On Labeling Problems of Graphs

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A Master Thesis Submitted to
the Graduate School
In Partial Fulfillment of the Requirements for
The Degree of Master of Science in
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Abstract

This thesis studies two types of graph labeling problems, namely binary labeling problems and $\mathbb{Z}_k$-magic labeling problems. We study general properties of both notions for general graph classes and in particular for regular graphs.

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Chapter 1

Introduction

In this thesis, unless otherwise stated, a graph means a finite undirected simple graph without multiple edges or loops. Notations not specifically defined here please see [7].

1.1 Binary Labeling

In 1995, M. Kong and S. M. Lee[3] initiated the study of the edge-balanced graphs, which may be inspired from the following situation. Let us imagine that in a two-party parliament of a democratic country, suppose the legislators from two parties are approximately the same (precisely, they differ by at most one). In order to form fairly committees with focuses on different affairs in the parliament, further suppose that every legislator participate exactly two committees at a time. By the majority rule, one would wonder that, what are possible arrangements of the legislators of parties such that a balanced situation for committees is reached? More precisely, how to make the numbers of committees dominated by either party are approximately the same (precisely, they differ by at most one)? We can naturally model this situation using so called edge-balanced graph labeling. Each vertex represents a committee and each edge represents a legislator from one specific party. Two vertices are joined by an edge if and only if the legislator(edge) takes part in these two committees(vertices). We seek a partition of the vertices into two sets which satisfies certain condition of balance, and hence the definition of edge-balanced graphs. In fact it is not hard to see that the concept of edge-balanced labeling may be applied to many other situations in practical life in which some type of balance is needed.

We will give the definition and some generzation in this paper.

1.2 A-magic Labeling

For any additive abelian group A, let $A^* = A - \{0\}$ where 0 is the additive identity element. Given a graph G, any mapping $f : E(G) \to A^*$ is called an edge labeling of G. A graph G is said to be ***A-magic*** if there exists an edge labeling such that the induced vertex labeling $f^+ : V(G) \to A$ defined by

$$f^+(v) = \sum_{uv \in E(G)} f(uv)$$

is a constant map. We call the constant a ***magic sum index*** of G with respect to A, an ***index*** for short, and $I_A(G) = \{r : G \text{ is } A\text{-magic with an index } r\}$ the ***magic sum spectrum***, or the ***index set*** for short, of G with respect to A. In general, a graph may admit more than one edge labeling to be an A-magic graph. No generally efficient algorithm is known for finding magic labeling for general graphs. A. Kotzig and A. Rosa used the same term in [10], their notion of magic labeling is different from what we consider here. It is well-known [22, 31, 40] that a graph G is $\mathbb{N}$-magic if and only if every edge of G is contained in a $\{1, 2\}$-factor and every pair of edges is separated by this $(1, 2)$-factor. For a list of properties of $\mathbb{N}$-magic graphs, see [21, 38, 39]. Richard Stanley studied $\mathbb{Z}$-magic graphs in [36, 37], and he demonstrated that the study of magic labelings can be reduced to solving a system of linear diophantine equations.

In this article, we shall refer A as the finite cyclic group $\mathbb{Z}_k$, and the magic sum index set $I_A(G)$ as $I_k(G)$. Note that the case of $\mathbb{Z}_2$-magicness is easy to settle. Since every edge of a $\mathbb{Z}_2$-magic graph must be labeled with 1, the magic sum is the degree of any vertex modulo 2.

However the discussion of $\mathbb{Z}_2$-magic graphs is completely different from that of $\mathbb{Z}_k$-magic graphs for $k \geq 3$. It is quite challenging, and still open thus far, to obtain similar or whatsoever characterizations for $\mathbb{Z}_k$-magic graphs for $k \geq 3$. Therefore the index sets of $I_k(G)$ is hard to calculate for $k \geq 3$. Throughout this article we study the $\mathbb{Z}_k$-magicness for all $k \geq 3$, unless otherwise stated. We remark that usually for the case of infinite cyclic group $\mathbb{Z}$ one may have similar results, when one obtains results over the finite cyclic group $\mathbb{Z}_k$.

In later chapter, we study the index sets of various graph classes. It is not hard to see that any regular graph is $\mathbb{Z}_k$-magic for $k \geq 3$, however it is not easy to determine completely the magic sum spectrum for a regular graph G. We show in this paper that a regular graph with a 1-factor has the full index set $\mathbb{Z}_k$ for all $k \geq 3$, and give examples of regular graphs without 1-factor whose index set is not full $\mathbb{Z}_k$ for some $k \geq 3$. We also show that the index set of complete bipartite graphs $K_{m,n}$ is the cyclic subgroups $\mathbb{Z}_d$ generated by $\frac{k}{d}$, where $d = gcd(m - n, k)$. Among others we determine completely the index sets of wheels W_n, fans F_n, and all circulant graphs. Some open problems are presented in the conclusion remarks.

Chapter 2

Binary Labeling

2.1 Terminology and Background

We define an edge friendly edge labeling f, and the edge-balance index for a graph G with respect to f as follows:

Definition 2.1.1 *A binary edge labeling f is a mapping from $E(G)$ to $\{0,1\}$, and the induced vertex labeling f^+ is a mapping from $V(G)$ to $\{0,1\}$ defined in the following way. $f^+(v) = 1$ if at the vertex v the number of incident edges labeled 1 is more than the number of incident edges of edges labeled 0. $f^+(v) = 0$ if at the vertex v the number of incident edges labeled 0 is more than the number of incident edges of edges labeled 1. $f^+(v)$ is undefined, if at the vertex v the numbers of incident edges labeled 0 and 1 respectively are the same. Let $e_f(i) = |\{e \in E(G) : f(e) = i\}|$, where $i = 0, 1$, and an edge labeling f is called* **edge-friendly** *if $|e_f(0) - e_f(1)| \leq 1$. For an edge-friendly labeling f, we denote $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$, where $i = 0, 1$, and $t_f = |\{v \in V(G) : f^+(v)$ is undefined$\}|$. We define $|v_f(0) - v_f(1)|$ to be an* **edge-balance index** *of G with respect to f. The set of all possible indices of G with respect to all possible edge-friendly labeling is called the* **edge-balance index set** *of G, and is denoted by $\mathrm{EBI}(G)$.*

A graph G is called **edge-balanced** if the edge-balance index of G is either 0 or 1. Equivalently, one graph is edge-balanced if there exists an edge labeling f such that $|e_f(0) - e_f(1)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$.

In [1] B.-L. Chen et al. proved that all connected simple graphs, except the star graphs $K_{1,2k+1}$, are edge-balanced, $k \geq 1$. In this article, we consider a more general notion edge-balance index set, and calculate the index sets of regular graphs and related graphs. The motivation to do the study is that, as indicated in the above imagined situation in a parliament, one may wonder the spectrum of various arrangements of the committees consisting of members of two parties, for example to the extreme in a monopolized situation, or somewhere in between, other than the balanced situation. This corresponds to the study of edge-balance index set of a graph. Also regular graphs represent the normal situation that every committee consists of the same number of legislators.

In the following are some known examples, as mentioned in [2].

Example 2.1.1 *The edge-balance index set* $\mathrm{EBI}(nK_2)$ *of the* 1*-regular graphs* nK_2 *is* $\{0\}$ *if* n *is even, and is* $\{2\}$ *if* n *is odd.*

Example 2.1.2 *The edge-balance index set* $\mathrm{EBI}(St(n))$ *of the star graph* $St(n)$ *with* n *pendant edges is* $\{0\}$ *if* n *is even, and is* $\{2\}$ *if* n *is odd.*

Example 2.1.3 $\mathrm{EBI}(P_n) = \begin{cases} \{2\}, & n=2 \\ \{0\}, & n=3 \\ \{1,2\}, & n=4 \\ \{0,1\}, & n \geq 5 \textit{ is odd} \\ \{0,1,2\}, & n \geq 6 \textit{ is even} \end{cases}$

2.2 Edge-Balance Indices of Cubic Graphs

A 3-regular graph is also called a cubic graph. We have the following observations for cubic graphs. A cubic graph G must have even order, and every vertex labeling f^+ induced from an edge friendly labeling f is defined, hence $t_f = 0$. Also for any $v \in V(G)$, we have $f^+(v) = 0$ if and only if there are at least two 0-edges incident with v. Note that for cubic graphs, the index $|v_f(0) - v_f(1)| = 2v_f(0) - |V(G)| = 2(p_2(G_0) + p_3(G_0)) - |V(G)|$ is even. In order to analyze the edge-balance indices of cubic graphs, we need the following two lemmas.

The first lemma gives the upper bound of the edge-balance index set of general cubic graphs. Let G be a cubic graph with p vertices, and let f be an edge-friendly labeling of G. Let G_i be a partial subgraph of G obtained from G and a friendly labeling f by deleting all edges labeled $1 - i$, where $i = 0, 1$. We denote $p_k(G_i)$ to be the number of vertices of degree k in G_i for $k = 0, 1, 2, 3$ and $i = 0, 1$. Then we have the following:

Lemma 2.2.1 *Let* G *be a cubic graph with* p *vertices. Then*

$$max(\mathrm{EBI}(G)) \leq 2\lceil \frac{3p}{4} \rceil - p$$

and the equality holds for the following conditions:

$max(\mathrm{EBI}(G)) = 2\lceil \frac{3p}{4} \rceil - p$ *if and only if*

$$p_0(G_0) = p - \lceil \frac{3p}{4} \rceil, p_1(G_0) = 0, p_2(G_0) = \lceil \frac{3p}{4} \rceil, \ \textit{and } p_3(G_0) = 0,$$

if and only if

$$p_0(G_1) = 0, p_1(G_1) = \lceil\frac{3p}{4}\rceil, p_2(G_1) = 0, \text{ and } p_3(G_1) = p - \lceil\frac{3p}{4}\rceil,$$

and if and only if the subgraph G_0 is a disjoint union of cycles with possibly isolated vertices.

The upper bound in the above lemma is optimal, since we easily have examples which achieve the upper bound.

The next natural question is that, can every even number strictly less than the upper bound be attained as an edge-friendly index of a cubic graph? It was proved in [43]:

Theorem 2.2.1 *Let G be a cubic graph with p vertices with a perfect matching M. Then every even number strictly less than the upper bound $2\lceil\frac{3p}{4}\rceil - p$ can be attained as an edge-balance index of G. That is, we have*

$$\{0, 2, 4, \cdots, 2\lceil\frac{3p}{4}\rceil - p - 2\} \subseteq \mathrm{EBI}(G).$$

Making use of the above result, one may conclude the calculation of the edge-balance index sets of many classes of graphs with perfect matching. Let us mention certain examples in the below.

First of all, from the well known Hall's Marriage Theorem one have that every k-regular graph has a perfect matching, therefore with the above Lemma 2.2.1 and Theorem 2.2.1, we may determine the edge-balance sets of bipartite cubic graphs with $8k + 4$ or $8k + 6$ vertices completely.

Theorem 2.2.2 *Let G be a bipartite cubic graphs with $p \equiv 4$ or 6 $(mod\ 8)$ vertices, we have* $\mathrm{EBI}(G) = \{0, 2, 4, \cdots, 2\lceil\frac{3p}{4}\rceil - p - 2\}$.

Remark 2.2.1 *In the following graph PC, which is of order 16 and size 24,* $\mathrm{EBI}(PC) = 0, 2, \cdots, 8$. *Since we always can find subgraphs of PC which are disjoint union of three components of types from H_1, H_2, H_3, where H_1 is of size 4, and $p_3(H_1)+p_2(H_1) = 4$, H_2 is of size 4, and $p_3(H_2)+p_2(H_2) = 3$, H_21 is of size 4, and $p_3(H_3)+p_2(H_3) = 2$. For any index $0 \leq 2x \leq 8$, $8 \leq v_0 \leq 12$, we always have such G_0 such that G_0 is of size 12, and $p_3(G_0) + p_2(G_0) = v_0$.*

This example shows that there exists a cubic graph without perfect matching which realizes all indices below the upper bound.

In order to realize the edge-balance indices of more regular graphs, let us define the circulant graph here.

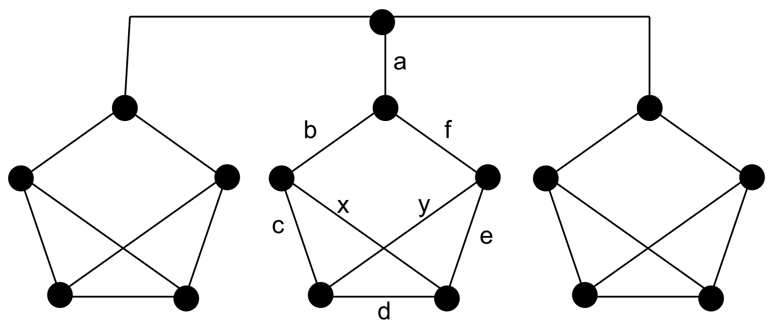

Figure 2.1: $PC1$

Definition 2.2.1 *A circulant graph $CIR_n(S)$ is defined by using the vertex set $V(CIR_n(S)) = \{0, 1, 2, \cdots, n-1\}$, and the edge set $E(CIR_n(S)) = \{ij : |i-j| \equiv s \pmod n, s \in S\}$, where $S \subset \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$.*

For example, $CIR_n(\{1\}) \cong C_n$, $CIR_n(\{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}) \cong K_n$, and $CIR_{2n}(\{1, n\}) \cong$ the n-Mӧbius ladder. A prism graph $Y(n)$ is a graph corresponding to the skeleton of an n-prism. Prism graphs $Y(n)$ are therefore both planar and polyhedral, and are isomorphic to the Cartesian product $C_n \times P_2$ which has $2n$ vertices and $3n$ edges. Note that the prism graphs are examples of cubic graphs with perfect matching, therefore it is straightforward to get the index sets if we use the result in this paper. In [44] we obtained the index set of the n-Mӧbius ladder $M(n)$ and the index set of prism graphs $Y(n)$:

Theorem 2.2.3 *For $n \geq 3$,* $\mathrm{EBI}(Y(n)) = \begin{cases} \{0, 2, \cdots, n\}, & n \equiv 0 \ (mod\ 4) \\ \{0, 2, \cdots, n+1\}, & n \equiv 1 \ (mod\ 4) \\ \{0, 2, \cdots, n-2\}, & n \equiv 2 \ (mod\ 4) \\ \{0, 2, \cdots, n+1\}, & n \equiv 3 \ (mod\ 4) \end{cases}$

Proof.

Since the prism graphs admit perfect matching, by Lemma 2.2.1, to obtain the index sets it is sufficient to determine the upper bounds.

CASE 1 $n = 4k$ Apply Lemma 2.2.1, the upper bound is $4k$. The G_0 is the cycle $u_0v_0v_1 \cdots v_{3k-1}u_{3k-1} \cdots u_1u_0$, therefore the upper bound is realized.

CASE 2 $n = 4k+1$ Apply Lemma 2.2.1, the upper bound is $4k+2$, The G_0 is the cycle $u_0v_0v_1 \cdots v_{3k}u_{3k} \cdots u_1u_0$, therefore the upper bound is realized.

