東海大學統計碩士班

碩士論文

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若干分配的常態近似研究

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摘要

在統計學中，中央極限定理(Central Limit Theorem)是一基礎且重要的定理。而在初等機率論和數理統計教科書中，所寫的都是在動差生成函數(Moment Generating Function)存在條件下，利用動差生成函數去證明中央極限定理成立，但是所運用的數學觀念是較艱難的。本文則是要運用 Proschan(2008)提出的想法去說明一維分配的中央極限定理成立，它是利用機率函數(Probability Function)來近似分配為基本概念，比較能讓統計學的初學者接受，然後再應用此方法，推廣到變數是多維度的情況，說明中央極限定理也是成立的。

目錄

第一章 緒論

第 1.1 節 背景及研究動機

中央極限定理(Central Limit Theorem)是機率理論及統計學中重要且常用的定理。主要是指從平均數為μ，標準差為σ的母體中，隨機地抽取大小為n的獨立樣本$X_1,\cdots,X_n$。當樣本數n很大時，其樣本平均$\bar{X}_n=\left(X_1+\cdots+X_n\right)/n$減掉平均數$\mu$再除以$\bar{X}_n$的標準差$\sigma/\sqrt{n}$，將會趨近平均數為 0，標準差為 1 的常態分配(Normal Distribution)。

我們在初等機率論和數理統計教科書中常見說明中央極限定理成立的方法是利用動差生成函數(Moment Generating Function)，但是需要動差生成函數存在才可運用，可參考 Hogg, Mckean, and Craig (2005, p.220-222) 。另外也有教科書是使用特徵函數(Characteristic Function)，它是利用複變函數(Complex Function)的基本概念去證明中央極限定理成立，而且任何隨機變數的特徵函數皆存在，是一嚴謹的證明，例如 Billingsley(1995, Section 27)。而以上兩種方法的數學觀念都過於艱難，對於統計學的初學者來說，只會運用中央極限定理解題目，卻不容易了解其概念，所以 Proschan(2008)提出利用機率函數(Probability Function)去近似分配的想法去說明，想法比前面兩個方法來的容易理解。不過 Proschan(2008)只利用二項分配(Binomial Distribution)去說明中央極限定理成立，而本篇是沿用 Proschan(2008)

的方法，除了在變數是一維度情況下說明中央極限定理成立，再利用變數是一維度推廣到多維度，也去說明多維度情況下中央極限定理會成立。上述這些將是本文接下來想要探討的部份。

第 1.2 節 研究架構

本文主要介紹 Proschan(2008)提出的方法說明一維及多維分配的中央極限定理成立。我們分成四章來討論。

第一章的第 1.1 節中簡單說明中央極限定理及本文的研究動機和目的；第 1.2 節敘述本文架構。

第二章是利用 Proschan(2008)方法說明一維分配的中央極限定理成立，第 2.1 節是屬於離散型的卜瓦松分配(Poisson Distribution)；第 2.2 節是屬於連續型的伽瑪分配(Gamma Distribution)。

第三章是利用 Proschan(2008)方法說明多維分配的中央極限定理成立，第 3.1 節是屬於離散型的多項式分配(Multinomial Distribution)；第 3.2 節是屬於連續型的多項伽瑪分配(Mathi and Moschopulos's Multivariate Gamma Distribution)。

第四章主要說明本篇使用的 Proschan(2008)方法與傳統上常用的動差生成函數法(Moment Generating Function)和特徵函數法(Characteristic Function)之間的差異，且分析各自的優缺點。

第二章 一維分配的定理說明

一維分配的中央極限定理是指從平均數為μ，標準差為σ的母體中，隨機地抽取大小為n的獨立樣本$X_1,\cdots,X_n$。當樣本數n很大時，其樣本平均$\bar{X}_n=\left(X_1+\cdots+X_n\right)/n$減掉平均數$\mu$再除以$\bar{X}_n$的標準差$\sigma/\sqrt{n}$，將會趨近平均數為 0，標準差為 1 的常態分配。

本章是利用 Proschan(2008)方法說明一維分配的中央極限定理成立，我們選擇了兩個分配做為舉例，第 2.1 節是屬於離散型的卜瓦松分配，而第 2.2 節是屬於連續型的伽瑪分配。

第 2.1 節 卜瓦松分配

首先我們利用 Proschan(2008)方法說明隨機變數是卜瓦松分配時，中央極限定理會成立，說明過程如下:

假設$Y_1,\cdots,Y_n$是一組獨立且具有相同分配的隨機變數服從卜瓦松分配，其平均數為λ，而Y_i的機率質量函數(Probability Mass Function)為

$$f\left(y_i\right)=\frac{e^{-\lambda}\lambda^{y_i}}{y_i\,!},\ y_i=0,1,2,\cdots$$

其中$i=1,\cdots,n$，參數$\lambda>0$，則$M_{Y_i}\left(t\right)=\exp\left(\lambda\left(e^t-1\right)\right)$是$Y_i$的動差生成函數，然後令$X=\sum_{i=1}^{n}Y_i$，因$Y_1,\cdots,Y_n$分配相同且獨立，所以$X$的動差生成函數可寫成

$$M_X(t)=E\left(e^{tX}\right)=E\left(e^{t\sum_{i=1}^{n}Y_i}\right)=\prod_{i=1}^{n}E\left(e^{tY_i}\right)=\exp\left(\lambda n\left(e^t-1\right)\right) \qquad (2.1)$$

由(2.1)式可以得出 X 的機率質量函數為

$$f_X(x)=\frac{e^{-n\lambda}(n\lambda)^x}{x!},\ x=0,1,2,\cdots \qquad (2.2)$$

其中參數 $\lambda>0$，所以可知 X 符合卜瓦松分配，其平均數為 $n\lambda$。

(2.2)式中的機率質量函數中有包含階層，會造成推導處理上的困難，所以我們考慮下列式子:

$$\frac{f_X(x+1)}{f_X(x)}=\frac{\dfrac{e^{-n\lambda}(n\lambda)^{x+1}}{(x+1)!}}{\dfrac{e^{-n\lambda}(n\lambda)^x}{x!}}=\frac{n\lambda}{x+1} \qquad (2.3)$$

把 X 變數標準化，即 $Z=\dfrac{X-n\lambda}{\sqrt{n\lambda}}$，再令 $z=\dfrac{x-n\lambda}{\sqrt{n\lambda}}$ 帶入(2.3)式可得

$$\frac{f_X(x+1)}{f_X(x)}=\frac{n\lambda}{n\lambda+z\sqrt{n\lambda}+1}=\frac{1}{1+\dfrac{z}{\sqrt{n\lambda}}+\dfrac{1}{n\lambda}} \qquad (2.4)$$

接著令 $\Delta=\dfrac{1}{\sqrt{n\lambda}}$，當 $n\to\infty$ 時，$\Delta=\dfrac{1}{\sqrt{n\lambda}}\to 0$，然後把 Δ 帶入(2.4)式可得

$$\frac{f_X(x+1)}{f_X(x)}=\frac{1}{1+\Delta z+\Delta^2} \qquad (2.5)$$

因為 $Z=\dfrac{X-n\lambda}{\sqrt{n\lambda}}$ 和 $z=\dfrac{x-n\lambda}{\sqrt{n\lambda}}$ 還有 $\Delta=\dfrac{1}{\sqrt{n\lambda}}$，可以得出

$f_X(X=x+1)=f_Z(Z=z+\Delta)$ 和 $f_X(X=x)=f_Z(Z=z)$，所以(2.5)式可改寫成

$$\frac{f_Z(Z=z+\Delta)}{f_Z(Z=z)}=\frac{1}{1+\Delta z+\Delta^2} \qquad (2.6)$$

接下來後對(2.6)式取 ln 可得

$$\ln\left(\frac{f_Z(Z=z+\Delta)}{f_Z(Z=z)}\right)=\ln\left(f_Z(Z=z+\Delta)\right)-\ln\left(f_Z(Z=z)\right)=\ln(1)-\ln\left(1+\Delta z+\Delta^2\right)$$

由Z的分配可以看出很類似鐘型分配，所以我們想要利用連續型的分配去近似Z的分配，接著假設$g(z)$是某一種連續型分配的機率密度函數，然後運用$g(z)$，讓$g(z+\Delta)/g(z)$值很靠近$\frac{f_Z(Z=z+\Delta)}{f_Z(Z=z)}$，所以可得$\ln(g(z+\Delta))-\ln(g(z))\approx\ln(f_Z(Z=z+\Delta))-\ln(f_Z(Z=z))$，用上面的條件去求出$g(z)$，然後考慮下列式子:

$$
\begin{aligned}
\frac{d}{dz}\ln g(z) &= \lim_{\Delta\to 0}\left(\frac{\ln(g(z+\Delta))-\ln(g(z))}{\Delta}\right) \\
&\approx \lim_{\Delta\to 0}\left(\frac{\ln(f_Z(Z=z+\Delta))-\ln(f_Z(Z=z))}{\Delta}\right) \\
&= \lim_{\Delta\to 0}\left(\frac{\ln(1)-\ln(1+\Delta z+\Delta^2)}{\Delta}\right) \\
&= \lim_{\Delta\to 0}\left(-\frac{z+2\Delta}{1+\Delta z+\Delta^2}\right) \\
&= -z \qquad (2.7)
\end{aligned}
$$

接著用(2.7)式對z變數積分可得

$$\ln g(z)=\left(-\frac{z^2}{2}\right)+k \qquad (2.8)$$

其中k是常數，再對(2.8)取 exp 可得

$$g(z)=\exp\left(\left(-\frac{z^2}{2}\right)+k\right) \qquad (2.9)$$

由(2.9)很直觀看出$g(z)$會等於標準常態的機率密度函數，所以可說明在 n 趨近無限大的情況下，標準化後的卜瓦松分配會近似標準常態分配。

第 2.2 節 伽瑪分配

接著利用 Proschan(2008) 方法說明隨機變數是伽瑪分配時,其中央極限定理會成立,說明過程如下:

假設 $Y_1,\cdots,Y_n$ 是一組獨立且具有相同分配的隨機變數服從伽瑪分配,其平均數為 $\alpha\beta$,變異數為 $\alpha\beta^2$,而 Y_i 的機率密度函數(Probability Density Function)是

$$f(y_i)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}y_i^{\alpha-1}e^{-\frac{y_i}{\beta}},\ 0<y_i<\infty$$

其中 $i=1,\cdots,n$,參數 $\alpha,\beta>0$,則可得出 $M_{Y_i}(t)=(1-\beta t)^{-\alpha}$ 是 Y_i 的動差生成函數,然後令 $X=\sum_{i=1}^{n}Y_i$,因 $Y_1,\cdots,Y_n$ 分配相同且獨立,所以 X 的動差生成函數可寫成

$$M_X(t)=E\left(e^{tX}\right)=E\left(e^{t\sum_{i=1}^{n}Y_i}\right)=\prod_{i=1}^{n}E\left(e^{tY_i}\right)=(1-\beta t)^{-\alpha n} \qquad (2.10)$$

由(2.10)式可得出 X 的機率質量函數為

$$f_X(x)=\frac{1}{\Gamma(\alpha n)\beta^{\alpha n}}x^{an-1}e^{-\frac{x}{\beta}},\ 0<x<\infty$$

