Abstract

This paper presents a new design method for μ-synthesis, from the lower bound of μ-norm. The spectral Nevanlinna-Pick interpolation theory is our main tool to design the robust controller. First the lower bound design problem is transformed into a spectral Nevanlinna-Pick problem $-$ spectral model matching problem. The solution technique is then construct to use spectral interpolstion theory for controller design. Due to the fact that the present result of spectral interpolation problem is only the 2 by 2 matrix case, some simple design examples of several typical models are presented for illustrative purpose.

Keywords : structured singular value, robust stability, robust performance, μ-synthesis, model matching problem, spectral Nevanlinna-Pick interpolation problem.

Contents

List of Figures

Notations

Symbol	Meaning
$\|\cdot\|$	a norm of a vector space.
$\|A\|_p$	the induced norm of a matrix A which is defined as $\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$
$\|x\|_p$	the vector p-norm of $x \in \mathbb{C}^n$
$\|F\|_s$	the spectral norm of matrix function F which is defined as $\|F\|_s = \sup_{\omega \in \mathbb{R}} \rho(F(j\omega))$
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}_+$	all complex numbers with positive real part
$\overline{\mathbb{C}}_+$	all complex numbers with nonnegative real part
$\mathbb{C}_-$	all complex numbers with negative real part
$\mathbb{C}^{n \times m}$	space of $n \times m$ complex matrices
$\mathbb{D}$	the open unit disk
$\overline{\mathbb{D}}$	the closed unit disk
f^*	the complex conjugate and transpose of f
$\|f\|_2$	$\|f\|_2 = \left(\int_{-\infty}^{\infty} \|f(t)\|^2 dt \right)^{1/2}$
$\|\hat{f}\|_\infty$	$\|\hat{f}\|_\infty = \sup_{w \in \mathbb{R}} \bar{\sigma}[\hat{f}(j\omega)]$
$\hat{f}(s)$	the Laplace transform of f, i.e., $\hat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$
$\mathcal{F}_\ell$	lower Linear Fraction Transformation
$\mathcal{F}_u$	upper Linear Fraction Transformation
$\mathcal{H}_2$	Hardy 2-space, $\mathcal{H}_2 = \{f : f \text{ is analytic in } \operatorname{Re} s > 0 \text{ and } \|f\|_2 < \infty.\}$

$\mathcal{H}_\infty$	Hardy ∞-space, $\mathcal{H}_\infty = \{f : f \text{ is analytic in Re } s > 0 \text{ and } \|f\|_\infty < \infty\}$
I	the identity matrix
Int Γ_2	$\{(s,p) : \lambda^2 - s\lambda + p = 0,\ \|\lambda\| < 1\}$.
$\mathcal{L}_1(0,\infty)$	class of Lebesque measurable complex-valued functions with $\int_0^\infty \|f(t)\| dt < \infty$
$\mathcal{L}_\infty(j\mathbb{R})$	$\{\hat{f}(s) : \|\hat{f}\|_\infty = \sup_{\omega\in\mathbb{R}} \bar{\sigma}[\hat{f}(j\omega)] < \infty\}$
LFT	linear fractional transformation
$\mathbb{N}$	the natural number
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^{n\times m}$	space of $n \times m$ real matrices
Re s	real part of s
RHP	the open right half plant
$\mathcal{RH}_\infty$	subspace of $\mathcal{H}_\infty$ which consists of all proper and real rational stable transfer functions in the open right half plane
$\mathcal{RH}_\infty^-$	subspace of $\mathcal{H}_\infty^-$ which consists of all proper and real rational stable transfer functions in the open left half plane
SNP	spectral Nevanlinna-Pick
$\mathbb{T}$	the unit circle
$\mathbb{V}$	vector space
$\bar{\sigma}(A)$	the largest singular value of A
$\rho(A)$	spectral radius of A, that is, $\max_i \|\lambda_i(A)\|$
$(u)_j$	the jth component of the vector u
Γ_2	the symmetrized bidisc, $\{(s,p) : \lambda^2 - s\lambda + p = 0,\ \|\lambda\| \leq 1\}$.
$\lambda(A)$	eigenvalues of A
μ	the structured singular value (SSV)

Chapter 1

Introduction

1.1 Motivation

For uncertain systems μ-synthesis procedure provides an effective method for controller design and it guarantees the robustness of both stability and performance. Due to the difficulty in computing the system's μ-norm [12], the solution now provided by μ-synthesis is only an approximation. There is a computational access of μ-synthesis, D-K iteration [5], which design the controller from the upper bound of system's μ-norm. There are numerous design examples using D-K iteration for robust controller design. In [6, 8, 19, 23], some design examples are presented.

Although μ-synthesis has been widely accepted as standard design method, there is no definite mathematical theory, in fact, μ-norm is not actually a norm. To achieve an analytical breakthrough for μ-synthesis problem is the main reason for us to study the lower bound design method.

First of all, the μ-synthesis problems for robust stability and robust performance are transformed into spectral model matching problem (first proposed by this paper). The solution of this type of problem relies on finding analytic functions which satisfy certain interpolation condition with its spectral norm to be less than one. This problem is called the **spectral Nevanlinna-Pick** (SNP) interpolation problem in literature [2, 3, 9, 18]. In fact, to find the general solution of SNP problem is still an open problem [10]. The so called SNP interpolation theory is developed to solve SNP problem. Till now only the problem with 2×2 matrix has been solved [3, 18]. Once the interpolation problem is solved, the corresponding analytical function is used to construct our robust controller. Detail algorithms for controller synthesis see Chapter 4.

In this paper, we try to design the controller from the lower bound of μ. This involves much of the spectral radius ρ and the spectral Nevanlinna-Pick interpolation theory. Recently, some works about the spectral interpolation theory are proposed [18]. We utilize the theory to carry out our design for simple uncertain systems.

1.2 Scope and Organization

This paper is organized as follows: Chapter 2 provides the mathematical background of robust control . Definitions of norms and spaces are given and the concept of structured singular value μ are discussed. μ-synthesis and D-K iteration are also introduced. Chapter 3 review the classical NP theory and the spectral NP theory. Necessary and sufficient conditions and solutions for these problems are presented. Chapter 4 contains the application of the spectral NP theory and the lower bound design method. Some algorithms and design examples are given here to illustrate our design method.

Chapter 2

Preliminaries

This chapter provides the mathematical background and basic concepts of robust control. Structured singular value μ is also introduced in section 2.2.

2.1 Mathematical Preliminary

In this section, we survey the definitions of norms and spaces and some useful tools of linear system. To describe the performance specifications of a control system, to measure the size of signals of interest, norm is indispensable. Which norm is the most appropriate depends on the situation we meet and for this purpose some basic knowledge of *Hardy space*, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ spaces are needed.

Definition 2.1 *A real-valued function* $\|\cdot\|$ *defined on a vector space* $\mathbb{V}$ *is said to be a norm on* $\mathbb{V}$ *if it satisfies the following properties:*

(1) $\|x\| \geq 0$

(2) $\|x\| = 0$ *if and only if* $x = 0$

(2) $\|ax\| = |a| \; \|x\|$, *for any scalar* a

(3) $\|x + y\| \leq \|x\| + \|y\|$

for any $x \in \mathbb{V}$ *and* $y \in \mathbb{V}$.

Define the vector p-norm of $x \in \mathbb{C}^n$ as

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \text{ for } 1 \le p \le \infty.$$

When $p = \infty$ we have

$$\|x\|_\infty := \max_{1 \le i \le n} |x_i|.$$

Definition 2.2 *Let A be a $m \times n$ matrix and $x \in \mathbb{C}^n$, the matrix induced p-norm of A is defined as*

$$\|A\|_p := \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\|x\|_p = 1} \|Ax\|_p$$

This matrix norm is induced by vector p-norm.

The induced 2-norm of A is usually denoted simply as $\|A\|$ and from the definition we have

$$\begin{aligned} \|A\| = \|A\|_2 &= \text{ the largest singular value of } A \\ &= \bar{\sigma}(A) \end{aligned}$$

The norm of matrices has the following properties:

$$\|A\|_p \|x\|_p \le \|A\|_p \|x\|_p;$$

$$\|A + B\|_p \le \|A\|_p + \|B\|_p;$$

$$\|AB\|_p \le \|A\|_p \|B\|_p.$$

Now we consider norms of the signal in time domain which maps $(-\infty, \infty)$ to $\mathbb{R}^n$.

Definition 2.3 *The 1-norm, 2-norm, ∞-norm of a signal $f(t)$ in time domain are defined as*

$$\|f\|_1 := \int_{-\infty}^{\infty} \|f(t)\| dt.$$

$$\|f\|_2 := \left(\int_{\infty}^{\infty} \|f(t)\|^2 dt \right)^{1/2}.$$

$$\|f\|_\infty := \sup_t \|f(t)\|_\infty.$$

The following Theorem for matrix is important.

Theorem 2.1 *Let A be any $n \times n$ matrix and $\varepsilon > 0$ be given. Then there exists an induced matrix norm such that*

$$\rho(A) \leq \|A\| \leq \rho(A + \varepsilon)$$

Proof see [15].

For the frequency domain signal

$$\hat{f}(j\omega) = \int_{-\infty}^{\infty} f(j\omega) e^{-j\omega t} dt$$

Definition 2.4 *The 2-norm, ∞-norm of $\hat{f}(j\omega)$ are defined as*

$$\|\hat{f}\|_2 := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}^*(j\omega) \hat{f}(j\omega) \right)^{1/2}$$
$$\|\hat{f}\|_\infty := \sup_{\omega \in \mathbb{R}} |\hat{f}(j\omega)|$$

Definition 2.5 *$\mathcal{L}_2(-\infty, \infty)$ is a Hilbert space which consists of all functions (matrix-valued or scalar-valued) defined on the interval $(-\infty, \infty)$ with*

$$\|f\|_2 = \left(\int_{-\infty}^{\infty} \|f(t)\|^2 dt \right)^{1/2} < \infty$$

and the inner product of this Hilbert space is defined as

$$< f, g > := \int_{-\infty}^{\infty} \text{trace}[f^*(t) g(t)] dt$$

Definition 2.6 *$\mathcal{L}_2(j\mathbb{R})$ space is a Hilbert space of matrix-valued (or scalar-valued) functions on $j\mathbb{R}$ which consists of all complex matrix functions $\hat{f}$ such that*

$$\int_{-\infty}^{\infty} \text{trace}[\hat{f}^*(j\omega) \hat{f}(j\omega)] dt < \infty$$

The inner product for this Hilbert space is defined as

$$< \hat{f},\ \hat{g} > := \int_{-\infty}^{\infty} \text{trace}[\hat{f}^*(j\omega) \hat{g}(j\omega)] d\omega$$

Definition 2.7 *The Hardy 2-space $\mathcal{H}_2$ is a subspace of $\mathcal{L}_2(j\mathbb{R})$ with matrix functions $\hat{f}(s)$ analytic in* Re $s > 0$ *and the norm*

$$\|\hat{f}\|_2 := \left\{ \sup_{\sigma>0} \int_{-\infty}^{\infty} \text{trace}[\hat{f}^*(\sigma + j\omega)\hat{f}(\sigma + j\omega)]d\omega \right\}^{1/2}$$

is finite. That is,

$$\mathcal{H}_2 = \{\hat{f} : \hat{f} \text{ is analytic in Re } s > 0 \text{ and } \|\hat{f}\|_2 < \infty.\}$$

Definition 2.8 *$\mathcal{RH}_2$ is a subspace of $\mathcal{H}_2$ which consists of all strictly proper and real rational stable transfer functions.*

Definition 2.9 *$\mathcal{H}_\infty$ is a space with functions that are analytic and bounded in the open right-half plane* (RHP)*. The $\mathcal{H}_\infty$ norm is defined as*

$$\|\hat{f}\|_\infty := \sup_{\text{Re } s>0} \bar{\sigma}[\hat{f}(s)] = \sup_{\omega\in\mathbb{R}} \bar{\sigma}[\hat{f}(j\omega)].$$

Hence

$$\mathcal{H}_\infty := \{\hat{f} : \hat{f} \text{ is analytic in Re } s > 0 \text{ and } \|\hat{f}\|_\infty < \infty\}.$$

Definition 2.10 *$\mathcal{RH}_\infty$ is a subspace of $\mathcal{H}_\infty$ which consists of all proper and real rational stable transfer functions.*

2.2 Linear Fractional Transformation

Linear Fraction Transformations (LFT) are powerful tools to represent uncertainty in matrices and systems.

Let M be a complex matrix of the form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{C}^{(p_1+p_2)\times(q_1+q_2)}$$

Suppose that $D_1 \subset \mathbb{C}^{q_2 \times p_2}$, the *lower linear fractional transformation* is defined by

$$\begin{aligned} \mathcal{F}_\ell(M,\cdot) : D_1 &\longrightarrow \mathbb{C}^{p_1 \times q_1} \\ \Delta &\longmapsto \mathcal{F}_\ell(M,\Delta) = M_{11} + M_{12}\Delta(I - M_{22}\Delta)^{-1}M_{21} \end{aligned}$$

provided that $(I - M_{22}\Delta)^{-1}$ exists. This transformation could be used to system interconnection as shown in Figure 2-1.

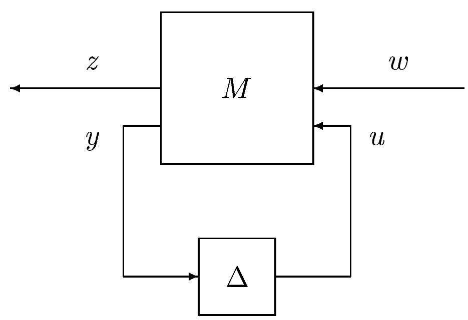


Figure 2-1: System interconnection using lower LFT

We can see that

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

and

$$u = \Delta y$$

Thus

$$z = \mathcal{F}_\ell(M,\Delta)w.$$

Similarly, let $D_1 \subset \mathbb{C}^{q_1 \times p_1}$ we define the *upper linear fraction transformation* as

$$
\begin{aligned}
\mathcal{F}_u(M,\cdot) : D_2 &\longrightarrow \mathbb{C}^{p_2 \times q_2} \\
\Delta &\longmapsto \mathcal{F}_u(M,\Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}
\end{aligned}
$$

It is clear that upper LFT exists if $(I - M_{22}\Delta)^{-1}$ exist. For the systems shown in Figure 2-2,

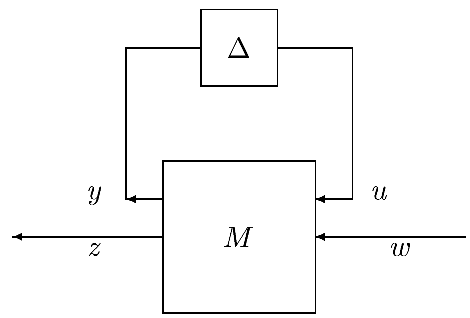


Figure 2-2: System interconnection using upper LFT

$$
\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}
$$

and

$$u = \Delta y$$

then

$$z = \mathcal{F}_u(M,\Delta)w$$

Parametric uncertainty can also be represented by using LFT. For example, let c be a parameter lies between 2.0 and 2.8. Write this as c=2.4+0.4δ_c where $\delta_c \in [-1,1]$. Let M be a matrix such that

$$
\begin{aligned}
c &= \mathcal{F}_\ell(M,\delta_c) \\
&= [M_{11} + M_{12}\delta_c(I - M_{22}\delta_c)^{-1}M_{21}]
\end{aligned}
$$

Thus

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} 2.4 & 0.4 \\ 1 & 0 \end{bmatrix}$$

Similarly,

$$c = \mathcal{F}_u\left(\begin{bmatrix} 0 & 1 \\ 0.4 & 2.4 \end{bmatrix}, \delta_c\right)$$

So when the the real parameteric uncertainty c appears in a block diagram, simply replace it with block as shown in Figure 2-3.

