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Stability Analysis on Nonselective Harvesting of a Predator-Prey Fishery System with Time Delays

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Abstract

In this thesis, we are interested in studying the stability of the unique positive equilibrium point of a nonselective harvesting of a predator-prey fishery system with time delay. Firstly, we state the formulation of the model. Secondly, we drive different conditions for local and global stability of the positive equilibrium of the system, respectively. Finally, we illustrate our results by some examples.

Keywords: Predator-prey, Fishery, Nonselective harvesting, Time delay, Holling-type II.

1 Introduction

Bioeconomic modelling is concerned with exploitation of renewable resources like fisheries and forestries. In recent years, the problems related to harvesting of multispecies fisheries have been investigated by many authors. Clark [6] studied the problem of combined harvesting of two independent fish species governed by the logistic law of growth. Brauer and Soudack [2][3], Dai and Tang[7], Myerscough et al.[18] discussed the effects of constant rate harvesting and stocking in predator-prey systems. Chaudhuri [4] studied the effect of harvesting both species in Gause's model. Purohit and Chaudhuri [19] studied the dynamical behavior of a harvesting predator-prey system. In recent years, time delays of one type or another have been incorporated into biological models by many reserchers. For example, Gopalsamy[8][9] , Kuang[16], He[10], Ho and Liang[11] and Ho and Tsai[12] studied on delayed predator-prey systems. Kar[15], Martin and Ruan[17] studied predator-prey harvesting model with time delay. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could case a stable equilibrium to become unstable and cause the populations to fluctuate.

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Chaudhuri and SahaRay [5] have considered the problem of exploiting a predator-prey community in which the growth of both predator and prey obey the logistic law of growth. And Kar and Chaudhuri[14] have studied the problem of exploiting a predator-prey community in which the growth of both predator and pre obeys the logistic law of growth, and each predators functional response to Holling type II. In this paper, we consider the problem of exploiting a predator-prey community in which both prey and predator obey the logistic law of growth with time delay, and each predators functional response to Holling type II. The main purpose of this paper is to analyze the stability of the unique positive equilibrium point of the system. Of course those results of reference[14] are special cases of our results. In section 2, we state the model, and the equilibrium point of the system. In section 3, we analyze uniform persistence of the system. In section 4, we analyze the local and global stability by constructing different Lyapunov function, respectively. In section 5, we illustrate our results by some examples.

2 Formulation and Equilibrium Points of the Model

The combined harvesting of predator-pre system with time delays is of the form

$$\begin{aligned} \dot{x_1}(t) &= r_1 x_1(t)\left[1-\frac{x_1(t-\tau_1)}{k_1}\right] - x_2(t)\frac{bx_1(t)}{a+x_1(t)} - q_1 e x_1(t) \\ \dot{x_2}(t) &= r_2 x_2(t)\left[1-\frac{x_2(t-\tau_2)}{k_2}\right] + c x_2(t)\frac{bx_1(t)}{a+x_1(t)} - q_2 e x_2(t) \end{aligned} \tag{2.1}$$

with the initial conditions

$$\begin{aligned} & x_i(\theta) = \phi_i(\theta) > 0 \ , \ \theta \in [-\tau, 0] \ , \ \phi_i \in C([-\tau, 0], R) \\ & \tau = \max\{\tau_1, \tau_2\} \ , \ i = 1, 2 \end{aligned} \tag{2.2}$$

where r_1, r_2, k_1, k_2, c, a, b, q_1, q_2 and e are all positive constants. Here r_1 and r_2 denote the natural growth rates, k_1 and k_2 are the environmental carrying capacity of the two species, and c is conversion factor $(c < 1)$. The natural growth of x_1 and x_2 species obey the time delay growth. The delay τ_1 and τ_2 are constants representing the prey's and predator's growth are affected by population density only after a fixed period of time. In addition, we have taken the predation term as $bx_1(t)/[a+x_1(t)]$ due to Holling [13] . And e denote the combined harvesting effort, q_1 and q_2 are catchability coefficients of the species. The catch-rate function $q_1 e x_1$ and $q_2 e x_2$ are based on the CPUE(catch-per-unit-effort) hypothesis [6] . All we want to discuss is biological population, so we just consider the first quadrant in the x_1-x_2 plane.

Clearly, $\overline{E} \equiv (0,0)$ is an equilibrium point of the system (2.1). And all possible equilibrium points of the system (2.1) are

$$\widetilde{E} \equiv (\widetilde{x}_1, 0), \;\; \widetilde{x}_1 = \frac{k_1}{r_1}(r_1 - q_1 e)$$
$$\widehat{E} \equiv (0, \widehat{x}_2), \;\; \widehat{x}_2 = \frac{k_2}{r_2}(r_2 - q_2 e)$$

and

$$E^* \equiv (x_1^*, x_2^*)$$

where x_1^* and x_2^* satisfy

$$\begin{aligned} r_1(1 - \frac{x_1^*}{k_1}) - \frac{b x_2^*}{a + x_1^*} - q_1 e = 0 \\ r_2(1 - \frac{x_2^*}{k_2}) + \frac{c b x_1^*}{a + x_1^*} - q_2 e = 0. \end{aligned} \tag{2.3}$$

The ratio r_1/q_1 of the biotic potential r_1 to the catchability coefficient q_1 is known as the biotechnical productivity (BTP) of the species [6]. It is easy to see that the equilibrium point $\widetilde{E}$ exists if $e < r_1/q_1$, i.e. the harvesting effort is less than the biotechinical productivity (BTP_{x_1}). Similarly, $\widehat{E}$ exists if $e < r_2/q_2$, i.e. the harvesting effort is less than the biotechinical productivity (BTP_{x_2}).

Remark *2.1 Let x_1^* and x_2^* satisfy the equation (2.3), and B, C and D satisfy ${x_1^*}^3 + B{x_1^*}^2 + Cx_1^* + D = 0$, where*

$$\begin{aligned} B &= \frac{q_1 e k_1}{r_1} + 2a - k_1 \\ C &= \frac{b k_1 k_2}{r_1 r_2}(r_2 + bc - q_2 e) + r_1 r_2 a(a - 2k_1) + \frac{2a q_1 e k_1}{r_1} \\ D &= \frac{a b k_1 k_2}{r_1 r_2}(r_2 - q_2 e) - \frac{a^2 k_1}{r_1}(r_1 - q_1 e). \end{aligned}$$

If one of the following holds

$$B > 0, \; C - \frac{D}{B} > 0, and \; D < 0$$
$$B > 0, \; C - \frac{D}{B} < 0, and \; D < 0$$
$$B < 0, \; C - \frac{D}{B} < 0, and \; D < 0$$

and

$$r_2 - q_2 e + \frac{bcx_1^*}{a + x_1^*} > 0$$

then $E^* = (x_1^*, x_2^*)$ *is the unique positive equilibrium point of the system (2.1).*

3 Uniform Persistence

System (2.1) has a unique positive equilibrium point if Remark 2.1 holds. In the following, we always assume that such a positive equilibrium exists and denote it by $E^*(x_1^*, x_2^*)$. The following lemmas are elementary concerned with the qualitative nature of solutions of the system (2.1).

Lemma 3.1 *All solutions of the system* (2.1) *with the initial conditions* (2.2) *are positive for all* $t \geq 0$.

Proof : It is true because

$$\begin{aligned} x_1(t) &= x_1(0)exp\left\{\int_0^t \left[r_1\left(1 - \frac{x_1(s-\tau_1)}{k_1}\right) - \frac{bx_2(s)}{a + x_1(s)} - q_1 e\right] ds\right\} \\ x_2(t) &= x_2(0)exp\left\{\int_0^t \left[r_2\left(1 - \frac{x_2(s-\tau_2)}{k_2}\right) + \frac{cbx_1(s)}{a + x_1(s)} - q_2 e\right] ds\right\} \end{aligned} \quad (3.1)$$

and $x_i(0) > 0 (i = 1,2)$. Therefore, we obtain that all solutions $(x_1(t), x_2(t))$ of the system (2.1) with the initial conditions (2.2) are positive. ∎

Lemma 3.2 *Let* $(x_1(t), x_2(t))$ *denote the solution of (2.1) with the initial conditions (2.2), then*

$$0 < x_i(t) \leq M_i, \ for \ \ i = 1,2 \quad (3.2)$$

eventually for all large t, where

$$M_1 = k_1 e^{r_1 \tau_1} \quad (3.3)$$

$$M_2 = (k_2 + \frac{k_2 \widetilde{M_1}}{r_2}) exp[(r_2 + \widetilde{M}_1)\tau_2], \ \widetilde{M_1} = \frac{cbM_1}{a + M_1}. \quad (3.4)$$

Proof : By Lemma 3.1, we know that solutions of the system (2.1) are positive , and hence, by first equation of the system (2.1)

$$\begin{aligned} \frac{dx_1(t)}{dt} &= r_1 x_1(t)\left[1 - \frac{x_1(t-\tau_1)}{k_1}\right] - x_2 \frac{bx_1(t)}{a + x_1(t)} - q_1 e x_1(t) \\ &\leq r_1 x_1(t)\left[1 - \frac{x_1(t-\tau_1)}{k_1}\right] \\ &= x_1(t)\left[r_1 - \frac{r_1}{k_1} x_1(t-\tau_1)\right]. \end{aligned} \quad (3.5)$$

Taking $M_1^* = k_1(1+b_1),\ \ 0 < b_1 < e^{r_1\tau_1} - 1$. Suppose $x_1(t)$ is not oscillatory about M_1^*. That is, there exists a $T_0 > 0$ such that either

$$x_1(t) \le M_1^* \qquad \text{for} \quad t > T_0 \tag{3.6}$$

or

$$x_1(t) > M_1^* \qquad \text{for} \quad t > T_0. \tag{3.7}$$

If (3.6) holds, then for $t > T_0$

$$x_1(t) \le M_1^* = k_1(1+b_1) < k_1 e^{r_1\tau_1} = M_1.$$

That is, (3.2) holds for $i = 1$. Suppose (3.7) holds, equation (3.5) implies that, for $t > T_0 + \tau_1$

$$\begin{aligned} \dot{x}_1(t) &\le r_1 x_1(t)\left[1 - \frac{x_1(t-\tau_1)}{k_1}\right] \\ &< r_1 x_1(t)\left[1 - \frac{M_1^*}{k_1}\right] \\ &= -b_1 r_1 x_1(t). \end{aligned}$$

It follows that

$$\begin{aligned} \int_{T_0+\tau_1}^{t} \frac{\dot{x}_1(s)}{x_1(s)} ds &< \int_{T_0+\tau_1}^{t} -b_1 r_1 ds \\ &= -b_1 r_1 (t - T_0 - \tau_1). \end{aligned}$$

