}$$

$$+\left(\frac{x_1^3}{a} + u^*\right)\left(\frac{\partial L}{\partial u^*} - x_2\frac{\partial^2 L}{\partial x_2 \partial u^*}\right) \tag{4.14c}$$

Therefore, a special subset of solutions can be found by considering parts (b) and (c) in (4.14) to be zeros, i.e.,

$$\begin{aligned} 2\frac{\partial L}{\partial u^*} + x_1\frac{\partial^2 L}{\partial x_1 \partial u^*} &= \frac{1}{x_1}\frac{\partial}{\partial x_1}\left(x_1^2\frac{\partial L}{\partial u^*}\right) = 0 \\ \frac{\partial L}{\partial u^*} - x_2\frac{\partial^2 L}{\partial x_2 \partial u^*} &= -x_2^2\frac{\partial}{\partial x_2}\left(\frac{1}{x_2}\frac{\partial L}{\partial u^*}\right) = 0 \end{aligned}$$

Thus, we suggest $\partial L/\partial u^*$ to be of the form

$$\frac{\partial L}{\partial u^*} = \left.\frac{\partial L}{\partial u}\right|_{u^*} = C_1\frac{x_2}{x_1^2}, \qquad C_1 \in \mathbb{R}$$

it follows that

$$\frac{\partial L}{\partial u} = C_1\frac{x_2}{x_1^2} + \bar{C}_2(x_1, x_2, u)(u - u^*)$$

Integrating about u gives us

$$L(x_1, x_2, u) = C_1\frac{x_2}{x_1^2}(u - u^*) + C_2(x_1, x_2, u)(u - u^*)^2 + \bar{C}_3(x_1, x_2)$$

where C_2 and $\bar{C}_3$ are unknown functions to be determined. Substituting L back into the part (a) in (4.14) leads to

$$-\bar{C}_3(x_1, x_2) + x_2\frac{\partial \bar{C}_3}{\partial x_2} = x_2^2\frac{\partial}{\partial x_2}\left(\frac{1}{x_2}\bar{C}_3\right) = 0$$

i.e., we can choose $\bar{C}_3(x_1, x_2) = x_2C_3(x_1)$. Since the value of C_1 will not affect the optimization process, choose $C_1 = 1$ for simplicity. Therefore the Lagrangian may have the form

$$L(x_1, x_2, u) = \frac{x_2}{x_1^2}(u - u^*) + C_2(x_1, x_2, u)(u - u^*)^2 + x_2C_3(x_1) \tag{4.15}$$

Thus the action functional is given by

$$I(u) = \int_0^{\theta_f} \left(\frac{x_2}{x_1^2}(u - u^*) + C_2(x_1, x_2, u)(u - u^*)^2 + x_2 C_3(x_1) \right) d\theta \tag{4.16}$$

We have the following result:

Theorem 4.1 *Let $C_2(x_1, x_2, u)$ be a nonnegative function. Suppose there exists a minimal value for action functional*

$$I(u) = \int_0^{\theta_f} \left(\frac{x_2}{x_1^2}(u + Kx_1^2) + C_2(x_1, x_2, u)(u + Kx_1^2)^2 + x_2 C_3(x_1) \right) d\theta \tag{4.17}$$

subjected to the system

$$\frac{dx_1}{d\theta} = x_2, \qquad x_1(0) = x_{10},$$
$$\frac{dx_2}{d\theta} = -\left(x_1 + \frac{a}{x_1^2} u \right), \qquad x_2(0) = x_{20},$$

then the corresponding controller is given by

$$u^* = -Kx_1^2$$

Moreover, the least action is

$$I^* = \min_{u \in U} I(u) = C_4(x_1^*(\theta_f)) - C_4(x_{10})$$

where C_4 is an antiderivative of C_3 and

$$x_1^*(\theta) = x_{20} \sin\theta + (x_{10} - aK)\cos\theta + aK.$$

Proof: Let λ_1 and λ_2 be the associated Lagrange multipliers, the Hamiltonian for this optimization problem is given by

$$\begin{aligned} H(x_1, x_2, u, \lambda_1, \lambda_2) &= \frac{x_2}{x_1^2}(u + Kx_1^2) + C_2(x_1, x_2, u)(u + Kx_1^2)^2 + x_2 C_3(x_1) \\ &\quad + \lambda_1 x_2 - \lambda_2 \left(x_1 + \frac{a}{x_1^2} u \right) \end{aligned}$$