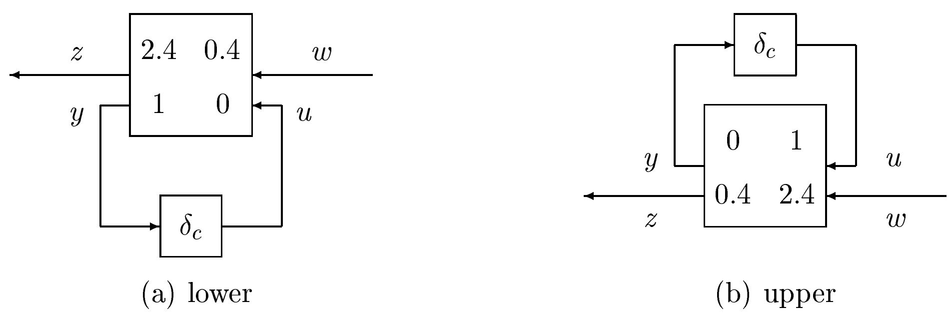


Figure 2-3: Lower LFT for parameteric uncertainty.

The cascade connections, parallel connections and feedback connections of LFTs preserve the LFT structure. The interested reader can refer to chapter 4 in [5] for more detail information about the interconnections between LFTs.

2.3 μ-Analysis

In this paragraph, we give an introduction of μ-theory and the relative given results. To see more details, the reader is referred to [5, 13, 25]. Before the illustration of structured singular value μ, we should have clear knowing to uncertainties of system.

Uncertainty for systems could be classified into two types, real parameter and unmodeled dynamics. Or in other word, the parameteric uncertainty and dynamical uncertainty. The parameteric uncertainty is as the form of

$$c = 2.4 + 0.4\delta_c, \quad \delta_c \in [-1, 1].$$

We could use LFT to represent it as we did in previous section. The dynamical uncertainty is applied when the uncertain system have a wide variety of plant variations and could not only present by parameteric uncertainty. For instance, define the uncertainty

$$\Delta := \frac{\widetilde{G} - G}{GW_u}$$

where $\widetilde{G}$ is the perturbed plant and G is the nominal plant. Similarly, it could be represented as an LFT. Besides the type of perturbation, we also care the dimension of the perturbation and the independent locations that particular uncertainty occurs.

Now we introduce the basic concept of the structure singular value $\mu(\cdot)$. Here we treat the uncertainty as unmodeled dynamics for conservation. Consider a matrix $M \in \mathbb{C}^{n\times n}$ and structured uncertainty $\mathbf{\Delta} \subset \mathbb{C}^{n\times n}$ with

$$\mathbf{\Delta} = \{\mathrm{diag}[\delta_1 I_{r1}, \cdots, \delta_s I_{rs}, \Delta_1, \cdots, \Delta_F] : \delta_i \in \mathbb{C}, \quad \Delta_j \in \mathbb{C}^{m_j \times m_j}\}$$

where $\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n$. Two nonnegative integers S and F represent the number of *repeated scalar* blocks and the number of *full* blocks. Let $\bar{\sigma}(M)$ and $\rho(M)$ denote the maximum singular value and the spectral radius of M respectively. Then

$$B\mathbf{\Delta} = \{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) \leq 1\}$$

Definition 2.11 *For $M \in \mathbb{C}^{n\times n}$, the structure singular value $\mu_\Delta(M)$ is defined as*

$$\mu_{\mathbf{\Delta}} = \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \quad \det(I - M\Delta) = 0\}}.$$

with $\mu_{\mathbf{\Delta}} = 0$ if no $\Delta \in \mathbf{\Delta}$ solves $\det(I - M\Delta) = 0$

$\mu_{\mathbf{\Delta}}(M)$ is the measure of the smallest Δ that causes"instability" of the feedback loop shown below:

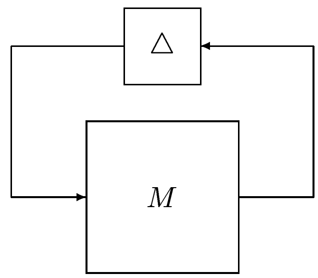


Figure 2-4: General uncertain closed-loop system

An alternative expression follows from the definition

$$\mu_{\mathbf{\Delta}}(M) = \max_{\Delta \in B\mathbf{\Delta}} \rho(M\Delta).$$

Since it is difficult to calculate $\mu_{\Delta}(M)$, we consider its bounds from two extreme cases of Δ. If $\Delta = \{\delta I : \delta \in C\}$ $(S = 1, F = 0, r_1 = n)$, $\mu_{\mathbf{\Delta}}(M) = \rho(M)$, and if $\Delta = \mathbb{C}^{n\times n}$ $(S = 0, F = 1, m_1 = n)$, $\mu_{\mathbf{\Delta}}(M) = \bar{\sigma}(M)$. Thus we have the following result

$$\rho(M) \leq \mu_{\mathbf{\Delta}}(M) \leq \bar{\sigma}(\Delta).$$

But this is not good enough. To find the better bounds of $\mu_{\mathbf{\Delta}}(M)$, define

$$\mathcal{U} = \{U \in \mathbf{\Delta} : \; UU^* = I_n\}$$

$$\mathcal{D} = \left\{ \begin{array}{c} \text{diag} \quad [D1, \ldots, D_S, d_1 I_{m1}, \ldots, D_{F-1} I_{m_{F-1}}, I_{m_F}] : \\ D_i \in C^{r_i \times r_i}, D_i = D_i^* > 0, \; d_j \in \mathbb{R}, \; d_j > 0 \end{array} \right\}.$$

and the bounds can be restricted to[25]

$$\max_{U \in \mathcal{U}} \rho(UM) \leq \mu_{\mathbf{\Delta}}(M) \leq \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}). \tag{2-1}$$

From [12], the lower bound is always an equality, that is,

$$\max_{U \in \mathcal{U}} \rho(UM) = \mu_{\mathbf{\Delta}}(M).$$

Since the set of $\rho(UM)$ is nonconvex, the quantity $\rho(UM)$ can have multiple local maxima that are not global, and the local search can only yield a lower bound. In [21], the lower bound power method is studied. Thus there is an alternative choice, the upper bound in inequality (2-1). From [20], for block struct0ures $\mathbf{\Delta}$ satisfying $2S + F \leq 3$, the upper bound is always equal to $\mu_{\mathbf{\Delta}}(M)$, and for block structures with $S + 2F > 3$, there exists matrices for which μ is less than the infimum. In this approach the problem will lead to the popular and effective method, "*D-K iteration*", which will be mentioned latter.

2.4 Robust Stability and Robust Performance with μ-Synthesis

Given a set of uncertainty model $\widetilde{G}$, suppose that $G \in \widetilde{G}$ is the nominal design model and K is the controller. K provides *robust stability* if it stabilize every plant G belongs to $\widetilde{G}$. *Robust performance* means the performance objectives are satisfied for every plant belongs to $\widetilde{G}$, of course, controller K must also satisfy the robust stability condition.

The following theorems provide the criteria of robust stability and robust performance with structured uncertainty. Suppose $M(s)$ is a stable, n_z inputs, n_w outputs transfer function of a linear system M. Let Δ be a block structure as we mentioned in section 2.3 and assume that the dimensions are such that $\mathbf{\Delta} \subset \mathbb{C}^{n_w \times n_z}$. Let $\mathbf{S}$ denote the set of real-rational, proper, stable transfer matrices. Associated with any block structure $\mathbf{\Delta}$, let $\mathbf{S}_{\mathbf{\Delta}}$ denote the set of all block diagonal, stable rational transfer functions, with block structure like

$$\mathbf{S}_{\mathbf{\Delta}} := \{\Delta \in \mathbf{S} : \Delta(s_o) \in \mathbf{\Delta} \text{ for all } s_o \in \overline{\mathbb{C}}_+\}$$

Theorem 2.2 [5] *Let $\beta > 0$. The loop shown in Figure 2-5 is well-posed and internally stable for all $\Delta \in \mathbf{S}_{\mathbf{\Delta}}$ with $\|\Delta\|_\infty < \frac{1}{\beta}$ if and only if*

$$\|M\|_\mu := \sup_{\omega \in \mathbb{R}} \mu_\Delta(M(j\omega)) \leq \beta$$

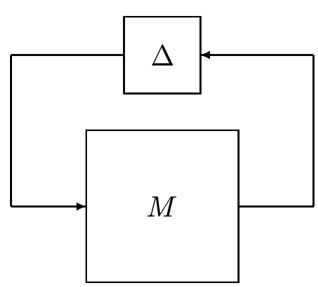


Figure 2-5: Robust stability

Assume that M is a stable, real-rational, proper transfer function, with $n_w + n_d$ inputs, and $n_z + n_e$ outputs. Partition M in the obvious manner so that M_{11} has n_w inputs and n_z outputs, and so on. Let $\Delta \subset \mathbb{C}^{n_w \times n_z}$ be a block structure and define an augmented block structure

$$\Delta_{RP} := \left\{ \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_F \end{bmatrix} : \Delta \in \mathbf{S_\Delta},\ \Delta_F \in \mathbb{C}^{n_d \times n_e} \right\}$$

By the setup of the extended uncertainty block Δ_{RP} we can reformulate the performance criteria as a robust stability problem like the previous theorem. The robust performance problem could be addressed as the loop shown Figure 2-6.

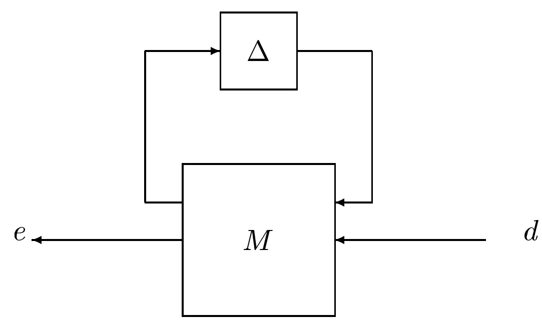


Figure 2-6: Robust performance

The perturbed transfer function from d to e is denoted by $\mathcal{F}_u(M, \Delta)$. Good performance means the quantity of $\mathcal{F}_u(M, \Delta)$ is small.

Theorem 2.3 [5] *Let $\beta < 0$. For all $\Delta(s) \in \mathbf{S_\Delta}$ with $\|\Delta\|_\infty < \frac{1}{\beta}$, the loop shown in Figure 2-6 is well-posed, internally stable, and $\|\mathcal{F}_u(M, \Delta)\|_\infty \leq \beta$ if and only if*

$$\|M\|_\mu := \sup_{\omega \in \mathbb{R}} \mu_{\Delta_{RP}}(M(j\omega)) \leq \beta.$$

This theorem says that the robust performance problem is equivalent to a robust stability problem with an augmented uncertainty Δ_{RP}, as shown in the following Figure 2-7.

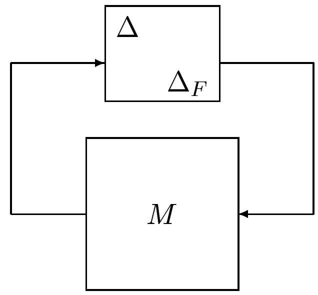


Figure 2-7: Robust performance reduced into robust stability

From Theorem 2.1 and Theorem 2.2, to achieve robust stability or robust performance is mathematically equivalent to this kind of problem

$$\|M\|_\mu := \sup_{\omega\in\mathbb{R}} \mu_\Delta(M(j\omega)) \leq 1. \tag{2-2}$$

Note that $\|\cdot\|_\mu$ is not a norm since the triangle inequality is not satisfied.

μ-synthesis procedure provides an effective method for controller design. Let $M = \mathcal{F}_\ell(P, K)$, P is the nominal plant and K is the controller. The problem is to find a stable controller $K(s)$ such that the closed-loop transfer function is internally stable and its infinity norm is less than one. That is, finding K satisfies inequality(2-2). Recently, the most popular approach for solving the problem of μ-synthesis is minimizing $\inf_{D\in\mathcal{D}} \bar{\sigma}(DMD^{-1})$ for either K or D while holding the other constant. This is the so called D-K iteration . As shown in Figure 2-8, solve

$$\min_K \inf_{D,D^{-1}\in\mathcal{H}_\infty} \|D\mathcal{F}_\ell(P,K)D^{-1}\|_\infty \tag{2-3}$$

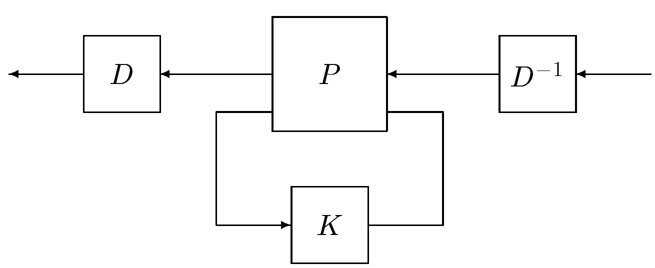


Figure 2-8: μ-synthesis scaling: D-K iteration

D-K iteration proceed by performing the two-parameter minimization in sequential fashion: first minimizing over K with D fixed, then minimizing pointwise over D with K fixed, then again over K, and again over D, etc. When D is fixed, the controller synthesis is a $\mathcal{H}_\infty$ optimization problem and can be solved using the well-known state-space method. That is, finding K such that $\|DMD^{-1}\|_\infty$ is a minimum. With fixed K, (2-3) can be minimized at each frequency as a convex optimization in $\ln(D)$. That is, finding D which minimizes $\bar{\sigma}(D(j\omega)M(j\omega)D^{-1}(j\omega))$ in the frequency domain. For more details of D-K iteration, see [5, 20], [5] also provides a graphic user interface to D-K iteration for μ-synthesis.

Chapter 3

Nevanlinna-Pick Interpolation Theory

As we know, the generalized control design problem could be modified to the following mathematical problem

$$\|M\|_\alpha \leq 1, \qquad \alpha = \infty,\ \mu, \text{or } s.$$

for certain complex matrix M. The case $\alpha = \infty$ was widely studied in 1980' and this could be solved by Nevanlinna-Pick (NP) interpolation theory which is one of the main theory in $\mathcal{H}_\infty$ control [14, 16]. To solve the case $\alpha = \mu$, we analyze this problem from the lower bound of μ-norm, i.e., $\|M\|_s$. Thus we survey the spectral norm and the spectral Nevanlinna-Pick interpolation problem.