Then

$0 < x_1(t) < x_1(T_0+\tau_1)e^{-b_1 r_1 (t-T_0-\tau_1)} \to 0$ as $t \to \infty$. That is, $\lim_{t\to\infty} x_1(t) = 0$ by the Squeeze Theorem. It contradics to (3.7). Therefore, there must exist a $T_1 > T_0$ such that $x_1(T_1) \le M_1^*$. If $x_1(t) \le M_1^*$ for all $t \ge T_1$, then (3.2) follows. If not, then there must exists a $T_2 > T_1$ such that T_2 be the first time which $x_1(T_2) > M_1^*$. Therefore, there exists a $T_3 > T_2$ such that $x_1(T_3) \le M_1^*$ by above discussion. By above, we know that $x_1(T_1) \le M_1^*,\ x_1(T_2) > M_1^*$, and $x_1(T_3) \le M_1^*$ where $T_1 < T_2 < T_3$. Then, by the Intermediate Value Theorem, there exists T_4 and T_5 such that

$$\begin{aligned} x_1(T_4) = M_1^* \quad &, \quad T_1 \le T_4 < T_2 \\ x_1(T_5) = M_1^* \quad &, \quad T_2 < T_5 \le T_3 \end{aligned}$$

and $x_1(t) > M_1^*$ for $T_4 < t < T_5$. Hence there is a $T_6 \in (T_4, T_5)$ such that $x_1(T_6)$ is local maximum, and it follows from (3.5) that

$$\begin{aligned} 0 = \dot{x_1}(T_6) &= r_1 x_1(T_6)\left[1 - \frac{x_1(T_6-\tau_1)}{k_1}\right] - \frac{bx_1(T_6)x_2(T_6)}{a+x_1(T_6)} - q_1 e x_1(T_6) \\ &\leq r_1 x_1(T_6)\left[1 - \frac{x_1(T_6-\tau_1)}{k_1}\right] \end{aligned} \tag{3.8}$$

and this implies

$$x_1(T_6-\tau_1) \leq k_1. \tag{3.9}$$

Integrating both sides of (3.8) on the interval
$[T_6-\tau_1, T_6]$, we have

$$\int_{T_6-\tau_1}^{T_6} \frac{\dot{x_1}(s)}{x_1(s)} ds \leq \int_{T_6-\tau_1}^{T_6} r_1\left[1 - \frac{x_1(s-\tau_1)}{k_1}\right] ds \leq r_1\tau_1. \tag{3.10}$$

By (3.9) and (3.10) imply

$$x_1(T_6) \leq x_1(T_6-\tau_1)\ exp(r_1\tau_1) \leq k_1\ exp(r_1\tau_1) = M_1.$$

Thus

$$x_1(t) \leq M_1,\ \ t \in [T_1, T_5].$$

Since any local maximum is less than or equal to M_1, thus there exist

$$x_1(t) \leq M_1,\ \ \text{for } t \geq T_6. \tag{3.11}$$

That is, (3.2) holds. Suppose $x_1(t)$ is oscillatory about M_1^*, for this case, the proof is similarly to above one. And we can conclude that there exists a $\hat{T} \geq T_6$, such that $x_1(t) \leq M_1$, for all $t \geq \hat{T}$.

Now, we want to show that $x_2(t)$ is bounded above by M_2 eventually for all large t. We have directly from second equation of the system (2.1) and $x_1(t) \leq M_1$ (for $t \geq \hat{T}$) that

$$\begin{aligned} \dot{x_2}(t) &= r_2 x_2(t)\left[1 - \frac{x_2(t-\tau_2)}{k_2}\right] + c x_2(t)\frac{bx_1(t)}{a+x_1(t)} - q_2 e x_2(t) \\ &\leq r_2 x_2(t)\left[1 - \frac{x_2(t-\tau_2)}{k_2}\right] + c x_2(t)\frac{bx_1(t)}{a+x_1(t)} \\ &\leq r_2 x_2(t)\left[1 - \frac{x_2(t-\tau_2)}{k_2}\right] + c x_2(t)\frac{bM_1}{a+M_1} \\ &= x_2(t)\left[(r_2 + \frac{cbM_1}{a+M_1}) - \frac{r_2}{k_2}x_2(t-\tau_2)\right]. \end{aligned} \tag{3.12}$$

By a similar argument, we can verify that there exists a
$T > \hat{T} + \tau_2$ such that

$$\begin{aligned} x_2(t) &\leq \frac{k_2}{r_2}\left(r_2 + \frac{cbM_1}{a+M_1}\right) exp[(r_2 + \frac{cbM_1}{a+M_1})\tau_2] \\ &= \left[k_2 + \frac{k_2 cbM_1}{r_2(a+M_1)}\right] exp[(r_2 + \frac{cbM_1}{a+M_1})\tau_2] \\ &\equiv (k_2 + \frac{k_2\widetilde{M}_1}{r_2}) exp[(r_2 + \widetilde{M}_1)\tau_2] \\ &\equiv M_2 \qquad \text{for } t \geq T. \end{aligned} \tag{3.13}$$

Thus

$$0 < x_i(t) \leq M_i, \;\; i = 1,2 \qquad \text{for } t \geq T. \tag{3.14}$$

This completes the proof.

Theorem 3.1 *Suppose that the system (2.1) satisfy*

$$r_1 - q_1 e - \frac{bM_2}{a} \quad > \quad 0 \tag{3.15}$$

and

$$r_2 - q_2 e \quad > \quad 0 \tag{3.16}$$

where $M_i(i = 1,2)$ *defined by (3.3) and (3.4). Then the system (2.1) is uniformly persistent. That is, there exist* $m_1,\ m_2$ *and* $T^* > 0$ *such that* $m_1 \leq x_1 \leq M_1$ *and* $m_2 \leq x_2 \leq M_2$ *for* $t \geq T^*$*, where*

$$\begin{aligned} m_1 &= \frac{k_1}{2r_1}(r_1 - \frac{bM_2}{a} - q_1 e) exp\{[r_1(1 - \frac{M_1}{k_1}) - \frac{bM_2}{a} - q_1 e]\tau_1\} \\ m_2 &= \frac{k_2}{2r_2}(r_2 - q_2 e) exp\{[r_2(1 - \frac{M_2}{k_2}) - q_2 e]\tau_2\}. \end{aligned}$$

Proof : By Lemma 3.2, equation (2.1) follows that for $t \geq T + \tau_1$

$$\dot{x}_1(t) = x_1(t)\left\{r_1\left[1 - \frac{x_1(t-\tau_1)}{k_1}\right] - \frac{bx_2(t)}{a + x_1(t)} - q_1 e\right\} \tag{3.17}$$

$$\geq x_1(t)\left[r_1\left(1 - \frac{M_1}{k_1}\right) - \frac{bM_2}{a} - q_1 e\right]. \tag{3.18}$$

Integrating both sides of (3.18) on $[t-\tau_1,t]$,where $t \geq T+\tau_1$, then we have

$$x_1(t) \geq x_1(t-\tau_1)exp\{[r_1(1-\frac{M_1}{k_1})-\frac{bM_2}{a}-q_1e]\tau_1\}.$$

That is

$$x_1(t-\tau_1) \leq x_1(t)exp\{-[r_1(1-\frac{M_1}{k_1})-\frac{bM_2}{a}-q_1e]\tau_1\}. \tag{3.19}$$

From (3.17) and (3.19), for $t \geq T+\tau_1$

$$\begin{aligned}
\dot{x}_1(t) &\geq x_1(t)\left\{r_1\{1-\frac{1}{k_1}x_1(t)exp\{-[r_1(1-\frac{M_1}{k_1})-\frac{bM_2}{a}-q_1e]\tau_1\}-\frac{bx_2(t)}{a+x_1(t)}-q_1e\right\} \\
&\geq x_1(t)\left\{(r_1-\frac{bM_2}{a}-q_1e)-\frac{x_1(t)}{\frac{k_1}{r_1}exp\{[r_1(1-\frac{M_1}{k_1})-\frac{bM_2}{a}-q_1e]\tau_1\}}\right\} \\
&= \left(r_1-\frac{bM_2}{a}-q_1e\right)x_1(t)\left\{1-\frac{x_1(t)}{\frac{k_1}{r_1}(r_1-\frac{bM_2}{a}-q_1e)exp\{[r_1(1-\frac{M_1}{k_1})-\frac{bM_2}{a}-q_1e]\tau_1\}}\right\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\liminf_{t\to\infty} x_1(t) &\geq \frac{k_1}{r_1}(r_1-\frac{bM_2}{a}-q_1e)exp\{[r_1(1-\frac{M_1}{k_1})-\frac{bM_2}{a}-q_1e]\tau_1\} \\
&\equiv \bar{m}_1
\end{aligned}$$

and $\bar{m}_1 > 0$ by (3.15). So, for large t, $x_1(t) > \bar{m}_1/2 \equiv m_1 > 0$. It follows from (2.1) that for $t \geq T+\tau_2$

$$\begin{aligned}
\dot{x_2}(t) &= r_2x_2(t)\left[1-\frac{x_2(t-\tau_2)}{k_2}\right]+cx_2(t)\frac{bx_1(t)}{a+x_1(t)}-q_2ex_2(t) \\
&= x_2(t)\left\{r_2\left[1-\frac{x_2(t-\tau_2)}{k_2}\right]+c\frac{bx_1(t)}{a+x_1(t)}-q_2e\right\} \\
&\geq x_2(t)\left\{r_2\left[1-\frac{x_2(t-\tau_2)}{k_2}\right]-q_2e\right\} \qquad (3.20) \\
&\geq x_2(t)\left\{r_2\left[1-\frac{M_2}{k_2}\right]-q_2e\right\}. \qquad (3.21)
\end{aligned}$$

Integrating both sides of (3.21) on $[t-\tau_2,t]$, where $t \geq T+\tau_2$,then we have

$$x_2(t) \geq x_2(t-\tau_2)exp\{[r_2(1-\frac{M_2}{k_2})-q_2e]\tau_2\}. \tag{3.22}$$

That is

$$x_2(t-\tau_2) \leq x_2(t)exp\{-[r_2(1-\frac{M_2}{k_2})-q_2e]\tau_2\}. \quad (3.23)$$

From (3.20) and (3.23)

$$\begin{aligned}
\dot{x_2}(t) &\geq x_2(t)\left\{r_2\{1-\frac{1}{k_2}x_2(t)exp\{-[r_2(1-\frac{M_2}{k_2})-q_2e]\tau_2\}\}-q_2e\right\} \\
&= x_2(t)\left[(r_2-q_2e)-\frac{x_2(t)}{\frac{k_2}{r_2}exp\{[r_2(1-\frac{M_2}{k_2})-q_2e]\tau_2\}}\right] \\
&\geq (r_2-q_2e)x_2(t)\left[1-\frac{x_2(t)}{\frac{k_2}{r_2}(r_2-q_2e)exp\{[r_2(1-\frac{M_2}{k_2})-q_2e]\tau_2\}}\right].
\end{aligned}$$

It follows that

$$\liminf_{t\to\infty} x_2(t) \geq \frac{k_2}{r_2}(r_2-q_2e)exp\{[r_2(1-\frac{M_2}{k_2})-q_2e]\tau_2\} \equiv \bar{m}_2$$

and $\bar{m}_2 > 0$ by (3.16). So, for large t, $x_2(t) > \bar{m}_2/2 \equiv m_2 > 0$. Hence, for $t > T+\tau$ where $\tau = \max\{\tau_1, \tau_2\}$, and t is large enough. Let

$$D = \{(x_1(t), x_2(t)) | m_1 \leq x_1(t) \leq M_1, m_2 \leq x_2(t) \leq M_2\}.$$

Then D is bounded compact region in R^2_+ that positive distance from coordinate hyperplancds. Hence we obtain that there exists a $T^* > 0$ such that if $t \geq T^*$, then every positive solution of system (2.1) with the initial conditions (2.2) eventually enters and remains in the region D, that is, system (2.1) is uniformly persistent. ■

4 Stability

In this chapter, we discuss the local stability and the global stability of the equilibrium point E^* of the system (2.1).

