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A Note on the Sums of Powers of Consecutive Integers

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Abstract

Let n, k be positive integers ($k > 1$), and let $S_n(k)$ be the sum of the n-th powers of positive integers up to $k-1$. Following an idea due to Jacques Bernoulli, we explore a formula for $S_n(k)$:

$$S_n(k) = \frac{1}{n+1} \sum_{i=0}^{n} \binom{n+1}{i} B_i k^{n+1-i},$$

where B_i are the Bernoulli numbers.

1 Observations and Conjectures

Let n, k be positive integers ($k > 1$), and let

$$S_n(k) = 1^n + 2^n + 3^n + \cdots + (k-1)^n.$$

Our objective is to find

A formula for $S_n(k)$.

It is well-known that

$$\begin{aligned} S_1(k) &= \frac{1}{2}k^2 - \frac{1}{2}k, \\ S_2(k) &= \frac{1}{3}k^3 - \frac{1}{2}k^2 + \frac{1}{6}k, \\ S_3(k) &= \frac{1}{4}k^4 - \frac{1}{2}k^3 + \frac{1}{4}k^2. \end{aligned}$$

Obviously, this strongly suggests the following three conjectures:

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(a) $S_n(k)$ is a polynomial in k of degree $n+1$ with leading coefficient $\frac{1}{n+1}$,

(b) The constant term of $S_n(k)$ is 0, i.e., $S_n(0) = 0$,

(c) The coefficient of k^n in $S_n(k)$ is $-\frac{1}{2}$.

In other words, $S_n(k)$ is a polynomial in k of the form

$$S_n(k) = \frac{1}{n+1}\, k^{n+1} - \frac{1}{2}\, k^n + a_{n-1}k^{n-1} + \cdots + a_1 k,$$

and hence we have

$$\frac{d}{dk} S_n(k) = k^n - \frac{n}{2}\, k^{n-1} + \cdots \ .$$

To make life easier, we put the first two conjectures together and we reach the following conjecture, which is what Jacques Bernoulli (1654–1705) claimed more than three hundred years ago.

Bernoulli's Claim: There exists a unique monic polynomial of degree n, say $B_n(x)$, such that

$$S_n(k) = 1^n + 2^n + 3^n + \cdots + (k-1)^n = \int_0^k B_n(x)\, dx\ . \tag{1.1}$$

To prove Bernoulli's claim, we observe that

$$(j+1)^{n+1} - j^{n+1} = \sum_{i=0}^{n} \binom{n+1}{i} j^i\ .$$

Summing over j from 0 to $k-1$, the left-hand side becomes k^{n+1}, but the right-hand side is a linear combination of $S_i(k)$ as follows:

$$k^{n+1} = \sum_{i=0}^{n} \binom{n+1}{i} S_i(k) = \sum_{i=0}^{n-1} \binom{n+1}{i} S_i(k) + (n+1) S_n(k)\ .$$

Therefore, we have a recursion formula for $S_n(k)$ as follows:

$$S_n(k) = \frac{1}{n+1} k^{n+1} - \frac{1}{n+1} \sum_{i=0}^{n-1} \binom{n+1}{i} S_i(k)\ . \tag{1.2}$$

The result of Bernoulli's claim follows immediately from (1.3) by induction, and so do the above three conjectures. Therefore our objective becomes to find

A formula for the Bernoulli polynomials $B_n(x)$.

Equations (1.1) and (1.3) yield a recursion formula for $B_n(k)$ as follows:

$$B_n(k) = k^n - \frac{1}{n+1}\sum_{i=0}^{n-1}\begin{pmatrix} n+1 \\ i \end{pmatrix} B_i(k) \ . \tag{1.3}$$

However, this is not what we want, since our objective is to find a real formula not just a recursion formula for $B_n(k)$.

2 A Formula for $S_n(k)$

It is quite obvious that equation (1.1) is the only resource we have. By the additivity of the definite integrals we obtain

$$\int_k^{k+1} B_n(x)\,dx = \int_0^{k+1} B_n(x)\,dx - \int_0^k B_n(x)\,dx = S_n(k+1) - S_n(k) = k^n.$$

Replacing k by x we have

$$\int_x^{x+1} B_n(t)dt = x^n \ . \tag{2.1}$$

Next is a big step. Differentiating with respect to x on both sides of equation (2.1) by applying the fundamental theorem of calculus yields

$$B_n(x+1) - B_n(x) = nx^{n-1}. \tag{2.2}$$

Summing over x from 0 to $k-1$ we obtain

$$B_n(k) - B_n(0) = nS_{n-1}(k) = n\int_0^k B_{n-1}(x)\,dx. \tag{2.3}$$

The first equality in equation (2.3) implies that

$$S_n(k) = \frac{1}{n+1}\left(B_{n+1}(k) - B_{n+1}(0)\right), \tag{2.4}$$

while the second equality yields an almost first-order recursion formula for the Bernoulli polynomials $B_n(x)$ that

$$B_n(k) = n\int_0^k B_{n-1}(x)\,dx + B_n(0). \tag{2.5}$$

Note that $B_n(0)$ are just the constant terms of the Bernoulli polynomials, which are called the Bernoulli numbers and are denoted by B_n. Therefore

$$B_n = B_n(0) = S_n'(0).$$

Let $B_0(x) = 1$ and write equation (2.5) in the following form:

$$B_n(x) = n\int_0^x B_{n-1}(t)dt + B_n, \quad n = 1,2,3,\cdots .$$

It can be shown that the first three Bernoulli polynomials are

$$\begin{aligned} B_1(x) &= x + B_1, \\ B_2(x) &= x^2 + 2B_1x + B_2, \\ B_3(x) &= x^3 + 3B_1x^2 + 3B_2x + B_3. \end{aligned}$$

By induction, we obtain a formula for $B_n(x)$ which reads

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}. \tag{2.6}$$

Putting (2.4) and (2.6) together, we finally have a formula for $S_n(k)$ as follows.

$$\boxed{S_n(k) = \frac{1}{n+1}\sum_{i=0}^{n} \binom{n+1}{i} B_i k^{n+1-i}}$$

Alternatively, we may use (1.1) instead of (2.4) to reach the above formula.

3 Bernoulli Numbers

Now, it remains to calculate the Bernoulli numbers B_n, but this can be done via equation (2.2). Let $x = 0$, then $B_n(1) - B_n(0) = 0$ and therefore

$$B_n = B_n(0) = B_n(1) = \sum_{i=0}^{n} \binom{n}{i} B_i.$$

Hence we have

$$B_{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} B_i = \sum_{i=0}^{n} \binom{n+1}{i} B_i + B_{n+1} \iff \sum_{i=0}^{n} \binom{n+1}{i} B_i = 0,$$

and we get a recursion formula for the Bernoulli numbers B_n as follows:

$$B_n = -\frac{1}{n+1}\sum_{i=0}^{n-1} \binom{n+1}{i} B_i. \tag{3.1}$$

Alternatively, formula (3.1) can be derived from equation (1.3) by putting $k = 0$ and noting that $B_i = B_i(0)\ \forall i$.

This completes our very brief journey into the exploration of the sums of powers of consecutive integers.

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連續正整數之次方和公式的探討

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Abstract

設n,k 為正整數($k>1$)，且令$S_n(k)$ 為前$k-1$個正整數的n次方和。隨著Jacques Bernoulli的腳步，我們探討$S_n(k)$ 的一個公式:

$$S_n(k)=\frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}B_i\,k^{n+1-i},$$

此處B_i為Bernoulli 數。

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