CASE 3 $n = 4k+2$ Apply Lemma 2.2.1, the upper bound is $4k+2$, and the G_0 is 2-regular and of size $6k+3$. On the other hand, Note that $Y(4k+2)$ is a bipartite graph, hence contains no odd cycles and every 2-regular subgraph of $Y(4k+2)$ is of even order. Therefore the upper bound is not realized.

CASE 4 $n = 4k+3$ Apply Lemma 2.2.1, the upper bound is $4k+4$, to realize $4k+4$ in $EBI(Y(4k+3))$, consider an edge friendly labeling f as following:

$$f(u_iv_i) = 1, i = 1, 3, \cdots, 4k+1, f(u_{i-1}u_i) = 1, i = 1, 5, \cdots, 4k+1$$

$$f(v_{j-1}v_j) = 1, j = 3, 7, \cdots, 4k-1$$

and

$$f(v_{4k+2}v_0) = 1, f(e) = 0 \text{ for other edges e.}$$

The upper bound is realized.

Q.E.D.

And similarly, we also have edge balanced index sets of n-Mӧbius ladder $M(n)$:

Theorem 2.2.4 *For* $n \geq 3$, $EBI(M_n) = \begin{cases} \{0, 2, \cdots, n\}, & n \equiv 0 \ (mod\ 4) \\ \{0, 2, \cdots, n+1\}, & n \equiv 1 \ (mod\ 4) \\ \{0, 2, \cdots, n\}, & n \equiv 2 \ (mod\ 4) \\ \{0, 2, \cdots, n-1\}, & n \equiv 3 \ (mod\ 4) \end{cases}$

In 1950 H. S. M. Coxeter introduced a family of graphs generalizing the Petersen graph. These graphs are now called generalized Petersen graphs, a name given to them in 1969 by Mark Watkins. In Watkins' notation:

Definition 2.2.2 *The generalized Petersen graph $G(n, k)$ is a graph with vertex set*

$$\{u_0, u_1, \cdots, u_{n-1}, v_0, v_1, ..., v_{n-1}\}$$

and edge set

$$\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} : i = 0, ..., n-1\},$$

where subscripts are to be read modulo n and $k < n/2$.

In Coxeter's notation for the generalized Petersen graph would be $\{n\} + \{n/k\}$. The Petersen graph itself is $G(5, 2)$ or $\{5\} + \{5/2\}$.

The generalized Petersen graph $G(n, k)$ is a graph consisting of an inner star polygon (circulant graph $CIR_n(k)$) and an outer regular polygon (cycle graph C_n) with corresponding vertices in the inner and outer polygons connected with edges. $G(n, k)$ has $2n$ vertices and $3n$ edges, and is bipartite if and only if n is even and k is odd. The famous example of generalized Petersen graphs is the Petersen graph $G(5, 2)$. The example of the prism graphs $Y(n) \equiv G(n, 1)$, and we have the following result of index sets for $G(n, 2)$:

Theorem 2.2.5 *For* $n \geq 3$, $\mathrm{EBI}(G(n, 2)) = \{0, 2, 4, \cdots, \lceil \frac{3n}{2} \rceil\}$.

2.3 General Regular Graphs

For a general regular graph G, we have the following observations. If G is odd regular, then G is of even order, and very vertex labeling f^+ induced from an edge-friendly labeling f is defined. Also for any $v \in V(G)$, we have $f^+(v) = 0$ if and only if there are at least $\lceil p/2 \rceil$ 0-edges incident with v. Note that we may assume $v_f(0) \geq v_f(1)$. Therefore, the edge-balance index associated with an edge-friendly labeling f of an odd regular graph G is $2v_f(0) - |V(G)|$, and thus every index in $\mathrm{EBI}(G)$ is even for odd regular graphs. As for the case of even regular graphs, it is more complicated since there could exist undefined vertices.

In order to analyze the edge-balance indices of regular graphs, we start with the following lemma of integer programming nature:

Lemma 2.3.1 *Suppose $a_n x_n + a_{n-1}x_{n-1} + ... + a_2 x_2 + x_1 = T$, where $a_n > a_{n-1} > \ldots > a_2 > 1$ fixed positive integers, T a fixed nonnegative integer, x_i for $1 \leq i \leq n$ are nonnegative integer variables, then*

1. *$x_n + x_{n-1} + ... + x_m \leq \lfloor \frac{T}{a_m} \rfloor$ for $m \geq 2$. In case a_m is a divisor of T, equality holds if and only if $x_m = \frac{T}{a_m}$, and $x_i = 0$, for $i \neq m$.*

2. *When $a_i = i$, $i = 2, \ldots, n$, we have $2(x_n + x_{n-1} + ... + x_m) + x_{m-1} \leq \lfloor \frac{T}{m} \rfloor + \lfloor \frac{T+1}{m} \rfloor$, for $m \geq 2$.*

Let $\delta_0 = \lfloor \frac{\delta}{2} + 1 \rfloor$, where δ is the minimum degree of G. We have that $deg_{G_0}(v) \geq \frac{1}{2}(\lfloor deg_G(v) \rfloor + 1)$ if and only if v is labeled by 0 in G. Note that for the vertices v labeled 0, we have that $deg_{G_0}(v) \geq \delta_0$. However the converse statement is not necessarily true in general. Apply the Lemma 2.3.1, we have $2v_0 + h_f \leq 2\sum_{i=\delta_0}^{\Delta} p_i + p_{\delta_0 - 1} \leq \lfloor \frac{2q_0}{\delta_0} \rfloor + \lfloor \frac{2q_0+1}{\delta_0} \rfloor$, and if G is an odd graph, hence no undefined vertex labeling, the bound may be more precise. In the case of n-regular graphs, $q_0 = \lceil \frac{np}{4} \rceil$. Therefore we are in a position to give the upper bounds of edge-balance indices for odd and even regular graphs.

Lemma 2.3.2 *Let G be an n-regular (n odd) graph with p vertices, and let f be an edge-friendly labeling of G. Then*

$$\max(\mathrm{EBI}(G)) \leq 2\lfloor 2\lceil \frac{n}{2} \rceil^{-1} \lceil \frac{np}{4} \rceil \rfloor - p.$$

Proof.

Note that $\delta_0 = \lceil \frac{n}{2} \rceil$. Apply Lemma 2.3.1, we have that $v_f(0) \leq \lfloor 2\lceil \frac{n}{2} \rceil^{-1} \lceil \frac{np}{4} \rceil \rfloor$, and for any index $r \in \mathrm{EBI}(G)$, $r = 2v_f(0) - p \leq 2\lfloor 2\lceil \frac{n}{2} \rceil^{-1} \lceil \frac{np}{4} \rceil \rfloor - p$.

Q.E.D.

Lemma 2.3.3 *Let G be an n-regular graph (n even) with p vertices, and let f be an edge-friendly labeling of G. Then*

$$max(\mathrm{EBI}(G)) \le \left\lfloor \frac{4}{n+2}\left\lceil \frac{np}{4}\right\rceil \right\rfloor + \left\lfloor \frac{2}{n+2}(2\left\lceil \frac{np}{4}\right\rceil + 1)\right\rfloor - p.$$

Proof.

Note that $\delta_0 = \lceil \frac{n}{2} + 1 \rceil$. Apply Lemma 2.3.1,

$$\max \mathrm{EBI}(G) \le \left\lfloor \frac{4}{n+2}\left\lceil \frac{np}{4}\right\rceil \right\rfloor + \left\lfloor \frac{2}{n+2}(2\left\lceil \frac{np}{4}\right\rceil + 1)\right\rfloor - p$$

.

Q.E.D.

Basic techniques used here are from previous lemmas. Now we determine the edge-balance index sets of complete graphs K_n in the following.

Theorem 2.3.1 *For $n = 4k$ or $n = 4k+2$, $\mathrm{EBI}(K_n) = \{0, 2, \cdots, n-2\}$, where k is a positive integer.*

Proof.

Note that K_{4k} is $(4k-1)$-regular, and has $2k(4k-1)$ edges. By Lemma 2.4.1, we have $4k-2$ as the upper bound of $\mathrm{EBI}(K_{4k})$. To realize $4k-2$ in $\mathrm{EBI}(K_{4k})$, we consider a subgraph of K_{4k} as follows:

$$G_0 \cong CIR_{4k-1}([1, k]) \cup \{v\}$$

Therefore, $|E(G_0)| = \frac{|E(K_{4k})|}{2} = k(4k-1)$ and G_0 is $2k$-regular. Let f be the labeling such that $f(e) = 0$ if and only if $e \in G_0$ and $f(e) = 1$ otherwise. Then $f^+(v) = 0$ for $4k-1$ vertices under the labeling f, and the upper bound is attained.

As for $0 \le 2t \le 4k-2$, we realize each $2t$ as an edge-balance index using the circulant graphs as follows. Consider a subgraph of K_{4k} using graphs as follows: $CIR_{4k-1}([2, k]) \cup \{v\}$ with $(k-1)(4k-1)$ edges, a path $P_{2k+t} = \{0, 1, 2, ..., 2k+t\}$ with $(2k+t)$ edges, and $E_t = \{1v, 2v, ..., (2k+t-1)v\}$ with $2k+t-1$ edges. Let $G_{2t} = CIR_{4k-1}[2, k] \cup \{v\} \cup P_{2k+t} \cup E_t$, then $|E(G_{2t})| = k(4k-1) = \frac{|E(K_{4k})|}{2}$. Let f_t be the labeling such that $f(e) = 0$ if and only if $e \in G_0$ and $f(e) = 1$ otherwise. Therefore $f_t^+(v) = 0$ for $2k+t$ vertices under the labeling f, and the index $r = 2v_f(0) - |V(G)| = 4k+2t-4k = 2t$ is realized.

The case for $n = 4k+2$ is similarly obtained by taking $G_0 \cong CIR_{4k+1}([1, k]) \bigcup E_0$, where the E_0 is an edge set consists of $\{0v, (1)(2k+1), 2(2k+2), \cdots, (2k)(4k)\}$, and realize other indices by deleting cycle on CIR_{4k+1} and add edges back.

Q.E.D.

Using the same method, that is, to find G_0 which contains a cycle formed by all vertices of degree $\frac{n}{2}+1$, delete the cycle, and add edges back such that the number of vertices of degree more than $\frac{n}{2}$ is $\frac{n}{2}+t$, and number of vertices of degree $\frac{n}{2}$ is 1 or 0.

Note that K_5 and K_7 are exceptional, one may still have the following results:

Lemma 2.3.4 *For* $\mathrm{EBI}(K_5) = \{0, 1\}$ *and* $\mathrm{EBI}(K_7) = \{0, 1, 2, 3, 4\}$.

Proof.

Since K_5 has 10 edges, G_0 is of size 5, take $G_0 \cong K_4 \setminus \{e\}$, which is a diamond, we have $v_0 = 2, v_1 = 1$.

Since K_7 has 21 edges, G_0 is of size 11 take $G_0 \cong K_5 \bigcup \{ux, uy\}$, where $x, y \in V(K_7), u \notin V(K_5)$, we have $v_0 = 5, v_1 = 1$.

Q.E.D.

Theorem 2.3.2 *For* $n = 4k+1$ *or* $4k+3$, $k \geq 2$, $\mathrm{EBI}(K_n) = \{0, 1, \cdots, n-4\}$.

Proof.

For $n = 4k+1, k \geq 2$, take $G_0 \cong CIR_{4k-1}([1,k]) \bigcup E_0$, where the E_0 is an edge sert consists of $\{0v, (1)(2k-2), 2(2k-1), \cdots, (2k-1)(4k-3)\}$.

For $n = 4k-1, k \geq 2$, take $G_0 \cong CIR_{4k-2}(\{1\} \bigcup CIR_{4k-3}([2,k])$.

Q.E.D.

2.4 Regular Graphs Join with Null Graphs

Let us consider almost regular graphs by using the **join product** of regular graphs .

Lemma 2.4.1 *Let G be an n-regular graph (n even) with p vertices, let $G+K_m^c$ be the join of G and m points if $m+n \leq p$.*

$$\max(\mathrm{EBI}(G + K_m^c)) \leq 2\lfloor 2\lceil \frac{n+1}{2} \rceil^{-1} \lceil \frac{n+2mp}{4} \rceil \rfloor - p - 1.$$

Proof.

Note that $\delta_0 = \frac{n+m}{2}$ and $q_0 = \frac{n+2mp}{2}$, by Lemma 2.3.1 it follows.

Q.E.D.

Lemma 2.4.2 *Let G be an n-regular graph with p vertices, let $G+K_m^c$ be the join of G and one point v if $m+n \leq p$ is even Then*

$$\max(\mathrm{EBI}(G+K_m^c)) \leq \left\lfloor \frac{4}{n+m+2} \left\lceil \frac{np}{4} \right\rceil \right\rfloor + \left\lfloor \frac{2}{n+2}(2 \left\lceil \frac{n+2m}{p} 4 \right\rceil + 1) \right\rfloor - p - 1$$

.

Proof.

Note that $\delta_0 = \frac{n+m}{2}$ and $q_0 = \frac{n+2mp}{2}$, by Lemma 2.3.1 it follows.

Q.E.D.

Theorem 2.4.1 *For a generalized wheel $GW_n = \{v_0\} + G$, where G is a 2-regular graph. Let $V(GW_n) = \{v\} \bigcup \{v_1, \cdots, v_n\}$ and $E(GW_n) = \{vv_i : i = 1, \cdots, n\} \bigcup E(C_n)$. Then*

$$EBI(GW_n) = \begin{cases} \{0, 2, \cdots, 2i, \cdots, n-1\}, & \text{if } n \text{ is odd,} \\ \{1, 3, \cdots, 2i+1, \cdots, n-1\} \bigcup \{0, 2, \cdots, 2(\lceil \frac{n}{4} \rceil - 1)\}, & \text{if } n \text{ is even.} \end{cases}$$

Proof.

Since $G+v$ has $n+1$ vertices and $2n$ edges, hence $q_0 = n$, $\delta_0 = 2$ here, and any index $x = 2v_0 + h_f - (n+1)$.

For n odd, each vertex labeling is defined and even, hence $h_f = 0$. Let $G_0 \cong G$, we realize index $n-1$. For $0 \leq 2t \leq n-3$, pick a path union cycles of size $\frac{n+1}{2} + t$, say $P_{2t} = u_0 u_1 \cdots u_m$, where $m = \frac{n+1}{2} + t$. Construct G_0 by joining v to $u_0, u_1, \cdots, u_{n-m}$, then we realize index $2t$.

For n even, if $h_f = 0$, the odd index is realized similarly by above method. For any even index $x = 2v_0 + h_f - (n+1)$, $h_f = 1$ and $deg_{G_0}(v) = \frac{n}{2}$, $G_0 \backslash \{v\}$ contains $n - \frac{n}{2} = \frac{n}{2}$ edges on the 2-regular graph G. Choose these edges such that they are independent if they are adjacent to vv_i, and the others form cycles union path.

Q.E.D.

2.5 Open Problems of Edge-Balance Index Sets

There are more open problems left as follows from our work:

- Characterize the graphs G with edge-balance index r, where $r \leq |V(G)| - 2$.
- In particular, characterize those graphs G which represents the majority, in the sense of the parliament and committee model mentioned, that $v_f(0) = r$, $t_f = |V(G)| - r$, and $v_f(1) = 0$. Note that the cases $r = |V(G)|$, $|V(G)| - 1$, and $|V(G)| - 2$ have been done in this article.