3.1 Classical NP Problems

3.1.1 Scalar Case

The standard NP problem is stated as follows:

Problem 3.1 *Let* $\{a_1, a_2, \ldots, a_n\}$ *be a set of distinct points in the unit disk* $\mathbb{D}$ *and* $\{b_1, b_2, \ldots, b_n\}$ *in the closed unit disk* $\overline{\mathbb{D}}$ *, i.e.,* $|a_i| < 1$*,* $|b_i| \leq 1$ *for* $i = 1, \ldots, n$*. The problem is to find an analytic function* $\varphi \in \mathcal{RH}_\infty$ *such that the following two conditions are satisfied:*

(1) $\varphi(a_i) = b_i, \quad i = 1, 2, \ldots, n.$

(2) $\|\varphi\|_\infty \leq 1$

We call $\{a_1, , \ldots, a_n, b_1, \ldots, b_n\}$ the data set of NP problem in latter text. Given $\xi_n = \{\varphi \in \mathcal{RH}_\infty | \varphi(a_i) = b_i, \ i = 1, 2, \ldots, n.\}$. Define a $n \times n$ matrix P_n with its (i, j) element given by

$$(P_n)_{ij} = \frac{1 - b_i \bar{b_j}}{1 - a_i \bar{a}_j}$$

P_n is called the *Pick matrix*. Sufficient and necessary condition for the solvability of NP problem is stated as follows:

Theorem 3.1 [4] $\xi_n \neq \emptyset$ *if and only if the corresponding Pick matrix* P_n *is semi-positive definite.*

Proof see [4].

The following definition of functions will be used in the solution of NP problem.

Definition 3.1 *A Möbius transformation is any function of the form* $\frac{az+b}{cz+d}$ *with the restriction that* $ad \neq bc$*.*

Note that the Möbius transformation maps the unit disk onto the unit disk. If let the *Möbius function* $\mathcal{M}_\beta : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ with $|\beta| < 1$, and

$$\mathcal{M}_\beta(z) = \frac{z - \beta}{1 - \bar{\beta} z}$$

then $\mathcal{M}_\beta(z)$ is a Möbius transformation and is analytic in $\mathbb{D}$. Moreover,

$$\mathcal{M}_\beta^{-1}(z) = \frac{z + \beta}{1 + \bar{\beta} z} = \mathcal{M}_{-\beta}(z)$$

is also a Möbius transformation.

Lemma 3.1 *Given data set* $\{a_1, b_1\}$ *and* $a_1 \in \mathbb{D},\ b_1 \in \overline{\mathbb{D}}$. *The set of all solutions of NP problem is*

$$\xi_1 = \{\varphi : \varphi(z) = \mathcal{M}_{-b_1}[\mathcal{M}_{a_1}(z)g]\}$$

where g *is a stable, rational, proper function with* $\|g\|_\infty \leq 1$.

Proof Since $\mathcal{M}_{a_1}(a_1)$=0 and $\mathcal{M}_{-b_1}(0)$=b_1

$$\varphi(a_1) = \mathcal{M}_{-b_1}[\mathcal{M}_{a_1}(a_1)] = \mathcal{M}_{-b_1}(0) = b_1.$$

For any stable, proper function g with $\|g\|_\infty < 1$,

$$\|\mathcal{M}_{-b_1}[\mathcal{M}_{a_1}(z)g]\|_\infty \leq 1$$

Hence complete the proof. ♯

For n data points, reduce this to $n-1$ points as $\{a_2, \ldots, a_n, b'_2, \ldots, b'_n\}$ where

$$b'_i = \mathcal{M}_{b_1}(b_i)/\mathcal{M}_{a_1}(a_i) \quad for\ i = 2, \ldots, n.$$

We call this reduced $n-1$ case the NP′ problem. Thus by reduction, the solution is as follows.

Lemma 3.2 *Given data set* $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ *and* $a_i \in \mathbb{D},\ , b_i \in \overline{\mathbb{D}}$ *for* $1 \leq i \leq n$. *The set of all solutions of NP problem is*

$$\xi_n = \{\varphi : \varphi(z) = \mathcal{M}_{-b_1}[\mathcal{M}_{a_1}(z)\psi(z)]\}$$

where $\psi(z)$ *is the solution set of the corresponding NP′ problem for data set* $\{a_2, \ldots, a_n, b'_2, \ldots, b'_n\}$.

If the NP problem is given that a_i are distinct points of right half plane(RHP) for $i = 1, \ldots, n$. The corresponding Pick matrix will be

$$(P_n)_{ij} = \frac{1 - b_i\bar{b_j}}{a_i + \bar{a}_j}$$

We will need the following function to solve the problem.

Definition 3.2 *An all-pass function is a function with its magnitude equals to 1 at all points on the imaginary axis.*

If let $A_a(s) = \dfrac{s-a}{s+\bar{a}}$ with Re $a > 0$ (a is in the RHP), then $A_a(s)$ maps the points in RHP to the unit disk $\mathbb{D}$ and $A_a(s)$ is an all-pass function. Now we consider the solution of NP problem. First consider the case $n = 1$, the following lemma gives the solution.

Lemma 3.3 [11] *Given data set* $\{a_1, b_1\}$, $a_1 \in RHP$, $b_1 \in \overline{\mathbb{D}}$. *The set of all solutions of this problem is*

$$\xi_1 = \{\varphi : \varphi(s) = \mathcal{M}_{-b_1}[A_{a_1}(s)g(s)]\}$$

where g *is a stable, rational, proper function with* $\|g\|_\infty \leq 1$.

If there are n data points (a_i, b_i) $i = 1, \ldots, n$ and the problem is solvable, we reduce it to the case of $n-1$ data points. We call the reduced problem the NP$'$ problem with the data set $\{a_2, \ldots, a_n, b'_2, \ldots, b'_n\}$ where

$$b'_i := \mathcal{M}_{b_1}(b_i)/A_{a_1}(a_i) \quad i = 2, \ldots, n.$$

Lemma 3.4 [11] *Given data set* $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ *and* $a_i \in RHP$, $b_i \in \bar{\mathbb{D}}$ *for* $1 \leq i \leq n$. *The set of all solutions of NP problem is*

$$\xi_n = \{\varphi : \varphi(s) = \mathcal{M}_{-b_1}[\psi(z)g]\}$$

where ψ *is the set of all solutions of the NP$'$ problem for the data set* $\{a_2, \ldots, a_n, b'_2, \ldots, b'_n\}$ *and* g *is a stable,rational,proper function with* $\|g\|_\infty \leq 1$.

The NP theory could be used to solve the *model matching problem* : Given stable, proper functions $T_1,\ T_2,\ T_3$, find the stable Q such that

$$\|T_1 - T_2 Q T_3\|_\infty \leq 1.$$

We discuss the simpler case, $|T_1 - T_2 Q\|_\infty \leq 1$. Let $\varphi := T_1 - T_2 Q$, if the right half plane zeros of T_2 are z_i, we have $\varphi(z_i) = T_1(z_i)$. Thus the problem becomes a standard NP interpolation problem . Given z_i and $T_1(z_i)$, find the mapping Q such that $\varphi(z_i) = T_1(z_i)$ and $\|\varphi\|_\infty \leq 1$. Solve P by NP theory and hence $Q = \frac{T_1 - \varphi}{T_2}$. Note that Q is stable since it has no RHP pole. If T_2 has no RHP zero , we could simply let $Q = \frac{T_1 - g}{T_2}$ where g is arbitrary stable, proper function with $\|g\| \leq 1$. For more details of the model matching problem, see [24].

3.1.2 Matrix Case

Now consider the matrix NP problem:

Problem 3.2 *Let* $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{D}$, $W_1, W_2, \ldots, W_n$ *be* $m \times m$ *complex matrices with* $\|W_i\|_\infty \leq 1, \quad i = 1, 2, \ldots, n$. *We want to find a matrix valued function* φ *such that*
(1)$\Phi(\lambda_i) = W_i$
(2)$\|\Phi\|_\infty \leq 1$

Similarly, the solvable condition of this problem is dependent on the *Pick matrix.*

Theorem 3.2 [4, 22] *Define a partitioned matrix*

$$P = \begin{bmatrix} P_{11} & \cdots\cdots & P_{1n} \\ P_{21} & \cdots\cdots & P_{2n} \\ \vdots & \cdots\cdots & \vdots \\ P_{n1} & \cdots\cdots & P_{nn} \end{bmatrix}$$

with

$$P_{ij} = \frac{1}{1 - \lambda_i \bar{\lambda}_j}(I - W_i W_j^*) \quad , i, j = 1, 2, \ldots, n.$$

The matrix NP problem is solvable if and only if $P \geq 0$.

We give a brief description of the solution to Problem 3.2. Suppose that $E \in \mathbb{C}^{n\times n}$, $\|E\| < 1$ and

$$A = (I - EE^*)^{-\frac{1}{2}} \qquad B = -(I - EE^*)^{-\frac{1}{2}}E$$
$$C = -(I - E^*E)^{-\frac{1}{2}}E^* \qquad D = (I - E^*E)^{-\frac{1}{2}}$$

Definition 3.3 *Let $x \in \mathbb{C}^{n\times n}$, define*

$$T_E(x) = (Ax + B)(Cx + D)^{-1}$$

There is an important properties associated with T_E[22]:

$$\|T_E(x)\| \leq 1 \text{ if and only if } \|x\| \leq 1 \tag{3-1}$$
$$T_E(E) = 0 \tag{3-2}$$

By definition, T_E can be rewritten as

$$T_E(x) = (I - EE^*)^{-1/2}(x - E)(I - E^*x)^{-1}(I - E^*E)^{1/2}$$

thus we have

$$T_E(x) = W \text{ if and only if } x = (A - WC)^{-1}(WD - B)$$

For the matrix NP problem, let

$$T_i = T_{W_1}(W_i) = (AW_i + B)(CW_i + D)^{-1}, \quad i = 1, 2, \dots, n$$

where

$$A_1 = (I - W_1W_1^*)^{-\frac{1}{2}} \qquad B_1 = -(I - W_1W_1^*)^{-\frac{1}{2}}W_1$$
$$C_1 = -(I - W_1^*W_1)^{-\frac{1}{2}}W_1^* \qquad D_1 = (I - W_1^*W_1)^{-\frac{1}{2}}$$

and let

$$y_1(z) = \frac{|\lambda_1|}{\lambda_1}\frac{z - \lambda_1}{1 - \bar{\lambda}_1 z}, \quad G_i = \frac{1}{y(\lambda_i)}T_i, \qquad i = 2, 3, \dots, n.$$

If the function ψ_1 is the solution of the matrix NP problem with data set $\{\lambda_2,\ \lambda_3, \dots, \lambda_n\}$, i.e.,

$$\begin{aligned} \psi_1(\lambda_i) &= G_i, \quad i = 1, 2, \dots, n. \\ \|\psi_1\|_\infty &\le 1 \end{aligned}$$

Then the solution for Problem 3.2 is given by

$$\Phi(z) = (A_1 - y_1(z)\psi_1(z)C_1)^{-1}(y_1(z)\psi_1(z)D_1 - B_1)$$

When $n = 1$, we choose ψ_1 to be any compatible matrix G such that $\|G\|_\infty \le 1$, then

$$\Phi(z) = (A_1 - y_1(z)GC_1)^{-1}(y_1(z)GD_1 - B_1)$$

with

$$\|\Phi(z)\|_\infty \le 1, \qquad \Phi(\lambda_1) = -A_1^{-1}B_1 = W_1.$$

When $n = 2$, let

$$y_1(z) = \frac{|\lambda_1|}{\lambda_1}\frac{z - \lambda_1}{1 - \bar{\lambda} z}, \qquad y_2(z) = \frac{|\lambda_2|}{\lambda_2}\frac{z - \lambda_2}{1 - \bar{\lambda}_2 z}, \qquad T_2 = T_{W_1}(W_2), \qquad G_2 = \frac{1}{y(\lambda_2)T_2}.$$

We want to find ψ_1 such that $\psi_1(\lambda_2) = G_2$ and $\|\psi_1\|_\infty < 1$.Thus let

$$\begin{aligned} A_2 &= (I - G_2G_2^*)^{-\frac{1}{2}},\ B_2 = -(I - G_2G_2^*)^{-\frac{1}{2}}G_2, \\ C_2 &= -(I - G_2^*G_2)^{-\frac{1}{2}}G_2^*,\ D_2 = (I - G_2^*G_2) \end{aligned}$$

Let G be any compatible matrix such that $\|G\|_\infty \le 1$. Then

$$\psi_1(z) = (A_2 - y_2(z)GC_2)^{-1}(y_2(z)GD_2 - B_2)$$

i.e.

$$\psi_1(\lambda_2) = G_2, \quad \|\psi_1\|_\infty \le 1.$$

Therefore

$$\Phi(z) = (A_1 - y_1(z)\psi_1(z)C_1)^{-1}(y_1(z)\psi_1(z)D_1 - B_1)$$

is our solution.

What we discussed here are only the main types of NP problem. There are still rich contents in classical NP interpolation problem and the interested reader could refer to [4, 22].

3.2 Spectral Nevanlinna-Pick Problem

The spectral NP problem is likely to be an approach to compute μ-norm. The following results are working by Alger, Yeh and Young [3, 18]. We will use these results to design a controller with μ synthesis. This problem is similar with the classical NP problem but replaced with the spectral norm $\|\cdot\|_s$. That is, given points λ_i in open unit disk and W_i be $n\times n$ matrices for $i=1,\dots,n$. We want to find the analytic matrix function F such that $F(\lambda_i)=W_i$ for $i=1,\dots,n$ and $\rho(F(\lambda))\le 1$ for all $\lambda\in\mathbb{D}$. Note that the scalar case of SNP problem is equivalent to classical NP problem since for a scalar a, $\rho(a)=|a|$. Now we consider the problem :

Problem3.3 *Let $\lambda_1,\ \lambda_2,\dots,\ ,\lambda_n$ be distinct points in open unit disk $\mathbb{D}$, $W_1,\ W_2,\dots,W_n\in\mathbb{C}^{n\times n}$. Find an analytic $n\times n$ matrix function F on $\mathbb{D}$ such that*
(1) $\rho(F(\lambda))\le 1$ for all $\lambda\in\mathbb{D}$
(2) $F(\lambda_i)=W_i,\quad i=1,2,\dots,n.$

Note that $\rho(\cdot)$ denote the spectral radius of a matrix. This is a complicated problem since the *unit spectral ball*

$$\Sigma_n \triangleq \{W\in\mathbb{C}^{n\times n}:\rho(W)\le 1\}$$

is not convex, not smooth and unbounded. There is an important theorem to check the existence of the solution of SNP problem.

Theorem 3.3 [9] *Given $\lambda_1,\dots,\lambda_n$ be distinct points in $\mathbb{D}$ and $W_1,\dots,W_n$ be $m\times m$ matrices. The matrix function F satisfies that $F(\lambda_i)=W_i$ and $\rho(F(\lambda))\le 1$ for all $\lambda\in\mathbb{D}$. Such a function F exists if and only if there exists invertible $m\times m$ matrices M_i for $i=1,\dots,n$ such that*

$$\left[\frac{I-M_iW_iM_i^{-1}(M_jW_jM_j^{-1})^*}{1-\lambda_i\bar{\lambda}_j}\right]_{i,j=1}^{n}\ge 0 \tag{3-3}$$

For the case $m=2$, Alger and Young[3] provided a simpler method to check the solvability.