其中參數 $\alpha,\beta>0$,由此可知 X 符合伽瑪分配,其平均數為 $\alpha n\beta$,變異數為 $\alpha n\beta^2$。接著把 X 變數標準化,令 $Z=\frac{X-\alpha\beta n}{\sqrt{\alpha n}\beta}$ 帶入(2.13)式可得

$$f_Z(z)=\frac{1}{\Gamma(\alpha n)\beta^{\alpha n}}\left(\alpha\beta n+z\sqrt{\alpha n}\beta\right)^{\alpha n-1}e^{-\frac{\alpha\beta n+z\sqrt{\alpha n}\beta}{\beta}}\sqrt{\alpha n}\beta,\ -\sqrt{\alpha n}<z<\infty$$

我們想要利用另一種連續型的分配去近似 Z 的分配,所以假設

$g(z)$是某一種連續型分配的機率密度函數,然後運用$g(z)$,讓$\lim_{n\to\infty} f_Z(z)$的值會很靠近$g(z)$，所以可得$\lim_{n\to\infty} f_Z(z) \approx g(z)$，用上面的條件去求出$g(z)$，再來考慮下列式子:

$$
\begin{aligned}
\frac{d}{dz}\ln\left(g(z)\right) &\approx \frac{d}{dz}\ln\left(\lim_{n\to\infty} f_Z(z)\right) \\
&= \lim_{n\to\infty}\left(\frac{d}{dz}\ln f_Z(z)\right) \\
&= \lim_{n\to\infty}\left(\frac{(\alpha n-1)\sqrt{\alpha n}\beta}{\alpha\beta n+z\sqrt{\alpha n}\beta}-\sqrt{\alpha n}\right) \\
&= \lim_{n\to\infty}\left(\frac{(\alpha n-1)\sqrt{\alpha n}\beta-\alpha\beta n\sqrt{\alpha n}-z\sqrt{\alpha n}\beta\sqrt{\alpha n}}{\alpha\beta n+z\sqrt{\alpha n}\beta}\right) \\
&= \lim_{n\to\infty}\left(\frac{-\sqrt{\alpha n}\beta-z\alpha\beta n}{\alpha\beta n+z\sqrt{\alpha n}\beta}\right) \\
&= \lim_{n\to\infty}\left(\frac{-\sqrt{\frac{\alpha}{n}}\beta-z\alpha\beta}{\alpha\beta+z\sqrt{\frac{\alpha}{n}}\beta}\right) \\
&= -z
\end{aligned}
\tag{2.11}
$$

然後對(2.11)式對z變數積分可得

$$\ln g(z)=\left(-\frac{z^2}{2}\right)+k \tag{2.12}$$

其中k是常數，再對(2.12)取 exp 可得

$$g(z)=\exp\left(\left(-\frac{z^2}{2}\right)+k\right) \tag{2.13}$$

由(2.13)很直觀看出$g(z)$會等於標準常態的機率密度函數，所以可說明在 n 趨近無限大的情況下,標準化後的伽瑪分配會近似標準常態分配。

第三章 多維分配的定理說明

假設數列$\{\mathbf{X}_1,\cdots,\mathbf{X}_n\}$由隨機向量$\mathbf{X}_i=(X_{i1},\cdots,X_{ik})$組成，$\mathbf{X}_1,\cdots,\mathbf{X}_n$分配相同且獨立，共同期望值向量是$(\mu_1,\cdots,\mu_k)$，共變異數矩陣則寫成

$\boldsymbol{\Sigma}=\begin{pmatrix}\sigma_{11} & \cdots & \sigma_{1k}\\ \vdots & \ddots & \vdots\\ \sigma_{k1} & \cdots & \sigma_{kk}\end{pmatrix}$，利用上述計算$\sum_{i=1}^{n}\mathbf{X}_i=(X_1,\cdots,X_k)$，所以$(X_1,\cdots,X_k)$的期望值向量是$(n\mu_1,\cdots,n\mu_k)$，共變異數矩陣是 n$\boldsymbol{\Sigma}$。運用 Hogg, Mckean, and Craig (2005 p.229)的定理，可得出多維分配的中央極限定理是

$\frac{1}{\sqrt{n}}\left(\frac{X_1-n\mu_1}{\sqrt{\sigma_{11}}},\cdots,\frac{X_k-n\mu_k}{\sqrt{\sigma_{kk}}}\right)$在 n 趨近無限大情況下，分配收斂到$N(\mathbf{0},\boldsymbol{\Sigma})$，其中$\mathbf{0}=(0,\cdots,0)_{1\times k}$，共變異數矩陣是

$$\boldsymbol{\Sigma}=\begin{pmatrix}1 & \cdots & \frac{\sigma_{1k}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{kk}}}\\ \vdots & \ddots & \vdots\\ \frac{\sigma_{k1}}{\sqrt{\sigma_{kk}}\sqrt{\sigma_{11}}} & \cdots & 1\end{pmatrix}$$

本章是推廣第二章使用的 Proschan(2008)方法，應用到分配是多維情況下，其中央極限定理會成立，我們選擇了兩個分配做為舉例，第 3.1 節是屬於離散型的多項式分配，而第 3.2 節是屬於連續型的多項伽瑪分配，可參考 Kotz, Balakrishnan, and Johnson (2000)。

第 3.1 節 多項式分配

首先我們利用 Proschan(2008)方法說明隨機向量是多項式分配時，其中央極限定理會成立，說明過程如下:

假設$(X_1,\cdots,X_k)$是一個隨機向量服從多項式分配，其參數為

$n,P_1,\cdots,P_k$，而$(X_1,\cdots,X_k)$的聯合機率質量函數為

$$f_{X_1,\cdots,X_k}(x_1,\cdots x_k)$$

$$=\frac{n!}{\left(\prod_{i=1}^{k}x_i!\right)\left(n-\sum_{i=1}^{k}x_i\right)!}\left(\prod_{i=1}^{k}P_i^{x_i}\right)\left(1-\sum_{i=1}^{k}P_i\right)^{n-\sum_{i=1}^{k}x_i},\ x_1,\cdots,x_k \text{ 為非負整數且 } x_1+\cdots+x_k\le n$$

其中$0<P_i<1$，$i=1,\cdots,k$且$0<\sum_{i=1}^{k}P_i<1$。而其聯合機率質量函數中有包含階層，會造成處理上的困難，所以我們考慮下列式子:

$$\frac{f_{X_1,\cdots,X_k}(x_1,\cdots,x_j+1,\cdots,x_k)}{f_{X_1,\cdots,X_k}(x_1,\cdots,x_j,\cdots,x_k)}$$

$$=\frac{\dfrac{n!}{\left(\prod_{i=1,i\neq j}^{k}x_i!\right)(x_j+1)!\left(n-\sum_{i=1}^{k}x_i-1\right)!}\left(\prod_{i=1,i\neq j}^{k}P_i^{x_i}\right)(P_j)^{x_j+1}\left(1-\sum_{i=1}^{k}P_i\right)^{n-\sum_{i=1}^{k}x_i-1}}{\dfrac{n!}{\left(\prod_{i=1}^{k}x_i!\right)\left(n-\sum_{i=1}^{k}x_i\right)!}\left(\prod_{i=1}^{k}P_i^{x_i}\right)\left(1-\sum_{i=1}^{k}P_i\right)^{n-\sum_{i=1}^{k}x_i}}$$

$$=\frac{\left(n-\sum_{i=1}^{k}x_i\right)P_j}{(x_j+1)\left(1-\sum_{i=1}^{k}P_i\right)} \tag{3.1}$$

其中$j=1,\cdots,k$，然後把$(X_1,\cdots,X_k)$變數標準化，即$Z_i=\dfrac{X_i-nP_i}{\sqrt{nP_i(1-P_i)}}$，且令

$z_i=\dfrac{x_i-nP_i}{\sqrt{nP_i(1-P_i)}}$，而$i=1,\cdots,j,\cdots,k$，把$z_i$帶入(3.1)式可得

$$\frac{f_{X_1,\cdots,X_k}(x_1,\cdots,x_j+1,\cdots,x_k)}{f_{X_1,\cdots,X_k}(x_1,\cdots,x_j,\cdots,x_k)}$$

$$=\frac{\left(n-\left(\sum_{i=1}^{k} z_i\sqrt{nP_i(1-P_i)}\right)-n\sum_{i=1}^{k}P_i\right)P_j}{\left(z_j\sqrt{nP_j(1-P_j)}+nP_j+1\right)\left(1-\sum_{i=1}^{k}P_i\right)}$$

$$=\frac{nP_j\left(1-\sum_{i=1}^{k}P_i\right)-P_j\left(\sum_{i=1}^{k}z_i\sqrt{nP_i(1-P_i)}\right)}{\left(nP_j\right)\left(1-\sum_{i=1}^{k}P_i\right)+\left(1-\sum_{i=1}^{k}P_i\right)+\left(z_j\sqrt{nP_j(1-P_j)}\right)\left(1-\sum_{i=1}^{k}P_i\right)}$$

$$=\frac{1-\dfrac{\left(\sum_{i=1}^{k}z_i\sqrt{P_i(1-P_i)}\right)}{\sqrt{n}\left(1-\sum_{i=1}^{k}P_i\right)}}{1+z_j\sqrt{\dfrac{1-P_j}{nP_j}+\dfrac{1}{nP_j}}} \tag{3.2}$$

接著令$\Delta_j=\frac{1}{\sqrt{nP_j(1-P_j)}}$，當$n\to\infty$時，$\Delta_j=\frac{1}{\sqrt{nP_j(1-P_j)}}\to 0$，然後把$\Delta_j$帶入(3.2)式可得

$$\frac{f_{X_1,\cdots,X_k}\left(x_1,\cdots,x_j+1,\cdots,x_k\right)}{f_{X_1,\cdots,X_k}\left(x_1,\cdots,x_j,\cdots,x_k\right)}=\frac{1-\dfrac{\Delta_j\sqrt{P_j(1-P_j)}\left(\sum_{i=1}^{k}z_i\sqrt{P_i(1-P_i)}\right)}{1-\sum_{i=1}^{k}P_i}}{1+z_j\Delta_j(1-P_j)+\Delta_j^2(1-P_j)} \tag{3.3}$$

因為$Z_i=\frac{X_i-nP_i}{\sqrt{nP_i(1-P_i)}}$、$z_i=\frac{x_i-nP_i}{\sqrt{nP_i(1-P_i)}}$和$\Delta_j=\frac{1}{\sqrt{nP_j(1-P_j)}}$，所以得出

$$f_{X_1,\cdots,X_k}\left(x_1,\cdots,x_j+1,\cdots,x_k\right)=f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j+\Delta_j,\cdots,z_k\right)$$

$$f_{X_1,\cdots,X_k}\left(x_1,\cdots,x_j,\cdots,x_k\right)=f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)$$

接著(3.3)式可改寫成

$$\frac{f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j+\Delta_j,\cdots,z_k\right)}{f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)}=\frac{1-\dfrac{\Delta_j\sqrt{P_j(1-P_j)}\left(\sum_{i=1}^{k}z_i\sqrt{P_i(1-P_i)}\right)}{1-\sum_{i=1}^{k}P_i}}{1+z_j\Delta_j(1-P_j)+\Delta_j^2(1-P_j)} \tag{3.4}$$