- Can the statement that, regular graphs with 1-factor admits all edge-balance indices except the maximum one, be extended or generalized in any way?
- Certain fundamental classes of graphs, as special cases, such as cycles C_n, complete graphs K_n, hypercubes Q_n, wheels W_n, fans F_n, complete bipartite graphs $K_{m,n}$, complete multipartite graphs $K_{n,n,\cdots,n}$ with parts of the same orders etc.
- All regular graphs with 1-factor, such as regular bipartite graphs, regular claw-free graphs, generalized Petersen graphs, cubic graphs with at most two bridges, some cases of circulant graphs etc.
- Certain regular graphs without 1-factor, such as Petersen's counterexample for cubic graphs with bridges admitting no 1-factor (that is our graph PC in the Figure 4.1), circulant graphs without 1-factor, etc.
- Is the sufficient condition admitting a 1-factor is also necessary for regular graphs to have the full index sets $\mathbb{Z}_k$, for all $k \geq 3$?

Chapter 3

Group Labeling with Zero Magic Sums

3.1 Introduction and Terminology

For a positive integer k, let $\mathbb{Z}_k = (\mathbb{Z}_k, +, 0)$ be the additive group of congruences modulo k with identity 0, and $\mathbb{Z}_k$ is the usual group of integers $\mathbb{Z}$ when $k = 1$. We call a finite simple graph $G = (V(G), E(G))$ to be **$\mathbb{Z}_k$-magic** if it admits an edge labeling $\ell : E(G) \to \mathbb{Z}_k \backslash \{0\}$ such that the induced vertex sum labeling $\ell^+ : V(G) \to \mathbb{Z}_k$ defined by $\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$ is constant. The constant is called a **magic sum index**, or an **index** for short, of G under the labeling ℓ, which follows R. Stanley. The **null set** of G, which is defined by E. Salehi as the set of all k such that G is $\mathbb{Z}_k$-magic with zero magic sum index, and is denoted by $\mathrm{Null}(G)$. For fix integer k, we consider the set of all possible magic sum indices r such that G is $\mathbb{Z}_k$-magic with a magic sum index r, and denote it by $I_k(G)$. We call $I_k(G)$ the **index set** of G with respect to $\mathbb{Z}_k$. In this paper, we investigate the properties and relations of the index sets $I_k(G)$ and the null sets $\mathrm{Null}(G)$ for $\mathbb{Z}_k$-magic graphs. Among others, we determine the null sets of generalized wheels and generalized fans, and also construct infinitely many examples of $\mathbb{Z}_k$-magic graphs with magic sum zero. Some open problems are presented.

For any additive abelian group A, let $A^* = A - \{0\}$ where 0 is the additive identity element. Given a graph G, any mapping $\ell : E(G) \to A^*$ is called an edge labeling of G. A graph G is said to be ***A-magic*** if there exists an edge labeling such that the induced vertex labeling $\ell^+ : V(G) \to A$ defined by

$$\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$$

is a constant map. We call the constant a ***magic sum index*** of G, an ***index*** for short, and $I_A(G) = \{r : G \text{ is } A\text{-magic with an index } r\}$ the ***index set*** of

G with respect to A. In this article we focus on $A = \mathbb{Z}_k$ and denote $I_A(G)$ by $I_k(G)$. The notion of index sets is first introduced and studied by C-M Lin and T-M Wang in [23]. A related notion is the ***null set*** of G, which is defined as the set of all k such that G is $\mathbb{Z}_k$-magic with index 0. The problems related to the null sets was studied by E. Salehi in [33, 32]. E. Salehi also studied a particular class of graphs called ***uniformly null***, which is defined as the graphs G with the property that, if G is $\mathbb{Z}_k$-magic, then the magic sum is zero only. He identified the class of complete bipartite graphs $K_{n,n+1}$ to be uniformly null in [33].

In general, a graph may admit more than one edge labeling to become an A-magic graph. At present, no generally efficient algorithm is known for finding magic labeling for general graphs. It is well-known [22, 31, 40] that a graph G is $\mathbb{N}$-magic if and only if every edge of G is contained in a $\{1, 2\}$-factor. For a list of properties of $\mathbb{N}$-magic graphs, see [21, 24, 26, 38, 39]. Stanley studied $\mathbb{Z}$-magic graphs in [36, 37]; he demonstrated that magic labelings can be found by solving a system of linear diophantine equations. Being $\mathbb{Z}$-magic is a weaker condition than being $\mathbb{N}$-magic. Given a graph G, the set of all k such that G is $\mathbb{Z}_k$-magic is defined as the ***integer-magic spectrum*** of G, and is denoted by $\mathrm{IM}(G)$. The integer-magic spectra of some families of graphs can be found in [25, 27, 28, 29]. The concepts of the index sets, the null sets, and the integer magic spectra of graphs are closely related. Note that the case of $\mathbb{Z}_2$-magicness is easy to settle. Since every edge must be labeled with 1, the magic sum is the degree of any vertex modulo 2. Therefore the degrees of the vertices must have the same parity. This leads to the following result.

Lemma 3.1.1 *A graph G is $\mathbb{Z}_2$-magic if and only if its degrees are all even or all odd.*

However the discussion of $\mathbb{Z}_2$-magic graphs is completely different from that of $\mathbb{Z}_k$-magic graphs for $k \geq 3$ or $k = 1$. It is quite challenging to obtain similar characterizations of $\mathbb{Z}_k$-magic graphs for $k \geq 3$ or $k = 1$.

In [33], E. Salehi introduced the null set of a graph, and obtained some interesting results regarding various classes of graphs. Formally we define the null set of a graph as follows.

Definition 3.1.1 *The null set of a graph G is the set of all possible positive integers k, such that G has a zero magic sum index under a $\mathbb{Z}_k$-magic edge labeling, and is denoted by* $\mathrm{Null}(G)$.

First of all we define precisely the ***magic sum spectrum*** with respect to $\mathbb{Z}_k$, which is referred as the ***index set*** in this paper, of a graph as follows:

Definition 3.1.2 *For a graph G, we define the set of all magic sum indices r such that G is $\mathbb{Z}_k$-magic with magic sum index r, to be the magic sum spectrum, namely the index set of G with respect to $\mathbb{Z}_k$, and denote such a set by $I_k(G)$.*

For the index sets of vertex disjoint union of graphs, we have the following formula.

Proposition 3.1.1 *Let H_1 and H_2 be vertex disjoint graphs. Then $I_k(H_1 \cup H_2) = I_k(H_1) \cap I_k(H_2)$. In particular, let nG be the vertex disjoint union of n copies of G, then nG has the same index set as G, that is, $I_k(nG) = I_k(G)$.*

Proof. On one hand, if $r \in I_k(H_1 \cup H_2)$, then $r \in I_k(H_1)$ and $r \in I_k(H_2)$ simply consider the restriction of the labeling of $H_1 \cup H_2$ over H_1 and H_2, respectively. On the other hand, if $r \in I_k(H_1)$ and $r \in I_k(H_2)$, then $r \in I_k(H_1 \cup H_2)$ since H_1 and H_2 are vertex disjoint, hence edge disjoint. □

We have the following basic observations over the indices of edge disjoint union of $\mathbb{Z}_k$-magic graphs. Let in particular $G = G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m$ be the edge disjoint union of spanning subgraphs $G_1, G_2, \cdots$ and G_m, that is, G admits a factorization with factors $G_1, G_2, \cdots, G_m$. Suppose for fix k the graphs G_i is $\mathbb{Z}_k$-magic with index r_i for $i = 1, \cdots, k$. Then we have $G = G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m$ is $\mathbb{Z}_k$-magic with an index $\sum_{i=1}^{m} r_i$. The reason is at each vertex v of $G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m$, the incident edges consist of the incident edges of v in $G_1, G_2, \cdots, G_m$. We organize these facts as follows.

Proposition 3.1.2 *Let $G = G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m$ be a factorization of spanning subgraphs $G_1, G_2, \cdots$ and G_m. Suppose for fix k the graphs G_i is $\mathbb{Z}_k$-magic with index r_i for $i = 1, \cdots, k$. Then we have:*

(1) $G = G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m$ is $\mathbb{Z}_k$-magic with an index $\sum_{i=1}^{m} r_i$.
(2) $I_k(G_1) + I_k(G_2) + \cdots\cdots + I_k(G_m) \subseteq I_k(G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m)$, and in particular, $I_k(G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m) = \mathbb{Z}_k$ if $I_k(G_i) = \mathbb{Z}_k$ for some i.

Remark. Note that in previous Proposition, $I_k(G_1) + I_k(G_2) + \cdots\cdots + I_k(G_m) \not\supseteq I_k(G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m)$ in general. That is, not necessarily every index of $G_1 \oplus G_2 \oplus \cdots\cdots \oplus G_m$ comes from the sum of indices of $G_1, G_2, \cdots$ and G_m. Examples could be found in later sections.

Remark. In case the graphs $G_1, G_2, \cdots$ and G_m are $\mathbb{Z}_k$-magic with indices 0 for each G_i, where $i = 1, \cdots, k$. Then we have the resulting arbitrary union graph $\bigcup_{i=1}^{k} G_i$, which is formed by attaching these m graphs in any way, is still $\mathbb{Z}_k$-magic with an index 0.

One direct application with the above observation is that the Cartesian product of $\mathbb{Z}_k$-magic graphs is still $\mathbb{Z}_k$-magic with an index of the sum of corresponding indices.

Proposition 3.1.3 *For fix k, let the graphs H_i be $\mathbb{Z}_k$-magic with index r_i for $i = 1, \cdots, s$. Then the Cartesian product $H_1 \times H_2 \times \cdots \times H_s$ has an index $\sum_{i=1}^{s} r_i$, and $I_k(H_1) + I_k(H_2) + \cdots\cdots + I_k(H_s) \subseteq I_k(H_1 \times H_2 \times \cdots \times H_s)$. In particular, the index set $I_k(H_1 \times H_2 \times \cdots \times H_s) = \mathbb{Z}_k$ whenever $I_k(H_i) = \mathbb{Z}_k$ for some $1 \leq i \leq s$.*

Proof.
It suffices to show that the Cartesian product of two $\mathbb{Z}_k$-magic graphs is $\mathbb{Z}_k$-magic with an index the sum of original indices. Let $|V(H_1)| = m_1$ and $|V(H_2)| = m_2$. Decompose $H_1 \times H_2$ as graphs K_1 and K_2 such that $V(K_1) = V(K_2) = V(H_1 \times H_2)$, where $E(K_1) = \{(u_i, v)(u_j, v)|u_i u_j \in E(H_1)\}$ and $E(K_2) = \{(u, v_i)(u, v_j)|v_i v_j \in E(H_2)\}$. Then $K_1 \cong m_2 H_1$ and $K_2 \cong m_1 H_2$, thus have indices r_1 and r_2, respectively. Then from previous fact, $H_1 \times H_2 = K_1 \oplus K_2$ has an index $r_1 + r_2$, and $I_k(H_1) + I_k(H_2) \subseteq I_k(H_1 \times H_2)$. Hence the proof is complete.

□

However in general, the index set will not be the full $\mathbb{Z}_k$. Note that the index set in general is a subset of $\mathbb{Z}_k$, not necessarily a subgroup. Note that if $r \in I_k(G)$ then $-r \in I_k(G)$, simply by reversing the sign of the labels on each edge. If one can show for any $r, r' \in I_k(G)$ we have $r + r' \in I_k(G)$, then $I_k(G)$ is a subgroup of $\mathbb{Z}_k$. We have the following necessary condition for a $\mathbb{Z}_k$-magic graph with magic sum r.

Proposition 3.1.4 *For any $\mathbb{Z}_k$-magic labeling f of a graph G with index r, we have*

$$2 \sum_{e \in E(G)} f(e) \equiv r \cdot |V(G)| \quad (mod\ k)$$

Proof. Summing the vertex sums will count each edge label twice. □

By the above necessary condition, for a graph G with $V(G)$ odd and for even k, it implies that r must be an even number, as a representative in the congruence class modulo k. Therefore we have the following fact.

Corollary 3.1.1 *For a graph G with odd vertices and for even k, we have that $I_k(G) \subseteq 2\mathbb{Z}_k \subsetneqq \mathbb{Z}_k$.*

Remark. For example, for $n \geq 1$ odd and $t \geq 0$, let G be $K_{n,n,\cdots,n}$, the complete multipartite graphs with $2t + 1$ parts of the same odd order n. By the above corollary we have the index set $I_k(G) = 2\mathbb{Z}_k \subsetneqq \mathbb{Z}_k$ for k even, since G is of odd order. Note that G is even regular, therefore by Lemma 4.1.10

later, every even index is assumed except K_1 and K_3. By the way, from the Theorem 4.1.2 in later section it follows that the remaining cases for the index set of $K_{n,n,\cdots,n}$ (even n, or parts of even orders) is full $\mathbb{Z}_k$ for all $k \geq 3$ since it admits 1-factor. Therefore we determine completely all the index sets of complete multipartite graphs $K_{n,n,\cdots,n}$ with parts of equal orders $n \geq 1$.

Remark. The converse statements of the Proposition 3.1.4 and Corollary 3.1.1 are not true in general, please see for example F_3 in Lemma 4.2.1 in later section, which provides with an example having the index set $I_k(G) \subseteqq 2\mathbb{Z}_k \subsetneqq \mathbb{Z}_k$ for even k, but with even number of vertices.

We have the following examples of the index sets of cycles and Cartesian product of cycles.

Proposition 3.1.5 *Let C_n be an n-cycle, where $n \geq 3$, and k be a positive integer. We have the following:*
(1) $I_k(C_n) = 2\mathbb{Z}_k^* = \{2x : \ x \neq 0, \ x \in \mathbb{Z}_k\}$, for n odd.
(2) $I_k(C_n) = \mathbb{Z}_k$, for n even.

Proof.
(1) Note that in any $\mathbb{Z}_k$-magic labeling of a cycle, the edges should alternatively be labeled the same. Therefore, for n odd, the labels on all edges are all the same. Therefore $I_k(C_n) = 2\mathbb{Z}_k^* = \{2x : \ x \neq 0, \ x \in \mathbb{Z}_k\}$.
(2) For n even, we label the edges $1, x-1, 1, x-1, \cdots, 1, x-1$ for $x \in \mathbb{Z}_k \backslash \{1\}$ to obtain the index x, and $2, -1, 2, -1, \cdots, 2, -1$ to obtain the index 1. Therefore $I_k(C_n) = \mathbb{Z}_k$.

□

Remark. Note that $2\mathbb{Z}_k^* = \mathbb{Z}_k^*$ for k is odd, and $2\mathbb{Z}_k^* = \{0, 2, \cdots, \frac{k}{2}\} = 2\mathbb{Z}_k$ for k is even. One may see from the above proposition that $0 \in I_k(C_n)$, for n even, and $0 \in I_k(C_n)$ if and only if k is even, for n odd.

As corollaries of the above observations, we have the following example for the index set of the hypercube graphs Q_n.

