Tunghai Science Vol. 5: 71−99
July, 2003

Persistence and Global Stability on Competition System with Time-Delay

Hsin-Chih Chen and Chao-Pao Ho*

Abstract

In this paper, we study the dynamical behavior of a competition system with time delay. We first use three different methods to analyze the global stability of the unique positive equilibrium point of the system without time delay. Secondly, it is shown that the system with time delay is uniform persistent under some appropriate conditions, and sufficient conditions are obtained for the global stability of the unique positive equilibrium point of the system with time delay. Finally, we illustrative our results by some examples.

Keywords:competition system, equilibrium stability, Dulac's criterion, Ponicaré-Bendixson theorem, limit cycle, uniform persistence, Lyapunov functional, global asymptotic stability.

1 Introduction

In recent years, the application of theories of functional differential equations in mathematical ecology has developed rapidly. Various mathematical models have been proposed in the study of population dynamics, ecology and epidemic. Some of them are described as autonomous delay differential equations. Many people are doing research on the dynamics of population with delays, which is useful for the control of the population of mankind, animals and the environment. One of the famous models for dynamics of population is the Lotka-Volterra competition system. Owing to its theoretical and practical significance, the Lotka-Volterra systems have been studied extensively [5,6,15]. There is a large volume of literature relevant to the theory of the Lotka-Volterra systems and methods and results can be found in Gopalsamy [6], Kuang [8], Takuechi [11] and the references therein.

Since time delays occur so often in nature, a number of models in ecology can be formulated as systems of differential equations with time delays. One of the most important problems for

*Department of Mathematics, Tunghai University, Taichung 407, TAIWAN

this type of system is to analyze the effect of time delays on the stability of the system. From the literature on ecological models with time delays, we have known that for some systems [3,15], the stability switches many times and the systems will eventually become unstable when time delays increase. While for other systems [12,14], there will be no change in uniform persistence or permanence of the systems when the time delays vary. Uniform persistence or permanence concerning the long time survival of a population is a more important concept of stability from the viewpoint of mathematical ecology.

In this paper, we consider the competition system with time delay,

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[r_1 - a_{11}x_1(t-\tau) - \tfrac{a_{12}x_2(t-\tau)}{1+x_2(t-\tau)}\right] \\ \dot{x}_2(t) &= x_2(t)\left[r_2 - a_{22}x_2(t-\tau) - \tfrac{a_{21}x_1(t-\tau)}{1+x_1(t-\tau)}\right] \end{aligned} \qquad (1.1)$$

with initial conditions

$$x_i(t) = \phi_i(t) \geq 0, t \in [-\tau, 0], \phi_i(0) > 0, i = 1,2 \qquad (1.2)$$

where $\cdot = d/dt$, r_i, a_{ij} $(i,j = 1,2)$, and τ are positive constants. $\phi_i(t)(i = 1,2)$ are continuous bounded functions on the interval $[-\tau, 0]$. $x_1(t)$ and $x_2(t)$ denote the population densities (or biomasses) of two species competing for a common pool of resources in a temporally uniform environment.

In this paper, we determine sufficient conditions on the parameters of the system that ensure uniform persistence and globally asymptotic stability of the system. The present paper is organized as follows.

In section 2 we analyze the system (2.1). We first propose our system, and obtain positivity and boundedness results. Next, the global stability property of the system (2.1) is established by the application of the Dulac's criterion plus Poincaré-Bendixson theorem, the construction of the Lyapunov function or stable limit cycle analysis.

In section 3 we analyze the system (3.1). First of all, we give conditions for uniform persistence to hold for the systems (3.1) and (3.2). Next, provides sufficient conditions for the unique positive equilibrium point of the system (3.1) to be globally asymptotically stable.

In section 4 we give two suitable examples to show the result in the sections 2 and 3, respectively.

2 The System without Time Delay

Consider the following competition system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[r_1 - a_{11}x_1(t) - \frac{a_{12}x_2(t)}{1+x_2(t)}\right] \equiv f_1(x_1,x_2) \\ \dot{x}_2(t) &= x_2(t)\left[r_2 - a_{22}x_2(t) - \frac{a_{21}x_1(t)}{1+x_1(t)}\right] \equiv f_2(x_1,x_2) \end{aligned} \tag{2.1}$$

where $\cdot = d/dt$, r_i, a_{ij} $(i,j=1,2)$ are positive constants. $x_1(t)$ and $x_2(t)$ denote the population densities (or biomasses) of two species competing for a common pool of resources in a temporally uniform environment.

2.1 Local Stability

Clearly, $E_0 \equiv (0,0)$, $E_1 \equiv (r_1/a_{11},0)$, and $E_2 \equiv (0,r_2/a_{22})$ are the equilibrium points of the system (2.1). On the other hand, let $E^* \equiv (x_1^*,x_2^*)$ be the unique positive equilibrium point of the system (2.1).

Remark *2.1* *If*

$$r_2 > (a_{21}-r_2)x_1^* \tag{2.2}$$

and

$$\alpha_1 < 0,\ \alpha_2 \neq 0,\ \alpha_3 > 0,\ \text{where} \tag{2.3}$$

$$\alpha_1 = a_{11}a_{21} - a_{11}a_{22} - r_2a_{11}$$

$$\alpha_2 = r_1a_{22} + r_1r_2 - r_2a_{12} - r_1a_{21} - r_2a_{11} - a_{11}a_{22} + a_{12}a_{21}$$

$$\alpha_3 = r_1a_{22} + r_1r_2 - r_2a_{12}$$

hold, then E^* is the unique positive equilibrium point of the system (2.1).

Firstly, we discuss the local stability of equilibrium points of the system (2.1) by the Hartman-Grobman theorem. Secondly, we use three different methods to analyze the global stability of the unique positive equilibrium point E^* of the system (2.1).

Now let us study the local behavior of the system (2.1) at equilibrium points E_0, E_1, E_2 and E^*. The Jacobian matrix of the system (2.1) take the form

$$J \equiv \begin{bmatrix} r_1 - 2a_{11}x_1 - \dfrac{a_{12}x_2}{1+x_2} & -\dfrac{a_{12}x_1}{(1+x_2)^2} \\ -\dfrac{a_{21}x_2}{(1+x_1)^2} & r_2 - 2a_{22}x_2 - \dfrac{a_{21}x_1}{1+x_1} \end{bmatrix}$$

The Jacobian matrix $J_0 \equiv J(0,0)$ of the system (2.1) at E_0 takes the form of

$$J_0 = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

Since $det(J_0) = r_1 r_2 > 0$ and $trace(J_0) = r_1 + r_2 > 0$, the equilibrium point E_0 is unstable.

Lemma 2.1

(a) If

$$r_2 \quad > \quad \frac{r_1 a_{21}}{r_1 + a_{11}} \tag{2.4}$$

then the equilibrium point E_1 is a saddle point. And we know

$$\Gamma_1 \quad = \quad \{(x_1,x_2)|\, x_1 > 0\ ,\ x_2 = 0\ \}$$

is the stable manifold of the equilibrium point E_1.

(b) If

$$r_2 \quad < \quad \frac{r_1 a_{21}}{r_1 + a_{11}} \tag{2.5}$$

then the equilibrium point E_1 is locally asymptotically stable.

Proof: The Jacobian matrix $J_1 \equiv J(r_1/a_{11},0)$ of the system (2.1) at E_1 takes the form of

$$J_1 = \begin{bmatrix} -r_1 & -\frac{r_1 a_{12}}{a_{11}} \\ 0 & r_2 - \frac{r_1 a_{21}}{r_1 + a_{11}} \end{bmatrix}$$

Since

$$det(J_1) \quad = \quad -r_1 \left(r_2 - \frac{r_1 a_{21}}{r_1 + a_{11}} \right)$$

and

$$trace(J_1) \quad = \quad -r_1 + r_2 - \frac{r_1 a_{21}}{r_1 + a_{11}}$$

Case 1. $r_2 > \dfrac{r_1 a_{21}}{r_1 + a_{11}}$

In this case,

$$det(J_1) = -r_1 \left(r_2 - \frac{r_1 a_{21}}{r_1 + a_{11}} \right) < 0$$

then the equilibrium point E_1 is a saddle point. And we know

$$\Gamma_1 \quad = \quad \{(x_1,x_2)|\, x_1 > 0\ ,\ x_2 = 0\ \}$$

is the stable manifold of the equilibrium point E_1.

Case 2. $r_2 < \dfrac{r_1 a_{21}}{r_1 + a_{11}}$

In this case,

$$det(J_1) = -r_1 \left(r_2 - \frac{r_1 a_{21}}{r_1 + a_{11}} \right) > 0$$

and

$$trace(J_1) = -r_1 + r_2 - \frac{r_1 a_{21}}{r_1 + a_{11}} < 0$$

then the equilibrium point E_1 is locally asymptotically stable. This completes the proof.

The Jacobian matrix $J_2 \equiv J(0, r_2/a_{22})$ of the system (2.1) at E_2 takes the form of

$$J_2 = \begin{bmatrix} r_1 - \dfrac{r_2 a_{12}}{r_2 + a_{22}} & 0 \\ -\dfrac{r_2 a_{21}}{a_{22}} & -r_2 \end{bmatrix}$$

From (2.3), we know

$$det(J_2) = -r_2 \left(r_1 - \frac{r_2 a_{12}}{r_2 + a_{22}} \right) < 0$$

then the equilibrium point E_2 is a saddle point. And we know

$$\Gamma_2 \quad = \quad \{(x_1, x_2) |\, x_1 = 0\ ,\ x_2 > 0\ \}$$

is the stable manifold of the equilibrium point E_2.

Lemma 2.2 *If*

$$(1 + x_1^*)^2 (1 + x_2^*)^2 \quad > \quad \frac{a_{12} a_{21}}{a_{11} a_{22}} \tag{2.6}$$

then the unique positive equilibrium point $E^* = (x_1^*, x_2^*)$ is locally asymptotically stable.