4.1 Local Stability

To investigate the local stability of the equilibrium point E^* we linearize the system (2.1). Let $y_1(t) = x_1(t) - x_1^*$, $y_2(t) = x_2(t) - x_2^*$ be the perturbed variables. After removing nonlinear terms, we obtain the linear variational system, by using equilibria conditions as

$$\begin{aligned} \frac{dy_1(t)}{dt} &= \frac{bx_1^* x_2^*}{(a+x_1^*)^2} y_1(t) - \frac{bx_1^*}{a+x_1^*} y_2(t) - \frac{r_1 x_1^*}{k_1} y_1(t-\tau_1) \\ \frac{dy_2(t)}{dt} &= \frac{cabx_2^*}{(a+x_1^*)^2} y_1(t) - \frac{r_2 x_2^*}{k_2} y_2(t-\tau_2). \end{aligned} \tag{4.1}$$

It is noticed that the asymptotical stability of E^* of the system (2.1) is determined by the asymptotical stability of the zero solution of the system (4.1)(see[1]).

Theorem 4.1 *Let $E^* = (x_1^*, x_2^*)$ be the unique equilibrium point of the system (2.1) and the delays τ_1 and τ_2 satisfy*

$$\alpha_1 - \alpha_2 \tau_1 - \alpha_3 \tau_2 > 0 \tag{4.2}$$

and

$$\beta_1 - \beta_2 \tau_1 - \beta_3 \tau_2 > 0 \tag{4.3}$$

where

$$\begin{aligned} \alpha_1 &= \frac{2r_1 x_1^*}{k_1} - \frac{2bx_1^* x_2^* + cabx_2^*}{(a+x_1^*)^2} - \frac{bx_1^*}{a+x_1^*}, & \beta_1 &= \frac{2r_2 x_2^*}{k_2} - \frac{bx_1^*}{a+x_1^*} - \frac{cabx_2^*}{(a+x_1^*)^2} \\ \alpha_2 &= \frac{2r_1 b {x_1^*}^2 x_2^*}{k_1(a+x_1^*)^2} + \frac{2r_1^2 {x_1^*}^2}{k_1^2} + \frac{r_1 b {x_1^*}^2}{k_1(a+x_1^*)}, & \beta_2 &= \frac{r_1 b {x_1^*}^2}{k_1(a+x_1^*)} \\ \alpha_3 &= \frac{r_2 cab {x_2^*}^2}{k_2(a+x_1^*)^2}, & \beta_3 &= \frac{r_2 cab {x_2^*}^2}{k_2(a+x_1^*)^2} + \frac{2r_2^2 {x_2^*}^2}{k_2^2} \end{aligned}$$

then the unique positive equilibrium point E^* of the system (2.1) is locally asymptotically stable.

Proof :

The equation (4.1) can be written as

$$\begin{aligned} \frac{d}{dt}\left[y_1(t) - \frac{r_1 x_1^*}{k_1}\int_{t-\tau_1}^{t} y_1(s)ds\right] &= \left[\frac{bx_1^* x_2^*}{(a+x_1^*)^2} - \frac{r_1 x_1^*}{k_1}\right] y_1(t) - \frac{bx_1^*}{a+x_1^*} y_2(t) \\ \frac{d}{dt}\left[y_2(t) - \frac{r_2 x_2^*}{k_2}\int_{t-\tau_2}^{t} y_2(s)ds\right] &= \frac{cabx_2^*}{(a+x_1^*)^2} y_1(t) - \frac{r_2 x_2^*}{k_2} y_2(t). \end{aligned} \tag{4.4}$$

Let

$$W_1(y(t)) = \left(y_1(t) - \frac{r_1 x_1^*}{k_1}\int_{t-\tau_1}^{t} y_1(s)ds\right)^2 + \left(y_2(t) - \frac{r_2 x_2^*}{k_2}\int_{t-\tau_2}^{t} y_2(s)ds\right)^2 \tag{4.5}$$

then

$$\begin{aligned}
\frac{dW_1(y(t))}{dt} &= 2\left(y_1(t) - \frac{r_1 x_1^*}{k_1}\int_{t-\tau_1}^{t} y_1(s)ds\right)\left[\left(\frac{bx_1^* x_2^*}{(a+x_1^*)^2} - \frac{r_1 x_1^*}{k_1}\right)y_1(t) - \frac{bx_1^*}{a+x_1^*}y_2(t)\right] \\
&\quad +2\left(y_2(t) - \frac{r_2 x_2^*}{k_2}\int_{t-\tau_2}^{t} y_2(s)ds\right)\left[\frac{cabx_2^*}{(a+x_1^*)^2}y_1(t) - \frac{r_2 x_2^*}{k_2}y_2(t)\right] \\
&= \left[\frac{2bx_1^* x_2^*}{(a+x_1^*)^2} - \frac{2r_1 x_1^*}{k_1}\right]y_1^2(t) - \frac{2bx_1^*}{a+x_1^*}y_1(t)y_2(t) \\
&\quad + \left[-\frac{2r_1 b {x_1^*}^2 x_2^*}{k_1(a+x_1^*)^2} + \frac{2r_1^2 {x_1^*}^2}{k_1^2}\right]\int_{t-\tau_1}^{t} y_1(t)y_1(s)ds \\
&\quad + \frac{2r_1 b {x_1^*}^2}{k_1(a+x_1^*)}\int_{t-\tau_1}^{t} y_2(t)y_1(s)ds \\
&\quad + \left[\frac{2cabx_2^*}{(a+x_1^*)^2}\right]y_1(t)y_2(t) - \frac{2r_2 x_2^*}{k_2}y_2^2(t) \\
&\quad + \left(\frac{-2r_2 cab {x_2^*}^2}{k_2(a+x_1^*)^2}\right)\int_{t-\tau_2}^{t} y_1(t)y_2(s)ds + \frac{2r_2^2 {x_2^*}^2}{k_2^2}\int_{t-\tau_2}^{t} y_2(t)y_2(s)ds \\
&\le \left[\frac{2bx_1^* x_2^*}{(a+x_1^*)^2} - \frac{2r_1 x_1^*}{k_1}\right]y_1^2(t) + \frac{bx_1^*}{a+x_1^*}2|y_1(t)||y_2(t)| \\
&\quad + \left[\frac{r_1 b {x_1^*}^2 x_2^*}{k_1(a+x_1^*)^2} + \frac{r_1^2 {x_1^*}^2}{k_1^2}\right]\int_{t-\tau_1}^{t} 2|y_1(t)||y_1(s)|ds \\
&\quad + \frac{r_1 b {x_1^*}^2}{k_1(a+x_1^*)}\int_{t-\tau_1}^{t} 2|y_2(t)||y_1(s)|ds \\
&\quad + \left[\frac{cabx_2^*}{(a+x_1^*)^2}\right]2|y_1(t)||y_2(t)| - \frac{2r_2 x_2^*}{k_2}y_2^2(t) \\
&\quad + \left(\frac{r_2 cab {x_2^*}^2}{k_2(a+x_1^*)^2}\right)\int_{t-\tau_2}^{t} 2|y_1(t)||y_2(s)|ds + \frac{r_2^2 {x_2^*}^2}{k_2^2}\int_{t-\tau_2}^{t} 2|y_2(t)||y_2(s)|ds
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \left[\frac{2bx_1^*x_2^*}{(a+x^*)^2} - \frac{2r_1x_1^*}{k_1} + \frac{bx_1^*}{a+x_1^*} + \frac{cabx_2^*}{(a+x_1^*)^2}\right] y_1^2(t) \\
& + \left[\frac{bx_1^*}{a+x_1^*} + \frac{cabx_2^*}{(a+x_1^*)^2} - \frac{2r_2x_2^*}{k_2}\right] y_2^2(t) \\
& + \left[\frac{r_1bx_1^{*2}x_2^*}{k_1(a+x_1^*)^2} + \frac{r_1^2x_1^{*2}}{k_1^2}\right] \int_{t-\tau_1}^{t} [y_1^2(t)t + y_1^2(s)]ds \\
& + \frac{r_1bx_1^{*2}}{k_1(a+x_1^*)} \int_{t-\tau_1}^{t} [y_2^2(t) + y_1^2(s)]ds \\
& + \frac{r_2cabx_2^{*2}}{k_2(a+x_1^*)^2} \int_{t-\tau_2}^{t} [y_1^2(t) + y_2^2(s)]ds \\
& + \frac{r_2^2x_2^{*2}}{k_2^2} \int_{t-\tau_2}^{t} [y_2^2(t) + y_2^2(s)]ds \\
\leq\ & \left[\frac{2bx_1^*x_2^* + cabx_2^*}{(a+x_1^*)^2} - \frac{2r_1x_1^*}{k_1} + \frac{bx_1^*}{a+x_1^*}\right] y_1^2(t) \\
& + \left[\frac{bx_1^*}{a+x_1^*} + \frac{cabx_2^*}{(a+x_1^*)^2} - \frac{2r_2x_2^*}{k_2}\right] y_2^2(t) \\
& + \left[\frac{r_1bx_1^{*2}x_2^*}{k_1(a+x_1^*)^2} + \frac{r_1^2x_1^{*2}}{k_1^2}\right] \tau_1 y_1^2(t) \\
& + \left[\frac{r_1bx_1^{*2}x_2^*}{k_1(a+x_1^*)^2} + \frac{r_1^2x_1^{*2}}{k_1^2}\right] \int_{t-\tau_1}^{t} y_1^2(s)ds \\
& + \frac{r_1bx_1^{*2}}{k_1(a+x_1^*)} \tau_1 y_2^2(t) + \frac{r_1bx_1^{*2}}{k_1(a+x_1^*)} \int_{t-\tau_1}^{t} y_1^2(s)ds \\
& + \left[\frac{r_2cabx_2^{*2}}{k_2(a+x_1^*)^2}\right] \tau_2 y_1^2(t) + \left[\frac{r_2cabx_2^{*2}}{k_2(a+x_1^*)^2}\right] \int_{t-\tau_2}^{t} y_2^2(s)ds \\
& + \frac{r_2^2x_2^{*2}}{k_2^2} \tau_2 y_2^2(t) + \frac{r_2^2x_2^{*2}}{k_2^2} \int_{t-\tau_2}^{t} y_2^2(s)ds \\
=\ & \left[\frac{2bx_1^*x_2^* + cabx_2^*}{(a+x_1^*)^2} - \frac{2r_1x_1^*}{k_1} + \frac{bx_1^*}{a+x_1^*}\right] y_1^2(t) \\
& + \left[\frac{bx_1^*}{a+x_1^*} + \frac{cabx_2^*}{(a+x_1^*)^2} - \frac{2r_2x_2^*}{k_2}\right] y_2^2(t) \\
& + \left\{ \left[\frac{r_1bx_1^{*2}x_2^*}{k_1(a+x_1^*)^2} + \frac{r_1^2x_1^{*2}}{k_1^2}\right] \tau_1 + \frac{r_2cabx_2^{*2}}{k_2(a+x_1^*)^2} \tau_2 \right\} y_1^2(t) \\
& + \left[\frac{r_1bx_1^{*2}}{k_1(a+x_1^*)} \tau_1 + \frac{r_2^2x_2^{*2}}{k_2^2} \tau_2\right] y_2^2(t) \\
& + \left[\frac{r_1bx_1^{*2}x_2^*}{k_1(a+x_1^*)^2} + \frac{r_1^2x_1^{*2}}{k_1^2} + \frac{r_1bx_1^{*2}}{k_1(a+x_1^*)}\right] \int_{t-\tau_1}^{t} y_1^2(s)ds \\
& + \left[\frac{r_2cabx_2^{*2}}{k_2(a+x_1^*)^2} + \frac{r_2^2x_2^{*2}}{k_2^2}\right] \int_{t-\tau_2}^{t} y_2^2(s)ds. \qquad (4.6)
\end{aligned}
$$