Let $(x_1^*, x_2^*, u^*, \lambda_1^*, \lambda_2^*)$ be the admissible solution, it follows that u^* must satisfy

$$\left. \frac{\partial H}{\partial u} \right|_{(x_1^*, x_2^*, u^*, \lambda_1^*, \lambda_2^*)} = \frac{x_2^*}{x_1^{*2}} - \frac{a}{x_1^{*2}} \lambda_2^* = 0, \tag{4.18}$$

(x_1^*, x_2^*) is the solution of

$$\frac{dx_1^*}{d\theta} = x_2^*, \qquad x_1^*(0) = x_{10}, \tag{4.19a}$$

$$\frac{dx_2^*}{d\theta} = -\left(x_1^* + \frac{a}{x_1^{*2}} u^* \right), \qquad x_2^*(0) = x_{20}, \tag{4.19b}$$

and $(\lambda_1^*, \lambda_2^*)$ is the solution of adjoint equation

$$\frac{d\lambda_1^*}{d\theta} = -\left.\frac{\partial H}{\partial x_1}\right|_{(x_1^*, x_2^*, u^*, \lambda_1^*, \lambda_2^*)} = \left(1 - \frac{2a}{x_1^{*3}} u^*\right) \lambda_2^* - x_2^* \left(\frac{2K}{x_1^*} + C_3'(x_1^*)\right), \tag{4.20a}$$

$$\frac{d\lambda_2^*}{d\theta} = -\left.\frac{\partial H}{\partial x_2}\right|_{(x_1^*, x_2^*, u^*, \lambda_1^*, \lambda_2^*)} = -C_3(x_1^*) - \lambda_1^*. \tag{4.20b}$$

and

$$\begin{aligned} H(x_1^*, x_2^*, u^*, \lambda_1^*, \lambda_2^*) &= \frac{x_2^*}{x_1^{*2}}(u^* + K x_1^{*2}) + C_2(x_1^*, x_2^*, u^*)(u^* + K x_1^{*2})^2 + x_2^* C_3(x_1^*) \\ &\quad + \lambda_1^* x_2^* - \lambda_2^* \left(x_1^* + \frac{a}{x_1^{*2}} u^*\right) \end{aligned}$$

Now,

$$\begin{aligned} & H(x_1^*, x_2^*, u, \lambda_1^*, \lambda_2^*) - H(x_1^*, x_2^*, u^*, \lambda_1^*, \lambda_2^*) \\ &= [C_2(x_1^*, x_2^*, u)(u + K x_1^2)^2 - C_2(x_1^*, x_2^*, u^*)(u^* + K x_1^2)^2] \\ &\quad + \left(\frac{x_2^*}{x_1^{*2}} - \frac{a}{x_1^{*2}} \lambda_2^*\right)(u - u^*) \\ &= C_2(x_1^*, x_2^*, u)(u + K x_1^2)^2 - C_2(x_1^*, x_2^*, u^*)(u^* + K x_1^2)^2 \end{aligned}$$