然後對(3.4)式取ln可得

$$
\begin{aligned}
&\ln\left(\frac{f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j+\Delta_j,\cdots,z_k\right)}{f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)}\right)\\
&=\ln\left(f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j+\Delta_j,\cdots,z_k\right)\right)-\ln\left(f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)\\
&=\ln\left(1-\frac{\Delta_j\sqrt{P_j(1-P_j)}\left(\sum_{i=1}^{k}z_i\sqrt{P_i(1-P_i)}\right)}{1-\sum_{i=1}^{k}P_i}\right)-\ln\left(1+z_j\Delta_j(1-P_j)+\Delta_j^2(1-P_j)\right)
\end{aligned}
$$

我們想要利用連續型的分配去近似$(Z_1,\cdots,Z_k)$的分配，所以先假設$g\left(z_1,\cdots,z_k\right)$是某一種連續型分配的聯合機率密度函數，然後讓

$$
\frac{g\left(z_1,\cdots,z_j+\Delta_j,\cdots,z_k\right)}{g\left(z_1,\cdots,z_j,\cdots,z_k\right)}\approx\frac{f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j+\Delta_j,\cdots,z_k\right)}{f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)}
$$

所以可得

$$
\begin{aligned}
&\ln\left(g\left(z_1,\cdots,z_j+\Delta_j,\cdots,z_k\right)\right)-\ln\left(g\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)\\
&\approx\ln\left(f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j+\Delta_j,\cdots,z_k\right)\right)-\ln\left(f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)
\end{aligned}
$$

接著利用上面的條件去求出$g\left(z_1,\cdots,z_k\right)$，然後我們考慮下列式子:

$$
\begin{aligned}
&\frac{\partial}{\partial z_j}\ln g\left(z_1,\cdots,z_j,\cdots,z_k\right)\\
&=\lim_{\Delta_j\to 0}\left(\frac{\ln\left(g\left(z_1,\cdots,z_j+\Delta_j,\cdots,z_k\right)\right)-\ln\left(g\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)}{\Delta_j}\right)\\
&\approx\lim_{\Delta_j\to 0}\left(\frac{\ln\left(f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j+\Delta_j,\cdots,z_k\right)\right)-\ln\left(f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)}{\Delta_j}\right)\\
&=\lim_{\Delta_j\to 0}\frac{\ln\left(1-\frac{\Delta_j\sqrt{P_j(1-P_j)}\left(\sum_{i=1}^{k}z_i\sqrt{P_i(1-P_i)}\right)}{1-\sum_{i=1}^{k}P_i}\right)-\ln\left(1+z_j\Delta_j(1-P_j)+\Delta_j^2(1-P_j)\right)}{\Delta_j}
\end{aligned}
$$

$$
= \lim_{\Delta_j \to 0} \frac{\ln\left(1 - \frac{\Delta_j \sqrt{P_j(1-P_j)}\left(\sum_{i=1}^{k} z_i \sqrt{P_i(1-P_i)}\right)}{1-\sum_{i=1}^{k} P_i}\right)}{\Delta_j} - \lim_{\Delta_j \to 0}\left(\frac{\ln\left(1 + z_j \Delta_j (1-P_j) + \Delta_j^2 (1-P_j)\right)}{\Delta_j}\right)
$$

$$
= \lim_{\Delta_j \to 0}\left(\frac{-\frac{\sqrt{P_j(1-P_j)}\left(\sum_{i=1}^{k} z_i \sqrt{P_i(1-P_i)}\right)}{1-\sum_{i=1}^{k} P_i}}{1 - \frac{\Delta_j \sqrt{P_j(1-P_j)}\left(\sum_{i=1}^{k} z_i \sqrt{P_i(1-P_i)}\right)}{1-\sum_{i=1}^{k} P_i}}\right) - \lim_{\Delta_j \to 0}\left(\frac{z_j(1-P_j) + 2\Delta_j(1-P_j)}{1 + z_j \Delta_j (1-P_j) + \Delta_j^2(1-P_j)}\right)
$$

$$
= -\frac{\sqrt{P_j(1-P_j)}\left(\sum_{i=1}^{k} z_i \sqrt{P_i(1-P_i)}\right)}{1-\sum_{i=1}^{k} P_i} - z_j(1-P_j)
$$

$$
= -\left(\frac{1}{P_j} + \frac{1}{1-\sum_{i=1}^{k} P_i}\right) P_j\left(1-P_j\right) z_j - \frac{\sum_{i=1, i\neq j}^{k} z_i \sqrt{P_i(1-P_i)P_j\left(1-P_j\right)}}{1-\sum_{i=1}^{k} P_i} \tag{3.5}
$$

然後用(3.5)式對 z_j 變數積分可得

$$
\ln g\left(z_1,\cdots,z_j,\cdots,z_k\right)
$$

$$
= -\left(\frac{1}{P_j} + \frac{1}{1-\sum_{i=1}^{k} P_i}\right) P_j\left(1-P_j\right)\left(\frac{z_j^2}{2}\right) - \frac{\sum_{i=1, i\neq j}^{k} z_i z_j \sqrt{P_i(1-P_i)P_j\left(1-P_j\right)}}{1-\sum_{i=1}^{k} P_i} + c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right) \tag{3.6}
$$

其中 $j = 1,\cdots,k$ ，$c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right)$ 是 $z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k$ 的函數，再對(3.6)式取 exp 可得

$$
g\left(z_1,\cdots,z_j,\cdots,z_k\right)
$$

$$=\exp\left(-\left(\frac{1}{P_j}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_j\left(1-P_j\right)\left(\frac{z_j^2}{2}\right)-\frac{\sum_{i=1,i\neq j}^{k}z_iz_j\sqrt{P_i(1-P_i)P_j\left(1-P_j\right)}}{1-\sum_{i=1}^{k}P_i}+c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right)\right)\quad(3.7)$$

從(3.7)式可知$g\left(z_1,\cdots,z_j,\cdots,z_k\right)=g\left(z_1,\cdots,z_{j'},\cdots,z_k\right)$，對所有的$j,j'=1,\cdots,k$

且$j'\neq j$，所以可得

$$-\left(\frac{1}{P_j}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_j\left(1-P_j\right)\left(\frac{z_j^2}{2}\right)-\frac{\sum_{i=1,i\neq j}^{k}z_iz_j\sqrt{P_i(1-P_i)P_j\left(1-P_j\right)}}{1-\sum_{i=1}^{k}P_i}+c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right)$$

$$=-\left(\frac{1}{P_{j'}}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_{j'}\left(1-P_{j'}\right)\left(\frac{z_{j'}^2}{2}\right)-\frac{\sum_{i=1,i\neq j'}^{k}z_iz_{j'}\sqrt{P_i(1-P_i)P_{j'}\left(1-P_{j'}\right)}}{1-\sum_{i=1}^{k}P_i}+c_{j'}\left(z_1,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k\right)\quad(3.8)$$

其中$c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right)$是$z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k$的函數，$c_{j'}\left(z_1,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k\right)$

是$z_1,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k$的函數，然後再討論$c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right)$，所以固

定j利用(3.8)式整理後可得

$$c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right)+\left(\frac{1}{P_{j'}}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_{j'}\left(1-P_{j'}\right)\left(\frac{z_{j'}^2}{2}\right)+\frac{\sum_{i=1,i\neq j,j'}^{k}z_iz_{j'}\sqrt{P_i(1-P_i)P_{j'}\left(1-P_{j'}\right)}}{1-\sum_{i=1}^{k}P_i}$$

$$=c_{j'}\left(z_1,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k\right)+\left(\frac{1}{P_j}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_j\left(1-P_j\right)\left(\frac{z_j^2}{2}\right)+\frac{\sum_{i=1,i\neq j,j'}^{k}z_iz_j\sqrt{P_i(1-P_i)P_j\left(1-P_j\right)}}{1-\sum_{i=1}^{k}P_i}\quad(3.9)$$

因為等式(3.9)左邊是$z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k$的函數，而等式(3.9)右邊是

$z_1,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k$的函數，所以可得

$$c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right)=-\left(\frac{1}{P_{j'}}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_{j'}\left(1-P_{j'}\right)\left(\frac{z_{j'}^2}{2}\right)-\frac{\sum_{i=1,i\neq j,j'}^{k}z_iz_{j'}\sqrt{P_i(1-P_i)P_{j'}\left(1-P_{j'}\right)}}{1-\sum_{i=1}^{k}P_i}$$

$$+k_{jj'}\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_{j'-1},z_{j'+1},\cdots,z_k\right) \qquad (3.10)$$

其中$k_{jj'}\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_{j'-1},z_{j'+1},\cdots,z_k\right)$是$z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_{j'-1},z_{j'+1},\cdots,z_k$的函數，然後由(3.10)看出，因為$j$固定，$j'=1,\cdots,k \quad j'\neq j$，所以有$k-1$個式子相等，然後兩兩解聯立方程式，但是去解$c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right)$最後帶回(3.7)式會得到一樣的$g\left(z_1,\cdots,z_j,\cdots,z_k\right)$，所以只需針對$c_1\left(z_2,\cdots,z_k\right)$做討論就好。

(3.10)式在$j=1$情況下可寫成

$$c_1\left(z_2,\cdots,z_k\right)$$
$$=-\left(\frac{1}{P_{j'}}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_{j'}\left(1-P_{j'}\right)\left(\frac{z_{j'}^2}{2}\right)-\frac{\sum_{i=2,i\neq j'}^{k}z_iz_{j'}\sqrt{P_i(1-P_i)P_{j'}\left(1-P_{j'}\right)}}{1-\sum_{i=1}^{k}P_i}+k_{1j'}\left(z_2,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k\right)$$

其中$j'=2,\cdots,k$，$c_1\left(z_2,\cdots,z_k\right)$是$z_2,\cdots,z_k$的函數，$k_{1j'}\left(z_2,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k\right)$是$z_2,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k$的函數，接下來我們可考慮下列步驟求解。

步驟1:先對$j'=2$和$j'=3$解聯立方程式

$$-\left(\frac{1}{P_2}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_2\left(1-P_2\right)\left(\frac{z_2^2}{2}\right)-\frac{\sum_{i=3}^{k}z_iz_2\sqrt{P_i(1-P_i)P_2\left(1-P_2\right)}}{1-\sum_{i=1}^{k}P_i}+k_{12}\left(z_3,\cdots,z_k\right)$$
$$=-\left(\frac{1}{P_3}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_3\left(1-P_3\right)\left(\frac{z_3^2}{2}\right)-\frac{\sum_{i=2,i\neq 3}^{k}z_iz_3\sqrt{P_i(1-P_i)P_3\left(1-P_3\right)}}{1-\sum_{i=1}^{k}P_i}+k_{13}\left(z_2,z_4,\cdots,z_k\right)$$