Now, we let

$$W_2(y(t)) = \left[\frac{r_1 b x_1^{*^2} x_2^*}{k_1(a+x_1^*)^2} + \frac{r_1^2 x_1^{*^2}}{k_1^2} + \frac{r_1 b x_1^{*^2}}{k_1(a+x_1^*)}\right] \int_{t-\tau_1}^{t} \int_{s}^{t} y_1^2(\rho) d\rho ds$$
$$+ \left[\frac{r_2 cab x_2^{*^2}}{k_2(a+x_1^*)^2} + \frac{r_2^2 x_2^{*^2}}{k_2^2}\right] \int_{t-\tau_2}^{t} \int_{s}^{t} y_2^2(\rho) d\rho ds \quad (4.7)$$

then

$$\frac{dW_2(y(t))}{dt} = \left[\frac{r_1 b x_1^{*^2} x_2^*}{k_1(a+x_1^*)^2} + \frac{r_1^2 x_1^{*^2}}{k_1^2} + \frac{r_1 b x_1^{*^2}}{k_1(a+x_1^*)}\right] \tau_1 y_1^2(t)$$
$$- \left[\frac{r_1 b x_1^{*^2} x_2^*}{k_1(a+x_1^*)^2} + \frac{r_1^2 x_1^{*^2}}{k_1^2} + \frac{r_1 b x_1^{*^2}}{k_1(a+x_1^*)}\right] \int_{t-\tau_1}^{t} y_1^2(s) ds$$
$$+ \left[\frac{r_2 cab x_2^{*^2}}{k_2(a+x_1^*)^2} + \frac{r_2^2 x_2^{*^2}}{k_2^2}\right] \tau_2 y_2^2(t)$$
$$- \left[\frac{r_2 cab x_2^{*^2}}{k_2(a+x_1^*)^2} + \frac{r_2^2 x_2^{*^2}}{k_2^2}\right] \int_{t-\tau_2}^{t} y_2^2(s) ds. \quad (4.8)$$

Now we define a Lyapunov functional $W(y(t))$ as

$$W(y(t)) = W_1(y(t)) + W_2(y(t)) \quad (4.9)$$

then we have from (4.6) and (4.8) that

$$
\begin{aligned}
\frac{dW(y(t))}{dt} &= \frac{dW_1(y(t))}{dt}+\frac{dW_2(y(t))}{dt} \\
&\le \left[\frac{2bx_1^*x_2^*+cabx_2^*}{(a+x_1^*)^2}-\frac{2r_1x_1^*}{k_1}+\frac{bx_1^*}{a+x_1^*}\right]y_1^2(t) \\
&\quad+\left[\frac{bx_1^*}{a+x_1^*}+\frac{cabx_2^*}{(a+x_1^*)^2}-\frac{2r_2x_2^*}{k_2}\right]y_2^2(t) \\
&\quad+\left\{\left[\frac{r_1bx_1^{*^2}x_2^*}{k_1(a+x_1^*)^2}+\frac{r_1^2x_1^{*^2}}{k_1^2}\right]\tau_1+\frac{r_2cabx_2^{*^2}}{k_2(a+x_1^*)^2}\tau_2\right\}y_1^2(t) \\
&\quad+\left[\frac{r_1bx_1^{*^2}}{k_1(a+x_1^*)}\tau_1+\frac{r_2^2x_2^{*^2}}{k_2^2}\tau_2\right]y_2^2(t) \\
&\quad+\left[\frac{r_1bx_1^{*^2}x_2^*}{k_1(a+x_1^*)^2}+\frac{r_1^2x_1^{*^2}}{k_1^2}+\frac{r_1bx_1^{*^2}}{k_1(a+x_1^*)}\right]\tau_1y_1^2(t) \\
&\quad+\left[\frac{r_2cabx_2^{*^2}}{k_2(a+x_1^*)^2}+\frac{r_2^2x_2^{*^2}}{k_2^2}\right]\tau_2y_2^2(t) \\
&\le -\left\{\left[\frac{2r_1x_1^*}{k_1}-\frac{2bx_1^*x_2^*+cabx_2^*}{(a+x_1^*)^2}-\frac{bx_1^*}{a+x_1^*}\right]\right. \\
&\quad\left.-\left[\frac{2r_1bx_1^{*^2}x_2^*}{k_1(a+x_1^*)^2}+\frac{2r_1^2x_1^{*^2}}{k_1^2}+\frac{r_1bx_1^{*^2}}{k_1(a+x_1^*)}\right]\tau_1-\frac{r_2cabx_2^{*^2}}{k_2(a+x_1^*)^2}\tau_2\right\}y_1^2(t) \\
&\quad-\left\{\left[\frac{2r_2x_2^*}{k_2}-\frac{bx_1^*}{a+x_1^*}-\frac{cabx_2^*}{(a+x_1^*)^2}\right]\right. \\
&\quad\left.-\frac{r_1bx_1^{*^2}}{k_1(a+x_1^*)}\tau_1-\left[\frac{r_2cabx_2^{*^2}}{k_2(a+x_1^*)^2}+\frac{2r_2^2x_2^{*^2}}{k_2^2}\right]\tau_2\right\}y_2^2(t) \\
&\equiv -\eta_1y_1^2(t)-\eta_2y_2^2(t). \qquad (4.10)
\end{aligned}
$$

Clearly, (4.2) and (4.3) implies that $\eta_1>0$ and $\eta_2>0$. Denote $\eta=\min\{\eta_1,\eta_2\}$, then (4.10) leads to

$$
W(t)+\eta\int_T^t[y_1^2(s)+y_2^2(s)]ds\le W(T)\ \text{ for }\ t\ge T \qquad (4.11)
$$

and which implies $y_1^2(t)+y_2^2(t)\in L_1[T,\infty)$. We can see from (4.1) and the boundedness of $y(t)$ that $y_1^2(t)+y_2^2(t)$ is uniform continuous and then, using Barbǎlat's Lemma(see[9]), we can conclude that $\lim_{t\to\infty}[y_1^2(t)+y_2^2(t)]=0$. Therefore the zero solution of (4.1) is asymptotically stable and this completes the proof.

Remark 4.1 *In Theorem 4.1, let* $\tau_1=\tau_2=0$, *then the result obtained in Theorem 4.1 is an extension of the result of [14].*

4.2 Global Stability

In this section, we drive sufficient conditions which guarantee that the positive equilibrium point E^* of the system (2.1) is globally asymptotically stable. Our method in the proof of the global asymptotic stability of the positive equilibrium E^* of the system (2.1) is to construct a suitable Lyapunov functinal.

Theorem 4.2 *Let E^* be the unique equilibrium point of the system (2.1) and the delays τ_1 and τ_2 satisfy*

$$r_1 - q_1 e - \frac{bM_2}{a} > 0 \quad and \quad r_2 - q_2 e > 0 \tag{4.12}$$

$$\gamma_1 - \gamma_2\tau_1 - \gamma_3\tau_2 > 0 \quad and \quad \delta_1 - \delta_2\tau_1 - \delta_3\tau_2 > 0 \tag{4.13}$$

where $M_i(i = 1,2)$ defined by (3.3), (3.4), and

$$\begin{aligned}
\gamma_1 &= \frac{-1}{a+m_1}\left[r_1x_1^* + \frac{bx_2^* + x_1^*(r_2+q_2e+cb)}{2} + \frac{r_1x_1^*(M_1+x_1^*)}{k_1} + \frac{r_2x_1^*(M_2+2x_2^*)}{2k_2}\right] \\
&\quad + \frac{1}{a+M_1}\left[q_1ex_1^* + \frac{r_1x_1^*(a+2x_1^*)}{k_1}\right] \\
\gamma_2 &= \frac{r_1M_1(a+x_1^*)}{k_1(a+m_1)^2}\left[x_1^*(r_1+q_1e) + \frac{bx_2^*}{2} + \frac{r_1x_1^*(a+M_1+3x_1^*)}{k_1}\right] \\
\gamma_3 &= \frac{r_2x_1^*M_2(a+x_1^*)}{2k_2(a+m_1)^2}\left(r_2+q_2e+cb+\frac{r_2x_2^*}{k_2}\right) \\
\delta_1 &= \frac{-1}{2(a+m_1)}\left[x_1^*(r_2+q_2e+cb) + bx_2^* + \frac{r_2x_1^*(M_2+2x_2^*)}{k_2}\right] + \frac{r_2x_2^*}{k_2(a+M_1)}(a+x_1^*) \\
\delta_2 &= \frac{r_1bx_2^*M_1(a+x_1^*)}{2k_1(a+m_1)^2} \\
\delta_3 &= \frac{r_2(a+x_1^*)M_2}{2k_2(a+m_1)^2}\left[x_1^*(r_2+q_2ex_1^*+cb) + \frac{r_2x_2^*(2a+2M_1+5x_1^*)}{k_2}\right]
\end{aligned}$$

then the unique positive equilibrium point E^* of the system (2.1) is globally asymptotically stable.