From Pontryagin Minimal Principle, the condition

$$H(x_1^*, x_2^*, u, \lambda_1^*, \lambda_2^*) - H(x_1^*, x_2^*, u^*, \lambda_1^*, \lambda_2^*) \geq 0, \qquad \forall u \in U,$$

leads to the unique optimal control law

$$u^* = -K x_1^2$$

for all nonnegative function C_2. Substituting the optimal control law into $L(x_1, x_2, u)$ gives us

$$L^* = L(x_1^*, x_2^*, u) = x_2^* C_3(x_1^*)$$

Therefore, the least action is

$$I^* = \min_{u \in U} I(u) = \int_0^{\theta_f} x_2^* C_3(x_1^*) d\theta = \int_{x_{10}}^{x_1^*(\theta_f)} C_3(x_1) dx_1 = C_4(x_1^*(\theta_f)) - C_4(x_{10})$$

where C_4 is an antiderivative of C_3. The optimal trajectory in (4.19) is computed by

$$\frac{d}{d\theta} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} + \begin{bmatrix} 0 \\ aK \end{bmatrix}$$

with its solution

$$x_1^*(\theta) = x_{20} \sin\theta + (x_{10} - aK) \cos\theta + aK, \tag{4.21a}$$

$$x_2^*(\theta) = x_{20} \cos\theta + (aK - x_{10}) \sin\theta. \tag{4.21b}$$

This concludes our proof. □

Furthermore, we discuss some properties of Theorem 4.1. Firstly, for $u^* = -Kx_1^{*2}$ the Hamiltonian along the optimal trajectory is given by

$$H(x_1^*, x_2^*, u^*, \lambda_1^*, \lambda_2^*) = x_2^* (\lambda_1^* + C_3(x_1^*)) - \lambda_2^* (x_1^* - aK) = 0 \tag{4.22}$$

And using (4.22) the optimal costate (4.20) becomes

$$\begin{aligned} \frac{d}{d\theta} [\lambda_1^* + C_3(x_1^*)] &= \lambda_2^*, \\ \frac{d\lambda_2^*}{d\theta} &= K - \frac{x_1^*}{a}, \end{aligned}$$

with its solution

$$\lambda_1^*(\theta) + C_3(x_1^*(\theta)) = \frac{x_1^*}{a} - K = \frac{x_{20}}{a} \sin\theta + \left(\frac{x_{10}}{a} - K\right) \cos\theta + aK, \tag{4.23a}$$

$$\lambda_2^*(\theta) = \frac{x_2^*(\theta)}{a} = \frac{x_{20}}{a} \cos\theta + \left(K - \frac{x_{10}}{a}\right) \sin\theta. \tag{4.23b}$$

This solution is as same as computed by using (4.18) and (4.22). From (4.21) and (4.23) we see that the optimal state and costate are periodic functions with period 2π, i.e.,

$$\begin{aligned} &x_1^*(2N\pi) = x_{10}, \qquad \lambda_1^*(2N\pi) + C_3(x_1^*(2N\pi)) = \frac{x_{10}}{a} - K, \\ &x_2^*(2N\pi) = x_{20}, \qquad \lambda_2^*(2N\pi) = \frac{x_{20}}{a}. \end{aligned}$$

for all integer N.

Next, from Theorem 4.1, the only condition on the function C_2 is it must be nonnegative. And the selection of C_3 function will not affect the optimal control law and the optimal trajectory. Once we choose certain type of function for C_3, the optimal costate λ_1^* is determined according to (4.23a).

Thirdly, the meaning of the action functional I can be clarified more clearly. Since I^* can be expressed as

$$I^* = J^* = \int_0^{\theta_f} H(x_1^*, x_2^*, u^*, \lambda_1^*, \lambda_2^*) d\theta - \int_0^{\theta_f} \left(\lambda_1^* x_1^{*\prime} + \lambda_2^* x_2^{*\prime}\right) d\theta$$

and with the aid of (4.22) we arrive at

$$\begin{aligned} I^* &= -\int_0^{\theta_f} \left(\lambda_1^* x_1^{*\prime} + \lambda_2^* x_2^{*\prime}\right) d\theta \\ &= -\int_0^{\theta_f} [(\lambda_1^* + C_3(x_1^*)) dx_1^*(\theta) + \lambda_2^* dx_2^*(\theta)] + \int_0^{\theta_f} C_3(x_1^*) dx_1^*(\theta) \end{aligned}$$