Proposition 3.1.6 *For fix k and $n \geq 2$, the index set of the hypercube is $I_k(Q_n) = \mathbb{Z}_k$. Note that $Q_n = \underbrace{P_2 \times P_2 \times \cdots \times P_2}_{n \text{ copies}}$.*

Proof. Note that $I_k(Q_2) = I_k(C_4) = \mathbb{Z}_k$ by Proposition 3.1.5. Then by Proposition 3.1.2, we have $I_k(Q_n) = \mathbb{Z}_k$. □

Moreover, we have the following examples for the index sets of toroidal grids $C_{m_1} \times C_{m_2} \times \cdots \times C_{m_t}$:

Proposition 3.1.7 *For fix k and $n \geq 2$, the index set of a toroidal grid $G = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_t}$ is*

$$I_k(G) = \begin{cases} \mathbb{Z}_k, & \text{if some } m_i \text{ is even,} \\ 2\mathbb{Z}_k, & \text{otherwise.} \end{cases}$$

Proof. Directly from the Proposition 3.1.3 and the Proposition 3.1.5. □

In general if G is $\mathbb{Z}_k$-magic, then it is $\mathbb{Z}_{mk}$-magic for k and m positive integers. Therefore it suffices to consider the cases of prime numbers when one studies the $\mathbb{Z}_k$-magicness of a graph. We have the following relationship of index sets with respect to k and mk:

Proposition 3.1.8 *Let m, k be positive integers. Then $m \cdot I_k(G) \subseteq I_{mk}(G)$ and $|I_k(G)| \leq |I_{mk}(G)|$.*

Proof. Note that if G is $\mathbb{Z}_k$-magic with index r $(mod\ k)$, then it is $\mathbb{Z}_{mk}$-magic with index mr $(mod\ mk)$, since we may simply use the $\mathbb{Z}_k$-magic labeling with index r, multiplying on each edge label by m, which turns out to be a $\mathbb{Z}_{mk}$-magic labeling with index mr. On the other hand, $mr_i \equiv mr_j$ $(mod\ mk)$ implies $r_i \equiv r_j$ $(mod\ k)$, thus $|I_k(G)| \leq |I_{mk}(G)|$. □

Remark. From the above observation, one may see that for checking the $\mathbb{Z}_k$-magicness of a graph, it suffices to look at the prime numbers k, instead of all positive integers k.

It is quite challenging to obtain similar characterizations as in $\mathbb{Z}_2$-magic cases for $\mathbb{Z}_k$-magic graphs for $k \geq 3$. However, we have the following observation for $\mathbb{Z}_3$-magic graphs. Let $G = (V, E)$ with $|V| = p$ and $|E| = q$, $V = \{v_1, v_2, \cdots, v_p\}$. Note that there are only two possible edge labels 1 and -1 in a $\mathbb{Z}_3$-magic labeling. Let i_k be the number of (-1)-edges incident with the vertex v_k, and j_k be the number of 1-edges incident with the vertex v_k for $1 \leq k \leq p$, respectively. Then $i_k + j_k = deg(v_k)$ for $1 \leq k \leq p$. Denote ν the total number of (-1)'s, hence $q - \nu$ is the total number of 1's, then we have $2\nu = \sum_{k=1}^{p} i_k = \sum_{k=1}^{p}(deg(v_k) - j_k) = 2q - \sum_{k=1}^{p} j_k$. Therefore in terms of the local information of the number of 1's or that of (-1)'s, we have the following.

Proposition 3.1.9 *Let G be a $\mathbb{Z}_3$-magic graph with p vertices and q edges. Then*

1. *If the magic sum index is 0 $(mod\ 3)$, then $3|q - 2\nu$.*
2. *If the magic sum index is 1 $(mod\ 3)$, then $3|2q - p - 4\nu$.*

3. If the magic sum index is $2 \pmod 3$*, then* $3|q - p - 2\nu$.

Proof. Directly from the Proposition 3.1.4. □

It is plausible but tedious to have similar necessary conditions for graphs being $\mathbb{Z}_k$-magic with the magic sum index r, in terms of local information of the graph, with the parameters the number of vertices and the number of edges.

3.2 Null Sets of Generalized Fans and Generalized Wheels

In this section, the null sets of windmills, fans, wheels, and their variants and generalizations are discussed and determined completely.

Note that at first for any $\mathbb{Z}_k$-magic labeling f of a graph G with index r, we have the following equation by summing all vertex sums:

$$2 \sum_{e \in E(G)} f(e) \equiv r \cdot |V(G)| \pmod k.$$

Hence we have $2\sum_{e\in E(G)} f(e) \equiv 0 \pmod k$ for any magic labeling with index 0, and also note that the sum of labels for all the incident edges with one single vertex is 0, therefore we have the following Lemma:

Lemma 3.2.1 *Let f be a $\mathbb{Z}_k$-magic labeling of G with an index 0, v be a vertex of G, and $G' = G - \{v\}$. Then we have:*

$$\sum_{e \in E(G)} f(e) = \begin{cases} \frac{k}{2} \text{ or } 0 \pmod k, & \text{for } k \text{ even,} \\ 0 \pmod k, & \text{for } k \text{ odd.} \end{cases}$$

and

$$\sum_{e \in E(G')} f(e) = \begin{cases} \frac{k}{2} \text{ or } 0 \pmod k, & \text{for } k \text{ even,} \\ 0 \pmod k, & \text{for } k \text{ odd.} \end{cases}$$

We will determine the null sets of generalized fan graphs, the null sets of generalized wheel graphs, and the null sets of generalized windmill graphs completely in this section.

Definition 3.2.1 *A* **fan** *graph $F_n = \{v\} + P_n$ is formed by adding a vertex v to the vertex set of P_n and joining this vertex to every vertex of P_n, and a* **wheel** *graph $W_n = \{v\} + C_n$ is formed by adding a vertex v to the vertex set of C_n and joining this vertex to every vertex of C_n, for $n \geq 3$. A* **generalized**

fan *graph is formed by joining one vertex to each vertex of a disjoint union of paths, and a* **generalized wheel** *graph is formed by joining one vertex to each vertex of a disjoint union of cycles. We call the vertex v the* **center***, and the edges connecting the center v and vertices of paths P_n or cycles C_n* **spokes***.*

Therefore, by the above Lemma 3.2.1, we observe that for the zero-sum $\mathbb{Z}_k$-magic labeling of $\{v\}+H$ (In particular $H = W_n$, $H = F_n$, or $H =$ disjoint union of paths or cycles), the sum of edge labels on H (that is cycles C_n, and paths P_n respectively) must be 0 or $\frac{k}{2}$ modulo an even k, and also the induced vertex sum of every vertex on H must be non-zero since the edge labels on spokes are non-zero.

Remark. We observe that fans F_n and wheels W_n for $n \geq 3$ have no $\mathbb{Z}_2$-magic labeling with index 0. Therefore $2 \notin \text{Null}(F_n)$ and $2 \notin \text{Null}(W_n)$.

Note that the null sets of wheels and fans have already been determined and presented at the 40th Southeastern International Conference on Combinatorics, Graph Theory and Computing, March 2-6, 2009(Please see the abstract of the talk "Zero-Sum Magic and Null Sets of Planar Graphs", E. Salehi and S. Hansen, University of Nevada Las Vegas). We work independently regarding this subject, and obtain the null sets of fans, wheels, and other graphs, respectively. Therefore we omit our proof and just put the result for the null set of the fans F_n for $n \geq 2$ in the following for reference. Note that the proof is based upon the induction and a method of subdivision.

Theorem 3.2.1 *For $n \geq 2$,*

$$\text{Null}(F_n) = \begin{cases} 2\mathbb{N}, & n = 2, \\ 2\mathbb{N}\backslash\{2\}, & n = 3, \\ \mathbb{N}\backslash\{2\}, & n > 3,\ n \equiv 1\ (mod\ 3), \\ \mathbb{N}\backslash\{2,3\}, & n > 3,\ n \equiv 0,2\ (mod\ 3). \end{cases}$$

More precisely, we define the generalized fans as follows.

Definition 3.2.2 *A generalized fan $F(n_1, n_2, \cdots, n_m) = \{v\} + (P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_m})$, where P_{n_i} are disjoint paths on $n_i \geq 2$ vertices, for $i = 1, \cdots, m$ and $m \geq 2$.*

We first deal with the $\mathbb{Z}_3$ case for a generalized fan $F(n_1, n_2, \cdots, n_m)$ where $n_i \geq 2$ vertices, for $i = 1, \cdots, m$:

Lemma 3.2.2 *For $n_i \geq 2$, $\forall i = 1, \cdots, m$, let $S = \{n_i : n_i \equiv 0$ or $2\ (mod\ 3)\}$. Then we have that $0 \in I_3(F(n_1, n_2, \cdots, n_m))$ if and only if $|S| \neq 1$.*

Proof. Suppose $0 \in I_3(F(n_1, n_2, \cdots, n_m))$. By Lemma 3.2.1, we note that every label restricted on each path is either 1 or -1 for any zero sum $\mathbb{Z}_3$-magic labeling of the generalized fan. If we label all 1s on some path P_{n_i}, then there are $(n_i - 2)$ 1's and two (-1)'s on the spokes incident with each vertex of such path. Therefore, the partial vertex sum of the center with respect to P_{n_i} is $n_i - 4 \equiv n_i - 1 \pmod 3$. On the other hand, if we choose to label all (-1)'s on the path P_{n_i}, the partial vertex sum of the center with respect to P_{n_i} is $4 - n_i \equiv 1 - n_i \pmod 3$.

Let the total vertex sum of the center be T, we have the following four cases:

Case 1: $|S| = 1$.

Then T is never zero (mod 3).

Case 2: $|S| = 3k,\ k \in \mathbb{N} \cup \{0\}$.

Label 1 on all the edges, then $T \equiv 3k \equiv 0 \pmod 3$.

Case 3: $|S| = 3k + 1,\ k \in \mathbb{N}$.

Label two fans such that the partial vertex sums of the center with respect to them are -1, and $3k-1$ fans such that the partial vertex sums with respect to them are 1, then $T \equiv (3k - 1) - 2 \equiv 0 \pmod 3$.

Case 4: $|S| = 3k + 2, k \in \mathbb{N}$.

Label one fans such that the partial vertex sum of the center is -1, and $3k + 1$ fans such that the partial vertex sums of the center with respect to them is 1, then $T \equiv (3k + 1) - 1 \equiv 0 \pmod 3$.

The converse is clear from the given labeling. □

Note that we may view $F(n_1, n_2, \cdots, n_m)$ as the one vertex union of F_{n_i}, for $i = 1, \cdots, m$, and write it as $F(n_1, n_2, \cdots, n_m) = F_{n_1} \odot F_{n_2} \cdots \odot F_{n_m}$. Since $N(F_n) \supset N \backslash \{2, 3\}$, for all $n \geq 4$, that is F_n $(n \geq 4)$ is $\mathbb{Z}_k$-magic with index 0 for all $k \geq 4$, therefore if we can show that $F(n_1, n_2, \cdots, n_m)$ with certain $n_i \leq 3$ admits a $\mathbb{Z}_k$-magic labeling with zero sum, for all $k \geq 4$, then the null set of $F(n_1, n_2, \cdots, n_m)$ is completely determined. We proceed with the following steps.

Lemma 3.2.3 *The double fans $F(2, m)$ and $F(3, m)$ admit a $\mathbb{Z}_k$-magic labeling with 0 index, for all $m \geq 4$ and $k \geq 4$.*

Proof. Clearly, $F(2, 4)$, $F(2, 5)$, $F(3, 4)$, and $F(3, 5)$ admit $\mathbb{Z}_k$-magic labeling with 0 index, for all $k \geq 4$, as shown in the Figure 3.1 and Figure 3.2.

Note that the given labeling has 1-edge and (-1)-edge over the F_4 and F_5 sides, then by inserting spokes labeled 2 and -2, we may get a $\mathbb{Z}_k$-magic labeling with 0 index for general $F(2, m+2)$ from $F(2, m)$, and get $F(3, m+2)$ from $F(3, m)$, respectively, for all $m \geq 4$ by induction. Please see the Figure 3.3 for the above mentioned method, which is also used in our proof in obtaining the

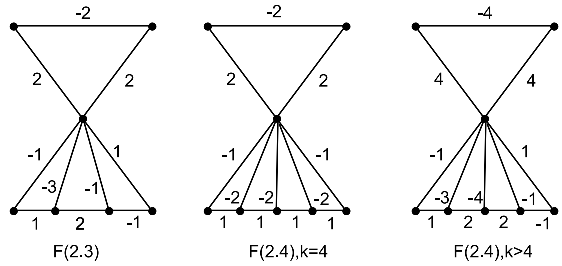

Figure 3.1: $F(2,4)$ and $F(2,5)$

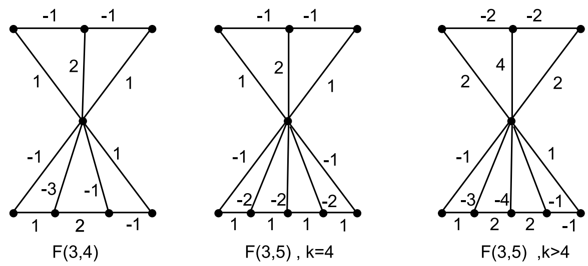

Figure 3.2: $F(3,4)$ and $F(3,5)$

null sets of fans. $\square$

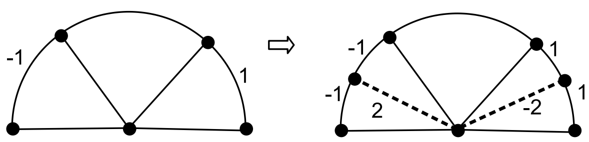

Figure 3.3: $F(*,m)$ to $F(*,m+2)$: a method of subdivision

Corollary 3.2.1 *For $k \geq 4$ and $m \geq 2$, suppose $\{n_1, n_2, \cdots, n_m\}$ contain only one 2 or only one 3. Then $F(n_1, n_2, \cdots, n_m)$ admits a $\mathbb{Z}_k$-magic labeling with zero sum index.*

Proof. If 2 (or 3) appears once in $\{n_1, n_2, \cdots, n_m\}$, then $F(n_1, n_2, \cdots, n_m)$ is a vertex union of $F(2,t)$ (or $F(3,t)$) for $t \geq 4$ and other fans F_j with $j \geq 4$. $\square$

Theorem 3.2.2 *The generalized fan $F(n_1, n_2, \cdots, n_m)$, $n_i \geq 2$, $\forall i = 1, \cdots, m$ and $m \geq 2$, admits a $\mathbb{Z}_k$-magic labeling with zero index, for all $k \geq 4$.*

Proof. For convenience and without loss of generality, we may express the generalized fan as the one vertex union of a copies of F, b copies of F_2, and c copies of F_3, where F is an a vertex union of fans F_j, $j \geq 4$. That is, we assume that $F(n_1, n_2, \cdots, n_m) = aF \odot bF_2 \odot cF_3$ where b, c non-negative and a is 0 or 1. Note that F_j, $j \geq 4$ admits $\mathbb{Z}_k$-magic labeling with zero sum for $k \geq 4$. Then we have the following cases:
Case 1. b, c are both even.

Note that this case can be reduced to $F(2,2)$ and $F(3,3)$. For $F(2,2)$, it admits a $\mathbb{Z}_k$-magic labeling with zero index, for $k \geq 4$, since it is Eulerian of even size. For $F(3,3)$, it admits a $\mathbb{Z}_k$-magic labeling with zero index, for $k \geq 4$, see the Figure 3.4. Therefore, $bF_2 \odot cF_3 = (\frac{b}{2}F_2 \odot \frac{b}{2}F_2) \odot (\frac{c}{2}F_3 \odot \frac{c}{2}F_3) = \frac{b}{2}F(2,2) \odot \frac{c}{2}F(3,3)$ admits a $\mathbb{Z}_k$-magic labeling with zero index, $k \geq 4$.