Proof :
Define

$$z(t) = (z_1(t), z_2(t))$$

by

$$z_1(t) = \frac{x_1(t) - x_1^*}{x_1^*},\ z_2(t) = \frac{x_2(t) - x_2^*}{x_2^*}.$$

From (2.1)

$$\begin{aligned}
\frac{dz_1(t)}{dt} &= (1+z_1(t))\left[\frac{r_1 x_1^* z_1(t)}{a + x_1^*(1+z_1(t))} - \frac{q_1 e x_1^* z_1(t)}{a + x_1^*(1+z_1(t))}\right. \\
&\quad - \frac{r_1 a x_1^* z_1(t-\tau_1)}{k_1[a + x_1^*(1+z_1(t))]} - \frac{r_1 {x_1^*}^2 z_1(t-\tau_1)}{k_1[a + x_1^*(1+z_1(t))]} - \frac{r_1 {x_1^*}^2 z_1(t) z_1(t-\tau_1)}{k_1[a + x_1^*(1+z_1(t))]} \\
&\quad \left. - \frac{r_1 {x_1^*}^2 z_1(t)}{k_1[a + x_1^*(1+z_1(t))]} - \frac{b x_2^* z_2(t)}{a + x_1^*(1+z_1(t))}\right]
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
\frac{dz_2(t)}{dt} &= (1+z_2(t))\left[\frac{r_2 x_1^* z_1(t)}{a + x_1^*(1+z_1(t))} - \frac{q_2 e x_1^* z_1(t)}{a + x_1^*(1+z_1(t))}\right. \\
&\quad - \frac{r_2 a x_2^* z_2(t-\tau_2)}{k_2[a + x_1^*(1+z_1(t))]} - \frac{r_2 x_1^* x_2^* z_2(t-\tau_2)}{k_2[a + x_1^*(1+z_1(t))]} - \frac{r_2 x_1^* x_2^* z_1(t) z_2(t-\tau_2)}{k_2[a + x_1^*(1+z_1(t))]} \\
&\quad \left. - \frac{r_2 x_1^* x_2^* z_1(t)}{k_2[a + x_1^*(1+z_1(t))]} + \frac{c b x_1^* z_1(t)}{a + x_1^*(1+z_1(t))}\right].
\end{aligned} \tag{4.15}$$

Let

$$V_1(z_t) = \{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\} \tag{4.16}$$

then we have from (4.14) and (4.15) that

$$\begin{aligned}
\frac{dV_1(z_t)}{dt} &= \frac{\dot{z_1}(t)z_1(t)}{1+z_1(t)} + \frac{\dot{z_2}(t)z_2(t)}{1+z_2(t)} \\
&= \frac{x_1^*(r_1-q_1e)}{a+x_1^*(1+z_1(t))}z_1^2(t) - \frac{r_1x_1^*(a+x_1^*)}{k_1[a+x_1^*(1+z_1(t))]}z_1(t)z_1(t-\tau_1) \\
&\quad - \frac{r_1{x_1^*}^2}{k_1[a+x_1^*(1+z_1(t))]}z_1^2(t)z_1(t-\tau_1) - \frac{r_1{x_1^*}^2}{k_1[a+x_1^*(1+z_1(t))]}z_1^2(t) \\
&\quad + \left\{ \frac{-bx_2^*+r_2x_1^*-q_2ex_1^*+cbx_1^*}{a+x_1^*(1+z_1(t))} - \frac{r_2x_1^*x_2^*}{k_2[a+x_1^*(1+z_1(t))]} \right\} z_1(t)z_2(t) \\
&\quad - \frac{r_2x_2^*(a+x_1^*)}{k_2[a+x_1^*(1+z_1(t))]}z_2(t)z_2(t-\tau_2) \\
&\quad - \frac{r_2x_1^*x_2^*}{k_2[a+x_1^*(1+z_1(t))]}z_1(t)z_2(t)z_2(t-\tau_2) \\
&\leq \frac{x_1^*(r_1-q_1e)}{a+x_1^*(1+z_1(t))}z_1^2(t) - \frac{r_1x_1^*(a+x_1^*)}{k_1[a+x_1^*(1+z_1(t))]}z_1(t)z_1(t-\tau_1) \\
&\quad - \frac{r_1{x_1^*}^2}{k_1[a+x_1^*(1+z_1(t))]}z_1^2(t)z_1(t-\tau_1) - \frac{r_1{x_1^*}^2}{k_1[a+x^*(1+z_1(t))]}z_1^2(t) \\
&\quad + \left\{ \frac{bx_2^*+r_2x_1^*+q_2ex_1^*+cbx_1^*}{a+x_1^*(1+z_1(t))} + \frac{r_2x_1^*x_2^*}{k_2[a+x_1^*(1+z_1(t))]} \right\} |z_1(t)||z_2(t)| \\
&\quad - \frac{r_2x_2^*(a+x_1^*)}{k_2[a+x_1^*(1+z_1(t))]}z_2(t)z_2(t-\tau_2) \\
&\quad - \frac{r_2x_1^*x_2^*}{k_2[a+x_1^*(1+z_1(t))]}z_1(t)z_2(t)z_2(t-\tau_2). \qquad (4.17)
\end{aligned}$$

By Theorem 3.1, there exist a $T^* > 0$ such that $m_1 \leq x_1^*[1+z_1(t)] \leq M_1$, and $m_2 \leq x_2^*[1+z_2(t)] \leq M_2$ for $t > T^*$. Then (4.17) implies that

$$
\begin{aligned}
\frac{dV_1(z_t)}{dt} \leq\ & \left(\frac{r_1x_1^*}{a+m_1}-\frac{q_1ex_1^*}{a+M_1}\right)z_1^2(t)\\
& -\frac{r_1x_1^*(a+x_1^*)}{k_1[a+x_1^*(1+z_1(t))]}z_1(t)\left[z_1(t)-\int_{t-\tau_1}^{t}\dot{z}_1(s)ds\right]\\
& +\frac{r_1x_1^{*^2}}{k_1[a+x_1^*(1+z_1(t))]}z_1^2(t)|z_1(t-\tau_1)|-\frac{r_1x_1^{*^2}}{k_1(a+M_1)}z_1^2(t)\\
& +\left\{\frac{bx_2^*+r_2x_1^*+q_2ex_1^*+cbx_1^*}{2(a+m_1)}+\frac{r_2x_1^*x_2^*}{2k_2(a+m_1)}\right\}z_1^2(t)\\
& +\left\{\frac{bx_2^*+r_2x_1^*+q_2ex_1^*+cbx_1^*}{2(a+m_1)}+\frac{r_2x_1^*x_2^*}{2k_2(a+m_1)}\right\}z_2^2(t)\\
& -\frac{r_2x_2^*(a+x_1^*)}{k_2[a+x_1^*(1+z_1(t))]}z_2(t)\left[z_2(t)-\int_{t-\tau_2}^{t}\dot{z}_2(s)ds\right]\\
& +\frac{r_2x_1^*x_2^*}{k_2[a+x_1^*(1+z_1(t))]}|z_1(t)||z_2(t)||z_2(t-\tau_2)|\\
\leq\ & \left\{\frac{r_1x_1^*}{a+m_1}-\frac{q_1ex_1^*}{a+M_1}-\frac{r_1x_1^*(a+2x_1^*)}{k_1(a+M_1)}\right.\\
& \left.+\frac{bx_2^*+r_2x_1^*+q_2ex_1^*+cbx_1^*}{2(a+m_1)}+\frac{r_2x_1^*x_2^*}{2k_2(a+m_1)}\right\}z_1^2(t)\\
& +\left\{\frac{bx_2^*+r_2x_1^*+q_2ex_1^*+cbx_1^*}{2(a+m_1)}+\frac{r_2x_1^*x_2^*}{2k_2(a+m_1)}-\frac{r_2x_2^*(a+x_1^*)}{k_2(a+M_1)}\right\}z_2^2(t)\\
& +\left\{\frac{r_1x_1^*(a+x_1^*)}{k_1[a+x_1^*(1+z_1(t))]}\right\}\int_{t-\tau_1}^{t}z_1(t)\dot{z}_1(s)ds\\
& +\left\{\frac{r_2x_2^*(a+x_1^*)}{k_2[a+x_1^*(1+z_1(t))]}\right\}\int_{t-\tau_2}^{t}z_2(t)\dot{z}_2(s)ds\\
& +\frac{r_1x_1^{*^2}|z_1(t-\tau_1)|}{k_1[a+x_1^*(1+z_1(t))]}z_1^2(t)+\frac{r_2x_1^*x_2^*|z_2(t-\tau_2)|}{2k_2[a+x_1^*(1+z_1(t))]}z_1^2(t)\\
& +\frac{r_2x_1^*x_2^*|z_2(t-\tau_2)|}{2k_2[a+x_1^*(1+z_1(t))]}z_2^2(t). \qquad (4.18)
\end{aligned}
$$