Theorem 3.4 [3] *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$ and $W_1, \ldots, W_n$ be 2×2 matrices for some $n \in \mathbb{N}$, none of them are scalar matrices, that is, none of them are scalar multiplicity of identity . The following are equivalent:*

1. *there exists an analytic $2 \times 2-$matrix function F on $\mathbb{D}$ such that $F(\lambda_j) = W_j, j = 1, \ldots, n$ and $\rho(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$.*

2. *there exist $b_1, \ldots, b_n, c_1, \ldots, c_n \in C$ such that*

$$\left[\frac{I - \begin{bmatrix} \frac{1}{2}s_i & b_i \\ c_i & -\frac{1}{2}s_i \end{bmatrix}^* \begin{bmatrix} \frac{1}{2}s_j & b_j \\ c_j & -\frac{1}{2}s_j \end{bmatrix}}{1 - \bar{\lambda}_i \lambda_j} \right]_{i,j=1}^{n} \geq 0 \tag{3-4}$$

where

$$s_j = \mathrm{trace}(W_j), \quad p_j = \det(W_j)$$

and

$$b_j c_j = p_j - \frac{s_j^2}{4}, 1 \leq j \leq n.$$

Note that Theorem 3.3 and Theorem 3.4 are different with the dimension of data set matrices W_i.

How to tackle this problem? Consider $n = 2$ and $m = 2$, that is, two interpolation conditions and $W \in \mathbb{C}^{2\times 2}$. When W is not a scalar matrix, by Schur Theorem there exists a non-singular matrix T such that W is similar to the companion matrix E,

$$E = \begin{bmatrix} 0 & 1 \\ -p & s \end{bmatrix} = TWT^{-1}.$$

Too see the advantage of the companion matrix, look at the characteristic polynomial

$$\begin{aligned}\det(\lambda I - W) &= \det(\lambda I - TWT^{-1}) \\ &= \det(\lambda I - E) \\ &= \lambda^2 - s\lambda + p\end{aligned}$$

in which $s = \text{trace}(W),\ p = \det(W)$. This is the motivation of the following definition.

Definition 3.4 *The symmetrized bidisc* $\Gamma_2 \subset \mathbb{C}^2$ *is defined as:*

$$\Gamma_2 \triangleq \{(s,p) : \lambda^2 - s\lambda + p = 0,\ |\lambda| \leq 1\}.$$

$W \in \Sigma_2$ if and only if $(\text{trace}(W), \det(W) \in \Gamma_2$. Thus the SNP interpolation problem on Σ_2 could be modified to Γ_2 by the following theorem.

Theorem 3.5 [3] *Let* $\lambda_1, \ldots, \lambda_n$ *be distinct in* $\mathbb{D}$ *and* $W_1, \ldots, W_n$ *be* 2×2 *matrices. Suppose that none of* $W_1, \ldots, W_n$ *are scalar matrices. The following are equivalent:*

1. *there exists an analytic* 2×2 *matrix function* F *in* $\mathbb{D}$ *such that* $F(\lambda_j) = W_j,\ j = 1, 2, \ldots, n$ *and* $\rho(F(\lambda)) \leq 1$ *for all* $\lambda \in \mathbb{D}$.

2. *there exists an analytic function* $f : \mathbb{D} \longrightarrow \Gamma_2$ *such that* $f(\lambda_j) = (\text{trace}(W_j), \det(W_j)), j = 1, 2, \ldots, n$.

Thus we will concentrate on the modified problem when $n = 2$:

Problem3.4 Given $\lambda_1,\ \lambda_2 \in \mathbb{D}$ and $(s_1, p_1),\ (s_2, p_2) \in \Gamma_2$. Find the analytic function

$$\begin{aligned}\varphi \ : \ & \mathbb{D} \longrightarrow \Gamma_2 \\ & \lambda \mapsto (s(\lambda), p(\lambda))\end{aligned}$$

such that $\varphi(\lambda_i) = (s_i, p_i), i = 1, 2$.

The following lemma is useful.

Lemma 3.5 [18] *For $s, p \in \mathbb{C}$, the following statements are equivalent:*

1. $(s,p) \in \Gamma_2$.

2. $|s - \bar{s}p| \le 1 - |p|^2$, *and* $|s| \le 2$.

3. $2|s - \bar{s}p| + |s^2 - 4p| \le 4 - |s|^2$, *and* $|p| \le 1$.

Before discussing the problem, we introduce some definitions of distance.

Definition 3.5 *For $z, w \in \mathbb{D}$, define the Poincaré distance d as*

$$d(z,w) = \frac{z - w}{1 - \bar{w}z}$$

Definition 3.6 *For $\Omega \subset \mathbb{C}^2$ and $z_1, z_2 \in \Omega$, the Carathéodory distance C_Ω on Ω is defined as*

$$C_\Omega(z_1, z_2) = \sup_G d(G(z_1), G(z_2))$$

where G is the set of all analytic function from Ω to $\mathbb{D}$.

From [1], when $\Omega =$Int Γ_2 and $|\omega| = 1$, that is, $\omega \in \mathbb{T}$, the function $G_\omega : \text{Int } \Gamma_2 \longrightarrow \mathbb{D}$

$$G_\omega(s,p) = \frac{2p - \omega s}{2 - \bar{\omega}s}$$

is the one which achieves the sup in Definition 3.6.

Definition 3.7 *If $|\omega_0| = 1$ and*

$$C_{\text{Int } \Gamma_2}((s_1,p_1),(s_2,p_2)) = d(G_{\omega_0}(s_1,p_1), G_{\omega_0}(s_2,p_2)),$$

$G_{\omega_0}(s,p)$ is called the C-extremal function.

Definition 3.8 *Let (s_1,p_1), $(s_2,p_2) \in \text{Int } \Gamma_2$ and $\varphi : \mathbb{D} \longrightarrow \text{Int } \Gamma_2$ the analytic function such that*

$$\begin{aligned} \varphi(\lambda_1) &= (s_1,p_1) \\ \varphi(\lambda_2) &= (s_2,p_2) \end{aligned}$$

The Kobayashi distance $K_{\text{Int }\Gamma_2}$ *is defined as*

$$K_{\text{Int }\Gamma_2}((s_1,p_1),(s_2,p_2)) = \inf_{\varphi} d(\lambda_1,\lambda_2)$$

Definition 3.9 *If* $C_{\text{Int }\Gamma_2} = K_{\text{Int }\Gamma_2}$ *and* $\varphi : \mathbb{D} \longrightarrow \text{Int }\Gamma_2$ *achieves the* inf *in Definition 3.8,* φ *is called the* K*-extremal function.*

Since the solution for Problem 3.4 is not available, we consider the simpler cases:

Problem3.5 Given $\lambda_1,\ \lambda_2 \in \mathbb{D}$ and $(s_1,0),\ (s_2,0) \in \Gamma_2$. Find the analytic function

$$\begin{aligned} \varphi \ &: \ \mathbb{D} \longrightarrow \Gamma_2 \\ &\quad \lambda \mapsto (s(\lambda),p(\lambda)) \end{aligned}$$

such that

$$\varphi(\lambda_i) = (s_i,0), \qquad i = 1,2.$$

Prblem3.6 Given $\lambda_2 \in \mathbb{D}$, $\lambda_2 \neq 0$ and $(s_2,p_2) \in \Gamma_2$, find an analytic function

$$\begin{aligned} \varphi \ &: \ \mathbb{D} \longrightarrow \Gamma_2 \\ &\quad \lambda \mapsto (s(\lambda),p(\lambda)) \end{aligned}$$

such that

$$\varphi(0) = (0,0), \qquad \varphi(\lambda_2) = (s_2,p_2)$$

What we want to find is all the K-extremal function φ, that is, all the analytic functions φ with the Carathéodory distance equal to the Kobayashi distance. For Problem 3.5, consider the following theorem.

Theorem 3.6 [18] *If* $C_{\text{Int }\Gamma_2}((s_1,0),(s_2,0)) = K_{\text{Int }\Gamma_2}((s_1,0),(s_2,0))$ *and the* C*-extremal function,* K*-extremal function are* $G_\omega : \text{Int }\Gamma_2 \longrightarrow \mathbb{D}$, $\varphi : \mathbb{D} \longrightarrow \text{Int }\Gamma_2$ *respectively. Then* $G_{\omega_0} \circ \varphi : \mathbb{D} \longrightarrow \mathbb{D}$ *is an analytic function and*

1. φ *is a isometry transformation of* $(\mathbb{D}, d) \longrightarrow (\varphi(\mathbb{D}), C_{\mathrm{Int}\ \Gamma_2})$.

2. $G_{\omega_0} \circ \varphi = Id_{\mathbb{D}}$ *(up to Möbius transformation).*

Proof see [18].

Let $\beta_i \triangleq G_{\omega_0}(s_i, 0) = \dfrac{-\omega_0 s_i}{2 - \bar{\omega}_0 s_i}$ and consider

$$h(\omega, \cdot) \triangleq G_{\omega_0} \circ \varphi : \mathbb{D} \longrightarrow \mathbb{D}$$

$h(\omega, \lambda_i) = \beta_i,\ i = 1, 2$. And h is analytic since G_ω and φ are both analytic. Since

$$\varphi(\lambda) \in \Gamma_2,\ \forall |\lambda| < 1 \Longleftrightarrow |h(\omega, \lambda)| \leq 1,\ \forall |\omega| \leq 1,\ |\lambda| \leq 1, \tag{3-5}$$

with Theorem 3.6 we know that $C_{\mathrm{Int}\ \Gamma_2}((s_1, 0), (s_2, 0)) = K_{\mathrm{Int}\ \Gamma_2}((s_1, 0), (s_2, 0))$ is the sufficient condition for our solution φ. For the C-extremal function G_{ω_0}, ω_0 satisfies the following equation which obtained directly from the definition

$$\min_{|\omega|=1} |2 - \bar{\omega} s_1 - \omega \bar{s}_2| = |2 - \bar{\omega}_0 s_1 - \omega_0 \bar{s}_2| \tag{3-6}$$

Now we try to construct φ. For simplicity, let $\lambda_i = \beta_i,\ i = 1, 2$, that is, $h(\omega_0, \lambda) := G_{\omega_0} \circ \varphi(\lambda) = \lambda$. Thus

$$G_{\omega_0}(s(\lambda), p(\lambda)) = \frac{2p(\lambda) - \omega_0 s(\lambda)}{2 - \bar{\omega}_0 s(\lambda)} = \lambda$$

and

$$s(\lambda) = 2\frac{p(\lambda) - \lambda}{\omega_0 - \bar{\omega}_0 \lambda}$$

Since $s(\lambda)$ need to be analytic, $p(\lambda)$ have to satisfy

$$p(\omega_0^2) = \omega_0^2$$

Moreover, $p(\lambda)$ satisfies

$$p(\lambda_1) = p(\lambda_2) = 0$$

$$\|p\|_\infty \le 1$$

Note that ω_0 must satisfies equation (3-6).

The solution is given as follows:

Theorem 3.7 [18] *If* $C_{\text{Int}\Gamma_2}((s_1,0),(s_2,0)) = K_{\text{Int}\Gamma_2}((s_1,0),(s_2,0))$, *there exists* $\omega_0 \in \mathbb{T}$, $(t,q) \in \text{Int}\Gamma_2$ *such that* $\omega_0\bar{t} - \omega^2\bar{q} \ge |q|$. *And*

$$p(\lambda) = \bar{\omega}_0^2 \frac{a\lambda^2 + b\lambda^2 + c}{\bar{c}\lambda^2 + \bar{b}\lambda + \bar{a}} \tag{3-7}$$

$$s(\lambda) = 2\frac{p(\lambda) - \lambda}{\omega_0 - \bar{\omega}_0\lambda} \tag{3-8}$$

where $t = s_1 + s_2,\ q = s_1 s_2$ *and* $a = (4 - 2\bar{\omega}_0 t + \bar{\omega}_0^2 q),\ b = 2\omega_0(t - \bar{\omega}_0 q),\ c = \omega_0^2 q$. *Then for all* $\lambda \in \mathbb{D}$,

$$\varphi(\lambda) = (s(\lambda), p(\lambda)) \in \text{Int}\Gamma_2$$

is an analytic function from $\mathbb{D}$ *to* $\text{Int}\Gamma_2$ *and satisfies*

$$\varphi(\lambda_1) = (s_1, 0)$$

$$\varphi(\lambda_2) = (s_2, 0)$$

Proof see[18].

We should realize that this solution satisfies only $\varphi(\beta_i) = (s_i, 0)$ since we have supposed that $h(\omega_0, \lambda) = \lambda$, i.e., $\lambda_i = \beta_i$. But the original interpolation condition is $\varphi(\lambda_i) = (s_i, 0)$. Practically $\lambda_i = \beta_i$ has a little chance to happen. Thus we have to do some work to match the interpolation condition by utilizing Möbius transformation. In next chapter (3-6) and (3-7) are used to carry out our design.

Similarly, for Problem 3.6 the result is in the following contents.

Lemma 3.6 [17] *Let* $\omega_0 \in \mathbb{T}$, *the unit circle, and* ω_0 *satisfies*

$$\left|\frac{2p_2 - \omega_0 s_2}{2 - \bar{\omega} s_2}\right| = \sup_{|\omega|=1} \left|\frac{2p_2 - \omega s_2}{2 - \bar{\omega} s_2}\right|$$

i.e., $C_{\text{Int }\Gamma_2} = K_{\text{Int }\Gamma_2}$, *then*

$$\left|\frac{2p_2 - \omega_0 s_2}{2 - \bar{\omega}_0 s_2}\right| = \frac{|4p_2 - s_2^2| + 2|s_2 - \bar{s}_2 p_2|}{4 - |s_2|^2}.$$

Let $\beta_1 = 0,\ \beta_2 \triangleq G_{\omega_0}(s_2, p_2) = \dfrac{2p_2 - \omega_0 s_2}{2 - \bar{\omega_0} s_2}$.

Theorem 3.8 [17] *Let* $\lambda_2 \in \mathbb{D},\ \ \lambda_2 \neq 0,\ (s_2,\ p_2) \in \text{Int }\Gamma_2$. *If there exists* $\omega_0 \in \mathbb{T}$ *which satisfies*

$$\left|\frac{2p_2 - \omega_0 s_2}{2 - \bar{\omega}_0 s_2}\right| = \frac{|4p_2 - s_2^2| + 2|s_2 - \bar{s}_2 p_2|}{4 - |s_2|^2}$$

and if

$$\lambda_2 = \beta_2 = \frac{2p_2 - \omega_0 s_2}{2 - \bar{\omega_0} s_2}.$$

Then there exists an analytic function $\varphi : \mathbb{D} \longrightarrow \Gamma_2$ *such that*

$$\varphi(0) = (0, 0), \quad \varphi(\lambda_2) = (s_2, p_2),$$

$\varphi(\lambda) = (s(\lambda), p(\lambda))$ *and*

$$s(\lambda) = \frac{2\alpha\lambda}{1 - \bar{\beta}\lambda} \tag{3-9}$$

$$p(\lambda) = \frac{\gamma\lambda(\lambda - \beta)}{1 - \bar{\beta}\lambda}. \tag{3-10}$$

where $\alpha,\ \beta,\ \gamma \in \mathbb{C}$, *and*

$$\alpha = \frac{\bar{\omega}k}{\bar{a}},\ \beta = \frac{-b}{a},\ \gamma = \frac{\bar{\omega}_0^2}{\bar{a}},\ k = \omega_0^2\bar{b} - a \in \mathbb{R},\ |k| = |a| - |b|$$

with

$$a = 1 - \omega_0^2 p_0 \bar{\alpha}_2,\ b = \omega_0^2 p_0 - \alpha_2,\ p_0 = \frac{p_2}{\alpha_2}.$$

Proof see [17].

Note that $a + b\bar{\omega}_0^2 \in \mathbb{R},\ k = \omega_0^2\bar{b} - a \in \mathbb{R}$, and $|k| = |a| - |b|$.