Proof: The Jacobian matrix J^* of the system (2.1) at E^* takes the form of

$$J^* = \begin{bmatrix} -a_{11} x_1^* & -\dfrac{a_{12} x_1^*}{(1 + x_2^*)^2} \\ -\dfrac{a_{21} x_2^*}{(1 + x_1^*)^2} & -a_{22} x_2^* \end{bmatrix}$$

Since

$$\begin{aligned} \det(J^*) &= (a_{11} x_1^*)(a_{22} x_2^*) - \frac{a_{21} x_2^*}{(1 + x_1^*)^2} \cdot \frac{a_{12} x_1^*}{(1 + x_2^*)^2} \\ &= a_{11} a_{22} x_1^* x_2^* - \frac{a_{12} a_{21} x_1^* x_2^*}{(1 + x_1^*)^2 (1 + x_2^*)^2} \\ &= x_1^* x_2^* \left[a_{11} a_{22} - \frac{a_{12} a_{21}}{(1 + x_1^*)^2 (1 + x_2^*)^2} \right] \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} \text{trace}(J^*) &= -a_{11}x_1^* - a_{22}x_2^* \\ &= -(a_{11}x_1^* + a_{22}x_2^*) \\ &< 0 \end{aligned}$$

the unique positive equilibrium point E^* is locally asymptotically stable. This completes the proof.

Lemma 2.3 *All solutions $(x_1(t), x_2(t))$ of the system (2.1) are positive and bounded.*

Proof: Firstly, we want to show that all solutions $(x_1(t), x_2(t))$ of the system (2.1) are positive. In other works, if the initial point $(x_1(0), x_2(0))$ is in the first quadrant, then $(x_1(t), x_2(t))$ is also in the first quadrant for all $t > 0$. Since x_1-axis and x_2-axis are the solutions of the system (2.1), then the trajectory of the solution $(x_1(t), x_2(t))$ with initial point $(x_1(0), x_2(0))$ in the first quadrant can not cross with x_1-axis and x_2-axis by the uniqueness of the solution. Hence all solutions $(x_1(t), x_2(t))$ of the system (2.1) are positive.

Secondly, we want to show that all solutions $(x_1(t), x_2(t))$ of the system (2.1) are bounded. That is, we want to show that $x_i(t) < K_i \equiv \max\{x_i(0), r_i/a_{ii}\}$ $(i = 1, 2)$ for all $t \geq 0$. Now, show that $x_1(t) < K_1$ for all $t \geq 0$. The proof for $x_2(t)$ is similar. For $x_1(t)$, suppose there exists $t^* > 0$ such that $x_1(t^*) = K_1$ and $\dot{x}_1(t^*) \geq 0$. Thus

$$\begin{aligned} \dot{x}_1(t^*) &= x_1(t^*)\left[r_1 - a_{11}x_1(t^*) - \frac{a_{12}x_2(t^*)}{1 + x_2(t^*)}\right] \\ &= K_1\left[r_1 - a_{11}K_1 - \frac{a_{12}x_2(t^*)}{1 + x_2(t^*)}\right] \\ &< K_1(r_1 - a_{11}K_1) \\ &\leq 0 \end{aligned}$$

This contradicts $\dot{x}_1(t^*) \geq 0$. Hence $x_1(t) < K_1$ for all $t \geq 0$. Similarly, we can show that $x_2(t) < K_2 \equiv \max\{x_2(0), r_2/a_{22}\}$ for all $t \geq 0$. Therefore, all solutions $(x_1(t), x_2(t))$ of the system (2.1) are bounded. This completes the proof.

2.2 Global Stability

In this section, we want to use three different methods to analyze the global stability of the unique positive equilibrium point E^* of the system (2.1).

(i) Dulac's criterion plus Poincaré-Bendixson theorem

(ii) Lyapunov function

(iii) Stable limit cycle analysis

At first, we use the method (i) to analyze the system (2.1).

Theorem 2.1 *If (2.3), (2.4) and (2.6) hold, then the unique positive equilibrium point E^* of the system (2.1) is globally asymptotically stable.*

Proof: From Lemma 2.3 the solution $(x_1(t), x_2(t))$ of the system (2.1) is positive and bounded. Let

$$H(x_1, x_2) = \frac{1}{x_1 x_2}, x_1, x_2 > 0$$

Then

$$\begin{aligned}
& \frac{\partial}{\partial x_1}(Hf_1) + \frac{\partial}{\partial x_2}(Hf_2) \\
= \; & \frac{\partial}{\partial x_1}\left\{H\left[x_1\left(r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1+x_2}\right)\right]\right\} \\
& + \frac{\partial}{\partial x_2}\left\{H\left[x_2\left(r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1+x_1}\right)\right]\right\} \\
= \; & \frac{\partial}{\partial x_1}\left[\frac{1}{x_2}\left(r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1+x_2}\right)\right] \\
& + \frac{\partial}{\partial x_2}\left[\frac{1}{x_1}\left(r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1+x_1}\right)\right] \\
= \; & -\frac{a_{11}}{x_2} - \frac{a_{22}}{x_1} \\
= \; & -\left(\frac{a_{11}}{x_2} + \frac{a_{22}}{x_1}\right) \\
< \; & 0
\end{aligned}$$

Hence by the Dulac's criterion, there is no closed orbit in the first quadrant. By Lemma 2.2, we know that the unique positive equilibrium point E^* is locally asymptotically stable. By Lemma 2.3 and the Poincaré-Bendixson theorem, it suffices to show that the unique positive equilibrium point E^* is globally asymptotically stable in the first quadrant. This completes the proof.

Secondly, we want to analyze the global stability of the unique positive equilibrium point E^* of the system (2.1) by using the method (ii).

Theorem 2.2 *If*

$$x_1^* \geq \frac{a_{21} - a_{22}}{a_{22}}, \qquad x_2^* \geq \frac{a_{12} - a_{11}}{a_{11}} \tag{2.7}$$

then the unique positive equilibrium point $E^* \equiv (x_1^*, x_2^*)$ of the system (2.1) is globally asymptotically stable.

Proof: Construct the following Lyapunov function

$$V(x_1, x_2) = V_1(x_1, x_2) + V_2(x_1, x_2)$$

where

$$
\begin{aligned}
V_1(x_1,x_2) &= \frac{1+x_2^*}{a_{12}}\left[x_1 - x_1^* - x_1^*\ln\frac{x_1}{x_1^*}\right]\\
V_2(x_1,x_2) &= \frac{1+x_1^*}{a_{21}}\left[x_2 - x_2^* - x_2^*\ln\frac{x_2}{x_2^*}\right]
\end{aligned}
$$

on $G = \{(x_1,x_2)|\ x_1 > 0\ ,\ x_2 > 0\ \}$. It is obvious that $V(x_1,x_2) \in \mathcal{C}^1(G,\mathbb{R})$, $V(x_1^*,x_2^*) = 0$ and $V(x_1,x_2) > 0$ for $(x_1,x_2) \in G - \{(x_1^*,x_2^*)\}$. Then the time derivative of $V_i(x_1,x_2)$, $i = 1,2$, along trajectory of the system (2.1) is given by

$$
\begin{aligned}
\dot{V}_1(x_1,x_2) &= \frac{1+x_2^*}{a_{12}}\left[\dot{x}_1 - x_1^*\cdot\frac{x_1^*}{x_1}\cdot\frac{\dot{x}_1}{x_1^*}\right]\\
&= \frac{1+x_2^*}{a_{12}}\cdot\frac{\dot{x}_1}{x_1}(x_1-x_1^*)\\
&= \frac{1+x_2^*}{a_{12}}\left(r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1+x_2}\right)(x_1-x_1^*)\\
&= \frac{1+x_2^*}{a_{12}}\left(a_{11}x_1^* + \frac{a_{12}x_2^*}{1+x_2^*} - a_{11}x_1 - \frac{a_{12}x_2}{1+x_2}\right)(x_1-x_1^*)\\
&= \frac{a_{11}(1+x_2^*)x_1^*x_1}{a_{12}} - \frac{a_{11}(1+x_2^*)(x_1^*)^2}{a_{12}} + x_1x_2^* - x_1^*x_2^* - \frac{a_{11}(1+x_2^*)x_1^2}{a_{12}}\\
&\quad + \frac{a_{11}(1+x_2^*)x_1^*x_1}{a_{12}} - \frac{(1+x_2^*)x_1x_2}{1+x_2} + \frac{(1+x_2^*)x_1^*x_2}{1+x_2}\\
&= -\frac{a_{11}(1+x_2^*)}{a_{12}}(x_1-x_1^*)^2 - \frac{(x_1-x_1^*)(x_2-x_2^*)}{1+x_2} \qquad (2.8)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{V}_2(x_1,x_2) &= \frac{1+x_1^*}{a_{21}}\left[\dot{x}_2 - x_2^*\cdot\frac{x_2^*}{x_2}\cdot\frac{\dot{x}_2}{x_2^*}\right]\\
&= \frac{1+x_1^*}{a_{21}}\cdot\frac{\dot{x}_2}{x_2}(x_2-x_2^*)\\
&= \frac{1+x_1^*}{a_{21}}\left(r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1+x_1}\right)(x_2-x_2^*)\\
&= \frac{1+x_1^*}{a_{21}}\left(a_{22}x_2^* + \frac{a_{21}x_1^*}{1+x_1^*} - a_{22}x_2 - \frac{a_{21}x_1}{1+x_1}\right)(x_2-x_2^*)\\
&= \frac{a_{22}(1+x_1^*)x_2^*x_2}{a_{21}} - \frac{a_{22}(1+x_1^*)(x_2^*)^2}{a_{21}} + x_1^*x_2 - x_1^*x_2^* - \frac{a_{22}(1+x_1^*)x_2^2}{a_{21}}\\
&\quad + \frac{a_{22}(1+x_1^*)x_2^*x_2}{a_{21}} - \frac{(1+x_1^*)x_1x_2}{1+x_1} + \frac{(1+x_1^*)x_1x_2^*}{1+x_1}\\
&= -\frac{a_{22}(1+x_1^*)}{a_{21}}(x_2-x_2^*)^2 - \frac{(x_1-x_1^*)(x_2-x_2^*)}{1+x_1} \qquad (2.9)
\end{aligned}
$$