Comparing with Theorem 4.1 leads to

$$\int_0^{\theta_f} (\lambda_1^* + C_3(x_1^*)) dx_1^*(\theta) + \lambda_2^* dx_2^*(\theta) = 0, \quad \forall \theta_f.$$

Let

$$E(x_1,x_2) = \int (\lambda_1 + C_3(x_1))dx_1 + \lambda_2 dx_2$$

then we have

$$\frac{dE}{d\theta} = \frac{\partial E}{\partial x_1}x_1' + \frac{\partial E}{\partial x_2}x_2' \triangleq (\lambda_1 + C_3(x_1))x_1' + \lambda_2 x_2'$$

i.e.,

$$\begin{aligned} \frac{\partial E}{\partial x_1} &= \lambda_1 + C_3(x_1) = \frac{x_1}{a} - K, \\ \frac{\partial E}{\partial x_2} &= \lambda_2 = \frac{x_2}{a}. \end{aligned}$$

Solving these two partial differential equations gives us

$$E(x_1,x_2) = \frac{1}{2}\frac{\ell^2}{m}(x_1^2 + x_2^2) - Kx_1 \tag{4.24}$$

after using use the relation $a = m/\ell^2$. Therefore for all θ_f the optimal control law $u^* = -Kx_1^2$ (i.e., the inverse square law) is to keep the function E be constant along the optimal trajectory

$$E(x_1^*(\theta_f), x_2^*(\theta_f)) = E(x_{10}, x_{20}) \triangleq E(\text{constant}).$$

What is the physical meaning of E? After introducing the notations $x_1 = 1/r$, $x_2 = \frac{d}{d\theta}\frac{1}{r}$, $\ell = mr^2\dot{\theta}$, and (3.4a) the function E can be expressed as

$$E = \frac{1}{2}\frac{\ell^2}{m}\left[\frac{1}{r^2} + \left(\frac{d}{d\theta}\frac{1}{r}\right)^2\right] - K\frac{1}{r} = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m\dot{r}^2 - \frac{K}{r}$$

hence E is the total energy of the system with

$$E = T(r,\dot{r}) + U(r)$$

in which

$$T(r,\dot{r}) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

is the kinematic energy and $U(r) = -K/r$ is the potential energy of the gravity field. The optimal control law u^* for least action is used by nature to keep the total energy being constant along the optimal orbit of motion . Therefore, we have the following conclusion.

Conclusion 1 *From optimal control theory point of view, the conservation of the total energy is automatically ensured by the inverse square law.*

The result is consistent with what we have learn from classical mechanics [4,5]. Thus when a new physical fact is observed from experiment, the inverse optimal control problem gives us the way to deal with the principle behind.

The following remarks the present study can also be conducted from Hamilton-Jacobi-Bellman (HJB) equation.

Remark 4.2 The results in Lemma 4.1 and Theorem 4.1 for arbitrary θ_f could also be obtained by solving the stationary HJB equation:

$$\begin{aligned} & L^* + x_2^* \frac{\partial I^*}{\partial x_1} - \left(x_1^* + \frac{a}{x_1^{*2}} u^* \right) \frac{\partial I^*}{\partial x_2} \\ = & \min_{u \in U} \left\{ L(x_1, x_2, u) + x_2 \frac{\partial I^*}{\partial x_1} - \left(x_1 + \frac{a}{x_1^2} u \right) \frac{\partial I^*}{\partial x_2} \right\} = 0 \end{aligned}$$

with

$$\frac{\partial}{\partial u} \left\{ L(x_1, x_2, u) + x_2 \frac{\partial I^*}{\partial x_1} - \left(x_1 + \frac{a}{x_1^2} u \right) \frac{\partial I^*}{\partial x_2} \right\} \Bigg|_{u^*} = 0.$$

5 Generalization for a Dynamical System

In previous section, the Lagrangian for least action to the motion of a particle must satisfy a second order partial differential equation. The corresponding optimal controller is really the inverse square law in gravity field. Within this section this idea is generalized to treat a dynamical system.