Chapter 4

Lower Bound Design for μ-synthesis

Unlike D-K iteration solves the μ-synthesis problem from the upper bound, the lower bound design method is presented in this chapter. The robust stability and performance problems are formulated as various types of Spectral Nevalinna-Pick problem. Till now, only two-dimensional problem can be solved. The SNP theory is applied to construct the robust controller. Some numerical examples are used to illustrate the solution process.

4.1 Problem Formulation

Consider a general uncertain system given in Figure 4-1 where Δ denotes the perturbations modelled as norm-bounded dynamical uncertainty or parametric uncertainty. The dynamical uncertainty is assumed as in Chapter 2,

$$\Delta(s) = \{\text{diag}[\delta_1 I_{r1}, \cdots, \delta_s I_{rs}, \Delta_1, \cdots, \Delta_F] : \quad \delta_i(s) \in \mathcal{RH}_\infty \ , \Delta_j(s) \in \mathcal{RH}_\infty\}$$

and $\|\delta_i\|_\infty < 1, \ \|\Delta_j\|_\infty < 1$.

Suppose P is the generalized plant, i.e.,

$$\begin{bmatrix} z \\ e \\ \hdashline y \end{bmatrix} = P \begin{bmatrix} w \\ d \\ \hdashline u \end{bmatrix}$$

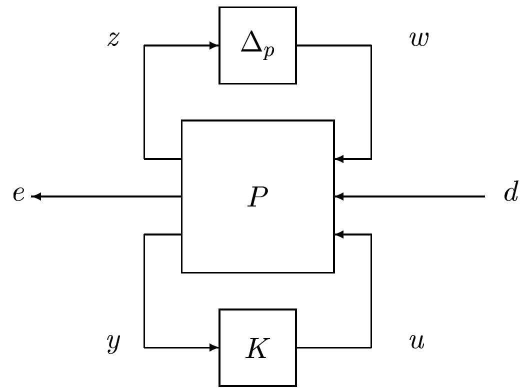


Figure 4-1: Generalized plant

with the partition as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

The robust stability problem is to design a controller K such that the system is stable under the influence of the uncertainty Δ_p when there are no signals d and e presented. Let the corresponding plant be

$$\begin{bmatrix} z \\ \hdashline y \end{bmatrix} = P_s \begin{bmatrix} w \\ \hdashline u \end{bmatrix}, \quad P_s = \begin{bmatrix} P_{s11} & P_{s12} \\ P_{s21} & P_{s22} \end{bmatrix}$$

Find K such that

$$\begin{aligned} \|\mathcal{F}_\ell(P_s, K)\|_\mu &= \sup_{\omega \in \mathbb{R}} \mu_{\Delta_p}(\mathcal{F}_\ell(P_s, K))(j\omega) \\ &= \sup_{\omega \in \mathbb{R}} \mu_{\Delta_p}(P_{s11} + P_{s12}K(I - P_{s22}K)^{-1}P_{s21})(j\omega) \\ &< 1. \end{aligned} \tag{4-1}$$

Let $Q = K(I - P_{s22}K)^{-1}$, $P_{s11} := T_{s1}$, $P_{s12} := T_{s2}$, $P_{s21} := T_{s3}$, the robust stability problem could be reduced to the form: find Q such that

$$\sup_{\omega \in \mathbb{R}} \mu_{\Delta_p}(T_{s1} + T_{s2}QT_{s3})(j\omega) \leq 1. \tag{4-2}$$

and hence $K = (I + QP_{s22})^{-1}Q$.

Robust performance problem is to design a controller K such that

$$\begin{aligned}\|\mathcal{F}_\ell(P,K)\|_\mu &:= \sup_{\omega\in\mathbb{R}} \mu_\Delta(\mathcal{F}_\ell(P,K)))(j\omega) \\ &= P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \\ &< 1. \end{aligned} \tag{4-3}$$

Note that the uncertainty set here is $\Delta = \{\text{diag}(\Delta_P,\ \Delta_F)\}$ where Δ_F is the fictitious block and if $H := \mathcal{F}_\ell(P,K) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$, $\mathcal{F}_\ell(P_s,K) = H_{11}$. Similarly the inequality (4-3) can also be written as the form of (4-2), hence the robust stability and performance problem can be reduced to: find Q such that

$$\sup_{\omega\in\mathbb{R}} \mu_\Delta(T_1 + T_2QT_3)(j\omega) \le 1.$$

and hence

$$K = (I + QP_{22})^{-1}Q.$$

Recall that

$$\rho(M) \le \max_{U\in\mathcal{U}} \rho(UM) \le \mu_\Delta(M) \le \inf_{D\in\mathcal{D}} \bar{\sigma}(DMD^{-1}) \le \bar{\sigma}(M). \tag{4-4}$$

We want to use this inequality to solve this problem from the lower bound approach. Thus we utilize spectral interpolation theory to solve the *spectral model matching problem*

$$\rho(T_1 + T_2QT_3) \equiv \rho(F) \le 1 \tag{4-5}$$

and hence design our controller. The interpolation condition may be constructed from the following statements. If p_i are poles of T_2 or T_3 in RHP, $F(p_i) = T_1(p_i)$ and $Q(p_i) = 0$. If z_i are zeros of T_2 or T_3 in RHP, $F(z_i) = T_1(z_i)$.

We summarize the problem in Table 4-1:

Classification	Robust stability	Robust performance
μ test: $\|M\|_\mu \leq 1$	$M = H_{11}$	$M = H$
Q parameterization:	$T_1 = P_{s11}$	$T_1 = P_{11}$
$\rho(F(j\omega)) \equiv \rho(T_1 + T_2QT_3)(j\omega)$	$T_2 = P_{s12}$	$T_2 = P_{12}$
$\leq \sup_{\omega \in R} \mu_\Delta(T_1 + T_2QT_3)(j\omega)$	$T_3 = P_{s21}$	$T_3 = P_{21}$
$\leq 1.$	$Q = K(I - P_{s22}K)^{-1}$	$Q = K(I - P_{22}K)^{-1}$
SNP interpolation condition:		
p_i RHP poles of T_2, T_3	$F(p_i) = T_1(p_i)$ and $Q(p_i) = 0$	
z_i RHP zeros of T_2, T_3	$F(z_i) = T_1(z_i)$	

Table 4-1: Problem formulation

4.2 Algorithms for Controller Synthesis via SNP Theory

Before the design procedure, we state the algorithm for the two by two spectral Nevanlinna-Pick interpolation problem, Problem 3.5. For given points $\lambda_1,\ \lambda_2$ in the open unit disk and 2 by 2 matrices $W_1,\ W_2$ with $\rho(W_i) \leq 1$, we will find the mapping F such that $F(\lambda_i) = W_i,\ i = 1,2$ and $\rho(F(\lambda)) \leq 1,\ \forall \lambda \in \mathbb{D}$. We solve this problem as follows:

Algorithm SNP

1. Reduce the problem to the *symmetrized bidisc* Γ_2 . And the problem turns to find a mapping ϕ such that $\phi(\lambda_i) = (\text{trace}(W_i), \det(W_i)) = (s_i, p_i),\ i = 1, 2.$

2. Find $|\omega_0| = 1$ such that $\min_{|\omega|=1} |2 - \bar{\omega}s_1 - \omega\bar{s_2}| = |2 - \bar{\omega}_0 s_1 - \omega_0 \bar{s_2}|$ and let $\beta_i = \frac{-\omega_0 s_i}{2 - \bar{\omega}_0 s_i},\ i = 1, 2.$

3. By Theorem 3.7, find the mapping $\varphi(\lambda) = (s'(\lambda), p'(\lambda))$, such that $\varphi(\beta_i) = (s_i, p_i),\ i = 1, 2.$

4. From Theorem 3.7, we could only find $\varphi(\lambda)$, but we want to find the mapping that maps λ_i

to (s_i, p_i). Thus find the Möbius function $\mathcal{M}_\beta$ which maps β_i to λ_i via classical NP theory, and then $\varphi \circ \mathcal{M}_{-\beta}(\lambda_i) = (s_i, p_i)$. Hence $\phi(\lambda) = (s'(\mathcal{M}_{-\beta}(\lambda)),\ p'(\mathcal{M}_{-\beta}(\lambda))) = (s(\lambda), p(\lambda))$ and $\phi(\lambda_i) = (s_i, p_i),\ i = 1, 2$. See Figure 4-2.

5. At last find the matrix function R which satisfies

$$W_i = R(\lambda_i) \begin{bmatrix} 0 & 1 \\ -p_i & s_i \end{bmatrix} R^{-1}(\lambda_i).$$

And hence

$$F(\lambda) = R(\lambda) \begin{bmatrix} 0 & 1 \\ -p(\lambda) & s(\lambda) \end{bmatrix} R^{-1}(\lambda)$$

will be the matrix function we want.

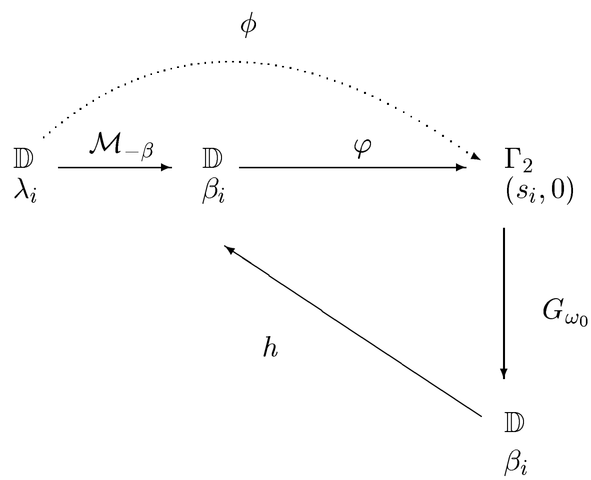


Figure 4-2: SNP problem solution procession

Note that there are infinitely many choice of F due to functions $\mathcal{M}_\beta$ and R for a fixed φ. Hence we have much freedom to adjust the function F in design procedure .

Now consider the system matrix $H := \mathcal{F}_\ell(P, K)$, if it could be rewritten to the model matching form $T_1 + T_2QT_3$ where $Q := K(I - P_{22}K)^{-1}$, we will design the controller K with the following algorithm:

Algorithm K

1. First find Q such that $\|T_1 + T_2QT_3\|_s < \gamma$ for a chosen $\gamma > 1$. We solve this with Spectral NP problem. Let $F := T_1 + T_2QT_3$. Suppose that $a_i,\ i = 1, \ldots, m$ and $b_i,\ i = 1, \ldots, \ell$ are the RHP zeros and poles of T_2 or T_3, since the algorithm for the interpolation problem is to find the mapping from unit disk to Γ_2, we have to transform the points in RHP to be in the unit disk. Let $A(a_i) = \frac{1+a_i}{1-a_i} = z_i$ and $A(b_j) = \frac{1+b_j}{1-b_j} = q_j$, this maps the RHP points $a_i,\ b_j$ to the unit disk point z_i, q_j. Transform all the functions $T_1(s),\ T_2(s)$ into unit disk by $s = \frac{1+\lambda}{1-\lambda}$. That is, substitute s as $\frac{1+\lambda}{1-\lambda}$.

2. We solve $\|\frac{T_1}{\gamma} + \frac{T_2}{\gamma}Q\frac{T_3}{\gamma}\|_s < 1$ instead. Let $F' = \frac{1}{\gamma}F = \frac{T_1}{\gamma} + \frac{T_2}{\gamma}Q\frac{T_3}{\gamma}$. Substitute $z_1,\ z_2$ into the equation and thus

$$\begin{aligned} F'(z_i) &= \frac{1}{\gamma}T_1(z_i) \\ F'(q_i) &= \frac{1}{\gamma}T_1(q_i), \qquad Q(q_i) = 0 \end{aligned}$$

are the interpolation conditions. Let

$$\begin{array}{llll} \lambda_i = z_i, & 1 \le i \le m, & \lambda_{m+j} = b_j, & 1 \le j \le \ell \\ W_i = T_1(z_i), & 1 \le i \le m, & W_{m+j} = T_1(q_j), & 1 \le j \le \ell \end{array}$$

and let $n = m + \ell$, then the interpolation problem id to find F' such that

$$\begin{aligned} F'(\lambda_i) &= \frac{1}{\gamma}W_i, \quad i = 1, \ldots, n \\ \|F'\|_s &\le 1 \end{aligned}$$

3. Check if this interpolation problem is solvable by Theorem 3.3, Theorem 3.4 or Lemma 3.5. If the solution doesn't exist, pick another γ.

4. Solve the SNP problem by algorithm SNP with data set λ_i and $\frac{1}{\gamma}W_i$. We have the mapping F' satisfies the conditions and hence $F(z) = \gamma F'(z)$. Transform the points back to s-domain by $\lambda = \frac{s-1}{s+1}$.

5. Since $\rho(H) \leq \max_{U \in \mathcal{U}} \rho(UH) \leq \mu_\Delta(H)$, test if $\|F\|_\mu < 1$. If $\|F\|_\mu < 1$, solve Q and hence K is designed.

6. If $\|F\|_\mu > 1$, pick a smaller γ and repeat the process.

Remark 1. Since spectral NP theory is still unfull-fledged, the Spectral interpolation problem we could handle now is two by two case. Thus the plant we want to design via this access is restricted in single-input, single-output (H is a 2×2 matrix) case. But in some case the plant could be 2-input, 2-output, we will discuss this in the following section. Note that there are some constraints, first the interpolation condition is not easy to meet. Second, when the smaller γ is picked, the interpolation condition will be the reciprocal of γ and hence tends to be unsolvable.

Remark 2. If T_2, T_3 are square and invertible, $Q = T_2^{-1}(F - T_1)T_3^{-1}$. If T_2, T_3 are not square and T_2 is left inverse, T_3 is right inverse, we can always appropriately chose the controllability gain and the observability gain such that $T_2^* T_2 = T_3 T_3^* = I$, see [24]. Hence there exists $T_{2\perp}$ and $T_{3\perp}$ such that

$$\begin{bmatrix} T_2^* \\ T_{2\perp}^* \end{bmatrix} \begin{bmatrix} T_2 & T_{2\perp} \end{bmatrix} = \begin{bmatrix} T_3 \\ T_{3\perp} \end{bmatrix} \begin{bmatrix} T_3^* & T_{3\perp}^* \end{bmatrix} = I$$

and then

$$T_2QT_3 = \begin{bmatrix} T_2 & T_{2\perp} \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_3 \\ T_{3\perp} \end{bmatrix}$$

$$\begin{bmatrix} T_2^* \\ T_{2\perp}^* \end{bmatrix} (F - T_1) \begin{bmatrix} T_3^* & T_{3\perp}^* \end{bmatrix} = \begin{bmatrix} T_2^* \\ T_{2\perp}^* \end{bmatrix} \begin{bmatrix} T_2 & T_{2\perp} \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_3 \\ T_{3\perp} \end{bmatrix} \begin{bmatrix} T_3^* & T_{3\perp}^* \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$$

Hence we could find Q. It is similar if T_2 is right inverse and T_3 is left inverse.