Thus the derivative of $V(x_1,x_2)$ is given by

$$\begin{aligned}
\dot{V}(x_1,x_2) &= \dot{V}_1(x_1,x_2)+\dot{V}_2(x_1,x_2)\\
&= -\frac{a_{11}(1+x_2^*)}{a_{12}}(x_1-x_1^*)^2-\frac{(x_1-x_1^*)(x_2-x_2^*)}{1+x_2}\\
&\quad -\frac{a_{22}(1+x_1^*)}{a_{21}}(x_2-x_2^*)^2-\frac{(x_1-x_1^*)(x_2-x_2^*)}{1+x_1}\\
&= -\frac{a_{11}(1+x_2^*)}{a_{12}}(x_1-x_1^*)^2-\frac{a_{22}(1+x_1^*)}{a_{21}}(x_2-x_2^*)^2\\
&\quad -\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}\right)(x_1-x_1^*)(x_2-x_2^*)\\
&= -\frac{1}{2}\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}\right)\left[(x_1-x_1^*)^2+(x_2-x_2^*)^2+2(x_1-x_1^*)(x_2-x_2^*)\right]\\
&\quad +\frac{1}{2}\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}\right)(x_1-x_1^*)^2-\frac{a_{11}(1+x_2^*)}{a_{12}}(x_1-x_1^*)^2\\
&\quad +\frac{1}{2}\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}\right)(x_2-x_2^*)^2-\frac{a_{22}(1+x_1^*)}{a_{21}}(x_2-x_2^*)^2\\
&= -\frac{1}{2}\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}\right)[(x_1-x_1^*)+(x_2-x_2^*)]^2\\
&\quad -\left[\frac{a_{11}(1+x_2^*)}{a_{12}}-\frac{1}{2}\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}\right)\right](x_1-x_1^*)^2\\
&\quad -\left[\frac{a_{22}(1+x_1^*)}{a_{21}}-\frac{1}{2}\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}\right)\right](x_2-x_2^*)^2\\
&< -\frac{1}{2}\left(\frac{1}{1+x_1}+\frac{1}{1+x_2}\right)[(x_1-x_1^*)+(x_2-x_2^*)]^2\\
&\quad -\left[\frac{a_{11}(1+x_2^*)}{a_{12}}-1\right](x_1-x_1^*)^2-\left[\frac{a_{22}(1+x_1^*)}{a_{21}}-1\right](x_2-x_2^*)^2\\
&< 0
\end{aligned}$$

Hence $\dot{V}(x_1,x_2)<0$ on G. Therefore, it follows from Lyapunov-LaSalle theorem that the unique positive equilibrium point E^* of the system (2.1) is globally asymptotically stable on G. This completes the proof.

Finally, we introduce the method (iii) to prove the global stability of the unique positive equilibrium point E^* of the system (2.1).

Theorem 2.3 *If (2.3), (2.4) and (2.6) hold, then the unique positive equilibrium point E^* of the system (2.1) is globally asymptotically stable.*

Proof: It suffices to show that the system (2.1) has no closed orbit in the first quadrant. Suppose on the contrary that there is a T-periodic orbit $\Gamma=\{(x_1(t),x_2(t))|\ 0\le t\le T\}$ in the first quadrant.

Then

$$
\begin{aligned}
\Delta &= \int_\Gamma \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) ds \\
&= \int_0^T \left[\frac{\partial}{\partial x_1} f_1(x_1,x_2) + \frac{\partial}{\partial x_2} f_2(x_1,x_2)\right]\Bigg|_{\substack{x_1=x_1(t)\\ x_2=x_2(t)}} dt \\
&= \int_0^T \left\{\frac{\partial}{\partial x_1}\left[x_1\left(r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1+x_2}\right)\right]\right. \\
&\quad \left. + \frac{\partial}{\partial x_2}\left[x_2\left(r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1+x_1}\right)\right]\right\}\Bigg|_{\substack{x_1=x_1(t)\\ x_2=x_2(t)}} dt \\
&= \int_0^T \frac{\partial}{\partial x_1}\left[x_1\left(r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1+x_2}\right)\right]\Bigg|_{\substack{x_1=x_1(t)\\ x_2=x_2(t)}} dt \\
&\quad + \int_0^T \frac{\partial}{\partial x_2}\left[x_2\left(r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1+x_1}\right)\right]\Bigg|_{\substack{x_1=x_1(t)\\ x_2=x_2(t)}} dt \\
&= \int_0^T \left[\left(r_1 - a_{11}x_1 - \frac{a_{12}x_2}{1+x_2}\right) - a_{11}x_1\right]\Bigg|_{\substack{x_1=x_1(t)\\ x_2=x_2(t)}} dt \\
&\quad + \int_0^T \left[\left(r_2 - a_{22}x_2 - \frac{a_{21}x_1}{1+x_1}\right) - a_{22}x_2\right]\Bigg|_{\substack{x_1=x_1(t)\\ x_2=x_2(t)}} dt \\
&= \int_0^T \frac{\dot{x}_1(t)}{x_1(t)} dt - a_{11}\int_0^T x_1(t)dt + \int_0^T \frac{\dot{x}_2(t)}{x_2(t)} dt - a_{22}\int_0^T x_2(t)dt \\
&= \int_{x_1(0)}^{x_1(T)} \frac{1}{x_1} dx_1 - a_{11}\int_0^T x_1(t)dt + \int_{x_2(0)}^{x_2(T)} \frac{1}{x_2} dx_2 - a_{22}\int_0^T x_2(t)dt
\end{aligned}
$$

Since Γ is a T-periodic,

$$\int_{x_1(0)}^{x_1(T)} \frac{1}{x_1} dx_1 = 0 \text{ and } \int_{x_2(0)}^{x_2(T)} \frac{1}{x_2} dx_2 = 0$$

Hence we have

$$
\begin{aligned}
\Delta &= -\int_0^T \left[a_{11}x_1(t) + a_{22}x_2(t)\right] dt \\
&< 0
\end{aligned}
$$

This indicates that all closed orbits of the system (2.1) in the first quadrant are orbitally stable. Since every closed orbit is orbitally stable and then there is an unique stable limit cycle in the first quadrant. That is, the unique positive equilibrium point E^* is unstable. However, by Lemma 2.2, the unique positive equilibrium point E^* is locally asymptotically stable. Thus there is no closed orbit in the first quadrant. By Lemma 2.3 and the Poincaré-Bendixson theorem, it suffices to show that the unique positive equilibrium point E^* of the system (2.1) is globally asymptotically stable in the first quadrant. This completes the proof.

3 The System with Time Delay

Consider the following competition system with time delay

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left[r_1 - a_{11}x_1(t-\tau) - \frac{a_{12}x_2(t-\tau)}{1+x_2(t-\tau)}\right] \\ \dot{x}_2(t) &= x_2(t)\left[r_2 - a_{22}x_2(t-\tau) - \frac{a_{21}x_1(t-\tau)}{1+x_1(t-\tau)}\right]\end{aligned} \tag{3.1}$$

with initial conditions

$$x_i(t) = \phi_i(t) \geq 0, \quad t \in [-\tau, 0], \quad \phi_i(0) > 0, \quad i = 1,2 \tag{3.2}$$

where $\cdot = d/dt$, r_i, a_{ij} $(i, j = 1,2)$, and τ are positive constants. $\phi_i(t)(i = 1,2)$ are continuous bounded functions on the interval $[-\tau, 0]$. $x_1(t)$ and $x_2(t)$ denote the population densities (or biomasses) of two species competing for a common pool of resources in a temporally uniform environment. Clearly, the equilibrium points of the system (3.1) are the same as in the system (2.1).

3.1 Uniform Persistence

The following lemmas are elementary and are concerned with the qualitative nature of solutions of the system (3.1) with initial conditions (3.2).

Lemma 3.1 *Solutions of the system (3.1) with initial conditions (3.2) remains positive for all* $t \geq 0$.

Proof: We want to show that all solutions $(x_1(t), x_2(t))$ of the system (3.1) with initial conditions (3.2) are positive. In other works, if the initial point $(x_1(0), x_2(0))$ is in the first quadrant, then $(x_1(t), x_2(t))$ is also in the first quadrant for all $t > 0$. Since x_1-axis and x_2-axis are the solutions of the system (3.1) with initial conditions (3.2), then the trajectory of the solution $(x_1(t), x_2(t))$ with initial point $(x_1(0), x_2(0))$ in the first quadrant can not cross with x_1-axis and x_2-axis by the uniqueness of the solution. Hence all solutions $(x_1(t), x_2(t))$ of the system (3.1) with initial conditions (3.2) are positive. This completes the proof.

Lemma 3.2 *Let $(x_1(t), x_2(t))$ denote any solution of the system (3.1) with initial conditions (3.2). Then*

$$0 < x_i(t) \leq M_i \qquad (i = 1, 2) \tag{3.3}$$

eventually for all large t, where

$$\begin{aligned} M_1 &= \frac{r_1}{a_{11}} e^{r_1 \tau} \\ M_2 &= \frac{(r_2 + a_{21})M_1 + r_2}{a_{22}(1 + M_1)} \cdot exp\left\{ \frac{(r_2 + a_{21})M_1 + r_2}{1 + M_1} \tau \right\} \end{aligned}$$

Proof: By Lemma 3.1, we know that the solutions of the system (3.1) with initial conditions (3.2) are positive, and hence, by first equation of the system (3.1),

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) \left[r_1 - a_{11} x_1(t - \tau) - \frac{a_{12} x_2(t - \tau)}{1 + x_2(t - \tau)} \right] \\ &\leq x_1(t) \left[r_1 - a_{11} x_1(t - \tau) \right] \end{aligned} \tag{3.4}$$

Taking $M_1^* = \frac{r_1}{a_{11}}(h_1 + 1)$, where $0 < h_1 < e^{r_1 \tau} - 1$. Firstly, suppose $x_1(t)$ is not oscillatory about M_1^*. That is, there exists a $T^* > 0$ such that either

$$x_1(t) \quad < \quad M_1^* \qquad \text{for } t > T^* \tag{3.5}$$

or

$$x_1(t) \quad > \quad M_1^* \qquad \text{for } t > T^* \tag{3.6}$$

If (3.5) holds then (3.3) follows. Suppose (3.6) holds, then

$$\begin{aligned} \dot{x}_1(t) &\leq x_1(t) \left[r_1 - a_{11} x_1(t - \tau) \right] \\ &< x_1(t)(r_1 - a_{11} M_1^*) \\ &< -r_1 h_1 x_1(t) \qquad\qquad \text{for } t \geq T^* + \tau \end{aligned}$$