Consider a system with n degrees of freedom possesses n equations of nonrelativistic motion:

$$\begin{aligned} \dot{q}_1 &= f_1(q_1, \ldots, q_n, u_1, \ldots, u_m), \\ &\vdots \\ \dot{q}_n &= f_n(q_1, \ldots, q_n, u_1, \ldots, u_m), \end{aligned} \tag{5.1}$$

where $u_i,\ 1 \le i \le m$ be the internal or external forces to the system (which is recognized as the control input). For this system, the *optimal control problem* is to find an optimal control force

$$u_i = u_i^*(q_1, \ldots, q_n), \quad i = 1, 2, \ldots, m$$

such that the action

$$I(u) = \int_0^{t_f} L(q_1, \ldots, q_n, u_1, \ldots, u_m, t) dt$$

is minimized with q_i and u_i satisfying the constraint (5.1).

Suppose some experiments have been conducted such that the forces of the system are measured with the following relationship:

$$u_i = u_i^\circ(q_1, \ldots, q_n), \quad i = 1, 2, \ldots, m \tag{5.2}$$

where u_i°'s are known functions depending on the generalized coordinates $q_1,\ldots,q_n$. As before, it ask: "Is there exists a Lagrangian $L^\circ(q_1,\ldots,q_n,u_1,\ldots,u_m,t)$ such that the admissible solution u_i^* in the optimal control problem is equal to u_i° for all i?" Here u_i° represents the physical law in nature. Once there exists such function L°, the law u° is the outcome of least action taken by nature.

The Hamiltonian H for this optimization is given by

$$H = L(q_1,\ldots,q_n,u_1,\ldots,u_m,t) + \sum_{j=1}^{n} \lambda_j f_j(q_1,\ldots,q_n,u_1,\ldots,u_m) \tag{5.3}$$

where $\lambda_j,\ 1 \le j \le n$ are the Lagrange multipliers. The optimal control law must satisfy

$$\left.\frac{\partial H}{\partial u_i}\right|_{(u_1^*,\ldots,u_m^*)} = 0$$

or equivalently,

$$\left.\frac{\partial L}{\partial u_i}\right|_{(u_1^*,\ldots,u_m^*)} + \sum_{j=1}^{n} \lambda_j \left.\frac{\partial f_j}{\partial u_i}\right|_{(u_1^*,\ldots,u_m^*)} = 0 \tag{5.4}$$

A vector notation is adopted for simplicity. Denote the following symbols

$$\mathbf{q} = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}^T, \qquad \mathbf{f} = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix},$$

$$\mathbf{u} = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}^T, \qquad \lambda = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix},$$

$$\frac{\partial L}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial L}{\partial u_1} & \cdots & \frac{\partial L}{\partial u_m} \end{bmatrix}^T, \quad \frac{\partial H}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial H}{\partial q_1} & \cdots & \frac{\partial H}{\partial q_n} \end{bmatrix}^T, \tag{5.5}$$

$$\mathbf{J}(\mathbf{f},\mathbf{u}) = \left[\frac{\partial f_j}{\partial u_i}\right]_{(i,j)} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_1} \\ \vdots & & \\ \frac{\partial f_1}{\partial u_m} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}$$

Then (5.4) can be rewritten as

$$0 = \left.\frac{\partial L}{\partial \mathbf{u}}\right|_{\mathbf{u}^*} + \mathbf{J}(\mathbf{f},\mathbf{u})\big|_{\mathbf{u}^*}\lambda \triangleq \frac{\partial L}{\partial \mathbf{u}^*} + \mathbf{J}(\mathbf{f},\mathbf{u}^*)\lambda \tag{5.6}$$