4.3 Design Examples for μ-Synthesis via SNP Theory

4.3.1 System with Parametric Uncertainty

Example 4.1 If the uncertainty plant is $\widetilde{G} = \frac{1+\delta_1}{s-1-\delta_1}$ where $\delta_1 \in [-0.325, 0.425]$. We could represent $\delta_1 = -0.05 + 0.375\delta$, $|\delta| < 1$, and

$$\begin{aligned} \widetilde{G} &= \frac{1 - 0.05 + 0.375\delta}{s - (1 - 0.05 + 0.375\delta)} \\ &= P_{22} + \frac{P_{21}\delta P_{12}}{1 - P_{11}\delta} \\ &= \mathcal{F}_u(P, \delta) \end{aligned}$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P22 \end{bmatrix} = \frac{1}{s - 0.95} \begin{bmatrix} 0.375 & s \\ 0.375 & 0.95 \end{bmatrix}$$

The plant is shown as Figure 4-3.

This is a example of real parameter uncertainty. The uncertainty set $\mathbf{\Delta} = \{\delta : \delta \in \ [-1, 1]\}$ We design a controller K to achieve robust stability. Let

$$\begin{aligned} H = \mathcal{F}_\ell(P, K) &= P_{11} + P_{12}K(1 - P_{22}K)^{-1}P_{21} \\ &= \frac{0.375}{s - 0.95} + \frac{s}{s - 0.95}K(1 - \frac{0.95}{s - 0.95}K)^{-1}\frac{0.375}{s - 0.95} \\ &= \frac{0.375}{s - 0.95}(1 + \frac{sK}{s - 0.95 - 0.95K}) \end{aligned}$$

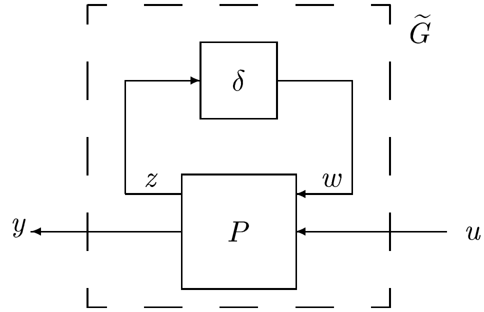


Figure 4-3: System with real parameter uncertainty

As in chapter 2 , $\mathcal{U} = \{U \in \mathbf{\Delta} : \ UU^* = I_n\}$ and we utilize the the relation $\max_{U\in\mathcal{U}} \rho(UH) \leq \mu(H)$. Since

$$\max_{U\in\mathcal{U}} \rho(UH) = \max_{U\in\mathcal{U}} |UH| = |H|$$

we chose a γ and solve K which satisfies $|H| < \gamma$ for all s. Then test if $\mu(H) < 1$, if not, pick a smaller γ and do this again. Thus we could design a robust stability controller.

Note that if we parameterize the controller, that is, let $Q = K(1 - P_{22}K)^{-1}$, $|H| < 1$ could be

$$\left|(\frac{0.375}{s - 0.95})(1 + sQ)\right| < 1$$

This is as the form of $\|T_1 + T_2Q\|_\infty < 1$.

4.3.2 System with Complex Structured Uncertainty

Now consider the following system

Given $G(s) = \frac{1}{s-1}$, $\quad W_p(s) = \frac{0.25s+0.6}{s+0.006}$, $\quad W_u(s) = \frac{4s+8}{s+32}$. This is a SISO system, from the diagram we have

$$\begin{bmatrix} z \\ e \\ \hdashline y \end{bmatrix} = P \begin{bmatrix} w \\ d \\ \hdashline u \end{bmatrix}$$

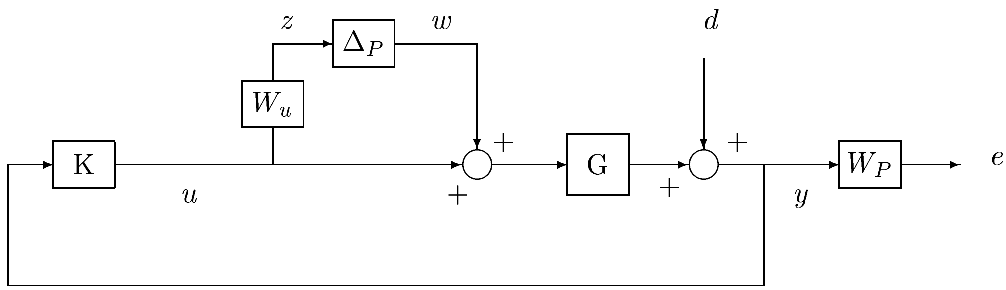


Figure 4-4: Closed-loop block diagram for a typical uncertain system Type I

where

$$P = \left[\begin{array}{cc:c} 0 & 0 & W_u \\ W_pG & W_p & W_pG \\ \hdashline G & I & G \end{array}\right]$$

Let $H := \mathcal{F}_\ell(P, K)$, then

$$\begin{aligned} H &= P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \\ &= \begin{bmatrix} \frac{W_uKG}{I-GK} & \frac{W_uK}{I-GK} \\ \frac{W_pG}{I-GK} & \frac{W_p}{I-GK} \end{bmatrix} \end{aligned} \tag{4-6}$$

Robust stability problem is to design K such that

$$\|H_{11}\|_\mu = \sup_{\omega\in\mathbb{R}} \mu_{\Delta_P}(H_{11}(j\omega)) < 1. \tag{4-7}$$

Robust performance robust problem is to design K such that

$$\|H\|_\mu = \sup_{\omega\in\mathbb{R}} \mu_{\Delta}(H(j\omega)) < 1. \tag{4-8}$$

The uncertainty set here is $\Delta = \{\text{diag}(\Delta_P,\ \Delta_F),\ \ \Delta_P \in \mathbb{C},\ \Delta_F \in \mathbb{C}\}$ where Δ_F is the fictitious block. To get the interpolation condition, we substitute zeros and poles of G into H, and we could find the interpolation condition.

First we transform the points with function $A(z) = \frac{1+z}{1-z}$, z is in the unit disk for s in RHP. Thus

$$G(z) = \frac{1+z}{2z}, \quad W_p(z) = -25\frac{17+7z}{-503+497z}, \quad W_u(z) = 4\frac{3+z}{33+31z}$$

Then try to find the interpolation condition. Substitute pole of $G(z)$ into H and we have

$$H(0) = \begin{bmatrix} -W_u(0) & 0 \\ \frac{W_p(0)}{k(0)} & 0 \end{bmatrix}.$$

The robust stability problem is to find

$$\varphi(z) = \frac{KW_uG}{1-GK}, \qquad \|\varphi\|_\infty \leq 1$$

which satisfies

$$\varphi(0) = -W_u(0)$$

and this is just a classical NP problem. K is solved by

$$K(z) = \frac{\varphi(z)}{G(z)(s(z)+W_u(z))}.$$

The robust performance problem is to find $\varphi(z) = (s(z), p(z)) = (\text{trace}(H), \det(H))$. But from (4-6) we have

$$s(z) = \text{trace}(H) = \frac{KW_uG + W_p}{1-GK}, \qquad p(z) = 0 \tag{4-9}$$

If we have $s(z)$ such that $\|s\|_\infty \leq 1$, then $\varphi(z) = (s(z), p(z)) \in \Gamma_2$ by Lemma 3.5. We only need to satisfy the condition $s(0) = -W_u(0)$ and this interpolation problem is just a classical NP problem. Once $s(z)$ is obtained, we could solve K directly from (4-9), that is,

$$K(z) = \frac{s(z) - W_P(z)}{G(z)(s(z)+W_u(z)}$$

Substitute $z = \frac{s-1}{s+1}$ to transform the unit disk point back. But this is the central spectral controller, i.e., this controller was generated from $\rho(H) < 1$. Thus it may not satisfy the robust specification since

$$\rho(H) \leq \max_{U\in\mathcal{U}} \rho(UH) \leq \mu_\Delta(H) \leq \inf_{D\in\mathcal{D}} \bar{\sigma}(DHD^{-1}) \leq \bar{\sigma}(H)$$

To guarantee the robustness, we have to use iteration as we mentioned in previous algorithm. We will solve this problem in next section. In fact, this system has an important property which lead this problem to a $\mathcal{H}_\infty$ problem. See next section.

4.4 Further Simplification for μ-Synthesis via SNP Theory

First we introduce two Lemmas of specified matrix and we will need them in the following contents.

Lemma 4.1 *For a partitioned matrix* $A=\begin{bmatrix} XG & X \\ YG & Y \end{bmatrix}$ *where* $X, Y,$*and* $G \in \mathbb{C}^{n\times n}$ *and* X *is nonsingular.* A *is similar to* $B = \begin{bmatrix} 0 & I \\ 0 & GX+Y \end{bmatrix}$.

Proof From simple computation we have

$$\begin{bmatrix} 0 & I \\ 0 & GX+Y \end{bmatrix} = \begin{bmatrix} X^{-1} & 0 \\ G & I \end{bmatrix} \begin{bmatrix} XG & X \\ YG & Y \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ G & I \end{bmatrix}^{-1}$$

and this is proved. ♯

Lemma 4.2 *For a* $2n \times 2n$ *system transfer matrix* $A=\begin{bmatrix} XG & X \\ YG & Y \end{bmatrix}$ *where* $X, Y,$*and* $G \in \mathbb{C}^{n\times n}$ *and* X *is nonsingular.*

$$\rho(A) = \rho(GX + Y).$$

Proof From previous lemma, A is similar to $B = \begin{bmatrix} 0 & I \\ 0 & GX+Y \end{bmatrix}$. Hence

$$\rho(A) = \rho\left(\begin{bmatrix} 0 & I \\ 0 & GX+Y \end{bmatrix}\right) = \rho(GX+Y).$$

$\sharp$

Now consider the system Type I in previous section,

$$H = \mathcal{F}_\ell(P, K) = \begin{bmatrix} XG & X \\ YG & Y \end{bmatrix}$$

where $X = W_u K (I - GK)^{-1}, \; Y = W_p (I - GK)^{-1}$.

The structured uncertainty $\Delta = \{\mathrm{diag}(\Delta_P, \; \Delta_F)\}$, and $\mathcal{U} = \{U \in \Delta, \quad U^* U = I \quad \}$. We want to use the equality $\mu_{\boldsymbol{\Delta}}(H) = \max_{U \in \mathcal{U}} (UH)$. Let

$$U = \begin{bmatrix} U_P & 0 \\ 0 & U_F \end{bmatrix}$$

where $U_P^* U_P = I, \quad U_F^* U_F = I$. Then

$$UH = \begin{bmatrix} U_P XG & U_P X \\ U_F YG & U_F Y \end{bmatrix}.$$

and $\det(UH) = 0$.

4.4.1 SISO Case

Let $S = \dfrac{1}{I - GK}$, $T = \dfrac{GK}{I - GK}$, we have the following result

Theorem 4.1 *For the SISO system in Figure 4-4,*

$$\mu_{\boldsymbol{\Delta}}(H) = |W_p S| + |W_u T| \tag{4-10}$$

Proof By [12], $\mu_{\mathbf{\Delta}}(H) = \max_{U \in \mathcal{U}} \rho(UH)$. From definition of U , we have $U = \begin{bmatrix} U_P & 0 \\ 0 & U_F \end{bmatrix}$ with

$U_P^* U_P = |U_P|^2 = 1, \quad U_F^* U_F = |U_F|^2 = 1.$ Let $\bar{R} = \begin{bmatrix} (U_P X)^{-1} & 0 \\ G & I \end{bmatrix}$, then

$$\begin{aligned} \rho(UH) &= \rho(\overline{R}UHR^{-1}) \\ &= \rho\left(\begin{bmatrix} 0 & I \\ 0 & GU_P X + U_F Y \end{bmatrix} \right) \\ &= \rho(GU_p X + U_F Y) \\ &= |GU_P X + U_F Y| \\ &\leq |GX| + |Y| \end{aligned}$$

Since all elements are scalar, if choose $U_F = |Y|Y^{-1}$, $U_P = G^{-1}|GX|X^{-1}$, then

$$\begin{aligned} |GU_P X + U_F Y| &= \|GX\| + \|Y\| \\ &= |GX| + |Y| \\ &= |W_p S| + |W_u T| \end{aligned}$$

Thus

$$\mu_{\Delta}(H) = \max_{U \in \mathcal{U}} \rho(UM) = |W_p S| + |W_u T|$$

♯

Hence in SISO case ,

$$\|H\|_{\mu} = \sup_{\omega} \mu_{\Delta}\left(H(j\omega)\right) = \| \ |W_p S| + |W_u T| \ \|_{\infty}$$

A different approach to this conclusion is given in [7].

To solve the controller K satisfies $\| \ |W_p S| + |W_u T| \ \|_{\infty} < 1$ is not easy but the so called *mixed sensitivity problem* $\| \ W_p S + W_u T \ \|_{\infty} < 1$ is valid. So we solve $\|W_p S + W_u T\|_{\infty} < \gamma$ and then use iteration. The algorithm is as follows:

1. Let $Q = K(I - GK)^{-1}$, then $K = (I + QG)^{-1}Q$, and$(I - GK)^{-1} = (I + GQ)$. Thus

$$\begin{aligned} |W_p S + W_u T| &= |W_p(I + GQ) + GW_u Q| \\ &= |W_p + (W_p G + GW_u)Q|. \end{aligned}$$

   First find γ_{opt} and then solve the following *model matching problem* via N-P theory.

$$\|W_p S + W_u T\|_\infty = \|W_p + (W_p G + GW_u)Q\|_\infty < \gamma.$$

   with $\gamma > \gamma_{opt}$.

2. For a chosen $\gamma > \gamma_{opt}$, we could find a set of controller K, then check if the controller satisfies

$$\||W_p S| + |W_u T|\|_\infty < 1$$

   If it does, the work is done.

3. If $\||W_p S| + |W_u T|\|_\infty > 1$, pick a smaller γ and repeat the process to find the controller.

Example 4.2 Suppose that $G(s) = \frac{1}{s-1}$, $W_p(s) = \frac{0.25s+0.6}{s+0.006}$, $W_u(s) = \frac{4s+8}{s+32}$.