Therefore,

$$0 < x_1(t) < x_1(0) e^{-r_1 h_1 t} \to 0 \qquad \text{as } t \to 0$$

That is, $\lim_{t \to \infty} x_1(t) = 0$, by the Squeeze Theorem. It contradicts to (3.6). So, there must exist a $T_1 \geq T^*$ such that $x_1(T_1) \leq M_1^*$. If $x_1(t) \leq M_1^*$ for all $t \geq T_1$, then (3.3) follows. If not, then there must exist a $T_2 > T_1$ such that $x_1(T_2) > M_1^*$. Therefore, there exists a $T_3 > T_2$ such that $x_1(T_3) \leq M_1^*$ by above discussion. By above, we know that $x_1(T_1) \leq M_1^*$, $x_1(T_2) > M_1^*$, and $x_1(T_3) \leq M_1^*$ where $T_1 < T_2 < T_3$. Then, by the Intermediate Value Theorem, there exist T_4 and T_5 such that $x_1(T_4) = x_1(T_5) = M_1^*$ and $x_1(t) > M_1^*$ for $T_4 < t < T_5$ with $T_1 < T_4 < T_2$ and $T_2 < T_5 < T_3$. Therefore, there is a $\overline{T} \in (T_4, T_5)$ such that $x_1(\overline{T})$ is any arbitrary local maximum of $x_1(t)$. Then it follows from (3.4) that

$$0 \quad = \quad \dot{x}_1(t)|_{t=\overline{T}} \ \leq \ x_1(\overline{T})\left[r_1 - a_{11}x_1(\overline{T}-\tau)\right] \tag{3.7}$$

This leads to

$$x_1(\overline{T}-\tau) \leq \frac{r_1}{a_{11}} \tag{3.8}$$

Integrating (3.4) on the interval $[\overline{T}-\tau, \overline{T}]$, we have

$$\begin{aligned} \ln\left[\frac{x_1(\overline{T})}{x_1(\overline{T}-\tau)}\right] &\leq \int_{\overline{T}-\tau}^{\overline{T}} [r_1 - a_{11}x_1(s-\tau)]\,ds \\ &\leq \int_{\overline{T}-\tau}^{\overline{T}} r_1\,ds \\ &= r_1\tau \end{aligned}$$

which implies that

$$x_1(\overline{T}) \leq x_1(\overline{T}-\tau)e^{r_1\tau} \leq \frac{r_1}{a_{11}}e^{r_1\tau} = M_1$$

Since $x_1(\overline{T})$ is an arbitrary local maximum of $x_1(t)$, we can conclude that there exists a $\widehat{T} \geq \overline{T}$ such that $x_1(t) \leq M_1$ for all $t > \widehat{T}$.

Secondly, suppose now that $x_1(t)$ is oscillatory about M_1^*. By a procedure similar to the discussion above, we can conclude that there exists a $\widetilde{T} \geq \widehat{T}$ such that $x_1(t) \leq M_1$ for all $t \geq \widetilde{T}$.

We indicate briefly the derivation of the estimation $x_2(t) \leq M_2$ eventually for all large t. We have directly from second equation of the system (3.1) and $x_1(t) \leq M_1$ (for $t \geq \widetilde{T}$) that

$$\begin{aligned} \dot{x}_2(t) &= x_2(t)\left[r_2 - a_{22}x_2(t-\tau) - \frac{a_{21}x_1(t-\tau)}{1+x_1(t-\tau)}\right] \\ &\leq x_2(t)\left[r_2 - a_{22}x_2(t-\tau) + \frac{a_{21}x_1(t-\tau)}{1+x_1(t-\tau)}\right] \\ &= x_2(t)\left\{r_2 - a_{22}x_2(t-\tau) + a_{21}\left[1 - \frac{1}{1+x_1(t-\tau)}\right]\right\} \\ &\leq x_2(t)\left\{r_2 - a_{22}x_2(t-\tau) + a_{21}\left[1 - \frac{1}{1+M_1}\right]\right\} \\ &= x_2(t)\left[r_2 - a_{22}x_2(t-\tau) + \frac{a_{21}M_1}{1+M_1}\right] \\ &= x_2(t)\left[\frac{r_2(1+M_1) + a_{21}M_1}{1+M_1} - a_{22}x_2(t-\tau)\right] \\ &= x_2(t)\left[\frac{(r_2+a_{21})M_1 + r_2}{1+M_1} - a_{22}x_2(t-\tau)\right] \end{aligned}$$

By a similar argument, we can verify that there exists a $T > \widetilde{T}$ such that

$$\begin{aligned} x_2(t) &\leq \frac{(r_2 + a_{21})M_1 + r_2}{a_{22}(1+M_1)} \cdot exp\left\{\frac{(r_2 + a_{21})M_1 + r_2}{1+M_1}\tau\right\} \\ &= M_2 \qquad\qquad \text{for } t > T \end{aligned}$$

Consequently

$$0 < x_1(t) \leq M_1 \ , \quad 0 < x_2(t) \leq M_2 \qquad\qquad \text{for } t \geq T \tag{3.9}$$

This completes the proof.

The following result shows that the system (3.1) is uniformly persistent.

Theorem 3.1 *Suppose that the system (3.1) satisfies the following:*

$$\begin{aligned} (r_1 - a_{12})M_2 + r_1 &> 0 \\ (r_2 - a_{21})M_1 + r_2 &> 0 \end{aligned} \tag{3.10}$$

in which M_i $(i = 1,2)$ is defined by (3.3). Then the system (3.1) is uniformly persistent.

Proof: Suppose $x(t) = (x_1(t), x_2(t))$ is a solution of the system (3.1) which satisfies (3.2). Then

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[r_1 - a_{11}x_1(t-\tau) - \frac{a_{12}x_2(t-\tau)}{1+x_2(t-\tau)}\right] \\ &\geq x_1(t)\left(r_1 - a_{11}M_1 - \frac{a_{12}M_2}{1+M_2}\right) \qquad \text{for } t \geq T + \tau \end{aligned} \tag{3.11}$$

Integrating (3.11) on the interval $[t-\tau, t]$, we have

$$\begin{aligned} \ln\left[\frac{x_1(t)}{x_1(t-\tau)}\right] &\geq \int_{t-\tau}^{t}\left(r_1 - a_{11}M_1 - \frac{a_{12}M_2}{1+M_2}\right)ds \\ &= \left(r_1 - a_{11}M_1 - \frac{a_{12}M_2}{1+M_2}\right)\tau \end{aligned}$$

which implies that, for $t \geq T + \tau$,

$$x_1(t-\tau) \leq x_1(t) \cdot exp\left\{-\left(r_1 - a_{11}M_1 - \frac{a_{12}M_2}{1+M_2}\right)\tau\right\}$$

If (3.10) holds, then

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)\left[r_1 - a_{11}x_1(t-\tau) - \frac{a_{12}x_2(t-\tau)}{1+x_2(t-\tau)}\right] \\
&\geq x_1(t)\left[r_1 - a_{11}x_1(t-\tau) - \frac{a_{12}M_2}{1+M_2}\right] \\
&\geq x_1(t)\left[r_1 - \frac{a_{12}M_2}{1+M_2} - a_{11}x_1(t)e^{-\left(r_1 - a_{11}M_1 - \frac{a_{12}M_2}{1+M_2}\right)\tau}\right] \\
&= x_1(t)\left[\frac{(r_1-a_{12})M_2+r_1}{1+M_2} - a_{11}x_1(t)e^{-\left(r_1 - a_{11}M_1 - \frac{a_{12}M_2}{1+M_2}\right)\tau}\right] \\
&= \frac{(r_1-a_{12})M_2+r_1}{1+M_2}x_1(t)\left[1 - \frac{x_1(t)}{\frac{\frac{(r_1-a_{12})M_2+r_1}{1+M_2}}{a_{11}}e^{\left(r_1 - a_{11}M_1 - \frac{a_{12}M_2}{1+M_2}\right)\tau}}\right] \\
&\equiv r_1^* x_1(t)\left[1 - \frac{x_1(t)}{K_1^*}\right] \qquad \text{for } t \geq T+\tau
\end{aligned}$$

which implies that

$$\liminf_{t\to\infty} x_1(t) \quad \geq \quad K_1^* \; = \; m_1$$

Thus, for large t, $x_1(t) \geq m_1$. By a procedure similar to the discussion above, we can verify that, for large t,

$$x_2(t) \geq m_2 = \frac{(r_2-a_{21})M_1+r_2}{a_{22}(1+M_1)} \cdot exp\left\{\left(r_2 - a_{22}M_2 - \frac{a_{21}M_1}{1+M_1}\right)\tau\right\}$$

Now, we let

$$\mathcal{D} = \{(x_1,x_2) |\; m_i \leq x_i \leq M_i\;,\; i=1,2\;\}$$

Then $\mathcal{D}$ is a bounded compact region in $\mathbb{R}_+^2$ which has positive distance from coordinate planes. From what has been discussed above, we obtain that there exists a $T^{**} > 0$, if $t > T^{**}$, then every positive solution of the system (3.1) with initial conditions (3.2) eventually enters and remains in the region $\mathcal{D}$. The proof is completed.

3.2 Global Asymptotic Stability

In this section, we derive sufficient conditions which guarantee that the unique positive equilibrium point E^* of the system (3.1) with initial conditions (3.2) is globally asymptotically stable. Our strategy in the proof is to construct a suitable Lyapunov functional. Before mention our result, we need the following notation:

$$\begin{aligned}
\beta_{11} &= a_{11}\left(1 - a_{11}M_1\tau\right)\ , \\
\beta_{12} &= -\frac{a_{12}\left(1+2x_2^*\right)}{1+x_2^*}\left(1 + a_{11}M_1\tau\right)\ , \\
\beta_{21} &= -\frac{a_{21}\left(1+2x_1^*\right)}{1+x_1^*}\left(1 + a_{22}M_2\tau\right)\ , \\
\beta_{22} &= a_{22}\left(1 - a_{22}M_2\tau\right).
\end{aligned}$$

where M_i $(i = 1,2)$ is defined by (3.3).