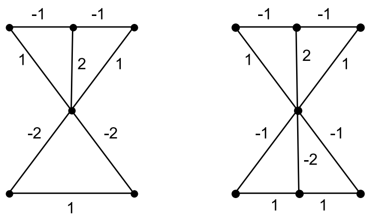

Figure 3.4: $F(2,3)$ and $F(3,3)$

Case 2. b even, c odd.

In case $b \geq 2$ it reduces to the case $F(3,2,2)$, see the Figure 3.5. In case $b = 0$ and $c = 1$, since $a \neq 0$, that is $F \neq \phi$, it reduces to the case $F(3,p)$, $p \geq 4$. In case $b = 0$ and $c \geq 3$ odd, it reduces to the case $F(3,3,3)$, see Figure 3.6.

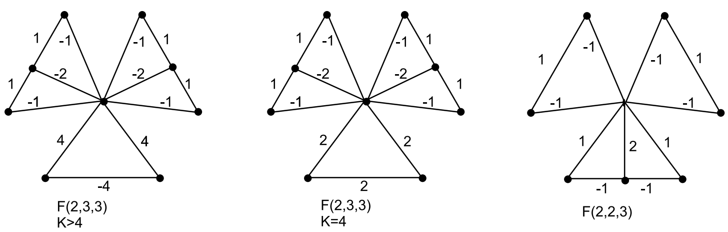

Figure 3.5: $F(2,3,3)$ and $F(2,2,3)$

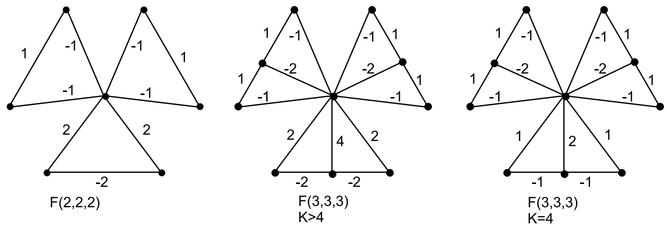

Figure 3.6: $F(2,2,2)$ and $F(3,3,3)$

Case 3. b odd, c even.

In case $b=1$, $c=0$. Then $a \neq 0$, that is $F \neq \phi$, it reduces to the case $F(2,p)$, $p \geq 4$. In case $b \geq 3$ odd, $c=0$, it reduces to the case $F(2,2,2)$, see Figure 3.6. In case $c \geq 2$, it reduces to the case $F(2,3,3)$, see the Figure 3.5.

Case 4. b, c are both odd.

In this case it reduces to the case $F(2,3)$, see the Figure 3.4.

□

Summarizing all up,

Theorem 3.2.3 *Let $F(n_1, n_2, \cdots, n_m)$ be a generalized fan, where $n_i \geq 2$, for $i = 1, \cdots, m$ and $m \geq 2$. The null set is*

$$\mathrm{Null}(F(n_1, n_2, \cdots, n_m)) = \begin{cases} \mathbb{N}, & \textit{if } n_1 = n_2 = \cdots = n_m = 2, \\ \mathbb{N}\backslash\{2,3\}, & \exists \textit{ unique } n_i \equiv 0 \textit{ or } 2 \ (\textit{mod } 3), \\ \mathbb{N}\backslash\{2\}, & \textit{otherwise.} \end{cases}$$

Remark. In particular, we have shown in the above Theorem, assuming $n_1 = n_2 = \cdots = n_m = 2$, that the **windmill graphs** (see the Figure 3.7) $WM_n = F(2,2,\cdots,2) = \{v\} + nP_2$, $n \geq 2$, admits $\mathbb{Z}_k$-magic zero sum labeling for all $k \in \mathbb{N}$, that is, the null set $\mathrm{Null}(WM_n) = \mathbb{N}$, $n \geq 2$.

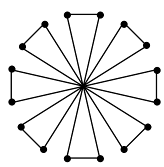

Figure 3.7: Windmill WM_n

Now we proceed to determine the null sets of generalized wheel graphs. First we put the results here without proof for the null set of the wheel graphs W_n, $n \geq 3$, for reference. Note that the proof is also based upon the induction and a method of subdivision as in the case of fans and generalized fans.

Theorem 3.2.4 *For $n \geq 3$,*

$$\mathrm{Null}(W_n) = \begin{cases} \mathbb{N}\backslash\{2\}, & n \equiv 0 \ (mod\ 3), \\ \mathbb{N}\backslash\{2,3\}, & n \equiv 1, 2 \ (mod\ 3). \end{cases}$$

To be more precise about the generalized wheel graphs, we define as follows:

Definition 3.2.3 *A generalized wheel graph $W(n_1, n_2, \cdots, n_m) = \{v\} + (C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_m})$, where C_{n_i} are disjoint cycles on $n_i \geq 3$ vertices, for $i = 1, \cdots, m$ and $m \geq 2$.*

Similar to the situation in the $\mathbb{Z}_3$-magic case of generalized fans, we have the following Lemma for the generalized wheels $W(n_1, n_2, \cdots, n_m)$. The proof is straightforward and similar to the one in Lemma 3.2.2, hence it is left to the reader.

Lemma 3.2.4 *Let $n_i \geq 3$, $\forall i = 1, \cdots, m$, and $S = \{n_i : \ n_i \equiv 1 \ or\ 2 \ (mod\ 3)\}$. Then we have that $0 \in I_3(W(n_1, n_2, \cdots, n_m))$ if and only if $|S| \neq 1$.*

Theorem 3.2.5 *Let $W(n_1, n_2, \cdots, n_m)$ be a generalized wheel, where $n_i \geq 3$ for $i = 1, \cdots, m$, and $m \geq 2$. The null set is*

$$\mathrm{Null}(W(n_1, n_2, \cdots, n_m)) = \begin{cases} \mathbb{N}\backslash\{2,3\}, & \exists \ unique \ n_i \equiv 1 \ or\ 2 \ (mod\ 3), \\ \mathbb{N}\backslash\{2\}, & otherwise. \end{cases}$$

Proof. Directly from Lemma 3.2.4 and Theorem 4.2.10. □

3.3 New Classes of Uniformly Null Graphs

In this section we study another class of graphs related to the null sets. A graph is **uniformly null** if every $\mathbb{Z}_k$-magic labeling induces 0 magic sum index, which was studied by E. Salehi in [33, 32]. Note that this definition implies all non-magic (that is non-$\mathbb{Z}_k$-magic for all k) graphs are uniformly null in Salehi's sense. He identified the class of complete bipartite graphs $K_{n,n+1}$ to be uniformly null.

Definition 3.3.1 *We call G an* **almost equi-bipartite graph** *if G is a bipartite graph (without isolated vertices) with bipartition (X, Y) and $||X| - |Y|| = 1$.*

We have the following observation for the index sets of almost equi-bipartite graphs:

Proposition 3.3.1 *Let G be an almost equi-bipartite graph with bipartition (X, Y), and $||X| - |Y|| = 1$. If G is $\mathbb{Z}_k$-magic, then it is uniformly null, that is, $I_k(G) = \{0\}$, $\forall k \geq 3$.*

Proof. Suppose G admits a $\mathbb{Z}_k$-magic labeling f with index r, and $|X| = m$, $|Y| = m + 1$. By adding all the vertex sums in X, and adding all the vertex sums in Y respectively, we have

$$mr = \sum_{e \in E(G)} f(e) = (m + 1)r,$$

which implies $r \equiv 0 \pmod{k}$. Thus the proof is complete.

□

Remark. If G is an almost equi-bipartite graph, and if moreover G is an even graph (that is in G every vertex is of even degree), then G is $\mathbb{Z}_k$-magic with an index 0 since it is a disjoint union of Eulerian graphs of even size. However conversely we have examples of uniformly null graphs which are not even graphs, namely, the complete almost equi-bipartite graphs $K_{n,n+1}$ for $n \geq 3$, as E. Salehi pointed out in [33].

3.3.1 C_4-Construction of Almost Equi-bipartite Graphs

The following is a construction with C_4 of an infinite family of almost equi-bipartite graphs G whose degrees are all even and for which $I_k(G) = \{0\}$, $\forall k \geq 3$. In fact, $I_k(G) = \{0\}$, for $k = 1, 2$ as well.

Note that since almost equi-bipartite graphs G with degree one vertices are not $\mathbb{Z}_k$-magic with zero sum, the minimum degree $\delta(G) \geq 2$ and hence G contains cycles and only even cycles. Therefore, we may have that the order $|V(G)| \geq 7$ and is odd. So the minimum order of such graphs is 7 as shown in the graph of Figure 3.8, which is isomorphic to the dumbbell graph $D(4, 4)$, one vertex union of two four-cycles. We denote it by B_7, and clearly $I_k(B_7) = 0$, $\forall k \geq 3$. Let $B_7 \in \beta_7$ be the first family of almost equi-bipartite graphs with the least order 7 and B_7 is the one with the least number of edges.

Construct families β_{n+2} from β_n using the following steps to obtain $B_{n+2} \in \beta_{n+2}$ from $B_n \in \beta_n$ for $n \geq 1$:

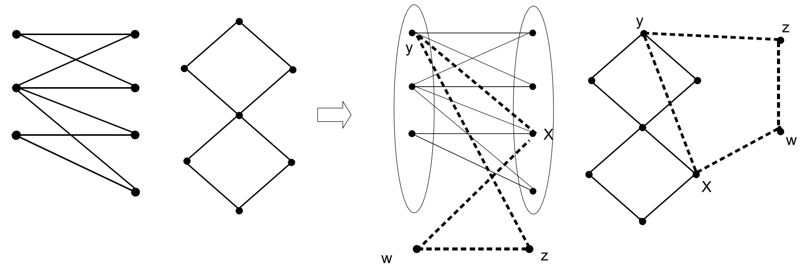

Figure 3.8: Construction from $B_7 \cong D(4,4)$ to $B_9 \in \beta_9$

Step 1. Choose vertices x, y in $B_n \in \beta_n$ such that their distance $d(x, y)$ is odd and strictly greater than 1, that is $d(x, y) \in \{3, 5, 7, \cdots\}$. Hence x, y are in different partite sets and non-adjacent.

Step 2. Then we add two new vertices w, z such that x, z and y, w are in the same partite sets, respectively, and join the edges to get xy, xw, yz, wz to create a graph $B_{n+2} \in \beta_{n+2}$.

In such a way we attach a C_4 to the chosen vertices x, y to create new graphs $B_{n+2} \in \beta_{n+2}$. Note that the new graphs are still Eulerian of even size and, in fact, are of the smallest size in β_{n+2} if the construction starts from $B_7 \cong D(4,4)$. Therefore, we obtain an infinitely family β of uniformly null almost equi-bipartite graphs via the above constructions.

3.3.2 One Point Union Construction of Almost Equi-bipartite Graphs

We have another infinite set of examples of uniformly null graphs, which is constructed by attaching even cycles in a particular way. The construction is as follows.

Step 1. Pick a path of even length, and make each edge to be two parallel edges between every pair of adjacent vertices.

Step 2. Insert even number (including none) of vertices of degree 2 in each edge so that the resulting graph is simple and a one point union of even cycles.

Then it is routine to check that the resulting graph is an almost equi-bipartite graph, therefore it is a uniformly null graph by the above Proposition 3.3.1. Please see Figure 3.9.

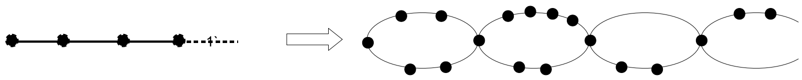

Figure 3.9: One Point Union of Even Cycles

3.3.3 Application to Integer Magic Spectrum of Corona Product

Let G and H be two graphs, where $|V(G)| = n$. Take one copy of G and n copies of H, for each i from 1 to n, join the ith vertex of G to each vertex in the ith copy of H. The resulting graph is called the ***corona product*** of G with H, which we shall denote $G©H$. On the other hand, given a graph G, the set of all k such that G is $\mathbb{Z}_k$-magic is defined as the ***integer-magic spectrum*** of G, and is denoted by $\mathrm{IM}(G)$. Please see [25, 27, 28, 34, 35].

We obtain the following criterion to get the integer magic spectra of the corona $G©N_m$ using the information of index sets of G in [23], where N_m is the null graph (empty graph) with m isolated points.

Proposition 3.3.2 *For fixed m and $k \geq 2$, let $d = gcd(k, 1-m)$. We analyze $\mathrm{IM}(G©N_m)$ in the following cases:*

Case 1. *$d > 1$, then $k \in \mathrm{IM}(G©N_m)$ if and only if $d|r_i$, for some $r_i \in I_k(G)$.*

Case 2. *$d = 1$, then $k \in \mathrm{IM}(G©N_m)$ if and only if $I_k(G)$ has an non-zero element.*

Therefore,

1. If $I_k(G)$ contains both 0 and another non-zero element, then $k \in \mathrm{IM}(G©N_m)$.

2. In particular if $0 \in I_k(G)$, then $k \in \mathrm{IM}(G©N_m)$.

3. Moreover, for all non-negative integer n, $G©N_{nk+1}$ is $\mathbb{Z}_k$-magic if and only if $0 \in I_k(G)$.

Hence we have the following observation for calculating the integer magic spectra of the corona products of uniformly null graphs with null graphs:

Proposition 3.3.3 *If $I_k(G) = \{0\}$ for all $k \geq 3$, then the integer magic spectrum*

$$\mathrm{IM}(G \circledcirc N_m) = \{k : \ gcd(1-m, k) > 1\} = \bigcup_{i=1}^{t} p_i \mathbb{N},$$

where $m - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ the prime divisor decomposition.

Therefore we have obtained the integer magic spectra of infinitely many examples of corona products of G with null graphs N_m by the above Proposition 3.3.3, where G could be any of previously constructed uniformly null graphs, say $K_{n,n+1}$, graphs in the family β by C_4-construction, or graphs constructed by one point union of even cycles.

3.4 Open Problems of Null Sets

Note that we have obtained the null sets of generalized wheels and generalized fans in this article. Also we have created infinitely many examples of uniformly null graphs using the different even cycle constructions. Therefore in particular we answer the open problems posted by E. Salehi in [33], which are finding the null sets of wheels and fans, and identifying families of uniformly null graphs other than the complete bipartite graphs $K_{n,n+1}$.