First find the optimal γ, γ_{opt}, so we could choose the γ we will use in iteration. Since $G(W_p+W_u)$ is unstable, let $T_2 := (W_p + W_u), T_1 = W_p$ and $\tilde{Q} := GQ$. We solve the problem

$$\|T_1 + T_2\tilde{Q}\|_\infty < \gamma.$$

There is no RHP zeros of T_2 , if chose $\tilde{Q} = \frac{-W_p}{(W_p+W_u)}$, $Q = \frac{-W_p}{G(W_p+W_u)}$ is stable and thus $\|W_p + (W_p + W_u)GQ\|_\infty = \|W_p - W_p\|_\infty = 0$, so $\gamma_{opt} = 0$. Now let

$$\tilde{Q} = \frac{T - W_p}{(W_p + W_u)},$$

we have

$$\|W_p + (W_pG + GW_u)Q\|_\infty = \|T\|_\infty.$$

Now we start the iteration. First chose $\gamma_1 = 2$, thus we could choose $T = 2\frac{s-1}{s+1}$. And hence

$$K_1 = (I+QG)^{-1}Q = \frac{875s^4 + 25706s^3 - 72295s^2 + 35922s + 9792}{2(1500s^3 + 18509s^2 - 13889s - 84)}.$$

From this we have $\||W_pS| + |W_uT|\|_\infty \simeq 3.2234 > 1$. Choose $T = \frac{s-1}{s+1}$, that is $\gamma_2 = 1$. Then

$$K_2 = \frac{500s^4 + 13556s^3 - 76867s^2 + 43515s + 19296}{(9000s^3 + 51554s^2 - 7691s - 48)}$$

and $\||W_pS| + |W_uT|\|_\infty \simeq 1.8258 > 1$. Choose $T = \frac{1}{2}\frac{s-1}{(s+1)}$, i.e. $\gamma_3 = 1/2$. Thus

$$K_3 = \frac{250s^4 + 6403s^3 - 50360s^2 + 24411s + 19296}{4500s^3 + 27527s^2 - 7835s - 48}$$

and $\||W_pS| + |W_uT|\|_\infty \simeq 1.1443 > 1$. Choose $T = \frac{1}{3}\frac{s-1}{s+1}$, $\gamma_4 = 1/3$.

$$K_4 = \frac{125s^4 + 2103s^3 - 59835s^2 + 28711s + 28896}{6500s^3 + 33539s^2 - 3799s - 24}$$

and $\||W_pS|+|W_uT|\|_\infty \simeq 0.9257 < 1$. Reduced the order such that $K_4 = \frac{2103s^3 - 59835s^2 + 28711s + 28896}{6500s^3 + 33539s^2 - 3799s - 24}$ and we have the result

$$\||W_pS| + |W_uT|\|_\infty \simeq 0.9255 < 1.$$

Hence K_4 satisfies the robust performance criteria. Figure 4-5 is the robust performance of K_2, K_3, K_4 and Figure 4-6 is the comparison with the controller K in [5], $K = \frac{-2.8s-1}{s}$. Note that Fig 4-6 is plotted with the Matlab command "`mu`".

Now compare the following two systems in Figure 4-7, 4-8, X, Y are defined as previous.

$$\text{Type II}: \quad H = \mathcal{F}_\ell(P,K) = \begin{bmatrix} GX & GX \\ Y & Y \end{bmatrix}$$

$$\text{Type III}: \quad H = \mathcal{F}_\ell(P,K) = \begin{bmatrix} X & X \\ Y & Y \end{bmatrix}$$

These systems are similar to Type I and we can use the same method to design K.

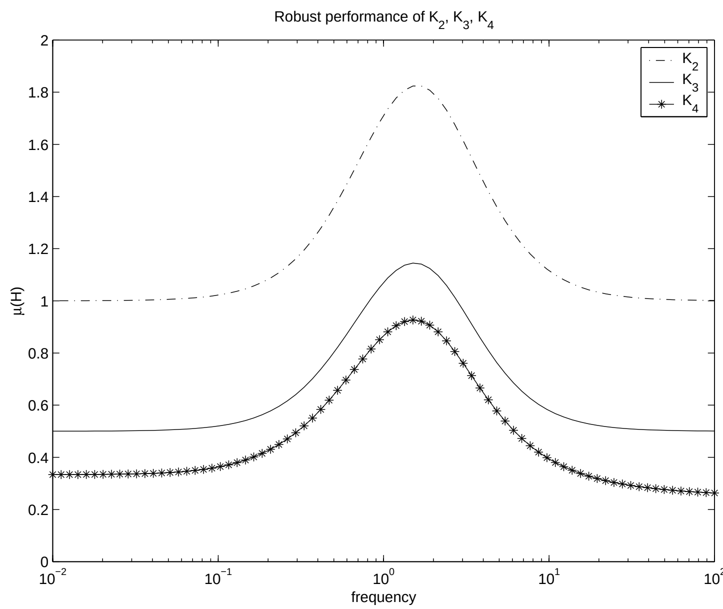


Figure 4-5: Robust performance of designed controllers.

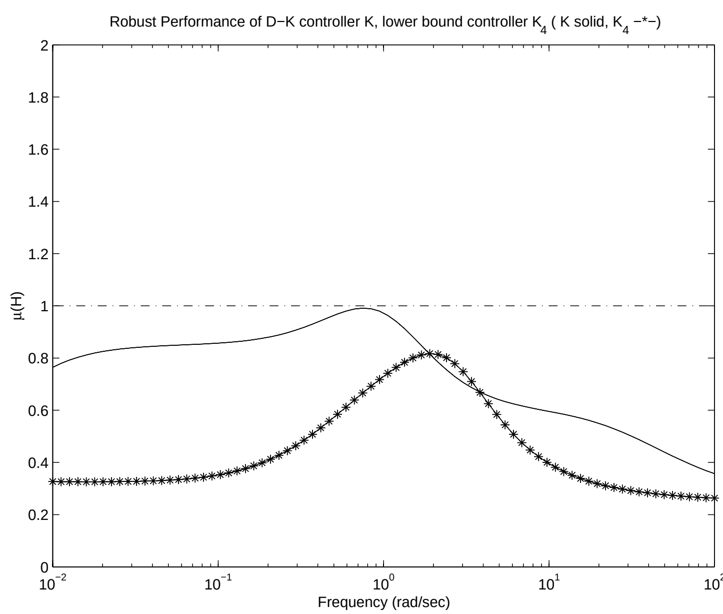


Figure 4-6: Comparison with the $\mu-$tool box controller.

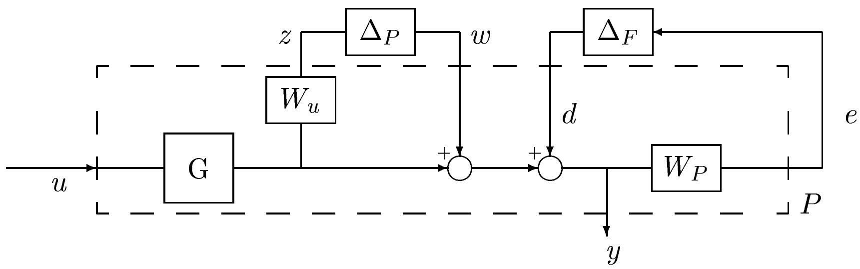


Figure 4-7: Closed-loop block diagram for a typical uncertain system Type II

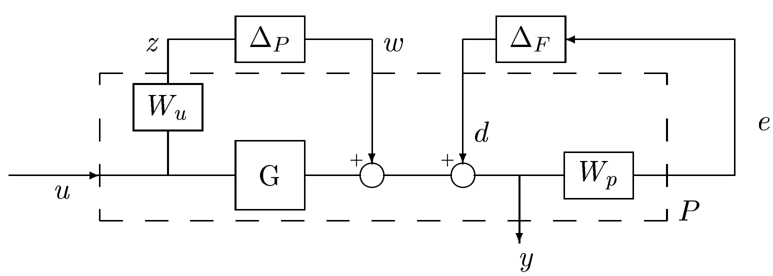


Figure 4-8: Closed-loop block diagram for a typical uncertain system Type III

From Lemma 4.2, we find that the spectral Nevanlinna-Pick interpolation problem in Theorem 3.7 could be used in closed loop systems with 2-input, 2-output. Suppose that closed loop system Type I in Figure 4-4 is 2-input, 2-output, that is, G, W_u, W_p are 2×2 matrices and the generalized transfer function H is a 4×4 matrix. By Lemma 4.2, $\rho(H) = \rho(GX + Y)$ and $GX + Y$ is a 2×2 matrix. Hence we could use spectral Nevanlinna-Pick problem to design a controller. Now we want to design K such that

$$\rho(GX + Y) = \rho(GW_uK(I - GK)^{-1} + W_p(I - GK)^{-1}) < 1.$$

With controller parameterized , our problem comes out to be

$$\rho(W_p + (W_pG + GW_u)Q) < 1$$

where $Q = K(I - GK)^{-1}$. Thus if the plant matches the requisite we have mentioned in Section 4.1, we could solve this problem with our algorithm.

Example 4.3 Given closed loop system Type I where

$$G(s) = (s-3)(s-1)\begin{bmatrix} \frac{1}{4(2s+1)(3s+1)} & \frac{1}{2(3s+1)^2} \\ \frac{1}{8(2s+1)^2} & \frac{2}{(3s+1)(5s+3)} \end{bmatrix}, \quad W_p = \begin{bmatrix} \frac{s-1}{3s+1} & 0 \\ 0 & \frac{s-3}{2(3s+1)} \end{bmatrix}, \quad W_u = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3s-1}{2(3s+1)} \end{bmatrix}$$

The uncertainty set is $\Delta_P = \{\mathrm{diag}[\delta I_2] : \ \delta \in \mathbb{C}\}$. Design a controller via spectral Nevanlinna-Pick theory.

Figure 4-9, 4-10 are the Bode diagrams of W_P and W_u. The following steps are carried out to design K from Algorithm K:

Step1. First let $\lambda = \frac{s-1}{s+1}$, thus $s = \frac{1+\lambda}{1-\lambda}$ and λ is in the unit disk for s in RHP.

$$G(\lambda) = \lambda(\lambda - \frac{1}{2})\begin{bmatrix} \frac{1}{(\lambda+2)(\lambda+3)} & \frac{1}{(\lambda+2)(\lambda+2)} \\ \frac{1}{(\lambda+3)(\lambda+3)} & \frac{1}{(\lambda+2)(\lambda+4)} \end{bmatrix}, \quad W_p(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda+2} & 0 \\ 0 & \frac{\lambda-\frac{1}{2}}{\lambda+2} \end{bmatrix}, \quad W_u(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\lambda+\frac{1}{2}}{\lambda+2} \end{bmatrix}$$

$T_1 = W_p$, $T_2 = (W_pG + GW_u)$ and zeros of $T_2(\lambda)$ are $z_1 = 0, z_2 = \frac{1}{2}$. We will compute $F(\lambda)$ directly and then transform λ back to s by $\lambda = \frac{s+1}{s-1}$.

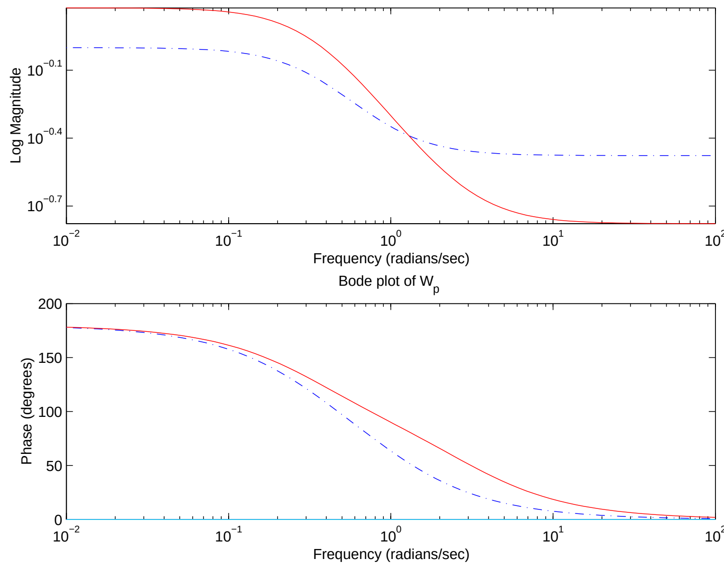


Figure 4-9: Bode diagram of W_p

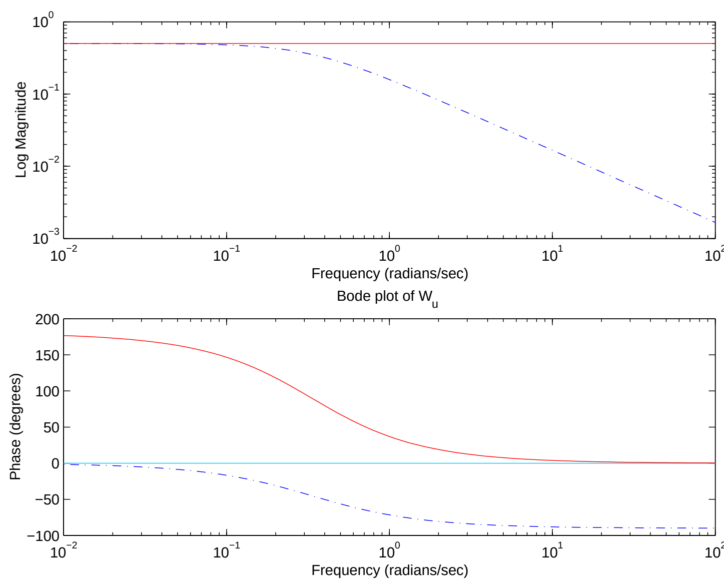


Figure 4-10: Bode diagram of W_u

Step2. First pick $\gamma_1 = 1$. $F_1 = T_1 + T_2 Q$, the interpolation conditions are

$$\begin{aligned} F_1(0) &= W_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix} = R(0) \begin{bmatrix} 0 & 1 \\ -p(0) & s(0) \end{bmatrix} R^{-1}(0) \\ F_1(\frac{1}{2}) &= W_2 = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{bmatrix} = R(\frac{1}{2}) \begin{bmatrix} 0 & 1 \\ -p(\frac{1}{2}) & s(\frac{1}{2}) \end{bmatrix} R^{-1}(\frac{1}{2}) \end{aligned}$$

and $s_1 = -\frac{1}{4}$, $p_1 = 0$, $s_2 = \frac{1}{5}$, $p_2 = 0$. From Theorem 3.7 we need to find ω_0 such that

$$|2 - \bar{\omega}_0 s_1 - \omega_0 \bar{s}_2| = \min_{|\omega|=1} |2 - \bar{\omega} s_1 - \omega \bar{s}_2|$$

and

$$\beta_i = \frac{2p_i - \omega_0 s_i}{2 - \omega_0 s_i}.$$

Thus $\omega_0 = -1$, and then $\beta_1 = -\frac{1}{7}$, $\beta_2 = \frac{1}{11}$. Hence

$$\begin{aligned} s'(\lambda) &= -2\frac{\lambda^2 + 74\lambda + 1}{\lambda^2 - 4\lambda - 77} \\ p'(\lambda) &= -\frac{77\lambda^2 + 4\lambda - 1}{\lambda^2 - 4\lambda - 77} \end{aligned}$$

and $s(\lambda) = s'(\mathcal{M}_{-\beta}(\lambda))$, $p(\lambda) = p'(\mathcal{M}_{-\beta}(\lambda))$ where $\mathcal{M}_{-\beta}(\lambda)$ is the Möbius function which maps z_i to β_i.

$$\mathcal{M}_{-\beta}(\lambda) = \frac{-(182\lambda^2 g - 103\lambda g + 97\lambda - 42\lambda^2 + 6g - 26)}{-182 + 103\lambda - 97\lambda g + 42g + 26\lambda^2 g - 6\lambda^2}$$

g is any stable, rational, proper function with $\|g\|_\infty \le 1$.