經過移項後

$$k_{12}\left(z_3,\cdots,z_k\right)+\left(\frac{1}{P_3}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_3\left(1-P_3\right)\left(\frac{z_3^2}{2}\right)+\frac{\sum_{i=4}^{k}z_iz_3\sqrt{P_i(1-P_i)P_3\left(1-P_3\right)}}{1-\sum_{i=1}^{k}P_i}$$

$$=k_{13}\left(z_2,z_4,\cdots,z_k\right)+\left(\frac{1}{P_2}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_2\left(1-P_2\right)\left(\frac{z_2^2}{2}\right)+\frac{\sum_{i=4}^{k}z_iz_2\sqrt{P_i(1-P_i)P_2\left(1-P_2\right)}}{1-\sum_{i=1}^{k}P_i}$$

上述等式左項為 $z_3,\cdots,z_k$ 的函數，而右項為 $z_2,z_4,\cdots,z_k$ 的函數，因此等式的左項和右項皆為 $z_4,\cdots,z_k$ 的函數，故可解出

$$k_{12}\left(z_3,\cdots,z_k\right)=-\left(\frac{1}{P_3}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_3\left(1-P_3\right)\left(\frac{z_3^2}{2}\right)-\frac{\sum_{i=4}^{k}z_iz_3\sqrt{P_i(1-P_i)P_3\left(1-P_3\right)}}{1-\sum_{i=1}^{k}P_i}+k_{123}\left(z_4,\cdots,z_k\right)$$

其中 $k_{12}\left(z_3,\cdots,z_k\right)$ 是 $z_3,\cdots,z_k$ 的函數，$k_{123}\left(z_4,\cdots,z_k\right)$ 是 $z_4,\cdots,z_k$ 的函數。

步驟 2:把 $k_{12}\left(z_3,\cdots,z_k\right)$ 帶回，可得

$$c_1\left(z_2,\cdots,z_k\right)$$

$$=-\sum_{i=2}^{3}\left(\frac{1}{P_i}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_i\left(1-P_i\right)\left(\frac{z_i^2}{2}\right)-\frac{\sum_{j'=2}^{3}\sum_{i>j'}^{k}z_iz_{j'}\sqrt{P_i(1-P_i)P_{j'}\left(1-P_{j'}\right)}}{1-\sum_{i=1}^{k}P_i}+k_{123}\left(z_4,\cdots,z_k\right)\quad(3.11)$$

步驟 3:再對(3.11)和 j'=4 解聯立方程式

$$-\sum_{i=2}^{3}\left(\frac{1}{P_i}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_i\left(1-P_i\right)\left(\frac{z_i^2}{2}\right)-\frac{\sum_{j'=2}^{3}\sum_{i>j'}^{k}z_iz_{j'}\sqrt{P_i(1-P_i)P_{j'}\left(1-P_{j'}\right)}}{1-\sum_{i=1}^{k}P_i}+k_{123}\left(z_4,\cdots,z_k\right)$$

$$=-\left(\frac{1}{P_4}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_4\left(1-P_4\right)\left(\frac{z_4^2}{2}\right)-\frac{\sum_{i=2,i\neq4}^{k}z_iz_4\sqrt{P_i(1-P_i)P_4\left(1-P_4\right)}}{1-\sum_{i=1}^{k}P_i}+k_{14}\left(z_2,z_3,z_5,\cdots,z_k\right)$$

經過移項後

$$k_{123}\left(z_4,\cdots,z_k\right)+\left(\frac{1}{P_4}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_4\left(1-P_4\right)\left(\frac{z_4^2}{2}\right)+\frac{\sum_{i=5}^{k}z_iz_4\sqrt{P_i(1-P_i)P_4\left(1-P_4\right)}}{1-\sum_{i=1}^{k}P_i}$$

$$=k_{14}\left(z_2,z_3,z_5,\cdots,z_k\right)+\sum_{i=2}^{3}\left(\frac{1}{P_i}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_i\left(1-P_i\right)\left(\frac{z_i^2}{2}\right)+\frac{\sum_{j'=2}^{3}\sum_{\substack{i>j'\\ i\neq 4}}^{k}z_iz_{j'}\sqrt{P_i(1-P_i)P_{j'}\left(1-P_{j'}\right)}}{1-\sum_{i=1}^{k}P_i}$$

上述等式左項為$z_4,\cdots,z_k$的函數，而右項為$z_2,z_3,z_5,\cdots,z_k$的函數，因此等式的左項和右項皆為$z_5,\cdots,z_k$的函數，故可解出

$$k_{123}\left(z_4,\cdots,z_k\right)=-\left(\frac{1}{P_4}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_4\left(1-P_4\right)\left(\frac{z_4^2}{2}\right)-\frac{\sum_{i=5}^{k}z_iz_4\sqrt{P_i(1-P_i)P_4\left(1-P_4\right)}}{1-\sum_{i=1}^{k}P_i}+k_{1234}(z_5,\cdots,z_k)$$

其中$k_{123}\left(z_4,\cdots,z_k\right)$是$z_4,\cdots,z_k$的函數，$k_{1234}\left(z_5,\cdots,z_k\right)$是$z_5,\cdots,z_k$的函數。

步驟 4:把$k_{123}\left(z_4,\cdots,z_k\right)$帶回(3.11)，可得

$$c_1\left(z_2,\cdots,z_k\right)=-\sum_{i=2}^{4}\left(\frac{1}{P_i}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_i\left(1-P_i\right)\left(\frac{z_i^2}{2}\right)-\frac{\sum_{j'=2}^{4}\sum_{i>j'}^{k}z_iz_{j'}\sqrt{P_i(1-P_i)P_{j'}\left(1-P_{j'}\right)}}{1-\sum_{i=1}^{k}P_i}+k_{1234}\left(z_5,\cdots,z_k\right)$$

接下來照以上步驟以此類推對j'=5,⋯,k 解聯立方程式，最後可得出

$$c_1\left(z_2,\cdots,z_k\right)$$

$$=-\sum_{i=2}^{k}\left(\frac{1}{P_i}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_i\left(1-P_i\right)\left(\frac{z_i^2}{2}\right)-\frac{\sum_{j'=2}^{k}\sum_{i>j'}^{k}z_iz_{j'}\sqrt{P_i(1-P_i)P_{j'}\left(1-P_{j'}\right)}}{1-\sum_{i=1}^{k}P_i}+c \quad (3.12)$$

其中$c_1\left(z_2,\cdots,z_k\right)$是$z_2,\cdots,z_k$的函數，$c$是常數，再把(3.12)帶回(3.7)式可得

$$g\left(z_1,\cdots,z_k\right)$$

$$=\exp\left(-\sum_{i=1}^{k}\left(\frac{1}{P_i}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_i\left(1-P_i\right)\left(\frac{z_i^2}{2}\right)-\frac{\sum_{j=1,i>j}^{k}z_iz_j\sqrt{P_i(1-P_i)P_j\left(1-P_j\right)}}{1-\sum_{i=1}^{k}P_i}+c\right) \quad (3.13)$$

其中c是常數。

假設$\left(Z_1',\cdots,Z_k'\right)$是一個隨機向量符合多維常態分配，其期望值向量是$\mathbf{0}$，共變異矩陣是

$$\Sigma=\begin{pmatrix} 1 & \frac{-P_1P_2}{\sqrt{P_1(1-P_1)P_2(1-P_2)}} & \cdots & \frac{-P_1P_{k-1}}{\sqrt{P_1(1-P_1)P_{k-1}(1-P_{k-1})}} & \frac{-P_1P_k}{\sqrt{P_1(1-P_1)P_k(1-P_k)}} \\ \frac{-P_1P_2}{\sqrt{P_1(1-P_1)P_2(1-P_2)}} & 1 & \cdots & \frac{-P_2P_{k-1}}{\sqrt{P_2(1-P_2)P_{k-1}(1-P_{k-1})}} & \frac{-P_2P_k}{\sqrt{P_2(1-P_2)P_k(1-P_k)}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-P_1P_{k-1}}{\sqrt{P_1(1-P_1)P_{k-1}(1-P_{k-1})}} & \frac{-P_2P_{k-1}}{\sqrt{P_2(1-P_2)P_{k-1}(1-P_{k-1})}} & \cdots & 1 & \frac{-P_{k-1}P_k}{\sqrt{P_{k-1}(1-P_{k-1})P_k(1-P_k)}} \\ \frac{-P_1P_k}{\sqrt{P_1(1-P_1)P_k(1-P_k)}} & \frac{-P_2P_k}{\sqrt{P_2(1-P_2)P_k(1-P_k)}} & \cdots & \frac{-P_{k-1}P_k}{\sqrt{P_{k-1}(1-P_{k-1})P_k(1-P_k)}} & 1 \end{pmatrix}$$

其聯合機率密度函數可寫成

$$f_{Z_1',\cdots,Z_k'}(z_1',\cdots,z_k')=\frac{1}{\left(2\pi\right)^{k/2}\left|\mathbf{\Sigma}\right|^{1/2}}\exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{\Sigma}^{-1}\mathbf{z}\right)$$

其中 $\mathbf{z}^T=\left(z_1'\cdots z_k'\right)$，而共變異數的反矩陣是

$$\Sigma^{-1}=\begin{pmatrix} \left(\frac{1}{P_1}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_1\left(1-P_1\right) & \frac{\sqrt{P_1(1-P_1)P_2\left(1-P_2\right)}}{1-\sum_{i=1}^{k}P_i} & \cdots & \frac{\sqrt{P_1(1-P_1)P_{k-1}\left(1-P_{k-1}\right)}}{1-\sum_{i=1}^{k}P_i} & \frac{\sqrt{P_1(1-P_1)P_k\left(1-P_k\right)}}{1-\sum_{i=1}^{k}P_i} \\ \frac{\sqrt{P_1(1-P_1)P_2\left(1-P_2\right)}}{1-\sum_{i=1}^{k}P_i} & \left(\frac{1}{P_2}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_2\left(1-P_2\right) & \cdots & \frac{\sqrt{P_2(1-P_2)P_{k-1}\left(1-P_{k-1}\right)}}{1-\sum_{i=1}^{k}P_i} & \frac{\sqrt{P_2(1-P_2)P_k\left(1-P_k\right)}}{1-\sum_{i=1}^{k}P_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\sqrt{P_1(1-P_1)P_{k-1}\left(1-P_{k-1}\right)}}{1-\sum_{i=1}^{k}P_i} & \frac{\sqrt{P_2(1-P_2)P_{k-1}\left(1-P_{k-1}\right)}}{1-\sum_{i=1}^{k}P_i} & \cdots & \left(\frac{1}{P_{k-1}}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_{k-1}\left(1-P_{k-1}\right) & \frac{\sqrt{P_{k-1}(1-P_{k-1})P_k\left(1-P_k\right)}}{1-\sum_{i=1}^{k}P_i} \\ \frac{\sqrt{P_1(1-P_1)P_k\left(1-P_k\right)}}{1-\sum_{i=1}^{k}P_i} & \frac{\sqrt{P_2(1-P_2)P_k\left(1-P_k\right)}}{1-\sum_{i=1}^{k}P_i} & \cdots & \frac{\sqrt{P_{k-1}(1-P_{k-1})P_k\left(1-P_k\right)}}{1-\sum_{i=1}^{k}P_i} & \left(\frac{1}{P_k}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_k\left(1-P_k\right) \end{pmatrix}$$