The corresponding optimal state is given by

$$\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q},\mathbf{u}^*) \triangleq \mathbf{f}^*, \tag{5.7}$$

and the corresponding Hamiltonian is

$$H^* = L(\mathbf{q},\mathbf{u}^*,t) + \lambda^T \mathbf{f}(\mathbf{q},\mathbf{u}^*) = L^* + \lambda^T \mathbf{f}^* = 0 \tag{5.8}$$

where L^* is used to denote $L(\mathbf{q},\mathbf{u}^*,t)$. And the costate is the solution of

$$\dot{\lambda} = -\frac{\partial H^*}{\partial \mathbf{q}} = -\frac{\partial L^*}{\partial \mathbf{q}} - \mathbf{J}(\mathbf{f}^*,\mathbf{q})\lambda, \tag{5.9}$$

respectively. Our purpose is to combine (5.6), (5.7), and (5.9) so as to obtain a partial differential equation containing L^* as its unknown.

Firstly, we differentiate the first term in (5.6) to obtain

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{u}^*} = \frac{\partial^2 L}{\partial t \partial \mathbf{u}^*} + \mathbf{J}\left(\frac{\partial L}{\partial \mathbf{u}^*},\mathbf{q}\right)^T \dot{\mathbf{q}} = \frac{\partial^2 L}{\partial t \partial \mathbf{u}^*} + \mathbf{J}\left(\frac{\partial L}{\partial \mathbf{u}^*},\mathbf{q}\right)^T \mathbf{f}^* \tag{5.10}$$

and for second term to obtain

$$\begin{aligned}\frac{d}{dt}[\mathbf{J}(\mathbf{f},\mathbf{u}^*)\lambda]) &= \mathbf{J}(\mathbf{f},\mathbf{u}^*)\dot{\lambda} + \frac{d}{dt}[\mathbf{J}(\mathbf{f},\mathbf{u}^*)]\lambda \\ &= -\mathbf{J}(\mathbf{f},\mathbf{u}^*)\frac{\partial L^*}{\partial \mathbf{q}} + \left(\frac{d}{dt}[\mathbf{J}(\mathbf{f},\mathbf{u}^*)] - \mathbf{J}(\mathbf{f},\mathbf{u}^*)\mathbf{J}(\mathbf{f}^*,\mathbf{q})\right)\lambda \end{aligned}\tag{5.11}$$

Then the derivative of (5.6) with respect to time is

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \mathbf{u}^*} + \frac{d}{dt}[\mathbf{J}(\mathbf{f},\mathbf{u}^*)\lambda] \tag{5.12}$$

Substitution of equations (5.10) and (5.11) into (5.12) leads to

$$\begin{aligned} 0 = &\frac{\partial^2 L}{\partial t \partial \mathbf{u}^*} + \mathbf{J}\left(\frac{\partial L}{\partial \mathbf{u}^*},\mathbf{q}\right)^T \mathbf{f}^* - \mathbf{J}(\mathbf{f},\mathbf{u}^*)\frac{\partial L^*}{\partial \mathbf{q}} \\ &+ \left(\frac{d}{dt}[\mathbf{J}(\mathbf{f},\mathbf{u}^*)] - \mathbf{J}(\mathbf{f},\mathbf{u}^*)\mathbf{J}(\mathbf{f}^*,\mathbf{q})\right)\lambda \end{aligned}\tag{5.13}$$