Hence if we pick $g = 1$,

$$\begin{aligned} s(\lambda) &= \frac{-\frac{1}{10}(1300\lambda^4 + 380\lambda^3 - 10219\lambda^2 + 380\lambda + 1300)}{6\lambda^3 - 119\lambda^2 - 46\lambda + 520} \\ p(\lambda) &= \frac{\lambda(520\lambda^3 - 46\lambda^2 - 119\lambda + 6)}{6\lambda^3 - 119\lambda^2 - 46\lambda + 520} \end{aligned}$$

If pick $g = 0$,

$$\begin{aligned} s(\lambda) &= \frac{36\lambda^2 - 612\lambda + 169}{4(18\lambda - 169)} \\ p(\lambda) &= \frac{-18\lambda(2\lambda - 1)}{18\lambda - 169} \end{aligned}$$

Chose

$$R(\lambda) = \begin{bmatrix} -\frac{\lambda-\frac{1}{2}}{\lambda+2} & 1 \\ -\frac{\lambda}{\lambda+2} & 1 \end{bmatrix} \tag{4-11}$$

For $g = 1$,

$$F_1(\lambda) = R(\lambda) \begin{bmatrix} 0 & 1 \\ -p(\lambda) & s(\lambda) \end{bmatrix} R^{-1}(\lambda) = \begin{bmatrix} F_1(\lambda)_{11} & F_1(\lambda)_{12} \\ F_1(\lambda)_{21} & F_1(\lambda)_{22} \end{bmatrix}$$

where

$$\begin{aligned}
F_1(\lambda)_{11} &= \frac{-\frac{1}{5}\lambda(6500\lambda^5 + 23380\lambda^4 + 7091\lambda^3 - 26463\lambda^2 + 2970\lambda + 240)}{(\lambda+2)(6\lambda^3 - 119\lambda^2 - 46\lambda + 520)} \\
F_1(\lambda)_{12} &= \frac{\frac{1}{10}\lambda(13000\lambda^5 + 45460\lambda^4 + 11142\lambda^3 - 42247\lambda^2 + 25863\lambda - 7010)}{(\lambda+2)(6\lambda^3 - 119\lambda^2 - 46\lambda + 520)} \\
F_1(\lambda)_{21} &= \frac{-\frac{1}{5}\lambda(6500\lambda^5 + 23380\lambda^4 + 7121\lambda^3 - 27058\lambda^2 + 2740\lambda + 2840)}{(\lambda+2)(6\lambda^3 - 119\lambda^2 - 46\lambda + 520)} \\
F_1(\lambda)_{22} &= \frac{\frac{1}{10}(13000\lambda^6 + 45460\lambda^5 + 11202\lambda^4 - 43467\lambda^3 + 25998\lambda^2 - 1580\lambda - 2600)}{(\lambda+2)(6\lambda^3 - 119\lambda^2 - 46\lambda + 520)}
\end{aligned}$$

Replace λ as $\frac{s-1}{s+1}$.

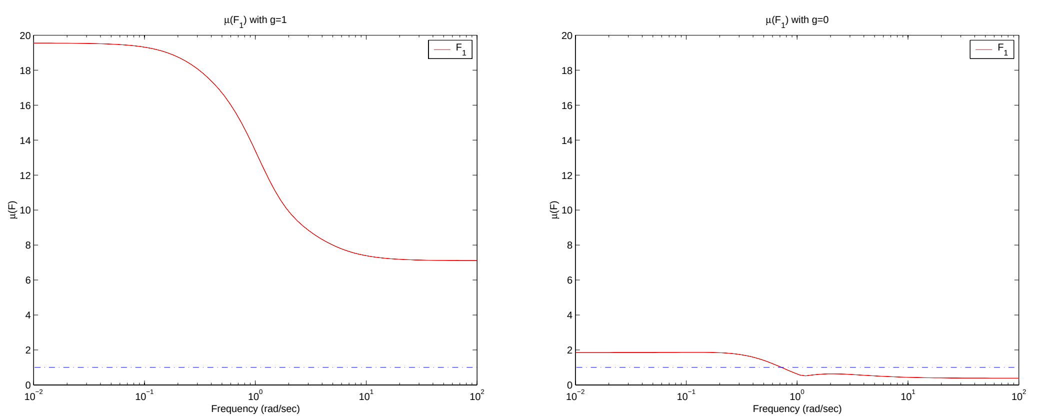
$$\begin{aligned}
F_1(s)_{11} &= \frac{-\frac{2}{5}(s-1)(6859s^5 + 79422s^3 - 63042s^4 + 22664s^2 - 32361s - 9702)}{(3s+1)(s+1)^2(361s^3 + 1615s^2 + 1743s + 441)} \\
F_1(s)_{12} &= \frac{\frac{1}{5}(s-1)(+23104s^5 - 106115s^4 + 132378s^3 - 27064s^2 - 107562s - 26901)}{(3s+1)(s+1)^2(361s^3 + 1615s^2 + 1743s + 441)} \\
F_1(s)_{21} &= \frac{-\frac{1}{5}(s-1)(+15523s^5 - 114399s^4 + 185514s^3 + 73038s^2 - 51597s - 17199)}{(3s+1)(s+1)^2(361s^3 + 1615s^2 + 1743s + 441)} \\
F_1(s)_{22} &= \frac{\frac{1}{10}(48013s^6 - 252168s^5 + 47187 - 371184s^3 + 468601s^4 - 231001s^2 + 124152s)}{(3s+1)(s+1)^2(361s^3 + 1615s^2 + 1743s + 441)}
\end{aligned}$$

and K has order 19 in each element.

For $g = 0$,

$$\begin{aligned}
F_1(s)_{11} &= \frac{1/2(-1+s)(521+161s+1351s^2+271s^3)}{(1+3s)(187+151s)(s+1)^2} \\
F_1(s)_{12} &= \frac{1/4(-3+s)(-1+s)(449+106s+377s^2)}{(1+3s)(187+151s)(s+1)^2} \\
F_1(s)_{21} &= \frac{-1/2(-1+s)(-147+889s-373s^2+31s^3)}{(1+3s)(187+151s)(s+1)^2} \\
F_1(s)_{22} &= \frac{1/4(-3+s)(-75+1393s+707s^2+679s^3)}{(1+3s)(187+151s)(s+1)^2}
\end{aligned}$$

Test $\|F_1\|_\mu$. Figure 4-11 compare the μ value with $g = 1$ and $g = 0$. We see that with $g = 0$ the performance is much better than $g = 1$. Here we find that the selection of g did has influence in the μ value, how to choose g is a question.

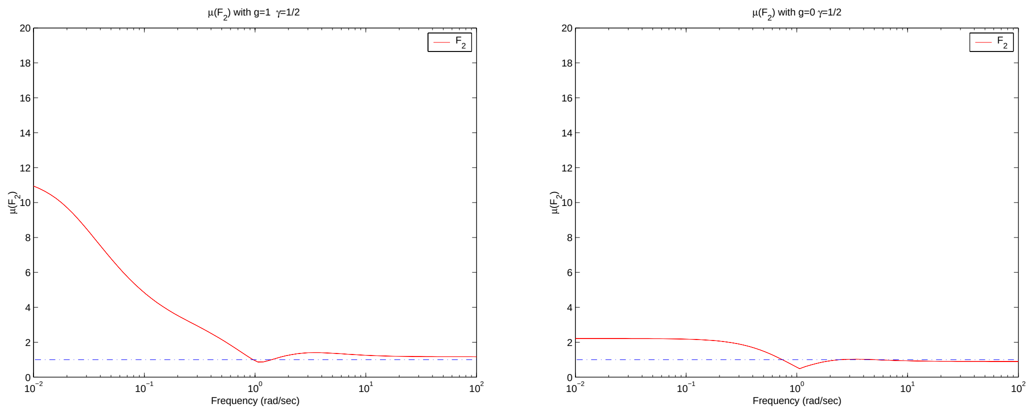


(a) $g = 1$ (b) $g = 0$

Figure 4-11: $\mu(F_1)$ with $g = 1$ and $g = 0$

Note that $s(\lambda)$, $p(\lambda)$ in Theorem 3.7 is not the whole solution set and $R(\lambda)$ is not unique, thus all of them could be adjusted. F_1 here is only one of the solution.

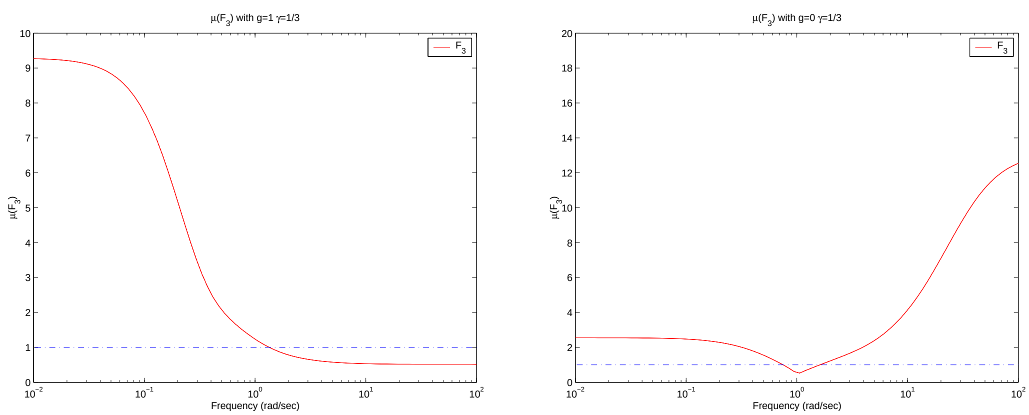
Step3. Pick $\gamma_2 = \frac{1}{2}$, and repeat previous steps, hence $s_1 = -\frac{1}{2}$, $s_2 = \frac{2}{5}$, $p_1 = p_2 = 0$ and $\omega_0 = -1$, $\beta_1 = -\frac{1}{3}$, $\beta_2 = \frac{1}{6}$. Then we can see the μ value of $F_2 = T_1 + T_2 Q$ with $\|F_2\|_s < \frac{1}{2}$ in Figure 4-12.



(a) $g = 1$ (b) $g = 0$

Figure 4-12: $\mu(F_2)$ with $g = 1$ and $g = 0$

Step5. Pick $\gamma_3 = \frac{1}{3}$, we have $s_1 = -\frac{3}{4}$, $s_2 = \frac{3}{5}$, $p_1 = p_2 = 0$. See Figure 4-13.



(a) $g = 1$ (b) $g = 0$

Figure 4-13: $\mu(F_3)$ with $g = 1$ and $g = 0$

Note that if we pick $\gamma = \frac{1}{4}$, the solution of this interpolation problem will not exist by Lemma 3.5, and hence the iteration hold on.

Chapter 5

Conclusions

5.1 Concluding Remarks and Discussion

With the development of spectral Nevanlinna-Pick interpolation theory, we have shown in this thesis the lower bound design for several typical closed loop systems in SISO case and MIMO case. The algorithms presented here give a different approach to μ- synthesis and this is a new attempt to solve this problem.

The model we could handle is restricted since the spectral NP problem can be solved now is still few. To my best knowledge, by far the dimension of spectral NP problem which was solved is not greater than 2. In addition, the interpolation condition of the system is a tricky part of design procedure. In 2 by 2 matrix functions , the general case $\rho(T_1 + T_2QT_3) \leq 1$ is still working and this is more practical to our problem. The higher dimension SNP problems' solutions will be great helpful to the lower bound design.

5.2 Future Directions

The controller we designed usually has very high order and is often not proper. This makes the analysis of μ value more difficult. If we could find the character to reduce the order of controllers during computing will be great helpful . How γ effects the μ value is also a good question for this design method. Example 4.3 show that γ can't be chosen arbitrarily small, it is restricted by the solvability of SNP problem.

Uncertainty set of the system is a key point to μ-norm computing. For D-K iteration, the scaling D matrix will be easy to produce when uncertainty set are full blocks. For our case, the lower bound design, uncertainty sets of the repeated scalar blocks or real parameters are preferred than full blocks, but the fictitious uncertainty block is always of full block type. This is a problem we have to overcome.

References

[1] J. Alger and N. J. Young, "A Schwarz Lemma for the Symmetrized Bidisc", *Bull. London Math. Soc*, 2001.

[2] J. Agler and N. J. Young, "The Two-Point Spectral Nevanlinna-Pick Problem", *Integral Equations Operator Theory*, Vol. 37, pp. 375-385, 2000.

[3] J. Agler and N. J. Young, "The Two-by-Two Spectral Nevanlinna-Pick Problem", submited for publication.

[4] J. A. Ball, I. Gohberg, and L. Rodman, *Interploation of Rational Matrix Functions*, Operator Theory: Advances and Application vol.45, Birkhaüser Verlag, Basekl, 1990.

[5] G. J. Balas, J. C. Doyle, K. Glover, A. Packard, and R. Smith, *μ-Analysis and Synthesis Toolbox User's Guide*, MUSYN Inc. and The MathWorks, Inc., Massachusetts, 2001.

[6] H. Bevrani, "Robust Load Frequency Controller in a Deregulated Environment: A μ-Synthesis Approach", *IEEE Proceedings of International Conference on Control Applications*, pp. 22-27, August, 1999.

[7] O. H. Bosgra, H. Kwakernaak and G. Meinsma, *Design Methods for Control Systems*, Lecture Notes for a DISC course, Winter, 2001-2002.

[8] S. Buso, "Design of a Robust Voltage Controller for a Buck-Boost Converter Using μ-Synthesis", *IEEE Transcations on control systems technology*, Vol. 7, No.2, March, 1999.

[9] H. Bercovici, C. Foias and A. Tannenhaum, "Spectral Radius Interpolation and Robust Control", *Preceedings of 28th Conference on Decision amd Control*, Florida, December, 1989.

[10] V. D. Blonded, E. D. Sontag, M.Vidyasagar and J. C. Willems, *Open Problems in Mathematical Systems and Control Theory*, pp. 217-220, Springer, London, 1999.

[11] J. C. Doyle, B. Francis and A. Tannenbaum, *Feedback Control Theory*, Macmillan Publishing Company, 1992.

[12] J. C. Doyle, "Analysis of feedback system with structure uncertainties", *IEE Proceedings*, Part D, Vol. 133, pp. 45-56, 1982.

[13] J. C. Doyle, "Structured Uncertainty in Control System Design", *Proceedings of 24th Conference On Decsion and Control*, Ft. Lauderdale, 1985.

[14] B. A. Francis, *A Course in H_∞ Control Theory*, Springer-Verlag, Berlin; New York, 1987.

[15] E. Isaacson and H. B. Keller, *Analysis of Numerical Methods*, John Wiely and Sons, New York, 1966.

[16] H. Kimura, "Direction Interpolation Approach to H^∞-Optimization and Robust Stabilition", *IEEE Transactions on Automatical Control* Vol. AC-32, No. 12, pp. 1085-1093, December, 1987.

[17] C. T. Lin, "Schwarz Lemma On Symmetrized Bidisc", Master Thesis, Department of Mathematics, Tunghai University, Taiwan, July, 2001.

[18] T. D. Lin, "Spectral Nevanlinna-Pick Interpolation On Symmetrized Bidisc", Master Thesis, Department of Mathematics, Tunghai University, Taiwan, July, 2001.

[19] R. Lind, G. Balas, and A. Packard, "Evaluating D-K Iteration for Control Design", *Proceedings of the American control Conference*, Baltimore, MD, 1994.

[20] A. Packard and J. C. Doyle, "The Complex Structured Singular Value", *Automatica*, Vol. 29, pp. 71-109, 1993.

[21] A. Packard, M. K. H. Fan, and J. Doyle, "A Power Method for Structured Singular Value", *Proceedings of 27th Conference On Decsion and Control*, Texas, December, 1989.

[22] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, Massachusetts, 1985.

[23] G. F. Wallis, and R. Tymersky, "Generalized Approach for μ Synthesis of Robust Switching Regulators", *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 36, No. 2, 2000.

[24] F. B. Yeh and C. D. Young, *Post Modern Control Theoy and Design*, Eurasia Book Company, Taiwan, 1991.

[25] K. Zhou and J. C. Doyle, *Essentials of Robust Control*, Prentice-Hall International, Inc., New Jersey, 1998.