所以把聯合機率密度函數展開後得

$$f_{Z_1',\cdots,Z_k'}(z_1',\cdots,z_k')$$

$$=\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\left(-\sum_{i=1}^{k}\left(\frac{1}{P_i}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_i(1-P_i)\left(\frac{(z_i')^2}{2}\right)-\frac{\sum_{j=1,i>j}^{k}z_i'z_j'\sqrt{P_i(1-P_i)P_j(1-P_j)}}{1-\sum_{i=1}^{k}P_i}\right)\quad(3.14)$$

而由(3.13)和(3.14)可得

$$g(z_1,\cdots,z_k)$$

$$=\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\left(-\sum_{i=1}^{k}\left(\frac{1}{P_i}+\frac{1}{1-\sum_{i=1}^{k}P_i}\right)P_i(1-P_i)\left(\frac{z_i^2}{2}\right)-\frac{\sum_{j=1,i>j}^{k}z_iz_j\sqrt{P_i(1-P_i)P_j(1-P_j)}}{1-\sum_{i=1}^{k}P_i}\right)\quad(3.15)$$

由(3.15)可以說明在 n 趨近無限大的情況下，標準化後的多項式分配會近似期望值向量是$\mathbf{0}$，共變異矩陣是$\mathbf{\Sigma}$的多維常態分配。

第 3.2 節 Mathi and Moschopulos's 多項伽瑪分配

此節繼續應用 Proschan(2008)方法，來說明隨機向量是 Mathi and Moschopulos's 多項伽瑪分配時，其中央極限定理成立，說明過程如下：

假設數列$\{\mathbf{Y}_1,\cdots,\mathbf{Y}_n\}$由隨機向量$\mathbf{Y}_l=(Y_{l1},\cdots,Y_{lk})$組成，其中$l=1,\cdots,n$，而$\mathbf{Y}_1,\cdots,\mathbf{Y}_n$分配相同且獨立，且$\mathbf{Y}_l=(Y_{l1},\cdots,Y_{lk})$符合 Mathi and Moschopulos's 多項伽瑪分配，而$(Y_{l1},\cdots,Y_{lk})$的聯合機率密度函數為

$$f_{Y_{l1},\cdots,Y_{lk}}(y_1,\cdots,y_k)=\frac{(y_1-\gamma_1)^{\alpha_1-1}}{\beta^{\sum_{i=1}^{k}\alpha_i}\prod_{i=1}^{k}\Gamma(\alpha_i)}\prod_{i=2}^{k}(y_i-y_{i-1}-\gamma_i)^{\alpha_i-1}e^{-\left(y_k-\left(\sum_{i=1}^{k}\gamma_i\right)\right)/\beta}$$

其中$y_1>\gamma_1$，$y_{i-1}<y_i-\gamma_i\ (i=2,\cdots,k)$，$y_k<\infty$，$\beta>0$，$\alpha_i>0$，$\gamma_i$是實數，$i=1,\cdots,k$，則可得出

$$M_{Y_{l1},\cdots,Y_{lk}}(t_1,\cdots,t_k)=\frac{e^{\gamma_1(t_1+\cdots+t_k)}}{(1-\beta(t_1+\cdots+t_k))^{\alpha_1}}\frac{e^{\gamma_2(t_2+\cdots+t_k)}}{(1-\beta(t_2+\cdots+t_k))^{\alpha_2}}\cdots\frac{e^{\gamma_k t_k}}{(1-\beta t_k)^{\alpha_k}}$$

是$(Y_{l1},\cdots,Y_{lk})$的聯合動差生成函數，然後令$(X_1,\cdots,X_k)=\sum_{l=1}^{n}\mathbf{Y}_l$，因$\mathbf{Y}_1,\cdots,\mathbf{Y}_n$分配相同且獨立，所以$(X_1,\cdots,X_k)$的動差生成函數可寫成

$$\begin{aligned}&M_{X_1,\cdots,X_k}(t_1,\cdots,t_k)\\&=E\left(e^{\sum_{i=1}^{k}t_iX_i}\right)\\&=E\left(e^{\sum_{i=1}^{k}t_i\sum_{l=1}^{n}Y_{li}}\right)\end{aligned}$$

$$=\prod_{l=1}^{n} E\left(e^{\sum_{i=1}^{k} t_i Y_{li}}\right)$$

$$=\frac{e^{\gamma_1 n(t_1+\cdots+t_k)}}{\left(1-\beta\left(t_1+\cdots+t_k\right)\right)^{\alpha_1 n}}\frac{e^{\gamma_2 n(t_2+\cdots+t_k)}}{\left(1-\beta\left(t_2+\cdots+t_k\right)\right)^{\alpha_2 n}}\cdots\frac{e^{\gamma_k n t_k}}{\left(1-\beta t_k\right)^{\alpha_k n}} \quad (3.16)$$

由(3.16)式可得出$(X_1,\cdots,X_k)$的聯合機率密度函數為

$$f_{X_1,\cdots,X_k}\left(x_1,\cdots,x_k\right)=\frac{\left(x_1-n\gamma_1\right)^{\alpha_1 n-1}}{\beta^{\sum_{i=1}^{k}\alpha_i n}\prod_{i=1}^{k}\Gamma\left(\alpha_i n\right)}\prod_{i=2}^{k}\left(x_i-x_{i-1}-\gamma_i n\right)^{\alpha_i n-1} e^{-\left(y_k-\left(\sum_{i=1}^{k}\gamma_i\right)n\right)\Big/\beta} \quad (3.17)$$

其中$x_1>n\gamma_1$，$x_{i-1}<x_i-n\gamma_i\ (i=2,\cdots,k)$，$x_k<\infty$。而$(X_1,\cdots,X_k)$的期望值為 $E(X_i)=(\alpha_i^*\beta+\gamma_i^*)$n，變異數$Var(X_i)=\alpha_i^* n\beta^2$，其中$i=1,\cdots,k$，$\alpha_i^*=\sum_{m=1}^{i}\alpha_m$，$\gamma_i^*=\sum_{m=1}^{i}\gamma_m$。

接著把$(X_1,\cdots,X_k)$變數標準化，令$Z_i=\dfrac{X_i-E(X_i)}{\sqrt{Var(X_i)}}$，其中$i=1,\cdots,k$，經由轉換公式可得$(Z_1,\cdots,Z_k)$的聯合機率密度函數為

$$f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_k\right)=$$

$$\prod_{i=1}^{k}\sqrt{\sum_{m=1}^{i}\alpha_m n}\beta^k \frac{\left(\sqrt{\alpha_1 n}\beta z_1+\alpha_1 n\beta\right)^{\alpha_1 n-1}}{\beta^{\sum_{i=1}^{k}\alpha_i n}\prod_{i=1}^{k}\Gamma\left(\alpha_i n\right)}\left(\sqrt{\left(\alpha_1+\alpha_2\right)n}\beta z_2-\sqrt{\alpha_1 n}\beta z_1+\alpha_2 n\beta\right)^{\alpha_2 n-1}\cdots$$

$$\left(\sqrt{\left(\sum_{i=1}^{k}\alpha_i\right)n}\beta z_k-\sqrt{\left(\sum_{i=1}^{k-1}\alpha_i\right)n}\beta z_{k-1}+\alpha_k n\beta\right)^{\alpha_k n-1} e^{-\left(\sqrt{\left(\sum_{i=1}^{k}\alpha_i\right)n}\beta z_k+\left(\sum_{i=1}^{k}\alpha_i\right)n\beta\right)\Big/\beta}$$

其中$z_i>-\sqrt{\sum_{m=1}^{i}\alpha_m n}$，$\alpha_i>0$，$\beta>0$，$i=1,\cdots,k$。

我們想要利用另一種連續型的分配去近似$(Z_1,\cdots,Z_k)$的分配，所以假設$g(z_1,\cdots,z_k)$是某一種連續型分配的機率密度函數，然後運用$g(z_1,\cdots,z_k)$，讓$\lim_{n\to\infty} f_{Z_1,\cdots,Z_k}(z_1,\cdots,z_k)$的值會很靠近$g(z_1,\cdots,z_k)$，所以可得$\lim_{n\to\infty} f_{Z_1,\cdots,Z_k}(z_1,\cdots,z_k)\approx g(z_1,\cdots,z_k)$，用上面的條件去求出$g(z_1,\cdots,z_k)$，再來考慮下列式子:

$$\frac{\partial}{\partial z_j}\ln\left(g\left(z_1,\cdots,z_j,\cdots,z_k\right)\right) \tag{3.18}$$

(3.18)式在$j=1$情況下

$$\frac{\partial}{\partial z_1}\ln\left(g(z_1,\cdots,z_k)\right)\approx\frac{\partial}{\partial z_1}\ln\left(\lim_{n\to\infty} f_{Z_1,\cdots,Z_k}(z_1,\cdots,z_k)\right)$$

$$=\lim_{n\to\infty}\left(\frac{(\alpha_1 n-1)\sqrt{\alpha_1 n}}{\sqrt{\alpha_1 n}z_1+\alpha_1 n}+\frac{(\alpha_2 n-1)\left(-\sqrt{\alpha_1 n}\right)}{\sqrt{(\alpha_1+\alpha_2)n}z_2+\alpha_2 n-\sqrt{\alpha_1 n}z_1}\right)$$

$$=\lim_{n\to\infty}\left(\left(\frac{\alpha_1 n\sqrt{\alpha_1 n}-\sqrt{\alpha_1 n}}{\sqrt{\alpha_1 n}z_1+\alpha_1 n}-\sqrt{\alpha_1 n}\right)+\left(\frac{-\alpha_2 n\sqrt{\alpha_1 n}+\sqrt{\alpha_1 n}}{\sqrt{(\alpha_1+\alpha_2)n}z_2+\alpha_2 n-\sqrt{\alpha_1 n}z_1}+\sqrt{\alpha_1 n}\right)\right)$$

$$=\lim_{n\to\infty}\left(\left(\frac{-\sqrt{\alpha_1 n}-\alpha_1 n z_1}{\sqrt{\alpha_1 n}z_1+\alpha_1 n}\right)+\left(\frac{\sqrt{\alpha_1 n}+\sqrt{(\alpha_1+\alpha_2)n}z_2\sqrt{\alpha_1 n}-\alpha_1 n z_1}{\sqrt{(\alpha_1+\alpha_2)n}z_2+\alpha_2 n-\sqrt{\alpha_1 n}z_1}\right)\right)$$

$$=\lim_{n\to\infty}\left(\left(\frac{-\sqrt{\frac{\alpha_1}{n}}-\alpha_1 z_1}{\sqrt{\frac{\alpha_1}{n}}z_1+\alpha_1}\right)+\left(\frac{\sqrt{\frac{\alpha_1}{n}}+\sqrt{(\alpha_1+\alpha_2)}z_2\sqrt{\alpha_1}-\alpha_1 z_1}{\sqrt{\frac{(\alpha_1+\alpha_2)}{n}}z_2+\alpha_2-\sqrt{\frac{\alpha_1}{n}}z_1}\right)\right)$$

$$=\frac{-\alpha_1 z_1}{\alpha_1}+\frac{\sqrt{(\alpha_1+\alpha_2)}z_2\sqrt{\alpha_1}-\alpha_1 z_1}{\alpha_2}$$

$$=-\frac{\alpha_1+\alpha_2}{\alpha_2}z_1+\frac{\sqrt{\alpha_1(\alpha_1+\alpha_2)}}{\alpha_2}z_2 \tag{3.19}$$