When $m = n$ and suppose $\mathbf{J}(\mathbf{f},\mathbf{u}^*)$ be nonsingular, from (5.6) it follows that

$$\lambda = -\mathbf{J}(\mathbf{f},\mathbf{u}^*)^{-1}\frac{\partial L}{\partial \mathbf{u}^*} \tag{5.14}$$

and (5.13) becomes a second-order partial differential equation for L^*:

$$\begin{aligned} 0 = &\frac{\partial^2 L}{\partial t \partial \mathbf{u}^*} + \mathbf{J}\left(\frac{\partial L}{\partial \mathbf{u}^*},\mathbf{q}\right)^T \mathbf{f}^* - \mathbf{J}(\mathbf{f},\mathbf{u}^*)\frac{\partial L^*}{\partial \mathbf{q}} \\ &- \left(\frac{d}{dt}[\mathbf{J}(\mathbf{f},\mathbf{u}^*)] - \mathbf{J}(\mathbf{f},\mathbf{u}^*)\mathbf{J}(\mathbf{f}^*,\mathbf{q})\right)\mathbf{J}(\mathbf{f},\mathbf{u}^*)^{-1}\frac{\partial L}{\partial \mathbf{u}^*} \end{aligned}$$

On the other hand, when $m \neq n$ or $\mathbf{J}(\mathbf{f},\mathbf{u}^*)$ with $m = n$ be singular, the costate should be eliminated by using (5.6) and (5.8). We recapitulate the above discussion as following theorem:

Theorem 5.1 *Consider the dynamical system*

$$\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q},\mathbf{u}), \quad \mathbf{q}(0) = \mathbf{q}_0 \tag{5.15}$$

driven by a known control force $\mathbf{u}^\circ(\mathbf{q})$ *where* $\mathbf{q},\ \mathbf{u} \in \mathbb{R}^n$ *and the Jacobian matrix* $\mathbf{J}(\mathbf{f},\mathbf{u})$ *is defined by*

$$\mathbf{J}(\mathbf{f},\mathbf{u}) = \left[\frac{\partial f_j}{\partial u_i}\right]_{(i,j)} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_1} \\ \vdots & & \\ \frac{\partial f_1}{\partial u_m} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}.$$

The Lagrangian $L^\circ(\mathbf{q},\mathbf{u},t)$ *for least action principle with optimal control law* $\mathbf{u}^* = \mathbf{u}^\circ$ *is the solution of the following partial differential equation*

$$\begin{aligned} 0 \;=\; & \frac{\partial^2 L}{\partial t \partial \mathbf{u}^*} + \mathbf{J}\left(\frac{\partial L}{\partial \mathbf{u}^*},\mathbf{q}\right)^T \mathbf{f}^* - \mathbf{J}(\mathbf{f},\mathbf{u}^*)\frac{\partial L^*}{\partial \mathbf{q}} \\ & - \left(\frac{d}{dt}\left[\mathbf{J}(\mathbf{f},\mathbf{u}^*)\right] - \mathbf{J}(\mathbf{f},\mathbf{u}^*)\mathbf{J}(\mathbf{f}^*,\mathbf{q})\right)\mathbf{J}(\mathbf{f},\mathbf{u}^*)^{-1}\frac{\partial L}{\partial \mathbf{u}^*} \end{aligned} \tag{5.16}$$

provided that $\mathbf{J}(\mathbf{f},\mathbf{u}^*) = \mathbf{J}(\mathbf{f},\mathbf{u})\big|_{\mathbf{u}^*}$ *is nonsingular, where*

$$\mathbf{f}^* = \mathbf{f}(\mathbf{q},\mathbf{u}^*), \quad \frac{\partial L}{\partial \mathbf{u}^*} = \frac{\partial L}{\partial \mathbf{u}}\bigg|_{u^*}$$

Example 5.2 Find the Lagrangian L such that the function $u^* = -2x$ is the optimal control law to minimize

$$J(u) = \int_0^\infty L(x,u)dt$$

subjected to the dynamical system

$$\dot{x}(t) = x(t) + u(t), \quad x(0) = x_0$$

with $x(t),\ u(t) \in \mathbb{R}$.