Then for $t \geq T^*+\tau \equiv \tilde{T},\ \tau = \max\{\tau_1,\tau_2\}$, we have from (4.18) that

$$
\begin{aligned}
\frac{dV_1(z_t)}{dt} \leq\ & \left\{ \frac{r_1 x_1^*}{a+m_1} - \frac{q_1 e x_1^*}{a+M_1} - \frac{r_1 x_1^*(a+2x_1^*)}{k_1(a+M_1)} + \frac{bx_2^* + r_2 x_1^* + q_2 e x_1^* + cbx_1^*}{2(a+m_1)} \right. \\
& \left. + \frac{r_2 x_1^* x_2^*}{2k_2(a+m_1)} + \frac{r_1 x_1^*(M_1+x_1^*)}{k_1(a+m_1)} + \frac{r_2 x_1^*(M_2+x_2^*)}{2k_2(a+m_1)} \right\} z_1^2(t) \\
& + \left\{ \frac{bx_2^* + r_2 x_1^* + q_2 e x_1^* + cbx_1^*}{2(a+m_1)} - \frac{r_2 x_2^*(a+x_1^*)}{k_2(a+M_1)} + \frac{r_2 x_1^*(M_2+2x_2^*)}{2k_2(a+m_1)} \right\} z_2^2(t) \\
& + \left\{ \frac{r_1 x_1^*(a+x_1^*)}{k_1[a+x_1^*(1+z_1(t))]} \right\} \int_{t-\tau_1}^{t} z_1(t)(1+z_1(s)) \left\{ \frac{x_1^*(r_1 - q_1 e) z_1(s)}{a + x_1^*(1+z_1(s))} \right. \\
& - \frac{r_1 x_1^*(a+x_1^*) z_1(s-\tau_1)}{k_1[a+x_1^*(1+z_1(s))]} - \frac{r_1 {x_1^*}^2 z_1(s) z_1(s-\tau_1)}{k_1[a+x_1^*(1+z_1(s))]} \\
& \left. - \frac{r_1 {x_1^*}^2 z_1(s)}{k_1[a+x_1^*(1+z_1(s))]} - \frac{bx_2^* z_2(s)}{a+x_1^*(1+z_1(s))} \right\} ds \\
& + \left\{ \frac{r_2 x_2^*(a+x_1^*)}{k_2[a+x_1^*(1+z_1(t))]} \right\} \int_{t-\tau_2}^{t} z_2(t)(1+z_2(s)) \left\{ \frac{x_1^*(r_2 - q_2 e - cb) z_1(s)}{a+x_1^*(1+z_1(s))} \right. \\
& \left. - \frac{r_2 x_2^*(a+x_1^*) z_2(s-\tau_2)}{k_2[a+x_1^*(1+z_1(s))]} - \frac{r_2 x_1^* x_2^* z_1(s) z_2(s-\tau_2)}{k_2[a+x_1^*(1+z_1(s))]} - \frac{r_2 x_1^* x_2^* z_1(s)}{k_2[a+x_1^*(1+z_1(s))]} \right\} ds \\
\leq\ & \left\{ \frac{r_1 x_1^*}{a+m_1} - \frac{q_1 e x_1^*}{a+M_1} - \frac{r_1 x_1^*(a+2x_1^*)}{k_1(a+M_1)} + \frac{bx_2^* + r_2 x_1^* + q_2 e x_1^* + cbx_1^*}{2(a+m_1)} \right. \\
& \left. + \frac{r_1 x_1^*(M_1+x_1^*)}{k_1(a+m_1)} + \frac{r_2 x_1^*(M_2+2x_2^*)}{2k_2(a+m_1)} \right\} z_1^2(t) \\
& + \left\{ \frac{bx_2^* + r_2 x_1^* + q_2 e x_1^* + cbx_1^*}{2(a+m_1)} - \frac{r_2 x_2^*(a+x_1^*)}{k_2(a+M_1)} + \frac{r_2 x_1^*(M_2+2x_2^*)}{2k_2(a+m_1)} \right\} z_2^2(t) \\
& + \frac{r_1 M_1(a+x_1^*)}{k_1(a+m_1)^2} \int_{t-\tau_1}^{t} \left\{ x_1^*(r_1+q_1 e)|z_1(t)||z_1(s)| + \frac{r_1 x_1^*(a+x_1^*)}{k_1} |z_1(t)||z_1(s-\tau_1)| \right. \\
& + \frac{r_1 {x_1^*}^2}{k_1} [|z_1(t)||z_1(s-\tau_1)||1+z_1(s)| + |z_1(t)||z_1(s-\tau_1)|] \\
& \left. + \frac{r_1 {x_1^*}^2}{k_1} |z_1(t)||z_1(s)| + bx_2^* |z_1(t)||z_2(s)| \right\} ds \\
& + \frac{r_2 M_2(a+x_1^*)}{k_2(a+m_1)^2} \int_{t-\tau_2}^{t} \left\{ x_1^*(r_2+q_2 e+cb)|z_2(t)||z_1(s)| + \frac{r_2 x_2^*(a+x_1^*)}{k_2} |z_2(t)||z_2(s-\tau_2)| \right. \\
& \left. + \frac{r_2 x_1^* x_2^*}{k_2} [|z_2(t)||z_2(s-\tau_2)||1+z_1(s)| + |z_2(t)||z_2(s-\tau_2)|] + \frac{r_2 x_1^* x_2^*}{k_2} |z_2(t)||z_1(s)| \right\} ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \Big\{\frac{r_1x_1^*}{a+m_1}-\frac{q_1ex_1^*}{a+M_1}-\frac{r_1x_1^*(a+2x_1^*)}{k_1(a+M_1)}+\frac{bx_2^*+r_2x_1^*+q_2ex_1^*+cbx_1^*}{2(a+m_1)}\\
&\quad+\frac{r_1x_1^*(M_1+x_1^*)}{k_1(a+m_1)}+\frac{r_2x_1^*(M_2+2x_2^*)}{2k_2(a+m_1)}\Big\}z_1^2(t)\\
&\quad+\Big\{\frac{bx_2^*+r_2x_1^*+q_2ex_1^*+cbx_1^*}{2(a+m_1)}-\frac{r_2x_2^*(a+x_1^*)}{k_2(a+M_1)}+\frac{r_2x_1^*(M_2+2x_2^*)}{2k_2(a+m_1)}\Big\}z_2^2(t)\\
&\quad+\Big\{\frac{r_1M_1(a+x_1^*)(r_1x_1^*+q_1ex_1^*+bx_2^*)}{2k_2(a+m_1)^2}+\frac{r_1^2x_1^*M_1(a+x_1^*)(a+M_1+3x_1^*)}{2k_1^2(a+m_1)^2}\Big\}\tau_1z_1^2(t)\\
&\quad+\Big\{\frac{r_2x_1^*M_2(a+x_1^*)(r_2+q_2e+cb)}{2k_2(a+m_1)^2}+\frac{r_2^2x_2^*M_2(a+x_1^*)(a+3x_1^*+M_1)}{2k_2^2(a+m_1)^2}\Big\}\tau_2z_2^2(t)\\
&\quad+\Big\{\frac{r_1x_1^*M_1(a+x_1^*)(r_1+q_1e)}{2k_1(a+m_1)^2}+\frac{r_1^2x_1^2M_1(a+x_1^*)}{2k_1^2(a+m_1)^2}\Big\}\int_{t-\tau_1}^{t}z_1^2(s)ds\\
&\quad+\frac{r_1bx_2^*M_1(a+x_1^*)}{2k_1(a+m_1)^2}\int_{t-\tau_1}^{t}z_2^2(s)ds\\
&\quad+\Big\{\frac{r_1^2x_1^*M_1(a+x_1^*)(a+M_1+2x_1^*)}{2k_1^2(a+m_1)^2}\Big\}\int_{t-\tau_1}^{t}z_1^2(s-\tau_1)ds\\
&\quad+\Big\{\frac{r_2x_1^*M_2(a+x_1^*)(r_2+q_2e+cb)}{2k_2(a+m_1)^2}+\frac{r_2^2x_1^*x_2^*M_2(a+x_1^*)}{2k_2^2(a+m_1)^2}\Big\}\int_{t-\tau_2}^{t}z_1^2(s)ds\\
&\quad+\Big\{\frac{r_2^2x_2^*M_2(a+x_1^*)(a+M_1+2x_1^*)}{2k_2^2(a+m_1)^2}\Big\}\int_{t-\tau_2}^{t}z_2^2(s-\tau_2)ds. \qquad (4.19)
\end{aligned}
$$

Let

$$
\begin{aligned}
V_2(z_t) &= \Big\{\frac{r_1x_1^*M_1(a+x_1^*)(r_1+q_1e)}{2k_1(a+m_1)^2}+\frac{r_1^2x_1^2M_1(a+x_1^*)}{2k_1^2(a+m_1)^2}\Big\}\int_{t-\tau_1}^{t}\int_{s}^{t}z_1^2(\rho)d\rho ds\\
&\quad+\frac{r_1bx_2^*M_1(a+x_1^*)}{2k_1(a+m_1)^2}\int_{t-\tau_1}^{t}\int_{s}^{t}z_2^2(\rho)d\rho ds\\
&\quad+\Big\{\frac{r_1^2x_1^*M_1(a+x_1^*)(a+M_1+2x_1^*)}{2k_1^2(a+m_1)^2}\Big\}\int_{t-\tau_1}^{t}\int_{s}^{t}z_1^2(\rho-\tau_1)d\rho ds\\
&\quad+\Big\{\frac{r_2x_1^*M_2(a+x_1^*)(r_2+q_2e+cb)}{2k_2(a+m_1)^2}+\frac{r_2^2x_1^*x_2^*M_2(a+x_1^*)}{2k_2^2(a+m_1)^2}\Big\}\int_{t-\tau_2}^{t}\int_{s}^{t}z_1^2(\rho)d\rho ds\\
&\quad+\Big\{\frac{r_2^2x_2^*M_2(a+x_1^*)(a+M_1+2x_1^*)}{2k_2^2(a+m_1)^2}\Big\}\int_{t-\tau_2}^{t}\int_{s}^{t}z_2^2(\rho-\tau_2)d\rho ds
\end{aligned}
\qquad (4.20)
$$

then

$$
\begin{aligned}
\frac{dV_2(z_t)}{dt} = {} & \left\{\frac{r_1x_1^*M_1(a+x_1^*)(r_1+q_1e)}{2k_1(a+m_1)^2}+\frac{r_1^2x_1^2M_1(a+x_1^*)}{2k_1^2(a+m_1)^2}\right\}\tau_1z_1^2(t) \\
& -\left\{\frac{r_1x_1^*M_1(a+x_1^*)(r_1+q_1e)}{2k_1(a+m_1)^2}+\frac{r_1^2x_1^2M_1(a+x_1^*)}{2k_1^2(a+m_1)^2}\right\}\int_{t-\tau_1}^{t}z_1^2(s)ds \\
& +\frac{r_1bx_2^*M_1(a+x_1^*)}{2k_1(a+m_1)^2}\tau_1z_2^2(t)-\frac{r_1bx_2^*M_1(a+x_1^*)}{2k_1(a+m_1)^2}\int_{t-\tau_1}^{t}z_2^2(s)ds \\
& +\left\{\frac{r_1^2x_1^*M_1(a+x_1^*)(a+M_1+2x_1^*)}{2k_1^2(a+m_1)^2}\right\}\tau_1z_1^2(t-\tau_1) \\
& -\left\{\frac{r_1^2x_1^*M_1(a+x_1^*)(a+M_1+2x_1^*)}{2k_1^2(a+m_1)^2}\right\}\int_{t-\tau_1}^{t}z_1^2(s-\tau_1)ds \\
& +\left\{\frac{r_2x_1^*M_2(a+x_1^*)(r_2+q_2e+cb)}{2k_2(a+m_1)^2}+\frac{r_2^2x_1^*x_2^*M_2(a+x_1^*)}{2k_2^2(a+m_1)^2}\right\}\tau_2z_1^2(t) \\
& -\left\{\frac{r_2x_1^*M_2(a+x_1^*)(r_2+q_2e+cb)}{2k_2(a+m_1)^2}+\frac{r_2^2x_1^*x_2^*M_2(a+x_1^*)}{2k_2^2(a+m_1)^2}\right\}\int_{t-\tau_2}^{t}z_1^2(s)ds \\
& +\left\{\frac{r_2^2x_2^*M_2(a+x_1^*)(a+M_1+2x_1^*)}{2k_2^2(a+m_1)^2}\right\}\tau_2z_2^2(t-\tau_2) \\
& -\left\{\frac{r_2^2x_2^*M_2(a+x_1^*)(a+M_1+2x_1^*)}{2k_2^2(a+m_1)^2}\right\}\int_{t-\tau_2}^{t}z_2^2(s-\tau_2)ds
\end{aligned}
\tag{4.21}
$$