Theorem 3.2 *If*

$$\beta_{ii} > 0\ , \quad i = 1,2, \tag{3.12}$$

and

$$\beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0\ , \tag{3.13}$$

then the unique positive equilibrium point E^* of the system (3.1) with initial conditions (3.2) is globally asymptotically stable.

Proof: Let $x(t) = (x_1(t), x_2(t))$ be any solution of the system (3.1) with initial conditions (3.2). Define

$$z(t) = (z_1(t), z_2(t))$$

by

$$z_i(t) = \ln \frac{x_i(t)}{x_i^*} \qquad (i = 1,2). \tag{3.14}$$

It follows from (3.1) and (3.14) that

$$
\begin{aligned}
\frac{dz_1(t)}{dt} &= \frac{\dot{x}_1(t)}{x_1(t)} \\
&= \frac{1}{x_1(t)} \cdot x_1(t)\left[r_1 - a_{11}x_1(t-\tau) - \frac{a_{12}x_2(t-\tau)}{1+x_2(t-\tau)}\right] \\
&= \left[a_{11}x_1^* + \frac{a_{12}x_2^*}{1+x_2^*}\right] - a_{11}x_1^* e^{z_1(t-\tau)} - \frac{a_{12}x_2^*}{1+x_2(t-\tau)} e^{z_2(t-\tau)} \\
&= -a_{11}x_1^*\left[e^{z_1(t-\tau)} - 1\right] - \frac{a_{12}x_2^*}{1+x_2(t-\tau)}\left[e^{z_2(t-\tau)} - 1\right] \\
&\quad + \frac{a_{12}(x_2^*)^2}{(1+x_2^*)[1+x_2(t-\tau)]}\left[e^{z_2(t-\tau)} - 1\right] \qquad (3.15)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dz_2(t)}{dt} &= \frac{\dot{x}_2(t)}{x_2(t)} \\
&= \frac{1}{x_2(t)} \cdot x_2(t)\left[r_2 - a_{22}x_2(t-\tau) - \frac{a_{21}x_1(t-\tau)}{1+x_1(t-\tau)}\right] \\
&= \left[a_{22}x_2^* + \frac{a_{21}x_1^*}{1+x_1^*}\right] - a_{22}x_2^* e^{z_2(t-\tau)} - \frac{a_{21}x_1^*}{1+x_1(t-\tau)} e^{z_1(t-\tau)} \\
&= -a_{22}x_2^*\left[e^{z_2(t-\tau)} - 1\right] - \frac{a_{21}x_1^*}{1+x_1(t-\tau)}\left[e^{z_1(t-\tau)} - 1\right] \\
&\quad + \frac{a_{21}(x_1^*)^2}{(1+x_1^*)[1+x_1(t-\tau)]}\left[e^{z_1(t-\tau)} - 1\right] \qquad (3.16)
\end{aligned}
$$

Equation (3.15) can be rewritten as

$$
\begin{aligned}
\frac{dz_1(t)}{dt} &= -a_{11}x_1^*\left[e^{z_1(t)} - 1\right] - \frac{a_{12}x_2^*}{1+x_2(t-\tau)}\left[e^{z_2(t-\tau)} - 1\right] + \frac{a_{12}(x_2^*)^2}{(1+x_2^*)[1+x_2(t-\tau)]}\left[e^{z_2(t-\tau)} - 1\right] \\
&\quad + a_{11}x_1^*\left[e^{z_1(t)} - e^{z_1(t-\tau)}\right] \\
&= -a_{11}x_1^*\left[e^{z_1(t)} - 1\right] - \frac{a_{12}x_2^*}{1+x_2(t-\tau)}\left[e^{z_2(t-\tau)} - 1\right] + \frac{a_{12}(x_2^*)^2}{(1+x_2^*)[1+x_2(t-\tau)]}\left[e^{z_2(t-\tau)} - 1\right] \\
&\quad + a_{11}x_1^*\int_{t-\tau}^{t} e^{z_1(s)}\frac{dz_1(s)}{ds}ds \\
&= -a_{11}x_1^*\left[e^{z_1(t)} - 1\right] - \frac{a_{12}x_2^*}{1+x_2(t-\tau)}\left[e^{z_2(t-\tau)} - 1\right] + \frac{a_{12}(x_2^*)^2}{(1+x_2^*)[1+x_2(t-\tau)]}\left[e^{z_2(t-\tau)} - 1\right] \\
&\quad + a_{11}x_1^*\int_{t-\tau}^{t} e^{z_1(s)}\left\{-a_{11}x_1^*\left[e^{z_1(s-\tau)} - 1\right] - \frac{a_{12}x_2^*}{1+x_2(s-\tau)}\left[e^{z_2(s-\tau)} - 1\right]\right. \\
&\qquad \left. + \frac{a_{12}(x_2^*)^2}{(1+x_2^*)[1+x_2(s-\tau)]}\left[e^{z_2(s-\tau)} - 1\right]\right\} ds
\end{aligned}
$$

(3.17)

Let

$$V_{11}(t) = |z_1(t)| \tag{3.18}$$

Then, by (3.17) and (3.18), the upper right Dini derivative of $V_{11}(t)$ along the solution of (3.15) and (3.16) is given by

$$\begin{aligned}
D^+V_{11}(t) &= \frac{dz_1(t)}{dt}\cdot \mathrm{sign} z_1(t)\\
&= \mathrm{sign} z_1(t)\cdot\Bigg(-a_{11}x_1^*\left[e^{z_1(t)}-1\right]-\frac{a_{12}x_2^*}{1+x_2(t-\tau)}\left[e^{z_2(t-\tau)}-1\right]\\
&\quad+\frac{a_{12}\,(x_2^*)^2}{(1+x_2^*)\,[1+x_2(t-\tau)]}\left[e^{z_2(t-\tau)}-1\right]\\
&\quad+a_{11}x_1^*\int_{t-\tau}^{t}e^{z_1(s)}\Bigg\{-a_{11}x_1^*\left[e^{z_1(s-\tau)}-1\right]-\frac{a_{12}x_2^*}{1+x_2(s-\tau)}\left[e^{z_2(s-\tau)}-1\right]\\
&\quad+\frac{a_{12}\,(x_2^*)^2}{(1+x_2^*)\,[1+x_2(s-\tau)]}\left[e^{z_2(s-\tau)}-1\right]\Bigg\}ds\Bigg)\\
&= -a_{11}x_1^*\left[e^{z_1(t)}-1\right]\cdot\mathrm{sign} z_1(t)-\frac{a_{12}x_2^*}{1+x_2(t-\tau)}\left[e^{z_2(t-\tau)}-1\right]\cdot\mathrm{sign} z_1(t)\\
&\quad+\frac{a_{12}\,(x_2^*)^2}{(1+x_2^*)\,[1+x_2(t-\tau)]}\left[e^{z_2(t-\tau)}-1\right]\cdot\mathrm{sign} z_1(t)\\
&\quad+\mathrm{sign} z_1(t)\cdot\Bigg(a_{11}x_1^*\int_{t-\tau}^{t}e^{z_1(s)}\Bigg\{-a_{11}x_1^*\left[e^{z_1(s-\tau)}-1\right]\\
&\quad-\frac{a_{12}x_2^*}{1+x_2(s-\tau)}\left[e^{z_2(s-\tau)}-1\right]\\
&\quad+\frac{a_{12}\,(x_2^*)^2}{(1+x_2^*)\,[1+x_2(s-\tau)]}\left[e^{z_2(s-\tau)}-1\right]\Bigg\}ds\Bigg)\\
&\le -a_{11}x_1^*\left|e^{z_1(t)}-1\right|+a_{12}x_2^*\left|e^{z_2(t-\tau)}-1\right|+\frac{a_{12}\,(x_2^*)^2}{1+x_2^*}\left|e^{z_2(t-\tau)}-1\right|\\
&\quad+a_{11}x_1^*\int_{t-\tau}^{t}e^{z_1(s)}\Bigg\{a_{11}x_1^*\left|e^{z_1(s-\tau)}-1\right|+a_{12}x_2^*\left|e^{z_2(s-\tau)}-1\right|\\
&\quad+\frac{a_{12}\,(x_2^*)^2}{1+x_2^*}\left|e^{z_2(s-\tau)}-1\right|\Bigg\}ds
\end{aligned} \tag{3.19}$$

By Theorem 3.1, we know that there exists a $T > 0$, such that $x_1^* e^{z_1(t)} = x_1(t) \le M_1$ for $t \ge T$. Hence for $t \ge T + \tau$, we have

$$\begin{aligned} D^+V_{11}(t) &\le -a_{11}x_1^*\left|e^{z_1(t)}-1\right| + \frac{a_{12}x_2^*(1+2x_2^*)}{1+x_2^*}\left|e^{z_2(t-\tau)}-1\right| \\ &\quad + a_{11}M_1\int_{t-\tau}^{t}\left\{a_{11}x_1^*\left|e^{z_1(s-\tau)}-1\right| + a_{12}x_2^*\left|e^{z_2(s-\tau)}-1\right|\right. \\ &\quad \left. + \frac{a_{12}(x_2^*)^2}{1+x_2^*}\left|e^{z_2(s-\tau)}-1\right|\right\}ds \end{aligned} \tag{3.20}$$