We conclude this paper by posting the following open problems left out of the discussion over these related topics:

1. Determine the index sets of the generalized fans and the generalized wheels.

2. Characterize the class of graphs G for which $I_k(G) = \{0\}, \ \forall k \geq 3$.

3. Characterize the class of almost equi-bipartite graphs G which are uniformly null.

Chapter 4

Group Labeling with Various Magic Sums

4.1 Index Sets of Regular Graphs

We may have many examples of regular graphs with index sets $\mathbb{Z}_k$ from the above observations for now. We have the following observations for the index sets of general r-regular graphs:

Lemma 4.1.1 *Let G be an r-regular graph, $r \geq 1$. Then the index set $I_k(G) \supseteq \mathbb{Z}_k^*$ for those integers k relatively prime with r, that is, $gcd(r,k) = 1$. In particular for 1-regular G, we have $I_k(G) = \mathbb{Z}_k^*$.*

Proof. Note that $r\mathbb{Z}_k^* = \mathbb{Z}_k^*$ if $gcd(r,k) = 1$. Then in order to finish the proof, one simply label all edges with $i \in \mathbb{Z}_k^*$, for $i = 1, \cdots, k-1$, respectively. □

Lemma 4.1.2 *Let G be an r-regular graph, $r \geq 2$, and let $gcd(r,k) = d > 1$. Then the index set $I_k(G) \supseteq \mathbb{Z}_{\frac{k}{d}}$ = the cyclic subgroup of $\mathbb{Z}_k$ generated by the element r. In particular $0 \in I_k(G)$.*

Proof. Let $r = r' \cdot d$ and $k = k' \cdot d$, where $gcd(r', k') = 1$. Label all edges with $i \in \mathbb{Z}_k^*$, for $i = 1, \cdots, k-1$, respectively, we have the magic sums $r = r'd, 2r = 2r'd, \cdots, k'r = k'r'd = r'k \equiv 0 \ (mod\ k), \cdots, (k-1)r, \cdots$, and may see then the pattern is repeated with the period $k' = \frac{k}{d}$. □

Remark. From the above lemmas we see that, for regular graphs only partial information is immediate for the calculation of the index sets. For example one is not sure and wonder that whether 0 is in $I_k(G)$ or not for a r-regular graph in case $gcd(r,k) = 1$. In fact we have an example later in Theorem 4.1.4 showing that sometimes $0 \notin I_k(G)$ may happen to regular G in such a situation. More generally, one will wonder that for a regular graph G, when is the index set $I_k(G) = \mathbb{Z}_k$? We show the answer is positive for regular graphs admitting a 1-factor in the following section.

4.1.1 Index Sets of Regular Graphs with 1-Factor

We start with several lemmas.

Lemma 4.1.3 *Any Eulerian graph with even size has a zero index.*

Proof. Label 1and -1 alternately on an Eulerian cycle. $\square$

Lemma 4.1.4 *For any cubic graph G with a 1-factor, $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$.*

Proof. Assume $k \geq 3$. Let M be the 1-factor of G. Since $I_k(M) = \mathbb{Z}_k^*$ and $G\backslash M$ is a 2-regular graph with index 2 via labeling 1 on all edges, then $2 + \mathbb{Z}_k^* = \mathbb{Z}_k\backslash\{2\} \subset I_k(G)$. It remains to show $2 \in I_k(G)$. Labeling -2 on all edges of M and 2 on all edges of $G\backslash M$, then it is easy to see that 2 is an index. $\square$

Lemma 4.1.5 *Let G be a regular graph. Suppose G has a factorization containing a 1-factor M_1 and a 2-factor M_2, then $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$.*

Proof. Assume that $k \geq 3$. Note that G may be factored as edge disjoint sum of two $\mathbb{Z}_k$-magic spanning subgraphs $G = (M_1 \oplus M_2) \oplus (G\backslash(M_1 \oplus M_2)$, where M_1 is a 1-factor and M_2 is a 2-factor. Therefore $I_k(M_1 \oplus M_2) = \mathbb{Z}_k$ by Lemma 4.1.4, and this lemma follows by $I_k(G) = I_k((M_1 \oplus M_2) \oplus (G\backslash(M_1 \oplus M_2)) \subseteq I_k((M_1 \oplus M_2)) + I_k((G\backslash(M_1 \oplus M_2)) = \mathbb{Z}_k$. $\square$

We are in a position for showing a sufficient condition to have the full index set $\mathbb{Z}_k$ for general r-regular graphs. It is well known one have the following theorem for even-regular graphs:

Theorem 4.1.1 (Petersen, [12] 1891) *A non-empty graph G admits 2-factorization if and only if G is $2m$-regular for some $m \geq 1$.*

Theorem 4.1.2 *Let G be a r-regular graph ($r \geq 2$) which admits a 1-factor, then $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$.*

Proof.

Assume that $k \geq 3$. Let M be the 1-factor. Note that $G = M \oplus (G\backslash M)$ and $G\backslash M$ is an $(r-1)$-regular graph. Since for the 1-factor $I_k(M) = \mathbb{Z}_k^*$, and $G\backslash M$ has indices $r-1$ and $1-r$ by labeling 1 and -1 respectively on edges, we have that $I_k(G) = I_k(M \oplus (G\backslash M)) \supseteq (r-1) + \mathbb{Z}_k^*$ and $I_k(G) = I_k(M \oplus (G\backslash M)) \supseteq (1-r) + \mathbb{Z}_k^*$. If $(r-1)+\mathbb{Z}_k^* \neq (1-r)+\mathbb{Z}_k^*$, then we are done with $I_k(G) = \mathbb{Z}_k$, since $(r-1)+\mathbb{Z}_k^* = \mathbb{Z}_k^*\backslash\{r-1\}$ and $(1-r)+\mathbb{Z}_k^* = \mathbb{Z}_k^*\backslash\{1-r\}$.

In case $(r-1) + \mathbb{Z}_k^* = (1-r) + \mathbb{Z}_k^*$, which implies $r-1 \equiv 1-r$ that is $2(r-1) \equiv 0 \ (mod\ k)$. Therefore, we have that $G\backslash M$ has index 0 by labeling 2 on edges. Hence $I_k(G) = I_k(M \oplus (G\backslash M)) \supseteq 0 + \mathbb{Z}_k^* = \mathbb{Z}_k^*$ in this case, and it remains to show that $0 \in I_k(G)$. For the case r is even, note that G has even order, hence G is an Eulerian graph of even size and $0 \in I_k(G)$ by Lemma 4.1.3. For the case r is odd, since $G\backslash M$ is a even-regular graph, hence by Petersen's Theorem 4.1.1 it has a 2-factorization. Then G has a 1-factor and a 2-factor, by Lemma 4.1.5, and $I_k(G) = \mathbb{Z}_k, \ k > 2$. Then we are done with the proof. □

From the above Theorem 4.1.2, immediately we obtain that the index sets I_k of Generalized Petersen graphs $P(n,k)$, Prisms, Mobius Ladders etc. are known to be the full $\mathbb{Z}_k, \ \forall k \geq 3$, since they admit 1-factors.

Corollary 4.1.1 *Let G be a regular bipartite graph. Then $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$.*

Proof. By Hall's Marriage Theorem, any regular bipartite graph has a 1-factor. It then follows by the above Theorem 4.1.2. □

A graph containing no $K_{1,3}$ as an induced subgraph is said to be claw-free. Another corollary is based upon the following result:

Theorem 4.1.3 (Sumner[19]) *If G is a connected claw-free graph of even order, then it has a 1-factor.*

Corollary 4.1.2 *Let G be a claw-free regular graphs with even vertices, then $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$.*

Proof. It follows by Theorem 4.1.2 and Theorem 4.1.3. □

Moreover, making use of the above Corollary 4.1.1, one may show that the index set of the lexicographic product of any graph with a $\mathbb{Z}_k$-magic graph is the full $\mathbb{Z}_k$. Given two graph $G = (V_1, E_1)$ and $H = (V_2, E_2)$. The **lexicographic product** of G and H is the graph $G \circ H$ with vertex set $V_1 \times V_2$ and for $(u_1, v_1), (u_2, v_2) \in V_1 \times V_2$, and (u_1, v_1) is adjacent with (u_2, v_2) whenever

(1)$u_1 = u_2$ and v_1 is adjacent to v_2 in H or (2)u_1 is adjacent to u_2 in G. The lexicographic product $G \circ H$ is also called the **composition** of G and H. Hence we have the following:

Corollary 4.1.3 *Let G and H be any two graphs, and H is $\mathbb{Z}_k$-magic. Then the index set of their lexicographic product is $I_k(G \circ H) = \mathbb{Z}_k$.*

Proof. Let $G(V_1, E_1)$ be a (p_1, q_1)-graph, where $|V_1| = p_1$, $|E_1| = q_1$, and $H = (V_2, E_2)$ be a (p_2, q_2)-graph, where $|V_2| = p_2$, and $|E_2| = q_2$. Then we notice that the lexicographic product $G \circ H$ can be decomposed into p_1 isomorphic copies of H, namely the subgraphs $H_i = \{(u_i, v_j)(u_i, v_k) : v_j v_k \in E_2\}$ for $1 \leq i \leq p_1$, together with q_1 isomorphic copies of complete bipartite graphs K_{p_2,p_2}, namely the subgraphs $K_j = \{(u_s, v_m)(u_t, v_n) : u_s u_t = e_j\}$ for $e_j \in E_1$, where $1 \leq j \leq q_1$. That is $G \circ H \cong p_1 H \oplus q_1 K_{p_2,p_2}$. Note that in the decomposition of $G \circ H$, there is one part of complete bipartite graphs with equi-bipartitions, hence regular bipartite. By Theorem 3.1.2 and Corollary 4.1.1, we see that $I_k(G \circ H) = \mathbb{Z}_k$. □

4.1.2 Index Sets of Regular Graphs without 1-Factor

Note that there are examples for regular graphs without 1-factor for which the index set is not full $\mathbb{Z}_k$, $\forall k \geq 3$. See the following example. Let PC be Petersen's example for a cubic bridgeless graph containing no 1-factors. Please see the Figure 4.1.

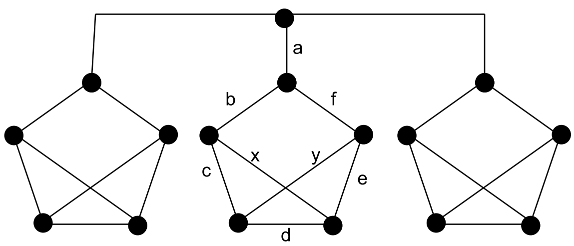

Figure 4.1: PC

Lemma 4.1.6 $\pm 1 \notin I_3(PC)$.

Proof. See Figure 4.1, we have $b + c = d + e$, $c + d = e + f$ which implies $b + f = 2d$. Suppose $1 \in I_3(PC)$. Without loss of generality, there exists one of three edge labels of the top claw to be 1, say $a = 1$. Therefore it implies $0 = b + f = 2d$ in $\mathbb{Z}_3$, the nonzero solution for d does not exist, a contradiction. □

Lemma 4.1.7 *$0 \notin I_4(PC)$, and $1, 2, 3 \in I_4(PC)$.*

Proof. See Figure 4.1, similarly we have $b + f = 2d$. Suppose $0 \in I_4(PC)$, then one of three edge labels of the top claw must be 1 or -1. Therefore it implies $\pm 1 = b + f = 2d$ in $\mathbb{Z}_4$, and the solution for d does not exist, a contradiction. On the other hand, since PC is 3-regular, by the Lemma 4.1.1, thus $I_4(G) \supseteq \mathbb{Z}_4^*$.

□

Lemma 4.1.8 *For $k \geq 5$, $0 \in I_k(PC)$*

Proof. See the Figure 4.2. □

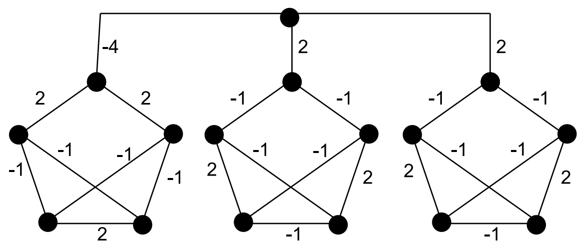

Figure 4.2: $\mathbb{Z}_k$-magic labeling with magic sum 0 of PC for $k \geq 5$

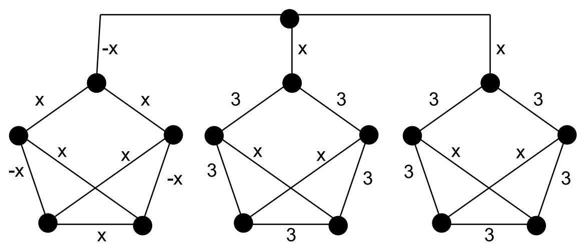

Figure 4.3: $\mathbb{Z}_6$-magic labeling of PC with a magic sum x of PC

Lemma 4.1.9 *For $x \in \mathbb{Z}_k^*$, $x \in I_k(PC)$, $k \geq 5$*

Proof. For the case of $\mathbb{Z}_6$, see the Figure 4.3. For the general case $k \geq 5$, $k \neq 6$,, see the Figure 4.4 and Figure 4.5, which give two ways of labeling making the magic sum index x. It remains to show at least one of them contains no edges labeled by zero. Suppose not, say $x-2 = 0$ or $x+2 = 0$ in the Figure 4.4,

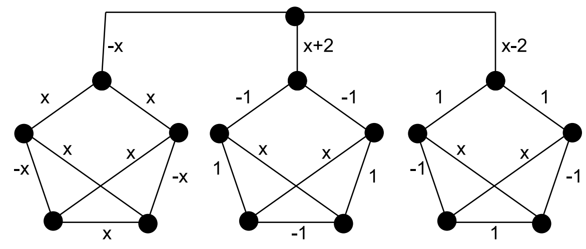

Figure 4.4: One $\mathbb{Z}_k$-magic labeling with magic sum x of PC for $k \geq 5$

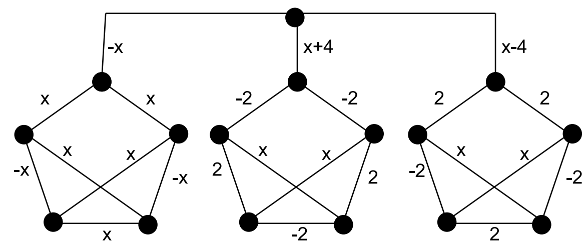

Figure 4.5: Another $\mathbb{Z}_k$-magic labeling with magic sum x of PC for $k \geq 5$

then in the Figure 4.5 either $x + 4 = 0$ or $x - 4 = 0$ in $\mathbb{Z}_6$, however this has been ruled out already.

□

To summarize from above lemmas, we have the following result for the index set of the graph PC.

Theorem 4.1.4

$$I_k(PC) = \begin{cases} \{0\}, & k = 3, \\ \mathbb{Z}_4 \backslash \{0\}, & k = 4, \\ \mathbb{Z}_k, & k \geq 5. \end{cases}$$

4.1.3 Index Sets of Circulant Graphs

To obtain more examples, we further determine the the index sets for the circulant graphs completely. Note that the class of circular graphs includes both the examples of regular graphs with 1-factor, and examples of regular graphs without 1-factor. Circulant graphs may be treated as a generalization of complete graphs, and contain many well known graph classes such as Möbius ladder graphs etc.