Comparing the notation presented in this section, we have

$$f(x,u) = x + u,$$

and

$$J(f,u) = \frac{\partial f}{\partial u} = 1, \quad J(f,x) = \frac{\partial f}{\partial x} = 1,$$

Along the optimal trajectory it follows that

$$J(f,u^*) = 1, \quad f^* = -x, \quad J(f^*,x) = -1,$$

and

$$\frac{d}{dt}J(f,u^*) = 0, \quad \frac{\partial^2 L}{\partial t \partial u^*} = 0, \quad J(\frac{\partial L}{\partial u^*},x) = \frac{\partial^2 L}{\partial x \partial u^*}.$$

Thus (5.16) becomes

$$0 = \frac{\partial L^*}{\partial x} + x\frac{\partial^2 L}{\partial x \partial u^*} + \frac{\partial L}{\partial u^*}$$

and it can be rewritten as follows

$$\begin{aligned} 0 &= \frac{\partial L^*}{\partial x} + \frac{\partial}{\partial x}\left(x \frac{\partial L}{\partial u^*}\right) \\ &= \frac{\partial}{\partial x}\left(L^* - \frac{u^*}{2}\frac{\partial L}{\partial u^*}\right) \\ &= \frac{\partial}{\partial x}\left(-\frac{u^3}{2}\frac{\partial}{\partial u}\left(\frac{L}{u^2}\right)\bigg|_{u^*}\right) \end{aligned}$$

Thus the Lagrangian must satisfy

$$-\frac{u^3}{2}\frac{\partial}{\partial u}\frac{L}{u^2}\bigg|_{u^*} = C_4 \text{ (constant)}$$

with all possible solution given by

$$L(x,u) = C_1(x,u)(u-u^*)^2 + C_2(x)u^2 + \frac{C_4}{u^2}$$

where C_1 and C_4 are arbitrary functions. Therefore a simplest solution is $L(x,u) = u^2$, and by direct verification the corresponding optimal control law is $u^* = -2x$.

6 Conclusions

Our paper provides the answer for the question: "Is the inverse square law in gravity field the outcome of some optimal action taken by nature?" As motivated by the assumption of Maupertuis, we take the action as time integral of some unknown Lagrangian and recognize the inverse square law in gravity field as an optimal control law for the motion of particles ensured by Newton's Second Law. As the result of applying Pontryagin Minimal Principle, the unknown Lagrangian must satisfy a second-order partial differential equation and some of its solution has been analytically constructed. There are infinitely many Lagrangian's which corresponds to the same optimal control law. For all these Lagrangian's we have shown that the optimal control law minimizing the action is exactly the inverse square law. And the minimization process is to maintain the level of total energy along the "least action" trajectory without change. At the meanwhile, this least action trajectory is a periodic orbit of period 2π. Furthermore, we generalize the formulation to give a necessary condition for Lagrangian under which an arbitrary nature law can be considered the optimal control law manipulated by nature.

There are topics for further study e.g., to answer: "Is the inverse square law in electromagnetic field also the outcome of some least action taken by nature?" or "What is the action corresponding to an optimal control law - inverse square law in the combination of gravity and electromagnetic fields?" This paper also raises an new research direction: inverse problem in mathematical optimization or optimal control theory.

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試問平方反比定律可否為大自然的最佳控制器?對應的行動能量為何?

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摘　　要

於古典力學範疇，通常先設定某一 Lagrangian 泛函為系統的行動能量，再利用變分法求其最小值而決定該系統的運動方程式。倘若我們假定平方反比定律為大自然的最佳控制手段，試問其對應的行動能量為何?本文以二個物體的平面運動為分析對象，來回答此一問題。首先利用 Pontryagin 最小原則，推導發現該行動能量須滿足特定的偏微分方程。此一方程的解雖有無限多組，然其中一組的解為該系統的行動能量是總能量，所顯示的物理意義為大自然若採取平方反比定律為最佳控制器，其目的在維持物體沿特定軌道運動的總能量為固定。文章最後，將此問題推廣到多體運動的情形，得到一組偏微分，在一般情形下不易求解。

關鍵詞：平方反比定律，Lagrangian 泛函，Pontryagin 最小原則，最佳控制，反問題。

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