Define

$$\begin{aligned} V_{12}(t) &= a_{11}M_1\int_{t-\tau}^{t}\int_{s}^{t}\left\{a_{11}x_1^*\left|e^{z_1(\theta-\tau)}-1\right| + a_{12}x_2^*\left|e^{z_2(\theta-\tau)}-1\right|\right. \\ &\quad \left. + \frac{a_{12}(x_2^*)^2}{1+x_2^*}\left|e^{z_2(\theta-\tau)}-1\right|\right\}d\theta ds \end{aligned} \tag{3.21}$$

then we have

$$\begin{aligned} D^+V_{12}(t) &= \tau\left[a_{11}^2M_1x_1^*\left|e^{z_1(t-\tau)}-1\right| + a_{11}a_{12}M_1x_2^*\left|e^{z_2(t-\tau)}-1\right|\right. \\ &\quad \left. + \frac{a_{11}a_{12}M_1(x_2^*)^2}{1+x_2^*}\left|e^{z_2(t-\tau)}-1\right|\right] \\ &\quad - a_{11}M_1\int_{t-\tau}^{t}\left\{a_{11}x_1^*\left|e^{z_1(s-\tau)}-1\right| + a_{12}x_2^*\left|e^{z_2(s-\tau)}-1\right|\right. \\ &\quad \left. + \frac{a_{12}(x_2^*)^2}{1+x_2^*}\left|e^{z_2(s-\tau)}-1\right|\right\}ds \end{aligned} \tag{3.22}$$

Define

$$\begin{aligned} V_{13}(t) &= a_{11}^2M_1x_1^*\tau\int_{t-\tau}^{t}\left|e^{z_1(s)}-1\right|ds \\ &\quad + \frac{a_{12}x_2^*(1+2x_2^*)}{1+x_2^*}(1+a_{11}M_1\tau)\int_{t-\tau}^{t}\left|e^{z_2(s)}-1\right|ds \end{aligned} \tag{3.23}$$

then we have

$$\begin{aligned} D^+V_{13}(t) &= a_{11}^2M_1x_1^*\tau\left[\left|e^{z_1(t)}-1\right| - \left|e^{z_1(t-\tau)}-1\right|\right] \\ &\quad + \frac{a_{12}x_2^*(1+2x_2^*)}{1+x_2^*}(1+a_{11}M_1\tau)\left[\left|e^{z_2(t)}-1\right| - \left|e^{z_2(t-\tau)}-1\right|\right] \end{aligned} \tag{3.24}$$

Now we define a Lyapunov functional $V_1(t)$ as

$$V_1(t) = V_{11}(t) + V_{12}(t) + V_{13}(t) \tag{3.25}$$

Then we have from (3.20)-(3.25) that for $t \geq T+\tau$

$$\begin{aligned}
D^+V_1(t) \;\leq\;& -a_{11}x_1^*\left|e^{z_1(t)}-1\right|+\frac{a_{12}x_2^*(1+2x_2^*)}{1+x_2^*}\left|e^{z_2(t-\tau)}-1\right|+a_{11}M_1\int_{t-\tau}^{t}\left\{a_{11}x_1^*\left|e^{z_1(s-\tau)}-1\right|+a_{12}x_2^*\left|e^{z_2(s-\tau)}-1\right|\right.\\
&\left.+\frac{a_{12}(x_2^*)^2}{1+x_2^*}\left|e^{z_2(s-\tau)}-1\right|\right\}ds+a_{11}^2M_1x_1^*\tau\left|e^{z_1(t-\tau)}-1\right|+a_{11}a_{12}M_1x_2^*\tau\left|e^{z_2(t-\tau)}-1\right|\\
&+\frac{a_{11}a_{12}M_1\tau(x_2^*)^2}{1+x_2^*}\left|e^{z_2(t-\tau)}-1\right|-a_{11}M_1\int_{t-\tau}^{t}\left\{a_{11}x_1^*\left|e^{z_1(s-\tau)}-1\right|+a_{12}x_2^*\left|e^{z_2(s-\tau)}-1\right|\right.\\
&\left.+\frac{a_{12}(x_2^*)^2}{1+x_2^*}\left|e^{z_2(s-\tau)}-1\right|\right\}ds+a_{11}^2M_1x_1^*\tau\left[\left|e^{z_1(t)}-1\right|-\left|e^{z_1(t-\tau)}-1\right|\right]\\
&+\frac{a_{12}x_2^*(1+2x_2^*)}{1+x_2^*}(1+a_{11}M_1\tau)\left[\left|e^{z_2(t)}-1\right|-\left|e^{z_2(t-\tau)}-1\right|\right]\\
=\;& -a_{11}(1-a_{11}M_1\tau)x_1^*\left|e^{z_1(t)}-1\right|+\frac{a_{12}(1+2x_2^*)}{1+x_2^*}(1+a_{11}M_1\tau)x_2^*\left|e^{z_2(t)}-1\right|\\
=\;& -\beta_{11}x_1^*\left|e^{z_1(t)}-1\right|-\beta_{12}x_2^*\left|e^{z_2(t)}-1\right| \qquad (3.26)
\end{aligned}$$

Equation (3.16) can be rewritten as

$$\begin{aligned}
\frac{dz_2(t)}{dt} \;=\;& -a_{22}x_2^*\left[e^{z_2(t)}-1\right]-\frac{a_{21}x_1^*}{1+x_1(t-\tau)}\left[e^{z_1(t-\tau)}-1\right]\\
&+\frac{a_{21}(x_1^*)^2}{(1+x_1^*)[1+x_1(t-\tau)]}\left[e^{z_1(t-\tau)}-1\right]+a_{22}x_2^*\left[e^{z_2(t)}-e^{z_2(t-\tau)}\right]\\
=\;& -a_{22}x_2^*\left[e^{z_2(t)}-1\right]-\frac{a_{21}x_1^*}{1+x_1(t-\tau)}\left[e^{z_1(t-\tau)}-1\right]\\
&+\frac{a_{21}(x_1^*)^2}{(1+x_1^*)[1+x_1(t-\tau)]}\left[e^{z_1(t-\tau)}-1\right]+a_{22}x_2^*\int_{t-\tau}^{t}e^{z_2(s)}\frac{dz_2(s)}{ds}ds\\
=\;& -a_{22}x_2^*\left[e^{z_2(t)}-1\right]-\frac{a_{21}x_1^*}{1+x_1(t-\tau)}\left[e^{z_1(t-\tau)}-1\right]\\
&+\frac{a_{21}(x_1^*)^2}{(1+x_1^*)[1+x_1(t-\tau)]}\left[e^{z_1(t-\tau)}-1\right]\\
&+a_{22}x_2^*\int_{t-\tau}^{t}e^{z_2(s)}\left\{-a_{22}x_2^*\left[e^{z_2(s-\tau)}-1\right]\right.\\
&\qquad-\frac{a_{21}x_1^*}{1+x_1(s-\tau)}\left[e^{z_1(s-\tau)}-1\right]\\
&\qquad\left.+\frac{a_{21}(x_1^*)^2}{(1+x_1^*)[1+x_1(s-\tau)]}\left[e^{z_1(s-\tau)}-1\right]\right\}ds \qquad (3.27)
\end{aligned}$$

Let

$$V_{21}(t) \quad = \quad |z_2(t)| \tag{3.28}$$

Then, by (3.27) and (3.28), the upper right Dini derivative of $V_{21}(t)$ along the solution of (3.15) and (3.16) is given by

$$\begin{aligned}
D^+V_{21}(t) \quad &= \quad \frac{dz_2(t)}{dt}\cdot \mathrm{sign} z_2(t) \\
&= \quad \mathrm{sign} z_2(t)\cdot\Bigg(-a_{22}x_2^*\left[e^{z_2(t)}-1\right]-\frac{a_{21}x_1^*}{1+x_1(t-\tau)}\left[e^{z_1(t-\tau)}-1\right] \\
&\quad +\frac{a_{21}\,(x_1^*)^2}{(1+x_1^*)\,[1+x_1(t-\tau)]}\left[e^{z_1(t-\tau)}-1\right] \\
&\quad +a_{22}x_2^*\int_{t-\tau}^{t}e^{z_2(s)}\Bigg\{-a_{22}x_2^*\left[e^{z_2(s-\tau)}-1\right]-\frac{a_{21}x_1^*}{1+x_1(s-\tau)}\left[e^{z_1(s-\tau)}-1\right] \\
&\qquad\qquad +\frac{a_{21}\,(x_1^*)^2}{(1+x_1^*)\,[1+x_1(s-\tau)]}\left[e^{z_1(s-\tau)}-1\right]\Bigg\}ds\Bigg) \\
&= \quad -a_{22}x_2^*\left[e^{z_2(t)}-1\right]\cdot \mathrm{sign} z_2(t)-\frac{a_{21}x_1^*}{1+x_1(t-\tau)}\left[e^{z_1(t-\tau)}-1\right]\cdot \mathrm{sign} z_2(t) \\
&\quad +\frac{a_{21}\,(x_1^*)^2}{(1+x_1^*)\,[1+x_1(t-\tau)]}\left[e^{z_1(t-\tau)}-1\right]\cdot \mathrm{sign} z_2(t) \\
&\quad +\cdot\,\mathrm{sign} z_2(t)\Bigg(a_{22}x_2^*\int_{t-\tau}^{t}e^{z_2(s)}\Bigg\{-a_{22}x_2^*\left[e^{z_2(s-\tau)}-1\right] \\
&\qquad\qquad -\frac{a_{21}x_1^*}{1+x_1(s-\tau)}\left[e^{z_1(s-\tau)}-1\right] \\
&\qquad\qquad +\frac{a_{21}\,(x_1^*)^2}{(1+x_1^*)\,[1+x_1(s-\tau)]}\left[e^{z_1(s-\tau)}-1\right]\Bigg\}ds\Bigg) \\
&\leq \quad -a_{22}x_2^*\left|e^{z_2(t)}-1\right|+a_{21}x_1^*\left|e^{z_1(t-\tau)}-1\right|+\frac{a_{21}\,(x_1^*)^2}{1+x_1^*}\left|e^{z_1(t-\tau)}-1\right| \\
&\quad +a_{22}x_2^*\int_{t-\tau}^{t}e^{z_2(s)}\Bigg\{a_{22}x_2^*\left|e^{z_2(s-\tau)}-1\right|+a_{21}x_1^*\left|e^{z_1(s-\tau)}-1\right| \\
&\qquad\qquad +\frac{a_{21}\,(x_1^*)^2}{1+x_1^*}\left|e^{z_1(s-\tau)}-1\right|\Bigg\}ds
\end{aligned} \tag{3.29}$$