Definition 4.1.1 *A circulant graph $CIR_n(S)$ with n vertices, with respect to $S \subset \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$, is defined as a graph with the vertex set $V(CIR_n(S)) =$*

$\{0, 1, 2, \cdots, n-1\}$, *and the edge set is formed by the following rule:*

$$E(CIR_n(S)) = \{ij : \ i - j \equiv \pm s \pmod n, \ s \in S\}.$$

For example, $CIR_n(\{1\}) \cong C_n$, the cycles, $CIR_n(\{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}) \cong K_n$, the complete graphs, and $CIR_{2n}(\{1, n\}) \cong$ the n-Möbius ladder graphs. Note that $CIR_n(S)$ is a class of regular graphs, and is even-regular if $\lfloor \frac{n}{2} \rfloor \notin S$, odd-regular otherwise. In order to obtain all the index set of the circulant graphs, we need the following lemmas:

Lemma 4.1.10 *Let G be a $2m$-regular graph. Then we have*

$$I_k(G) \supseteq \begin{cases} 2\mathbb{Z}_k^*, & m = 1, \\ 2\mathbb{Z}_k, & m \geq 2. \end{cases}$$

Proof. In case G is 2-regular, that is G is a single 2-factor, we have $2\mathbb{Z}_k^* \subseteq I_k(G)$ by Proposition 3.1.5.

In case $m \geq 2$, we proceed induction on m. By the Petersen's Theorem 4.1.1 G is 2-factorable, hence $G = F_1 \oplus F_2 \oplus \cdots \oplus F_m$, where F_i are 2-factors for $i = 1, \cdots, m$. For the case $m = 2$, $G = F_1 \oplus F_2$, and note that $I_k(F_i) \supseteq 2\mathbb{Z}_k^*$, for $i = 1, 2$. Therefore $I_k(G) = I_k(F_1 \oplus F_2) \supseteq 2\mathbb{Z}_k^* + 2\mathbb{Z}_k^* = 2\mathbb{Z}_k$. Assume the Lemma holds for $m = n$, consider the case $m = n + 1$ as follows: $G = (F_1 \oplus F_2 \oplus \cdots \oplus F_n) \oplus (F_{n+1})$. By induction hypothesis, $2\mathbb{Z}_k \subseteq I_k((F_1 \oplus F_2 \oplus \cdots \oplus F_n))$, then $I_k((F_1 \oplus F_2 \oplus \cdots \oplus F_n) \oplus (F_{n+1})) \supseteq I_k(F_1 \oplus F_2 \oplus \cdots \oplus F_n) + I_k(F_{n+1}) \supseteq 2\mathbb{Z}_k + 2\mathbb{Z}_k^* = 2\mathbb{Z}_k$. Therefore by induction we are done with the proof.

□

Let $S = \{a_1, a_2, \ldots\ldots, a_m\} \subseteq \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$, it is not hard to see that $CIR_n(S) = \bigoplus_{i=1}^{m} CIR_n(\{a_i\})$ is a factorization of circulant graphs with respect to one point sets $\{a_i\}$. Then we have the following characterization for circulant graphs with 1-factor:

Lemma 4.1.11 *The circulant graph $CIR_n(\{a_1, a_2, \cdots, a_m\})$ admits 1-factor if and only if $\frac{n}{d_i}$ is even for some i, where $d_i = gcd(a_i, n)$, for $i = 1, 2, \cdots, m$.*

Proof. Let G be the circulant graph $CIR_n(\{a_1, a_2, \cdots, a_m\})$. The sufficiency is clear since if $\frac{n}{d_i}$ is even for some i, then $CIR_n(\{a_i\}) \cong d_i \cdot C_{\frac{n}{d_i}}$, which is d_i copies of cycles of order $\frac{n}{d_i}$. Therefore G contains one 2-factor consisting of even cycles, thus a perfect matching of G is obtained.

For necessity, suppose $\frac{n}{d_i}$ is odd for all $i = 1, 2, \cdots, m$. Let $d = gcd(d_1, \cdots, d_m)$, and for $j = 1, 2, \cdots, d$, let $G_j = G[S_j]$ be the induced subgraph over $S_j \subseteq V(G)$, where $S_j = \{0 \leq t \leq n-1 : \ t \equiv j \ (mod \ d)\}$. Then $|S_j| = |V(G_j)| = \frac{n}{d}$ is odd for each $j = 1, 2, \cdots, d$. For $x \in S_j$ and $y \in S_k$, $j \neq k$, we have

$x - y \neq 0 \pmod d$ and hence $x - y \neq 0 \pmod n$. On the other hand, if x and y are adjacent in G, then $x - y = \pm a_i \pmod n$ for some i, which implies $x - y = 0 \pmod d$ since $d_i = gcd(a_i, n)$. Therefore G_j and G_k are disjoint for $j \neq k$. Thus G consists of disconnected parts $G_1, G_2, \cdots, G_d$ of odd orders, and hence G must contain odd components. Therefore G contains no 1-factor and we are done with the proof.

□

With the above lemmas, we are in a position to obtain the index set of all circulant graphs $I_k(CIR_n(S))$ as follows:

Theorem 4.1.5

$$I_k(CIR_n(S)) = \begin{cases} \mathbb{Z}_k, & \textit{if there exists some } a \in S \textit{ with } \frac{n}{gcd(a,n)} \textit{ even.} \\ 2\mathbb{Z}_k, & \textit{otherwise.} \end{cases}$$

Proof. First by applying Lemma 4.1.2 and Lemma 4.1.11, if there exists some $a \in S$ such that $\frac{n}{gcd(a,n)}$ is even, then $I_k(CIR_n(S)) = \mathbb{Z}_k$ since $CIR_n(S)$ is a regular graph admitting a 1-factor.

Otherwise, if $\frac{n}{d_r}$ is odd for all $r = 1, 2, \cdots k$, then as in previous Lemma 4.1.11, we can find in $CIR_n(S)$ disconnected parts $G_1, G_2, \cdots, G_d$ with odd orders, and hence some odd component, say P. Hence $I_k(G) \subseteq I_k(P) \subseteq 2\mathbb{Z}_k$ by the Proposition 3.1.1 and the Proposition 3.1.4. On the other hand, Lemma 4.1.10 implies $2\mathbb{Z}_k \subseteq I_k(G)$. Therefore we are done with $I_k(G) = 2\mathbb{Z}_k$ in this case.

□

Remark. Note that from the above observation, we have created many examples of regular graphs without 1-factor whose index sets are not full $\mathbb{Z}_k$ for some $k \geq 3$, just like the situation for previously mentioned example PC. For example, we mentioned earlier that $I_k(K_{2n+1}) = 2\mathbb{Z}_k \subsetneqq \mathbb{Z}_k$ for k even, where K_{2n+1} is the complete graphs of odd order. The complete graphs K_{2n+1} are special cases of circulant graphs without 1-factor, and note that the result regarding their index sets also follows from the necessary condition of being $\mathbb{Z}_k$-magic with magic sum r in the Proposition 3.1.4 and its corollary.

4.2 More Examples of Index Sets

We determine the index sets of complete bipartite graphs $K_{m,n}$ for $m, n \geq 1$, wheels W_n, $n \geq 3$, and fans F_n, $n \geq 3$, in this section. Note that these examples are not regular graphs in most cases. First of all we deal with the case of complete bipartite graphs $K_{m,n}$ for $m, n \geq 1$.

4.2.1 Index Sets of Complete Bipartite Graphs $K_{m,n}$

Note that the index x of a $\mathbb{Z}_k$-magic labeling for $K_{m,n}$ is the solution of the linear congruence equation $(m-n)x \equiv 0$ $(mod\ k)$, since mx and nx are both the total sum of all edge $\mathbb{Z}_k$-labels over the graph $K_{m,n}$, as seen by adding up m of the same indices x on one partite set, or n of the same indices x on the other partite set over the graph $K_{m,n}$.

If $m = 1$ or $n = 1$, $K_{m,n}$ is the star graph. Then by the solutions to above linear congruence equation, we have $I_k(K_{1,1}) = \mathbb{Z}_k^*$, and $I_k(K_{1,n}) = \{\frac{k}{d}, \frac{2k}{d}, \cdots, \frac{(d-1)k}{d}\}$ for $n \geq 2$ and $d = gcd(n-1, k) > 1$. Note that $I_k(K_{1,n}) = \emptyset$ for $n \geq 2$ and $gcd(n-1, k) = 1$, and in particular $0 \notin I_k(K_{1,n})$ in any case.

Now we study the case for $K_{m,n}$ for $m, n \geq 2$ in the following.

Proposition 4.2.1 *$0 \in I_k(K_{m,n})$ for $m, n \geq 2$ and all $k \geq 3$.*

Proof. We have the following two cases:
Case 1. m or n is even. Say m is even, we decompose $K_{m,n}$ into an edge disjoint union of copies of $K_{2,n}$. Then label on edges of $K_{2,n}$ by the following rules. Labeling 1 and -1 on each pair of edges incident to the first $n-1$ degree 2 vertices, then labeling, if necessary (whenever n is odd), 2 and -2 on last pair of edges. Therefore we get an index 0 over each copy of $K_{2,n}$, and hence an index 0 over whole $K_{m,n}$ for m or n is even.
Case 2. m and n are both odd. Note that $K_{3,3}$ can be labeled with ± 1 and ± 2 to obtain the magic sum index 0. In this case consider $K_{m,n}$ as the edge disjoint sum of $K_{3,3}$ and $K_{m-3,n-3}$. The part $K_{m-3,n-3}$ would have an index 0 by the **Case 1**, together with $K_{3,3}$ it give rise to an index 0. $\square$

Theorem 4.2.1 *For $m, n \geq 2$, and $k \geq 3$, the index set of complete bipartite graph $K_{m,n}$ is $I_k(K_{m,n}) = \langle \frac{k}{d} \rangle$, where $d = gcd(m-n, k)$, and $\langle a \rangle$ denote the additive subgroup of $\mathbb{Z}_k$ generated by the element a. Or equivalently,*

$$I_k(K_{m,n}) = \begin{cases} \{0\}, & \textit{if } gcd(m-n,k) = 1. \\ \mathbb{Z}_d \cong \langle \frac{k}{d} \rangle, & \textit{if } gcd(m-n,k) = d,\ 1 < d \leq k. \end{cases}$$

Proof. Note that the index x of a $\mathbb{Z}_k$-magic labeling for $K_{m,n}$ is the solution of the linear congruence equation $(m-n)x \equiv 0$ $(mod\ k)$.

In case $gcd(m-n, k) = 1$, then $(m-n)x \equiv 0$ $(mod\ k)$ has the unique solution 0, hence $I_k(K_{m,n}) = \{0\}$ since $0 \in I_k(K_{m,n})$ as shown in Proposition 4.2.1.

In case $gcd(m-n, k) = d > 1$, then there are d solutions $(mod\ k)$ for $(m-n)x \equiv 0$ $(mod\ k)$, namely, $0, \frac{k}{d}, \frac{2k}{d}, \cdots, \frac{(d-1)k}{d}$. Note that $\{0, \frac{k}{d}, \frac{2k}{d}, \cdots, \frac{(d-1)k}{d}\} = \langle \frac{k}{d} \rangle \cong \mathbb{Z}_d$. Then we may obtain all other possible magic sum index $x \in \mathbb{Z}_d^* \cong \{\frac{k}{d}, \frac{2k}{d}, \cdots, \frac{(d-1)k}{d}\}$ by the following observations. Without loss of generality

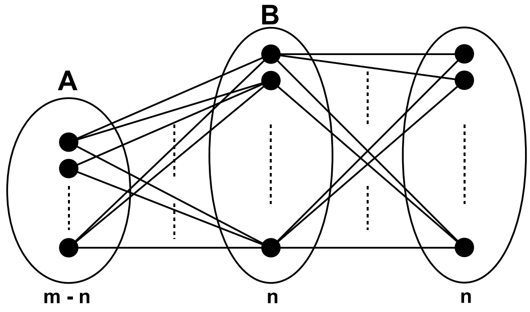

Figure 4.6: The decomposition of $K_{m,n}$ into $K_{m-n,n}$ and $K_{n,n}$

we may assume that $m \geq n \geq 2$. Decompose the graph $K_{m,n}$ into two parts, one is regular bipartite $K_{n,n}$, and the other is $K_{m-n,n}$. Note that we write the part $K_{m-n,n}$ as a bipartite graph with bipartition (A, B), as in the Figure 4.6.

We claim that there exists appropriate edge labeling on the part $K_{m-n,n}$ such that the vertex sums over the partite set A of $m-n$ vertices are $x \in \mathbb{Z}_d^*$, and the vertex sums over the partite set B of n vertices are 0, then since $I_k(K_{n,n}) = \mathbb{Z}_k$ for all $k \geq 3$ by the Theorem 4.1.2, the desired labeling is obtained by combining the labels from two parts. For proving the claim, we have two cases as follows:

Case 1. n is odd.

Note that the degrees of vertices in the partite set B are $m-n$. We label the edges of $K_{m-n,n}$ in an ordering of vertices in B by $\underbrace{x, \cdots, x}_{m-n}$, $\underbrace{-x, \cdots, -x}_{m-n}$, $\underbrace{x, \cdots, x}_{m-n}$, $\underbrace{-x, \cdots, -x}_{m-n}$, $\cdots$, until $\underbrace{x, \cdots, x}_{m-n}$. Then the partial vertex sums over the vertices in B with respect to A are either $(m-n) \cdot x$ or $(m-n) \cdot (-x)$, both $\equiv 0$ $(mod\ k)$ since x is a solution to $(m-n)x \equiv 0$ $(mod\ k)$. On the other hand, the vertex sums over the vertices in A are exactly x. Therefore we got the desired labeling in the claim in this case.

Case 2. n is even.

Subcase 2.1. $2x \neq 0$ $(mod\ k)$.

We label the edges as in previous case by $\underbrace{x, \cdots, x}_{m-n}$, $\underbrace{-x, \cdots, -x}_{m-n}$, $\underbrace{x, \cdots, x}_{m-n}$, $\underbrace{-x, \cdots, -x}_{m-n}$, $\cdots$, except the last two groups we use $\underbrace{2x, \cdots, 2x}_{m-n}$ and $\underbrace{-x, \cdots, -x}_{m-n}$ instead. Then the new labeling does the job.

Subcase 2.2. $2x = 0$ $(mod\ k)$.

In this case k is even and $x = \frac{k}{2} \neq 0$. We claim that $m-n$ must be even. Note that $x = \frac{k}{2} \neq 0$ is an element of order 2 in the group $\mathbb{Z}_k$, and hence also an element of order 2 in the group $\mathbb{Z}_d \cong \langle \frac{k}{d} \rangle$ since $d|k$. Therefore

$2|d = gcd(m-n, k)$, thus $m-n$ must be even.

Then in the following we may give a new labeling under the situation that n is even, $m-n$ is even, and k is even. In the partite set A, we pair off the vertices as $\{u_1, u_2\}, \{u_3, u_4\} \cdots, \{u_{m-n-1}, u_{m-n}\}$, and label on each 4-cycle containing these two-vertex sets $\{u_1, u_2\}, \{u_3, u_4\} \cdots, \{u_{m-n-1}, u_{m-n}\}$ respectively by $\frac{k}{2}-1, 1, -1, 1-\frac{k}{2}$ consecutively, as in the Figure 4.7. Then it is a direct computation that the partial vertex sums over $u_1, u_2, u_3, u_4, \cdots, u_{m-n-1}, u_{m-n}$ on these 4-cycles are all $\frac{k}{2} \equiv \frac{-k}{2}$ $(mod\ k)$, and the partial vertex sums over vertices related in the other partite set B are all zero.