(3.18)式在$j=2,\cdots,k-1$情況下

$$\frac{\partial}{\partial z_j}\ln\left(g\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)\approx\frac{\partial}{\partial z_j}\ln\left(\lim_{n\to\infty}f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)$$

$$=\lim_{n\to\infty}\left(\frac{\left(\alpha_j n-1\right)\left(\sqrt{\sum_{i=1}^{j}\alpha_i n}\right)}{\sqrt{\sum_{i=1}^{j}\alpha_i n z_j+\alpha_j n-\sqrt{\sum_{i=1}^{j-1}\alpha_i n z_{j-1}}}}+\frac{\left(\alpha_{j+1}n-1\right)\left(-\sqrt{\sum_{i=1}^{j}\alpha_i n}\right)}{\sqrt{\sum_{i=1}^{j+1}\alpha_i n z_{j+1}+\alpha_{j+1}n-\sqrt{\sum_{i=1}^{j}\alpha_i n z_j}}}\right)$$

$$=\lim_{n\to\infty}\left(\left(\frac{\alpha_j n\sqrt{\sum_{i=1}^{j}\alpha_i n}-\sqrt{\sum_{i=1}^{j}\alpha_i n}}{\sqrt{\sum_{i=1}^{j}\alpha_i n z_j+\alpha_j n-\sqrt{\sum_{i=1}^{j-1}\alpha_i n z_{j-1}}}}-\sqrt{\sum_{i=1}^{j}\alpha_i n}\right)+\left(\frac{-\alpha_{j+1}n\sqrt{\sum_{i=1}^{j}\alpha_i n}+\sqrt{\sum_{i=1}^{j}\alpha_i n}}{\sqrt{\sum_{i=1}^{j+1}\alpha_i n z_{j+1}+\alpha_{j+1}n-\sqrt{\sum_{i=1}^{j}\alpha_i n z_j}}}+\sqrt{\sum_{i=1}^{j}\alpha_i n}\right)\right)$$

$$=\lim_{n\to\infty}\left(\left(\frac{-\sqrt{\sum_{i=1}^{j}\alpha_i n}-\sum_{i=1}^{j}\alpha_i n z_j+\sqrt{\sum_{i=1}^{j-1}\alpha_i n}\sqrt{\sum_{i=1}^{j}\alpha_i n z_{j-1}}}{\sqrt{\sum_{i=1}^{j}\alpha_i n z_j+\alpha_j n-\sqrt{\sum_{i=1}^{j-1}\alpha_i n z_{j-1}}}}\right)+\left(\frac{\sqrt{\sum_{i=1}^{j}\alpha_i n}+\sqrt{\sum_{i=1}^{j+1}\alpha_i n}\sqrt{\sum_{i=1}^{j}\alpha_i n z_{j+1}}-\sum_{i=1}^{j}\alpha_i n z_j}{\sqrt{\sum_{i=1}^{j+1}\alpha_i n z_{j+1}+\alpha_{j+1}n-\sqrt{\sum_{i=1}^{j}\alpha_i n z_j}}}\right)\right)$$

$$=\lim_{n\to\infty}\left(\left(\frac{-\sqrt{\frac{\sum_{i=1}^{j}\alpha_i}{n}}-\sum_{i=1}^{j}\alpha_i z_j+\sqrt{\left(\sum_{i=1}^{j-1}\alpha_i\right)\left(\sum_{i=1}^{j}\alpha_i\right)}z_{j-1}}{\sqrt{\frac{\sum_{i=1}^{j}\alpha_i}{n}}z_j+\alpha_j-\sqrt{\frac{\sum_{i=1}^{j-1}\alpha_i}{n}}z_{j-1}}\right)+\left(\frac{\sqrt{\frac{\sum_{i=1}^{j}\alpha_i}{n}}+\sqrt{\left(\sum_{i=1}^{j+1}\alpha_i\right)\left(\sum_{i=1}^{j}\alpha_i\right)}z_{j+1}-\sum_{i=1}^{j}\alpha_i z_j}{\sqrt{\frac{\sum_{i=1}^{j+1}\alpha_i}{n}}z_{j+1}+\alpha_{j+1}-\sqrt{\frac{\sum_{i=1}^{j}\alpha_i}{n}}z_j}\right)\right)$$

$$=\frac{-\sum_{i=1}^{j}\alpha_i z_j+\sqrt{\left(\sum_{i=1}^{j-1}\alpha_i\right)\left(\sum_{i=1}^{j}\alpha_i\right)}z_{j-1}}{\alpha_j}+\frac{\sqrt{\left(\sum_{i=1}^{j+1}\alpha_i\right)\left(\sum_{i=1}^{j}\alpha_i\right)}z_{j+1}-\sum_{i=1}^{j}\alpha_i z_j}{\alpha_{j+1}}$$

$$=\frac{\sqrt{\left(\sum_{i=1}^{j-1}\alpha_i\right)\left(\sum_{i=1}^{j}\alpha_i\right)}z_{j-1}}{\alpha_j}-\left(\sum_{i=1}^{j}\alpha_i\right)\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right)z_j+\frac{\sqrt{\left(\sum_{i=1}^{j}\alpha_i\right)\left(\sum_{i=1}^{j+1}\alpha_i\right)}z_{j+1}}{\alpha_{j+1}} \qquad (3.20)$$

(3.18)式在 $j=k$ 情况下

$$\frac{\partial}{\partial z_k}\ln\left(g\left(z_1,\cdots,z_k\right)\right)\approx\frac{\partial}{\partial z_k}\ln\left(\lim_{n\to\infty}f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_k\right)\right)$$

$$=\lim_{n\to\infty}\left(\frac{(\alpha_k n-1)\left(\sqrt{\sum_{i=1}^{j}\alpha_i n}\right)}{\sqrt{\sum_{i=1}^{k}\alpha_i n z_k+\alpha_k n-\sqrt{\sum_{i=1}^{k-1}\alpha_i n z_{k-1}}}}-\sqrt{\sum_{i=1}^{k}\alpha_i n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{-\sqrt{\sum_{i=1}^{j}\alpha_i n}-\sum_{i=1}^{k}\alpha_i n z_k+\sqrt{\sum_{i=1}^{k-1}\alpha_i n}\sqrt{\sum_{i=1}^{k}\alpha_i n z_{k-1}}}{\sqrt{\sum_{i=1}^{k}\alpha_i n z_k+\alpha_k n-\sqrt{\sum_{i=1}^{k-1}\alpha_i n z_{k-1}}}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{-\sqrt{\frac{\sum_{i=1}^{j}\alpha_i}{n}}-\sum_{i=1}^{k}\alpha_i z_k+\sqrt{\left(\sum_{i=1}^{k-1}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i\right)}z_{k-1}}{\sqrt{\frac{\sum_{i=1}^{k}\alpha_i}{n}z_k+\alpha_k-\sqrt{\frac{\sum_{i=1}^{k-1}\alpha_i}{n}z_{k-1}}}}\right)$$

$$=-\frac{\left(\sum_{i=1}^{k}\alpha_i\right)z_k}{\alpha_k}+\frac{\sqrt{\left(\sum_{i=1}^{k-1}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i\right)}z_{k-1}}{\alpha_k} \tag{3.21}$$

經由(3.19)(3.20)(3.21)整理過後得

$$\frac{\partial}{\partial z_j}\ln\left(g\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)\approx\frac{\partial}{\partial z_j}\ln\left(\lim_{n\to\infty}f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)$$

$$=\begin{cases}-\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)z_1+\frac{\sqrt{\alpha_1\left(\alpha_1+\alpha_2\right)}z_2}{\alpha_2},j=1\\ \frac{\sqrt{\left(\sum_{i=1}^{j-1}\alpha_i\right)\left(\sum_{i=1}^{j}\alpha_i\right)}z_{j-1}}{\alpha_j}-\left(\sum_{i=1}^{j}\alpha_i\right)\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right)z_j+\frac{\sqrt{\left(\sum_{i=1}^{j}\alpha_i\right)\left(\sum_{i=1}^{j+1}\alpha_i\right)}z_{j+1}}{\alpha_{j+1}},j=2,\cdots,k-1\\ -\frac{\left(\sum_{i=1}^{k}\alpha_i\right)}{\alpha_k}z_k+\frac{\sqrt{\left(\sum_{i=1}^{k-1}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i\right)}z_{k-1}}{\alpha_k},j=k\end{cases} \tag{3.22}$$

然後對(3.22) 式對 z_j 變數積分可得

$$\ln\left(g\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)\approx\ln\left(\lim_{n\to\infty}f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)$$

$$=-\frac{\alpha_1+\alpha_2}{\alpha_2}\left(\frac{z_1^2}{2}\right)+\frac{\sqrt{\alpha_1\left(\alpha_1+\alpha_2\right)}z_1 z_2}{\alpha_2}+c_1\left(z_2,\cdots,z_k\right),j=1$$

$$\ln\left(g\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)\approx\ln\left(\lim_{n\to\infty}f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)$$

$$=\frac{\sqrt{\left(\sum_{i=1}^{j-1}\alpha_i\right)\left(\sum_{i=1}^{j}\alpha_i\right)}z_{j-1}z_j}{\alpha_j}-\left(\sum_{i=1}^{j}\alpha_i\right)\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right)\left(\frac{z_j^2}{2}\right)$$

$$+\frac{\sqrt{\left(\sum_{i=1}^{j}\alpha_i\right)\left(\sum_{i=1}^{j+1}\alpha_i\right)}z_jz_{j+1}}{\alpha_{j+1}}+c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right),\ j=2,\cdots,k-1$$

$$\ln\left(g_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)\approx\ln\left(\lim_{n\to\infty}f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)\right)$$

$$=-\frac{\left(\sum_{i=1}^{k}\alpha_i\right)}{\alpha_k}\left(\frac{z_k^2}{2}\right)+\frac{\sqrt{\left(\sum_{i=1}^{k-1}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i\right)}z_{k-1}z_k}{\alpha_k}+c_k\left(z_1,\cdots,z_{k-1}\right),\ j=k$$