By Theorem 3.1, we know that there exists a $T > 0$, such that $x_2^* e^{z_2(t)} = x_2(t) \le M_2$ for $t \ge T$. Hence for $t \ge T + \tau$, we have

$$\begin{aligned} D^+V_{21}(t) \quad \le \quad & -a_{22}x_2^* \left| e^{z_2(t)} - 1 \right| + \frac{a_{21}x_1^*(1+2x_1^*)}{1+x_1^*} \left| e^{z_1(t-\tau)} - 1 \right| \\ & + a_{22}M_2 \int_{t-\tau}^{t} \left\{ a_{22}x_2^* \left| e^{z_2(s-\tau)} - 1 \right| + a_{21}x_1^* \left| e^{z_1(s-\tau)} - 1 \right| \right. \\ & \left. + \frac{a_{21}(x_1^*)^2}{1+x_1^*} \left| e^{z_1(s-\tau)} - 1 \right| \right\} ds \end{aligned} \tag{3.30}$$

Define

$$\begin{aligned} V_{22}(t) \quad = \quad & a_{22}M_2 \int_{t-\tau}^{t} \int_{s}^{t} \left\{ a_{22}x_2^* \left| e^{z_2(\theta-\tau)} - 1 \right| + a_{21}x_1^* \left| e^{z_1(\theta-\tau)} - 1 \right| \right. \\ & \left. + \frac{a_{21}(x_1^*)^2}{1+x_1^*} \left| e^{z_1(\theta-\tau)} - 1 \right| \right\} d\theta ds \end{aligned} \tag{3.31}$$

then we have

$$\begin{aligned} D^+V_{22}(t) \quad = \quad & \tau \left[a_{22}^2 M_2 x_2^* \left| e^{z_2(t-\tau)} - 1 \right| + a_{21}a_{22}M_2 x_1^* \left| e^{z_1(t-\tau)} - 1 \right| + \frac{a_{21}a_{22}M_2 (x_1^*)^2}{1+x_1^*} \left| e^{z_1(t-\tau)} - 1 \right| \right] \\ & - a_{22}M_2 \int_{t-\tau}^{t} \left\{ a_{22}x_2^* \left| e^{z_2(s-\tau)} - 1 \right| + a_{21}x_1^* \left| e^{z_1(s-\tau)} - 1 \right| + \frac{a_{21}(x_1^*)^2}{1+x_1^*} \left| e^{z_1(s-\tau)} - 1 \right| \right\} ds \end{aligned} \tag{3.32}$$

Define

$$\begin{aligned} V_{23}(t) \quad = \quad & a_{22}^2 M_2 x_2^* \tau \int_{t-\tau}^{t} \left| e^{z_2(s)} - 1 \right| ds \\ & + \frac{a_{21}x_1^*(1+2x_1^*)}{1+x_1^*} (1 + a_{22}M_2\tau) \int_{t-\tau}^{t} \left| e^{z_1(s)} - 1 \right| ds \end{aligned} \tag{3.33}$$

then we have

$$\begin{aligned} D^+V_{23}(t) \quad = \quad & a_{22}^2 M_2 x_2^* \tau \left[\left| e^{z_2(t)} - 1 \right| - \left| e^{z_2(t-\tau)} - 1 \right| \right] \\ & + \frac{a_{21}x_1^*(1+2x_1^*)}{1+x_1^*} (1 + a_{22}M_2\tau) \left[\left| e^{z_1(t)} - 1 \right| - \left| e^{z_1(t-\tau)} - 1 \right| \right] \end{aligned} \tag{3.34}$$

Now we define a Lyapunov functional $V_2(t)$ as

$$V_2(t) \quad = \quad V_{21}(t) + V_{22}(t) + V_{23}(t) \tag{3.35}$$

Then we have from (3.30)-(3.35) that for $t \ge T + \tau$

$$
\begin{aligned}
D^{+}V_2(t) \quad \leq \quad & -a_{22}x_2^*\left|e^{z_2(t)}-1\right| + \frac{a_{21}x_1^*\,(1+2x_1^*)}{1+x_1^*}\left|e^{z_1(t-\tau)}-1\right| \\
& + a_{22}M_2\int_{t-\tau}^{t}\left\{a_{22}x_2^*\left|e^{z_2(s-\tau)}-1\right| + a_{21}x_1^*\left|e^{z_1(s-\tau)}-1\right|\right. \\
& \qquad\qquad \left. + \frac{a_{21}\,(x_1^*)^2}{1+x_1^*}\left|e^{z_1(s-\tau)}-1\right|\right\} ds \\
& + a_{22}^2M_2x_2^*\tau\left|e^{z_2(t-\tau)}-1\right| + a_{21}a_{22}M_2x_1^*\tau\left|e^{z_1(t-\tau)}-1\right| \\
& + \frac{a_{21}a_{22}M_2\tau\,(x_1^*)^2}{1+x_1^*}\left|e^{z_1(t-\tau)}-1\right| \\
& - a_{22}M_2\int_{t-\tau}^{t}\left\{a_{22}x_2^*\left|e^{z_2(s-\tau)}-1\right| + a_{21}x_1^*\left|e^{z_1(s-\tau)}-1\right|\right. \\
& \qquad\qquad \left. + \frac{a_{21}\,(x_1^*)^2}{1+x_1^*}\left|e^{z_1(s-\tau)}-1\right|\right\} ds \\
& + a_{22}^2M_2x_2^*\tau\left[\left|e^{z_2(t)}-1\right| - \left|e^{z_2(t-\tau)}-1\right|\right] \\
& + \frac{a_{21}x_1^*\,(1+2x_1^*)}{1+x_1^*}\,(1+a_{22}M_2\tau)\left[\left|e^{z_1(t)}-1\right| - \left|e^{z_1(t-\tau)}-1\right|\right] \\
= \quad & \frac{a_{21}\,(1+2x_1^*)}{1+x_1^*}\,(1+a_{22}M_2\tau)\,x_1^*\left|e^{z_1(t)}-1\right| \\
& - a_{22}\,(1-a_{22}M_2\tau)\,x_2^*\left|e^{z_2(t)}-1\right| \\
= \quad & -\beta_{21}x_1^*\left|e^{z_1(t)}-1\right| - \beta_{22}x_2^*\left|e^{z_2(t)}-1\right| \qquad (3.36)
\end{aligned}
$$

According to assumptions (3.12) and (3.13), we know that $\mathcal{B} \equiv (\beta_{ij})_{2\times 2}$ is an M-matrix (see [1]); hence there exist positive constants ρ_i $(i = 1,2)$ such that

$$
\rho_1\beta_{11} + \rho_2\beta_{21} = \delta_1 > 0\,, \qquad \rho_1\beta_{12} + \rho_2\beta_{22} = \delta_2 > 0 \qquad (3.37)
$$

Now define a Lyapunov functional $V(t)$ as

$$
V(t) = \rho_1V_1(t) + \rho_2V_2(t) \qquad (3.38)
$$

Then we have from (3.38), (3.26), (3.36), and (3.37) that for $t \geq T+\tau$

$$\begin{aligned} D^+V(t) &= \rho_1 D^+V_1(t) + \rho_2 D^+V_2(t) \\ &\leq \rho_1\left[-\beta_{11}x_1^*\left|e^{z_1(t)}-1\right| - \beta_{12}x_2^*\left|e^{z_2(t)}-1\right|\right] \\ &\quad +\rho_2\left[-\beta_{21}x_1^*\left|e^{z_1(t)}-1\right| - \beta_{22}x_2^*\left|e^{z_2(t)}-1\right|\right] \\ &= -(\rho_1\beta_{11}+\rho_2\beta_{21})x_1^*\left|e^{z_1(t)}-1\right| \\ &\quad -(\rho_1\beta_{12}+\rho_2\beta_{22})x_2^*\left|e^{z_2(t)}-1\right| \\ &= -\delta_1 x_1^*\left|e^{z_1(t)}-1\right| - \delta_2 x_2^*\left|e^{z_2(t)}-1\right| \end{aligned} \tag{3.39}$$

Since the system (3.1) is uniformly persistent, one can see that there exist positive constants m_i $(i=1,2)$ and a $T^* > T+\tau$ such that $x_i^* e^{z_i(t)} = x_i(t) \geq m_i$ $(i=1,2)$ for $t \geq T^*$. Using the Mean Value Theorem, one obtains $x_i^*\left|e^{z_i(t)}-1\right| = x_i^* e^{\theta_i(t)}|z_i(t)| \geq m_i|z_i(t)|$ $(i=1,2)$, where $x_i^* e^{\theta_i(t)}$ lies between $x_i(t)$ and x_i^*. Let $\delta = \min\{\delta_1 m_1, \delta_2 m_2\}$. Then it follows from (3.39) that for $t \geq T^*$

$$\begin{aligned} D^+V(t) &\leq -\delta_1 x_1^*\left|e^{z_1(t)}-1\right| - \delta_2 x_2^*\left|e^{z_2(t)}-1\right| \\ &\leq -\delta_1 m_1|z_1(t)| - \delta_2 m_2|z_2(t)| \\ &\leq -\delta(|z_1(t)|+|z_2(t)|) \end{aligned} \tag{3.40}$$

Let $w(s) = \delta s$, then w is nonnegative continuous and nondecreasing, $w(0) = 0$ and $w(s) > 0$ for $s > 0$. It follow from (3.40) that for $t \geq T^*$

$$\begin{aligned} D^+V(t) &\leq -\delta(|z_1(t)|+|z_2(t)|) \\ &\leq -\delta\sqrt{z_1^2(t)+z_2^2(t)} \\ &= -w(\|z(t)\|) \end{aligned} \tag{3.41}$$

Let $u(s) = \rho s$, where $\rho = \min\{\rho_1, \rho_2\}$, then u is nonnegative continuous and nondecreasing, $u(0) = 0$, $u(s) > 0$ for $s > 0$ and $\lim_{s\to\infty} u(s) = +\infty$. Now, we want to show that $V(t) \geq u(\|z(t)\|)$. It follow from (3.25), (3.35), and (3.38) that for $t \geq T^*$

$$\begin{aligned} V(t) &= \rho_1 V_1(t) + \rho_2 V_2(t) \\ &= \rho_1[V_{11}(t)+V_{12}(t)+V_{13}(t)] + \rho_2[V_{21}(t)+V_{22}(t)+V_{23}(t)] \\ &\geq \rho_1 V_{11}(t) + \rho_2 V_{21}(t) \\ &= \rho_1|z_1(t)| + \rho_2|z_2(t)| \\ &\geq \rho(|z_1(t)|+|z_2(t)|) \\ &\geq \rho\sqrt{z_1^2(t)+z_2^2(t)} \\ &= u(\|z(t)\|) \end{aligned} \tag{3.42}$$

we can conclude from the Lyapunov-Kransovskii theorem (see [8, Theorem 5.1, p.27; Corollary 5.2, p.30]) and (3.38) that the zero solution of (3.15) and (3.16) is globally asymptotically stable, and hence the unique positive equilibrium point E^* of the system (3.1) with initial conditions (3.2) is globally asymptotically stable. This completes the proof.