and then we have from (4.19) and (4.21) that for $t \geq \widehat{T}$

$$\begin{aligned}
\frac{dV_1(z_t)}{dt} + \frac{dV_2(z_t)}{dt} \leq & \left\{ \frac{r_1 x_1^*}{a+m_1} - \frac{q_1 e x_1^*}{a+M_1} - \frac{r_1 x_1^*(a+2x_1^*)}{k_1(a+M_1)} + \frac{bx_2^* + r_2 x_1^* + q_2 e x_1^* + cbx_1^*}{2(a+m_1)} \right. \\
& \left. + \frac{r_1 x_1^*(M_1 + x_1^*)}{k_1(a+m_1)} + \frac{r_2 x_1^*(M_2 + 2x_2^*)}{2k_2(a+m_1)} \right\} z_1^2(t) \\
& + \left\{ \frac{bx_2^* + r_2 x_1^* + q_2 e x_1^* + cbx_1^*}{2(a+m_1)} - \frac{r_2 x_2^*(a+x_1^*)}{k_2(a+M_1)} + \frac{r_2 x_1^*(M_2 + 2x_2^*)}{2k_2(a+m_1)} \right\} z_2^2(t) \\
& + \left\{ \frac{r_1 M_1 (a+x_1^*)(r_1 x_1^* + q_1 e x_1^* + bx_2^*)}{2k_1(a+m_1)^2} + \frac{r_1^2 x_1^* M_1 (a+x_1^*)(a+M_1+3x_1^*)}{2k_1^2(a+m_1)^2} \right\} \tau_1 z_1^2(t) \\
& + \left\{ \frac{r_2 x_1^* M_2 (a+x_1^*)(r_2 + q_2 e + cb)}{2k_2(a+m_1)^2} + \frac{r_2^2 x_2^* M_2 (a+x_1^*)(a+M_1+3x_1^*)}{2k_2^2(a+m_1)^2} \right\} \tau_2 z_2^2(t) \\
& + \left\{ \frac{r_1 x_1^* M_1 (a+x_1^*)(r_1 + q_1 e)}{2k_1(a+m_1)^2} + \frac{r_1^2 x_1^2 M_1 (a+x_1^*)}{2k_1^2(a+m_1)^2} \right\} \tau_1 z_1^2(t) \\
& + \frac{r_1 b x_2^* M_1 (a+x_1^*)}{2k_1(a+m_1)^2} \tau_1 z_2^2(t) \\
& + \left\{ \frac{r_1^2 x_1^* M_1 (a+x_1^*)(a+M_1+2x_1^*)}{2k_1^2(a+m_1)^2} \right\} \tau_1 z_1^2(t-\tau_1) \\
& + \left\{ \frac{r_2 x_1^* M_2 (a+x_1^*)(r_2 + q_2 e + cb)}{2k_2(a+m_1)^2} + \frac{r_2^2 x_1^* x_2^* M_2 (a+x_1^*)}{2k_2^2(a+m_1)^2} \right\} \tau_2 z_1^2(t) \\
& + \left\{ \frac{r_2^2 x_2^* M_2 (a+x_1^*)(a+M_1+2x_1^*)}{2k_2^2(a+m_1)^2} \right\} \tau_2 z_2^2(t-\tau_2).
\end{aligned} \tag{4.22}$$

Let

$$\begin{aligned}
V_3(z_t) = & \left\{ \frac{r_1^2 x_1^* M_1 (a+x_1^*)(a+M_1+2x_1^*)}{2k_1^2(a+m_1)^2} \right\} \tau_1 \int_{t-\tau_1}^{t} z_1^2(s)ds \\
& + \left\{ \frac{r_2^2 x_2^* M_2 (a+x_1^*)(a+M_1+2x_1^*)}{2k_2^2(a+m_1)^2} \right\} \tau_2 \int_{t-\tau_2}^{t} z_2^2(s)ds
\end{aligned} \tag{4.23}$$

then

$$\begin{aligned}
\frac{dV_3(z_t)}{dt} = & \left\{ \frac{r_1^2 x_1^* M_1 (a+x_1^*)(a+M_1+2x_1^*)}{2k_1^2(a+m_1)^2} \right\} \tau_1 z_1^2(t) \\
& - \left\{ \frac{r_1^2 x_1^* M_1 (a+x_1^*)(a+M_1+2x_1^*)}{2k_1^2(a+m_1)^2} \right\} \tau_1 z_1^2(t-\tau_1) \\
& + \left\{ \frac{r_2^2 x_2^* M_2 (a+x_1^*)(a+M_1+2x_1^*)}{2k_2^2(a+m_1)^2} \right\} \tau_2 z_2^2(t) \\
& - \left\{ \frac{r_2^2 x_2^* M_2 (a+x_1^*)(a+M_1+2x_1^*)}{2k_2^2(a+m_1)^2} \right\} \tau_2 z_2^2(t-\tau_2).
\end{aligned} \tag{4.24}$$

Now define a Lyapunov functional $V(z_t)$ as

$$V(z_t) \quad = \quad V_1(z_t)+V_2(z_t)+V_3(z_t) \tag{4.25}$$

then we have from (4.22) and (4.24) that for $t \geq \widetilde{T}$

$$\begin{aligned}
\frac{dV(z_t)}{dt} &= \frac{dV_1(z_t)}{dt}+\frac{dV_2(z_t)}{dt}+\frac{dV_3(z_t)}{dt} \\
&\leq -\left\{\frac{-1}{a+m_1}\left[r_1x_1^*+\frac{bx_2^*+x_1^*(r_2+q_2e+cb)}{2}+\frac{r_1x_1^*(M_1+x_1^*)}{k_1}+\frac{r_2x_1^*(M_2+2x_2^*)}{2k_2}\right]\right. \\
&\quad +\frac{1}{a+M_1}\left[q_1ex_1^*+\frac{r_1x_1^*(a+2x_1^*)}{k_1}\right] \\
&\quad -\frac{r_1M_1(a+x_1^*)\tau_1}{k_1(a+m_1)^2}\left[x_1^*(r_1+q_1e)+\frac{bx_2^*}{2}+\frac{r_1x_1^*(a+M_1+3x_1^*)}{k_1}\right] \\
&\quad \left.-\frac{r_2x_1^*M_2(a+x_1^*)\tau_2}{2k_2(a+m_1)^2}\left(r_2+q_2e+cb+\frac{r_2x_2^*}{k_2}\right)\right\}z_1^2(t) \\
&\quad -\left\{\frac{-1}{2(a+m_1)}\left[x_1^*(r_2+q_2e+cb)+bx_2^*+\frac{r_2x_1^*(M_2+2x_2^*)}{k_2}\right]\right. \\
&\quad +\frac{r_2x_2^*}{k_2(a+M_1)}(a+x_1^*)-\frac{r_1bx_2^*M_1(a+x_1^*)\tau_1}{2k_1(a+m_1)^2} \\
&\quad \left.-\frac{r_2(a+x_1^*)M_2\tau_2}{2k_2(a+m_1)^2}\left[x_1^*(r_2+q_2ex_1^*+cb)+\frac{r_2x_2^*(2a+2M_1+5x_1^*)}{k_2}\right]\right\}z_2^2(t) \\
&\equiv -\zeta_1z_1^2(t)-\zeta_2z_2^2(t).
\end{aligned} \tag{4.26}$$

Then it follows from (4.12) and (4.13) that $\zeta_1 > 0$ and $\zeta_2 > 0$. Let $w(s) = \zeta s^2$, where $\zeta = min\{\zeta_1,\zeta_2\}$, then w is nonnegative continuous on $[0,\infty), w(0) = 0$, and $w(s) > 0$ for $s > 0$. It follows from (4.26) that for $t \geq \widetilde{T}$

$$\dot{V}(z_t) \leq -\zeta[z_1^2(t)+z_2^2(t)] = -\zeta|z(t)|^2 = -w(|z(t)|). \tag{4.27}$$

Now, we want to find a function u such that $V(z_t) \geq u(|z(t)|)$. It follows from (4.16),(4.20), and (4.23) that

$$V(z_t) \quad \geq \quad \{z_1(t)-\ln[1+z_1(t)]\}+\{z_2(t)-\ln[1+z_2(t)]\}. \tag{4.28}$$

By the Taylor Theorem, we have that

$$z_i(t)-ln[1+z_i(t)] = \frac{z_i^2(t)}{2[1+\theta_i(t)]^2} \tag{4.29}$$

where $\theta_i(t) \in (0, z_i(t))$ or $(z_i(t), 0)$ for $i = 1, 2$.
Case 1: If $0 < \theta_i(t) < z_i(t)$ for $i = 1, 2.$,then

$$\frac{z_i^2(t)}{[1+z_i(t)]^2} < \frac{z_i^2(t)}{[1+\theta_i(t)]^2} < z_i^2(t). \tag{4.30}$$

By Theorem 3.1, it follows that for $t \geq T^*$

$$m_i \leq x_i^*[1+z_i(t)] = x_i(t) \leq M_i, \ \text{ for } \ i = 1, 2. \tag{4.31}$$

Then (4.30) implies that

$$\begin{aligned} (\frac{x_1^*}{M_1})^2 z_1^2(t) &\leq \frac{z_1^2(t)}{[1+\theta_1(t)]^2} < z_1^2(t) \\ (\frac{x_2^*}{M_2})^2 z_2^2(t) &\leq \frac{z_2^2(t)}{[1+\theta_2(t)]^2} < z_2^2(t). \end{aligned} \tag{4.32}$$

It follows that $(4.28), (4.29)$ and (4.32) that for $t \geq T^*$

$$\begin{aligned} V(z_t) &\geq \frac{z_1^2(t)}{2[1+\theta_1(t)]^2} + \frac{z_2^2(t)}{2[1+\theta_2(t)]^2} \\ &\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq min\left\{\frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2, \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2\right\}\left[z_1^2(t) + z_2^2(t)\right] \\ &\equiv \widetilde{m}|z(t)|^2. \end{aligned} \tag{4.33}$$