其中 $j=1,\cdots,k$ ，$c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right)$是 $z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k$ 的函數，再對上式取 exp 可得

$$g\left(z_1,\cdots,z_j,\cdots,z_k\right)\approx\lim_{n\to\infty}f_{Z_1,\cdots,Z_k}\left(z_1,\cdots,z_j,\cdots,z_k\right)$$

$$=\begin{cases}\exp\left(-\frac{\alpha_1+\alpha_2}{\alpha_2}\left(\frac{z_1^2}{2}\right)+\frac{\sqrt{\alpha_1\left(\alpha_1+\alpha_2\right)}z_1z_2}{\alpha_2}+c_1\left(z_2,\cdots,z_k\right)\right),\ j=1\\ \exp\left(\frac{\sqrt{\left(\sum_{i=1}^{j-1}\alpha_i\right)\left(\sum_{i=1}^{j}\alpha_i\right)}z_{j-1}z_j}{\alpha_j}-\left(\sum_{i=1}^{j}\alpha_i\right)\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right)\left(\frac{z_j^2}{2}\right)\right.\\ \quad\left.+\frac{\sqrt{\left(\sum_{i=1}^{j}\alpha_i\right)\left(\sum_{i=1}^{j+1}\alpha_i\right)}z_jz_{j+1}}{\alpha_{j+1}}+c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right)\right),\ j=2,\cdots,k-1\\ \exp\left(-\frac{\left(\sum_{i=1}^{k}\alpha_i\right)}{\alpha_k}\left(\frac{z_k^2}{2}\right)+\frac{\sqrt{\left(\sum_{i=1}^{k-1}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i\right)}z_{k-1}z_k}{\alpha_k}+c_k\left(z_1,\cdots,z_{k-1}\right)\right),\ j=k\end{cases}\tag{3.23}$$

因為去解$c_j\left(z_1,\cdots,z_{j-1},z_{j+1},\cdots,z_k\right)$ ，$j=1,\cdots,k$ ，最後帶回(3.23)式時會得到

一樣的$g\left(z_1,\cdots,z_j,\cdots,z_k\right)$，所以針對$c_1\left(z_2,\cdots,z_k\right)$做討論，而在(3.23)式中可知

$$g\left(z_1,\cdots,z_j,\cdots,z_k\right)=g\left(z_1,\cdots,z_{j'},\cdots,z_k\right)$$

其中 $j=1$ ， $j'=2,\cdots,k$ 。所以可得

$$\begin{aligned}
&-\frac{\alpha_1+\alpha_2}{\alpha_2}\left(\frac{z_1^2}{2}\right)+\frac{\sqrt{\alpha_1\left(\alpha_1+\alpha_2\right)}z_1z_2}{\alpha_2}+c_1\left(z_2,\cdots,z_k\right)\\
&=\frac{\sqrt{\left(\sum_{i=1}^{j'-1}\alpha_i\right)\left(\sum_{i=1}^{j'}\alpha_i\right)}z_{j'-1}z_{j'}}{\alpha_{j'}}-\left(\sum_{i=1}^{j'}\alpha_i\right)\left(\frac{1}{\alpha_{j'}}+\frac{1}{\alpha_{j'+1}}\right)\left(\frac{z_{j'}^2}{2}\right)\\
&\quad+\frac{\sqrt{\left(\sum_{i=1}^{j'}\alpha_i\right)\left(\sum_{i=1}^{j'+1}\alpha_i\right)}z_{j'}z_{j'+1}}{\alpha_{j'+1}}+c_{j'}\left(z_1,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k\right)\ \ j'=2,\cdots k-1\\
&=-\frac{\left(\sum_{i=1}^{k}\alpha_i\right)}{\alpha_k}\left(\frac{z_k^2}{2}\right)+\frac{\sqrt{\left(\sum_{i=1}^{k-1}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i\right)}z_{k-1}z_k}{\alpha_k}+c_k\left(z_1,\cdots,z_{k-1}\right),\ j'=k
\end{aligned}\tag{3.24}$$

其中 j'=$2,\cdots,k$ ， $c_1\left(z_2,\cdots,z_k\right)$是$z_2,\cdots,z_k$的函數，$c_{j'}\left(z_1,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k\right)$是$z_1,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k$的函數，經由(3.24)式可知

$$\begin{aligned}
&c_1\left(z_2,\cdots,z_k\right)+\left(\alpha_1+\alpha_2\right)\left(\frac{1}{\alpha_2}+\frac{1}{\alpha_3}\right)\left(\frac{z_2^2}{2}\right)-\frac{\sqrt{\left(\sum_{i=1}^{2}\alpha_i\right)\left(\sum_{i=1}^{3}\alpha_i\right)}z_2z_3}{\alpha_3}\\
&=c_2\left(z_1,z_3,\cdots,z_k\right)+\frac{\alpha_1+\alpha_2}{\alpha_2}\left(\frac{z_1^2}{2}\right),j'=2
\end{aligned}$$

$$\begin{aligned}
&c_1\left(z_2,\cdots,z_k\right)-\frac{\sqrt{\left(\sum_{i=1}^{j'-1}\alpha_i\right)\left(\sum_{i=1}^{j'}\alpha_i\right)}z_{j'-1}z_{j'}}{\alpha_{j'}}+\left(\sum_{i=1}^{j'}\alpha_i\right)\left(\frac{1}{\alpha_{j'}}+\frac{1}{\alpha_{j'+1}}\right)\left(\frac{z_{j'}^2}{2}\right)-\frac{\sqrt{\left(\sum_{i=1}^{j'}\alpha_i\right)\left(\sum_{i=1}^{j'+1}\alpha_i\right)}z_{j'}z_{j'+1}}{\alpha_{j'+1}}\\
&=c_{j'}\left(z_1,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k\right)+\frac{\alpha_1+\alpha_2}{\alpha_2}\left(\frac{z_1^2}{2}\right)-\frac{\sqrt{\alpha_1\left(\alpha_1+\alpha_2\right)}z_1z_2}{\alpha_2},\ j'=3,\cdots,k-1
\end{aligned}$$

$$c_1(z_2,\cdots,z_k)+\frac{\left(\sum_{i=1}^{k}\alpha_i\right)}{\alpha_k}\left(\frac{z_k^2}{2}\right)-\frac{\sqrt{\left(\sum_{i=1}^{k-1}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i\right)}z_{k-1}z_k}{\alpha_k}$$

$$=c_k(z_1,\cdots,z_{k-1})+\frac{\alpha_1+\alpha_2}{\alpha_2}\left(\frac{z_1^2}{2}\right)-\frac{\sqrt{\alpha_1(\alpha_1+\alpha_2)}z_1z_2}{\alpha_2},\ j'=k$$

把上述等式整理後可得

$$c_1(z_2,\cdots,z_k)$$

$$=\begin{cases}-(\alpha_1+\alpha_2)\left(\frac{1}{\alpha_2}+\frac{1}{\alpha_3}\right)\left(\frac{z_2^2}{2}\right)+\frac{\sqrt{\left(\sum_{i=1}^{2}\alpha_i\right)\left(\sum_{i=1}^{3}\alpha_i\right)}z_2z_3}{\alpha_3}+k_{12}(z_3,\cdots,z_k),j'=2\\ \frac{\sqrt{\left(\sum_{i=1}^{j'-1}\alpha_i\right)\left(\sum_{i=1}^{j'}\alpha_i\right)}z_{j'-1}z_{j'}}{\alpha_{j'}}-\left(\sum_{i=1}^{j'}\alpha_i\right)\left(\frac{1}{\alpha_{j'}}+\frac{1}{\alpha_{j'+1}}\right)\left(\frac{z_{j'}^2}{2}\right)+\frac{\sqrt{\left(\sum_{i=1}^{j'}\alpha_i\right)\left(\sum_{i=1}^{j'+1}\alpha_i\right)}z_{j'}z_{j'+1}}{\alpha_{j'+1}}\\ +k_{1j'}(z_2,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k)\ \ j'=3,\cdots,k-1\\ -\frac{\left(\sum_{i=1}^{k}\alpha_i\right)}{\alpha_k}\left(\frac{z_k^2}{2}\right)+\frac{\sqrt{\left(\sum_{i=1}^{k-1}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i\right)}z_{k-1}z_k}{\alpha_k}+k_{1k}(z_2,\cdots,z_{k-1}),j'=k\end{cases}$$

其中 $j'=2,\cdots,k$ ，$c_1(z_2,\cdots,z_k)$ 是 $z_2,\cdots,z_k$ 的函數，$k_{1j'}(z_2,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k)$ 是 $z_2,\cdots,z_{j'-1},z_{j'+1},\cdots,z_k$ 的函數，接下來我們可考慮下列步驟求解。

步驟 1:先對 $j'=2$ 和 $j'=3$ 解聯立方程式

$$-\left(\sum_{i=1}^{2}\alpha_i\right)\left(\frac{1}{\alpha_2}+\frac{1}{\alpha_3}\right)\left(\frac{z_2^2}{2}\right)+\frac{\sqrt{\left(\sum_{i=1}^{2}\alpha_i\right)\left(\sum_{i=1}^{3}\alpha_i\right)}z_2z_3}{\alpha_3}+k_{12}(z_3,\cdots,z_k)$$

$$=\frac{\sqrt{\left(\sum_{i=1}^{2}\alpha_i\right)\left(\sum_{i=1}^{3}\alpha_i\right)}z_2z_3}{\alpha_3}-\left(\sum_{i=1}^{3}\alpha_i\right)\left(\frac{1}{\alpha_3}+\frac{1}{\alpha_4}\right)\left(\frac{z_3^2}{2}\right)+\frac{\sqrt{\left(\sum_{i=1}^{3}\alpha_i\right)\left(\sum_{i=1}^{4}\alpha_i\right)}z_3z_4}{\alpha_4}+k_{13}(z_2,z_4,\cdots,z_k)$$

經過移項後

$$k_{12}\left(z_3,\cdots,z_k\right)+\left(\sum_{i=1}^{3}\alpha_i\right)\left(\frac{1}{\alpha_3}+\frac{1}{\alpha_4}\right)\left(\frac{z_3^2}{2}\right)-\frac{\sqrt{\left(\sum_{i=1}^{3}\alpha_i\right)\left(\sum_{i=1}^{4}\alpha_i\right)}z_3z_4}{\alpha_4}$$
$$=k_{13}\left(z_2,z_4,\cdots,z_k\right)+\left(\sum_{i=1}^{2}\alpha_i\right)\left(\frac{1}{\alpha_2}+\frac{1}{\alpha_3}\right)\left(\frac{z_2^2}{2}\right)$$

上述等式左項為$z_3,\cdots,z_k$的函數，而右項為$z_2,z_4,\cdots,z_k$的函數，因此等式的左項和右項皆為$z_4,\cdots,z_k$的函數，故可解出

$$k_{12}\left(z_3,\cdots,z_k\right)=-\left(\sum_{i=1}^{3}\alpha_i\right)\left(\frac{1}{\alpha_3}+\frac{1}{\alpha_4}\right)\left(\frac{z_3^2}{2}\right)+\frac{\sqrt{\left(\sum_{i=1}^{3}\alpha_i\right)\left(\sum_{i=1}^{4}\alpha_i\right)}z_3z_4}{\alpha_4}+k_{123}\left(z_4,\cdots,z_k\right)$$

其中$k_{12}\left(z_3,\cdots,z_k\right)$是$z_3,\cdots,z_k$的函數，$k_{123}\left(z_4,\cdots,z_k\right)$是$k_{123}\left(z_4,\cdots,z_k\right)$的函數。

步驟 2:然後把$k_{12}\left(z_3,\cdots,z_k\right)$帶回可得

$$c_1\left(z_2,\cdots,z_k\right)$$
$$=-\sum_{j=2}^{3}\left(\sum_{i=1}^{j}\alpha_i\right)\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right)\left(\frac{z_j^2}{2}\right)+\sum_{j=2}^{3}\left(\frac{\sqrt{\left(\sum_{i=1}^{j}\alpha_i\right)\left(\sum_{i=1}^{j+1}\alpha_i\right)}z_jz_{j+1}}{\alpha_{j+1}}\right)+k_{123}\left(z_4,\cdots,z_k\right) \quad (3.25)$$