4 Examples

In this section, we want to illustrate our results by some examples.

Example 4.1 *Consider the following system:*

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[1-2x_1(t)-\frac{x_2(t)}{1+x_2(t)}\right] \\ \dot{x}_2(t) &= x_2(t)\left[1-2x_2(t)-\frac{x_1(t)}{1+x_1(t)}\right] \end{aligned} \tag{4.1}$$

Comparing the system (4.1) with the system (2.1), we get $r_1 = r_2 = a_{12} = a_{21} = 1$ and $a_{11} = a_{22} = 2$. So the system (4.1) has the unique positive equilibrium point $E^* \equiv (\frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}-1}{2})$. It is easy to verify that the system (4.1) satisfies all assumptions in (2.4) and (2.6). From Lemma 2.2, we see that the unique positive equilibrium point E^* is locally asymptotically stable. Using Theorems 2.1 and 2.3, we know that the unique positive equilibrium point E^* is globally asymptotically stable.

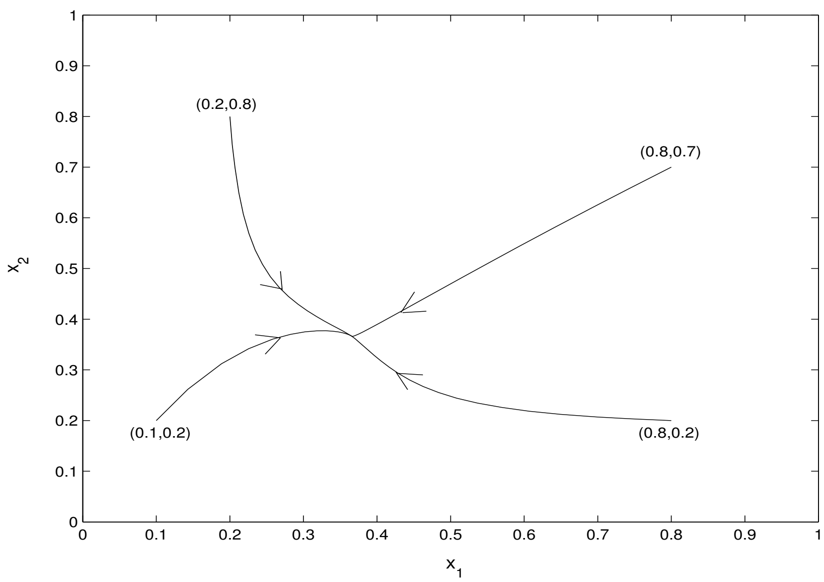

Figure 4.1: Phase portrait of the system (4.1).

Example 4.2 *Consider the following system:*

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[1 - 2x_1(t-\tau) - \frac{x_2(t-\tau)}{1+x_2(t-\tau)}\right] \\ \dot{x}_2(t) &= x_2(t)\left[1 - 2x_2(t-\tau) - \frac{x_1(t-\tau)}{1+x_1(t-\tau)}\right] \end{aligned} \tag{4.2}$$

Comparing the system (4.2) with the system (3.1), we get $r_1 = r_2 = a_{12} = a_{21} = 1$ and $a_{11} = a_{22} = 2$. So the system (4.2) has the unique positive equilibrium point $E^* \equiv (\frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}-1}{2})$. It is easy to verify that the system (4.2) satisfies all assumptions in (3.3), (3.10), (3.12) and (3.13). Using Theorem 3.1, we know that the system (3.2) is uniformly persistent. From Theorem 3.2, we know that the unique positive equilibrium point E^* is globally asymptotically stable provided that $\beta_{11} > 0$, $\beta_{22} > 0$, and $\beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$, where

$$\begin{aligned} \beta_{11} &= 2(1-2M_1\tau) \ , \\ \beta_{12} &= \left(\sqrt{3}-3\right)(1+2M_1\tau) \ , \\ \beta_{21} &= \left(\sqrt{3}-3\right)(1+2M_2\tau) \ , \\ \beta_{22} &= 2(1-2M_2\tau) \ , \\ M_1 &= \frac{1}{2}e^{\tau} \ , \\ M_2 &= \frac{2M_1+1}{2(1+M_1)} \cdot exp\left\{\frac{2M_1+1}{1+M_1}\tau\right\} . \end{aligned}$$

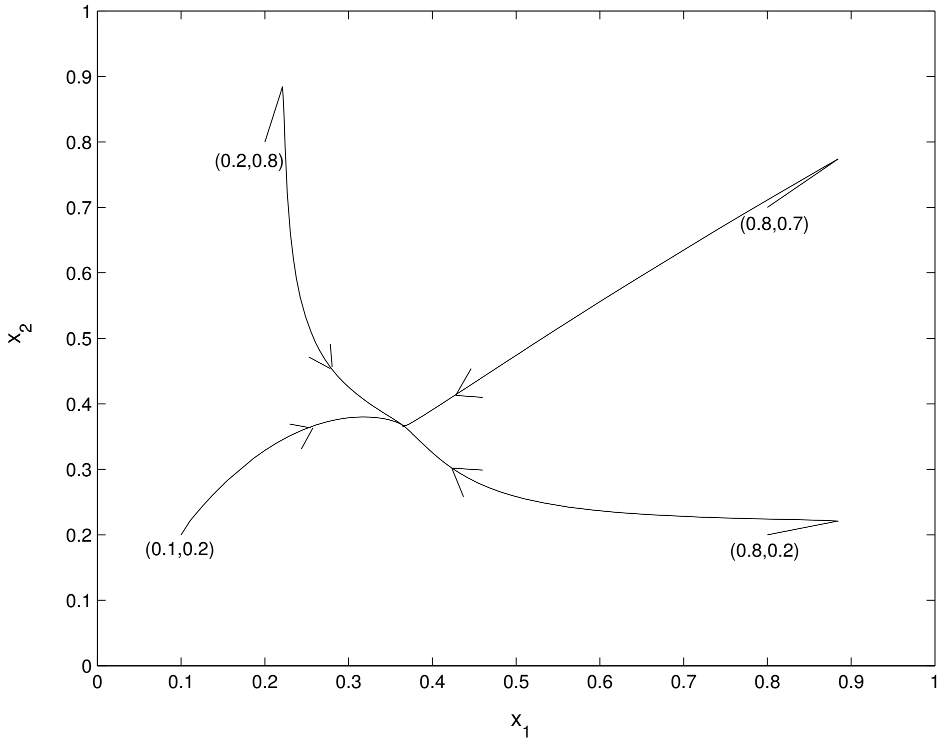

Figure 4.2: Phase portrait of the system (4.2) with $\tau = 0.1$.

References

[1] Araki, M. and Kondo, B. (1972), Stability and Transient Behavior of Composite Nonlinear Systems, *IEEE Trans. Automat. Control* **17** , 537-541.

[2] Bellman, R. and Cooke, K. L. (1963), *Differential Difference Equations*, Academic Press, New York.

[3] Blythe, S. P., Nisbet, R. M., Gurney, W. S. C., and Macdonald, N. (1985), Stability Switches in Distributed Delay Models, *J. Math. Anal. Appl.* **109** , 388-396.

[4] Cheng, K. S., Hsu, S. B. and Lin, S. S. (1981), Some Results on Global Stability of a Predator-Prey System, *J. Math. Biol.* **12** , 115-126.

[5] Freedman, H. I. and Waltman, P. (1985), Persistence in a Model of Three Competitive Populations, *Math. Biosci.* **73 , 89**.

[6] Gopalsamy, K. (1992), *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic, Dordrecht/Norwell, MA.

[7] Hale, J. K. (1969), *Ordinary Differential Equations*, New York: Wiley Interscience .

[8] Kuang, Y. (1993), *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston.

[9] Perko, L. (1991), *Differential Equations and Dynamical Systems*, Springer-Verlag New York, Inc. .

[10] Ray, W. O. (1988), *Real Analysis*, Prentice-Hall.

[11] Takeuchi, Y. (1996), *Global Dynamical Properties of Lotka-Volterra System*, World Scientific, Singapore.

[12] Wang, W. and Ma, Z. (1991), Harmless Delays for Uniform Persistence, *J. Math. Anal. Appl.* **158** , 256-268.

[13] Wiggins, S. (1990), *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer-Verlag New York, Inc.

[14] Zhang, J. and Chen, L. (1996), Permanence and Global Stability for a Two-species Cooperative System with Time Delays in a Two-patch Environment,*Comput. Math. Appl.***32** , 101-108.

[15] Zhien, M. (1986), Stability and Predation Models with Time Delays, *Appl. Anal.***22**, 169-192.

時滯式競爭系統之持久性和整體穩定性

陳信志* 何肇寶*

摘 要

本篇論文主要探討具時滯參數之競爭系統的動態行為。首先，我們利用三種不同方法分析不具時滯參數之競爭系統的整體穩定性。而後討論具時滯參數之競爭系統，在適當條件下系統之解是均勻持久的，進而得到此系統之唯一正均衡點的整體穩定性的充分條件。最後，我們舉例說明上述結果。

關鍵詞：競爭系統、均衡點穩定性、Dulac's 法則、Ponicaré-Bendixson 定理、極限環、均勻持久性、Lyapunov 函數、整體漸近穩定。

* 東海大學數學系