Case2 : If $-1 < z_i(t) < \theta_i(t) < 0$ for i=1,2 ,then

$$z_i^2(t) < \frac{z_i^2(t)}{[1+\theta_i(t)]^2} < \frac{z_i^2(t)}{[1+z_i(t)]^2}. \tag{4.34}$$

By $(4.31), (4.34)$ implies that

$$\begin{aligned} z_1^2(t) &< \frac{z_1^2(t)}{[1+\theta_1(t)]^2} \leq \left(\frac{x_1^*}{m_1}\right)^2 z_1^2(t) \\ z_2^2(t) &< \frac{z_2^2(t)}{[1+\theta_2(t)]^2} \leq \left(\frac{x_2^*}{m_2}\right)^2 z_2^2(t). \end{aligned} \tag{4.35}$$

It follows that $(4.28),(4.29)$ and (4.35) that for $t \geq T^*$

$$
\begin{aligned}
V(z_t) &\geq \frac{z_1^2(t)}{2[1+\theta_1(t)]^2} + \frac{z_2^2(t)}{2[1+\theta_2(t)]^2} \\
&> \frac{1}{2}z_1^2(t) + \frac{1}{2}z_2^2(t) \\
&\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\
&\geq \widetilde{m}\left[z_1^2(t) + z_2^2(t)\right] \\
&= \widetilde{m}|z(t)|^2. \qquad (4.36)
\end{aligned}
$$

Case3 : If $0 < \theta_1(t) < z_1(t)$ and $-1 < z_2(t) < \theta_2(t) < 0$, then it follows that (4.28), (4.29), (4.32) and (4.35) that for $t \geq T^*$

$$
\begin{aligned}
V(z_t) &\geq \frac{z_1^2(t)}{2[1+\theta_1(t)]^2} + \frac{z_2^2(t)}{2[1+\theta_2(t)]^2} \\
&> \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}z_2^2(t) \\
&\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\
&\geq \widetilde{m}\left[z_1^2(t) + z_2^2(t)\right] \\
&= \widetilde{m}|z(t)|^2. \qquad (4.37)
\end{aligned}
$$

Case4 : If $-1 < z_1(t) < \theta_1(t) < 0$ and $0 < \theta_2(t) < z_2(t)$, then it follows that (4.28),(4.29),(4.32) and (4.35) that for $t \geq T^*$

$$
\begin{aligned}
V(z_t) &\geq \frac{z_1^2(t)}{2[1+\theta_1(t)]^2} + \frac{z_2^2(t)}{2[1+\theta_2(t)]^2} \\
&> \frac{1}{2}z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\
&\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\
&\geq \widetilde{m}\left[z_1^2(t) + z_2^2(t)\right] \\
&= \widetilde{m}|z(t)|^2. \qquad (4.38)
\end{aligned}
$$

Let $u(s) = \widetilde{m}s^2$, then u is nonnegative continuous on $[0,\infty)$, $u(0) = 0, u(s) > 0$ for $s > 0$, and $\lim_{s\to\infty} u(s) = +\infty$. So, by case1 $\sim$ case4, we have

$$V(z_t) \geq u(|z(t)|) \text{ for } t \geq T^*. \tag{4.39}$$

So the unique equilibrium point E^* of the system (2.1) is globally asymptotically stable.■

Remark *4.2 In Theorem 4.2, let* $\tau_1 = \tau_2 = 0$*, then the result obtained in Theorem 4.2 is different with the result of [14].*

5 Examples

In this section, we present one simple example to illustrate the procedures of applying our results.

Example 5.1

Consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= 100x_1(t)\left[1-\frac{x_1(t-0.001)}{100}\right]-x_2(t)\frac{100x_1(t)}{1000+x_1(t)}-\frac{1}{20}1000x_1(t) \\ \dot{x}_2(t) &= 105x_2(t)\left[1-\frac{x_2(t-0.0015)}{105}\right]+0.01x_2(t)\frac{100x_1(t)}{1000+x_1(t)}-\frac{3}{50}1000x_2(t). \end{aligned} \tag{5.1}$$

Comparing the system (5.1) with the system (2.1), we get $r_1 = 100$; $r_2 = 105$; $k_1 = 100$; $k_2 = 105$; $b = 100$; $c = 0.01$; $a = 1000$; $q_1 = 0.05$; $q_2 = 0.06$; $e = 1000$; $\tau_1 = 0.001$; and $\tau_2 = 0.0015$. So the system (5.1) has a unique positive equilibrium point $E^* = (49.57, 45)$
And

$$\alpha_1 - \alpha_2\tau_1 - \alpha_3\tau_2 = 93.6852 > 0$$
$$\beta_1 - \beta_2\tau_1 - \beta_3\tau_2 = 83.4331 > 0.$$

Then we conclude that the unique positive equilibrium point E^* of the system (5.1) is local asymptotically stable by Theorem 4.1. The trajectory of the system (5.1) is depicted in Figure 5.1.

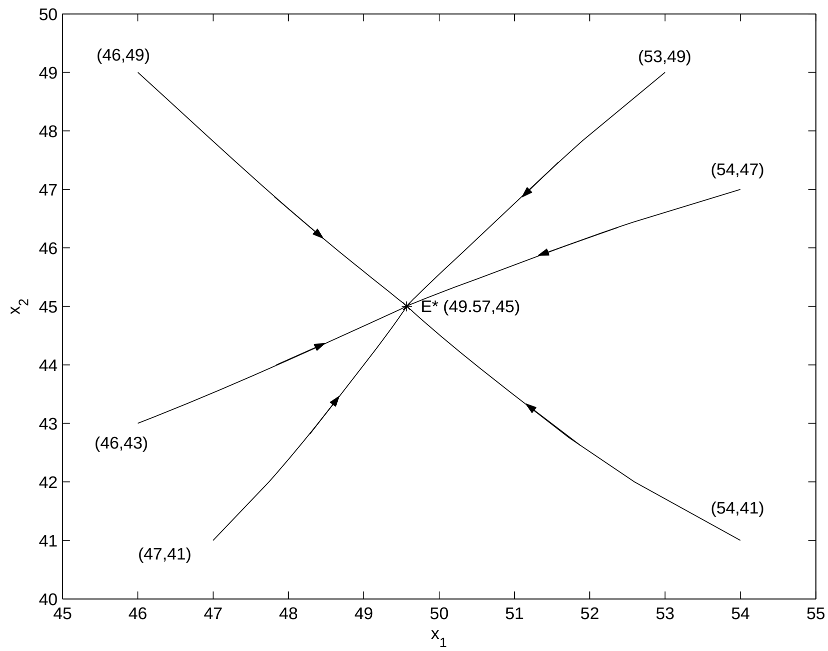

Figure 5.1: The trajectory of the system (5.1).

Example 5.2

Consider the following system:

$$\begin{aligned}
\dot{x}_1(t) &= 100x_1(t)\left[1-\frac{x_1(t-0.001)}{100}\right]-x_2(t)\frac{100x_1(t)}{1000+x_1(t)}-\frac{1}{20}1000x_1(t) \\
\dot{x}_2(t) &= 105x_2(t)\left[1-\frac{x_2(t-0.0015)}{105}\right]+0.01x_2(t)\frac{100x_1(t)}{1000+x_1(t)}-\frac{3}{50}1000x_2(t).
\end{aligned} \tag{5.2}$$

Comparing the system (5.2) with the system (2.1), we get $r_1 = 100$; $r_2 = 105$; $k_1 = 100$; $k_2 = 105$; $b = 100$; $c = 0.01$; $a = 1000$; $q_1 = 0.05$; $q_2 = 0.06$; $e = 1000$; $\tau_1 = 0.001$; and $\tau_2 = 0.0015$. So the system (5.2) has a unique positive equilibrium point $E^* = (49.57, 45)$

And

$$\begin{aligned}
r_1 - q_1 e - \frac{bM_2}{a} &= 48.7708 > 0 \\
r_2 - q_2 e &= 45 > 0 \\
\gamma_1 - \gamma_2\tau_1 - \gamma_3\tau_2 &= 20.5047 > 0 \\
\delta_1 - \delta_2\tau_1 - \delta_3\tau_2 &= 8.7389 > 0.
\end{aligned}$$

Then we conclude that the unique positive equilibrium point E^* of the system (5.2) is global asymptotically stable by Theorem 4.2. The trajectory of the system (5.2) is depicted in Figure 5.2.

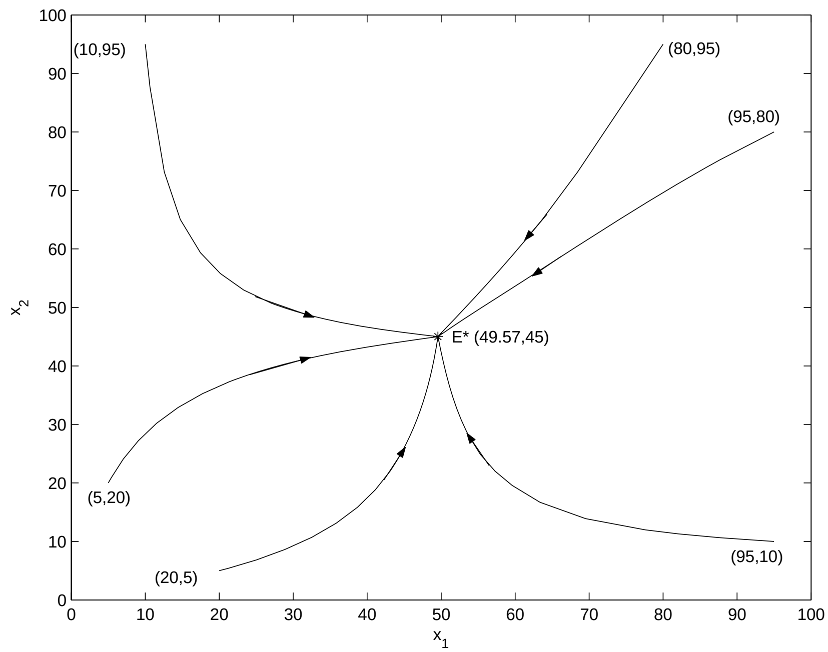

Figure 5.2: The trajectory of the system (5.2).

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具時滯參數的捕食漁獲系統之穩定性分析

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摘 要

本篇論文主要是研究具時滯參數的非選擇性捕食漁獲系統之整體穩定性。首先，我們先介紹這個模型；而後推導出系統的正均衡點之局部穩定性和整體穩定性的充分條件；最後，我們舉例說明上述結果。

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