步驟 3:再對(3.25)和 j'=4解聯立方程式

$$-\sum_{j=2}^{3}\left(\sum_{i=1}^{j}\alpha_i\right)\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right)\left(\frac{z_j^2}{2}\right)+\sum_{j=2}^{3}\left(\frac{\sqrt{\left(\sum_{i=1}^{j}\alpha_i\right)\left(\sum_{i=1}^{j+1}\alpha_i\right)}z_jz_{j+1}}{\alpha_{j+1}}\right)+k_{123}\left(z_4,\cdots,z_k\right)$$

$$=\frac{\sqrt{\left(\sum_{i=1}^{3}\alpha_i\right)\left(\sum_{i=1}^{4}\alpha_i\right)}z_3z_4}{\alpha_4}-\left(\sum_{i=1}^{4}\alpha_i\right)\left(\frac{1}{\alpha_4}+\frac{1}{\alpha_5}\right)\left(\frac{z_4^2}{2}\right)+\frac{\sqrt{\left(\sum_{i=1}^{4}\alpha_i\right)\left(\sum_{i=1}^{5}\alpha_i\right)}z_4z_5}{\alpha_5}+k_{14}\left(z_2,z_3,\cdots,z_k\right)$$

經過移項後

$$k_{123}(z_4,\cdots,z_k)+\left(\sum_{i=1}^{4}\alpha_i\right)\left(\frac{1}{\alpha_4}+\frac{1}{\alpha_5}\right)\left(\frac{z_4^2}{2}\right)-\frac{\sqrt{\left(\sum_{i=1}^{4}\alpha_i\right)\left(\sum_{i=1}^{5}\alpha_i\right)}z_4z_5}{\alpha_5}$$

$$=k_{14}(z_2,z_3,\cdots,z_k)+\sum_{j=2}^{3}\left(\sum_{i=1}^{j}\alpha_i\right)\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right)\left(\frac{z_j^2}{2}\right)-\frac{\sqrt{\left(\sum_{i=1}^{2}\alpha_i\right)\left(\sum_{i=1}^{3}\alpha_i\right)}z_2z_3}{\alpha_3}$$

上述等式左項為$z_4,\cdots,z_k$的函數，而右項為$z_2,z_3,z_5,\cdots,z_k$的函數，因此等式的左項和右項皆為$z_5,\cdots,z_k$的函數，故可解出

$$k_{123}(z_4,\cdots,z_k)=-\left(\sum_{i=1}^{4}\alpha_i\right)\left(\frac{1}{\alpha_4}+\frac{1}{\alpha_5}\right)\left(\frac{z_4^2}{2}\right)+\frac{\sqrt{\left(\sum_{i=1}^{4}\alpha_i\right)\left(\sum_{i=1}^{5}\alpha_i\right)}z_4z_5}{\alpha_5}+k_{1234}(z_5,\cdots,z_k)$$

其中$k_{123}(z_4,\cdots,z_k)$是$z_4,\cdots,z_k$的函數，$k_{1234}(z_5,\cdots,z_k)$是$z_5,\cdots,z_k$的函數。

步驟 4:把$k_{123}(z_4,\cdots,z_k)$帶回(3.25)，可得

$$c_1(z_2,\cdots,z_k)$$

$$=-\sum_{j=2}^{4}\left(\sum_{i=1}^{j}\alpha_i\right)\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right)\left(\frac{z_j^2}{2}\right)+\sum_{j=2}^{4}\left(\frac{\sqrt{\left(\sum_{i=1}^{j}\alpha_i\right)\left(\sum_{i=1}^{j+1}\alpha_i\right)}z_jz_{j+1}}{\alpha_{j+1}}\right)+k_{1234}(z_5,\cdots,z_k)$$

接下來照以上步驟以此類推對 $j'=5,\cdots,$k 解聯立方程式，最後可得出

$$c_1(z_2,\cdots,z_k)$$

$$=-\sum_{j=2}^{k-1}\left(\sum_{i=1}^{j}\alpha_i\right)\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right)\left(\frac{z_j^2}{2}\right)-\left(\sum_{i=1}^{k}\alpha_i\right)\left(\frac{1}{\alpha_k}\right)\left(\frac{z_k^2}{2}\right)+\sum_{j=2}^{k-1}\left(\frac{\sqrt{\left(\sum_{i=1}^{j}\alpha_i\right)\left(\sum_{i=1}^{j+1}\alpha_i\right)}z_jz_{j+1}}{\alpha_{j+1}}\right)+c \quad (3.26)$$

其中$c_1(z_2,\cdots,z_k)$是$z_2,\cdots,z_k$的函數，c是常數，再把(3.26)帶回(3.23)式可得

$g(z_1,\cdots,z_k)$

$$=\exp\left(-\sum_{j=1}^{k-1}\left(\sum_{i=1}^{j}\alpha_i\right)\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right)\left(\frac{z_j^2}{2}\right)-\left(\sum_{i=1}^{k}\alpha_i\right)\left(\frac{1}{\alpha_k}\right)\left(\frac{z_k^2}{2}\right)+\sum_{j=1}^{k-1}\left(\frac{\sqrt{\left(\sum_{i=1}^{j}\alpha_i\right)\left(\sum_{i=1}^{j+1}\alpha_i\right)}z_j z_{j+1}}{\alpha_{j+1}}\right)+c\right)(3.27)$$

其中c是常數。

假設$(Z_1',\cdots,Z_k')$是一個隨機向量符合多維常態分配，其期望值向量是$\mathbf{0}$，共變異矩陣是

$$\Sigma=\begin{pmatrix} 1 & \sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2}} & \cdots & \sqrt{\frac{\alpha_1}{\sum_{i=1}^{k-1}\alpha_i}} & \sqrt{\frac{\alpha_1}{\sum_{i=1}^{k}\alpha_i}} \\ \sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2}} & 1 & \cdots & \sqrt{\frac{\alpha_1+\alpha_2}{\sum_{i=1}^{k-1}\alpha_i}} & \sqrt{\frac{\alpha_1+\alpha_2}{\sum_{i=1}^{k}\alpha_i}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{\frac{\alpha_1}{\sum_{i=1}^{k-1}\alpha_i}} & \sqrt{\frac{\alpha_1+\alpha_2}{\sum_{i=1}^{k-1}\alpha_i}} & \cdots & 1 & \sqrt{\frac{\sum_{i=1}^{k-1}\alpha_i}{\sum_{i=1}^{k}\alpha_i}} \\ \sqrt{\frac{\alpha_1}{\sum_{i=1}^{k}\alpha_i}} & \sqrt{\frac{\alpha_1+\alpha_2}{\sum_{i=1}^{k}\alpha_i}} & \cdots & \sqrt{\frac{\sum_{i=1}^{k-1}\alpha_i}{\sum_{i=1}^{k}\alpha_i}} & 1 \end{pmatrix}$$

其聯合機率密度函數可寫成

$$f_{Z_1',\cdots,Z_k'}(z_1',\cdots,z_k')=\frac{1}{(2\pi)^{k/2}|\mathbf{\Sigma}|^{1/2}}\exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{\Sigma}^{-1}\mathbf{z}\right)$$

其中 $\mathbf{z}^T=(z_1'\cdots z_k')$ ，而共變異數的反矩陣是

$$\Sigma^{-1}=\begin{pmatrix} (\alpha_1)\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right) & -\sqrt{\frac{\alpha_1(\alpha_1+\alpha_2)}{\alpha_2}} & \cdots & 0 & 0 \\ -\sqrt{\frac{\alpha_1(\alpha_1+\alpha_2)}{\alpha_2}} & \left(\sum_{i=1}^{2}\alpha_i\right)\left(\frac{1}{\alpha_2}+\frac{1}{\alpha_3}\right) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \left(\sum_{i=1}^{k-1}\alpha_i\right)\left(\frac{1}{\alpha_{k-1}}+\frac{1}{\alpha_k}\right) & -\sqrt{\frac{\sum_{i=1}^{k-1}\alpha_i\left(\sum_{i=1}^{k}\alpha_i\right)}{\alpha_k}} \\ 0 & 0 & \cdots & -\sqrt{\frac{\sum_{i=1}^{k-1}\alpha_i\left(\sum_{i=1}^{k}\alpha_i\right)}{\alpha_k}} & \left(\sum_{i=1}^{k}\alpha_i\right)\left(\frac{1}{\alpha_k}\right) \end{pmatrix}$$

所以把聯合機率密度函數展開後得

$$f_{Z_1',\cdots,Z_k'}(z_1',\cdots,z_k')$$

$$=\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\left(-\sum_{j=1}^{k-1}\left(\sum_{i=1}^{j}\alpha_i\right)\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right)\left(\frac{(z_j')^2}{2}\right)-\left(\sum_{i=1}^{k}\alpha_i\right)\left(\frac{1}{\alpha_k}\right)\left(\frac{(z_k')^2}{2}\right)+\sum_{j=1}^{k-1}\left(\frac{\sqrt{\left(\sum_{i=1}^{j}\alpha_i\right)\left(\sum_{i=1}^{j+1}\alpha_i\right)}z_j'z_{j+1}'}{\alpha_{j+1}}\right)\right) \quad (3.28)$$

我們由(3.27)和(3.28)可得

$$g(z_1,\cdots,z_k)$$

$$=\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}}\exp\left(-\sum_{j=1}^{k-1}\left(\sum_{i=1}^{j}\alpha_i\right)\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right)\left(\frac{z_j^2}{2}\right)-\left(\sum_{i=1}^{k}\alpha_i\right)\left(\frac{1}{\alpha_k}\right)\left(\frac{z_k^2}{2}\right)+\sum_{j=1}^{k-1}\left(\frac{\sqrt{\left(\sum_{i=1}^{j}\alpha_i\right)\left(\sum_{i=1}^{j+1}\alpha_i\right)}z_jz_{j+1}}{\alpha_{j+1}}\right)\right) \quad (3.29)$$

由(3.29) 可以說明在 n 趨近無限的情況下，標準化後的 Mathi and Moschopulos's 多項伽瑪分配會近似期望值向量是**0**，共變異矩陣是**Σ**的多維常態分配。

第四章 結論

我們運用 Proschan(2008)方法說明了一維與多維分配的中央極限定理成立，如果和傳統上常用的兩種證明法，動差生成函數法與特徵函數法做比較，可發現動差生成函數法與特徵函數法雖然證明過程較為艱難，但是不需要分配的假設，Proschan(2008)方法說明較容易理解，但是過程繁瑣而且需要分配的假設。綜觀這三種方法，每個都各有優缺點，而且提供了我們以不同的想法去說明或證明中央極限定理。雖然無法說 Proschan(2008)方法是否真的優於傳統的動差生成函數法與特徵函數法，因為它也有分配假設的問題需要克服，但我們可以說 Proschan(2008)方法是比較容易且使用的觀念都較為基礎，比較適合統計學的初學者來了解中央極限定理。

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