At last, note that the remaining edges not used in the above C_4 constructions induce an even subgraph (in which every degree is even) of $K_{m-n,n}$, which is Eulerian of even size in each component. Then as in the Lemma 4.1.3 we label 1 and -1 consecutively on the edges of an Eulerian cycle, which would give partial vertex sums of zero over the even subgraph. Combining all the above partial sums together, we obtain a new labeling as desired on the part $K_{m-n,n}$.

□

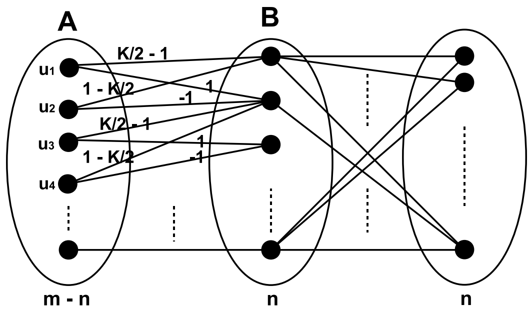

Figure 4.7: C_4 construction when n is even, $m-n$ is even, and k is even

4.2.2 Index Sets of Fans F_n

In [33], the concept of null sets of a graph is defined and studied. The null set of G is the set of all $k's$ such that G is $\mathbb{Z}_k$-magic with index 0. it is clear that the concepts null sets, index sets, and $\mathbb{Z}_k$-magicness are closely related each other. We most recently studied and determined completely the null sets of generalized fans and generalized wheels in [23] among other results. Note that the notion index sets is more general than the null sets, and naturally harder to settle.

In order to obtain the whole index sets of fans and wheels, we first describe a ***subdivision method*** which is commonly used in this and later sections for the construction of $\mathbb{Z}_k$-magic labeling with the same magic sum index of an infinitely family of graphs, in particular fans and wheels.

Subdivision Method:

Let G be a graph with index r under a $\mathbb{Z}_k$-magic labeling f, using the subdivision method we may obtain a new graph G' with larger order, and a new $\mathbb{Z}_k$-magic f' on G' with the same index r, based upon G and f. We proceed by choosing in G a vertex v and edges e_1, e_2 with labels $f(e_1) = a$, and $f(e_2) = b$, which are not incident with v. Then subdivide these two edges by inserting new vertices of degree 2, join them to v respectively. Please see the Figure 4.8. Now then we may construct a new labeling f' on G' by keeping the labels on G unchanged, and labeling $r - 2a, r - 2b$ on two newly inserted edges respectively. Note that if $(r - 2a) + (r - 2b) \equiv 0$, $r - 2a \neq 0$, and $r - 2b \neq 0$ $(mod\ k)$, then the new labeling f' on G' is still $\mathbb{Z}_k$-magic with index r, and $f'(E(G')) = f(E(G)) \cup \{r - 2a, r - 2b\}$.

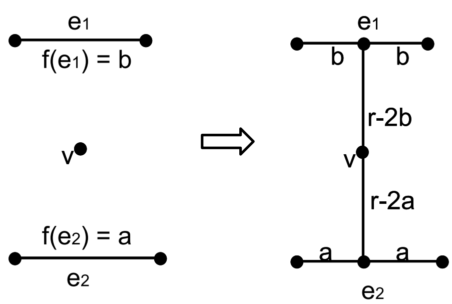

Figure 4.8: Subdivision Method

We calculate the index sets of fans F_n, and start with a lemma for $\mathbb{Z}_3$-magicness with zero sums. Note that the cases were already obtained in [23], however for completeness, we still provide with proofs here for fans, also for wheels in next section.

Note that the index set of 3-fan F_3 is special and different from other cases, as we show in the following.

Lemma 4.2.1 *For the 3-fan F_3, we have the following facts:*

1. *F_3 is $\mathbb{Z}_k$-magic if and only if k is even and $k \geq 2$. Therefore $I_k(F_3) = \emptyset$ for k odd.*

2. *$I_4(F_3) = \{0, 2\}$.*

3. For $k \geq 5$, $I_k(F_3) = \mathbb{Z}_k$, for k even.

Proof. In F_3, as in the Figure 4.9, we have that if r is an index, then there exists nonzero elements x and y in $\mathbb{Z}_k$ such that $2(x+y-r) = 0$ and $x+y-r \neq 0$, $r - x \neq 0$, $r - y \neq 0$ $(mod\ k)$, hence k is even since $x + y - r$ is an element of order 2. Therefore F_3 is $\mathbb{Z}_k$-magic if and only if k is even and $k \geq 2$.

Suppose $r = 1 \in I_4(F_3)$, we have the labels on the path are $1-x \neq 0, 1-y \neq 0$, and the labels on spokes are $x \neq 0, y \neq 0, 1 - x - y = x + y - 1 \neq 0$. Then we have $2(x + y) = 2$, hence $x + y = -1$, which is a contradiction to $x \neq 0, 1$ and $y \neq 0, 1$. Therefore $\pm 1 \notin I_4(F_3)$.

For even $k \geq 5$, in order to realize all indices in F_3 we take $a = 1, b = r - 1 + \frac{k}{2}$ for the index $r \neq 1$, and $a = 2, b = r - 2 + \frac{k}{2}$, for $r = 1$.

□

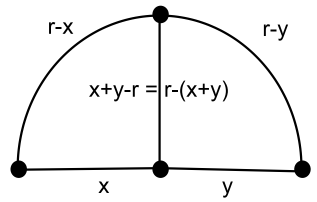

Figure 4.9: Magic Sum r of F_3

Lemma 4.2.2 *For $n \geq 3$, $0 \in I_3(F_n)$ if and only if $n \equiv 1$ $(mod\ 3)$.*

Proof. Assume $0 \in I_3(F_n)$. First observe that the edge labels on the path of the F_n have to be all the same, either all $1's$ or all $(-1)'s$. If not, then the there will be 0-edges over the spokes. Without loss of generality, suppose the edge labels on the path are all $1's$. Thus, the labels on the spokes are two $(-1)'s$, and $n - 2$ of them are $1's$. Then by calculating the vertex sum of the center, we have $(n - 2) - 2 \equiv n - 1 \equiv 0$ $(mod\ 3)$. Conversely follows from the given labeling.

□

Lemma 4.2.3 $0 \in I_k(F_n)$, *for $n \geq 4$ and $k \geq 4$.*

Proof. Please see the Figure 4.10, we have the labeling for F_4, F_5, F_6 such that the vertex sum is 0. Use the above subdivision method, we construct $\mathbb{Z}_k$-magic labeling from F_n to F_{n+2} by subdividing one pair of 1-edge and (-1)-edge in F_5 and F_6, respectively.

□

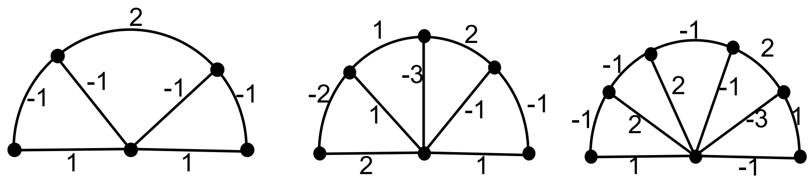

Figure 4.10: $\mathbb{Z}_k$-Magic Sum 0 of F_4, F_5, F_6, $k \geq 4$

For other cases of fans F_n, see the following. We split the discussion into cases $\mathbb{Z}_3$-magic, $\mathbb{Z}_4$-magic, and $\mathbb{Z}_k$-magic for $k \geq 5$.

Lemma 4.2.4 $\pm 1 \in I_3(F_n)$ *if and only if* $n \geq 6$.

Proof. Without loss of generality it suffices to consider the case of index 1. The labeling on the edges incident to vertices of degree 2 must be both -1. One may easily check that $\pm 1 \notin I_3(F_n)$ for $n < 6$, and we have $\mathbb{Z}_3$-magic labeling with index 1 of F_6, F_7, F_8 as in the Figure 4.11.

To obtain $\mathbb{Z}_3$-magic labeling with index 1 from F_n to F_{n+3} for $n \geq 6$, we insert three vertices of degree two on some 1-edge on the path of F_n, join them to the center, and label -1 on the newly added spokes, as in the Figure 4.11. Then the resulting labeling will do the job.

□

Note that F_n has $n+1$ vertex, thus by the necessary condition Proposition 3.1.4, $\pm 1 \notin I_4(F_n)$ for n even. Therefore we consider the following $\mathbb{Z}_4$-magicness of F_n.

Lemma 4.2.5 $\pm 1 \in I_4(F_n)$ *for* $n \geq 5$ *is odd.*

Proof. For $n \geq 5$ is odd, we have $1 \in I_4(F_5)$ as the labeling given in the Figure 4.12. Since there are edges labeled by 1 and 2 over the path of F_n respectively, the subdivision method mentioned above may be used by subdividing the 1-edge and 2-edge and connect the newly inserted degree 2 vertices to the center. Then we may construct $\mathbb{Z}_4$-magic labeling with index 1, hence -1, for $n \geq 5$ odd. □

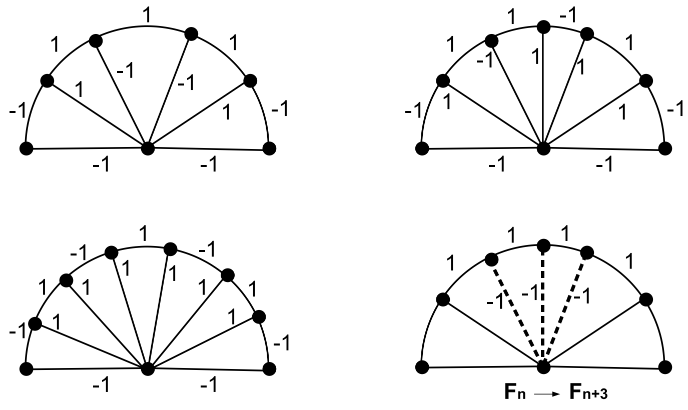

Figure 4.11: $\mathbb{Z}_3$-Magic Sum 1 of F_6, F_7, F_8, and construction from F_n to F_{n+3} for $n \geq 6$

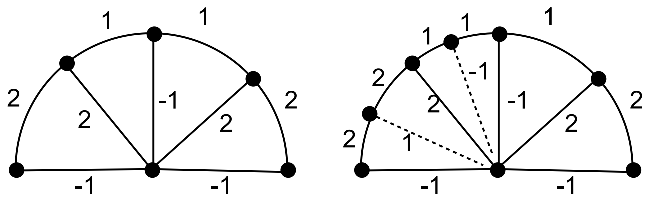

Figure 4.12: $\mathbb{Z}_4$-magic with index ± 1 for F_n for $n \geq 5$ odd

Lemma 4.2.6 $2 \in I_4(F_n)$ *for* $n \geq 3$.

Proof. As in the Figure 4.13, first of all we label $1, -1, 1, -1 \cdots$ alternatively on the path of F_n. For n odd, in order to obtain the index 2, the labeling on spokes has to be one of 1, $(n-2)$ of 2's, and one of -1. For n even, the labeling on spokes has to be two of 1's and $(n-2)$ of 2's. In both cases, the vertex sums are constat 2 and we are done.

□

In the following we deal with the $\mathbb{Z}_k$-magic case for $k \geq 5$. Again by the necessary condition of being $\mathbb{Z}_k$-magic with index r in Proposition 3.1.4, we have that $I_k(F_n) \subset 2\mathbb{Z}_k$, for n even. We will realize every nonzero index $r = 2a$, where $a \in \mathbb{Z}_k^*$ in the following lemma.

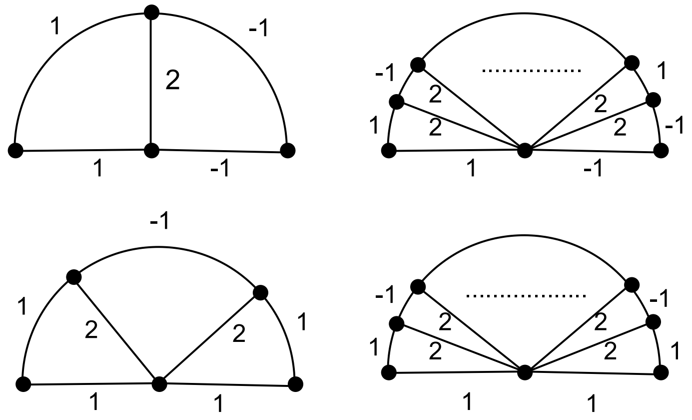

Figure 4.13: $\mathbb{Z}_4$-magic with index 2 for F_n

Lemma 4.2.7 *For all $a \in \mathbb{Z}_k^*$, we have $2a \in I_k(F_n)$, for all $n \geq 4$, $k \geq 5$, and n even.*

Proof. Let $n = 2t$ and $F_n = \{v\} + \{u_1 u_2 \cdots u_{2t}\}$, we first delete the edges $u_{2i}u_{2i+1}$, $i = 1, 2, \cdots, t-1$ alternatively, and obtain a windmill M, which is Eulerian since every vertex is of even degree, as in the Figure 4.14.

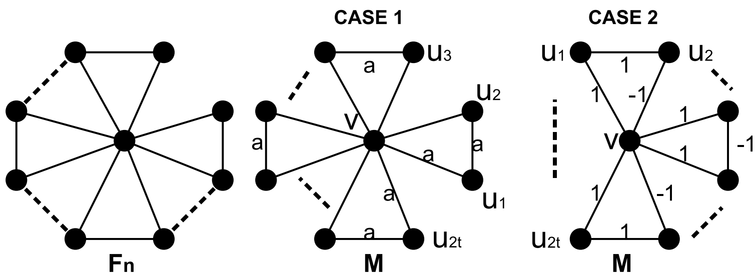

Figure 4.14: Index Sets of Fans F_n, n even

Case 1: M contains even number of triangles.

Hence M is of even size, and we may labels ± 1 on one Eulerian tour of M such that vertex sum is 0 on each vertex. Add extra a to the ± 1 labeling on the edges vu_1, vu_{2t}, and $u_{2j-1}u_{2j}$, $j = 1, 2, \cdots, t$, then we have labels $a \pm 1$ on M such that vertex sums on v, u_1, u_{2t} are $2a$, and are a on other vertices, please see the Figure 4.14. Now put the edges $u_{2i}u_{2i+1}$, $i = 1, 2, \cdots, t-1$, back and label a on each of them, then we have a labeling, call it L_1, on F_n with $a, a \pm 1$ such that each vertex sum is $2a$.

By similar method as above, except labeling ± 2 instead on the Eulerian tour of M, we then have another labeling L_2 on F_n with $a, a \pm 2$ such that each vertex sum is $2a$. Note that for $a \neq 0$ and $k > 3$, if $a \pm 1 = 0$ in L_1, then we use the nonzero $\mathbb{Z}_k$-labeling L_2. If $a \pm 2 = 0$ in L_2, we use the nonzero $\mathbb{Z}_k$-labeling L_1. Thus we are done in this case.

Case 2: M contains odd numbers (at least three) of triangles.

Take three triangles containing vertices u_1 and u_{2t} of M, and label them as in the Figure 4.14 such that the partial vertex sums on v, u_1, and u_{2t} are 2 respectively, and 0 on other vertices. Note that the other triangles form an Euler graph of even size, and we label ± 1 on them such that partial vertex sum is 0 for each vertex on one Eulerian tour, and remains 2 on the center. Put the edges $u_{2i}u_{2i+1}, i = 1, 2, \cdots, t-1$ back and label 2 on them, we then have labels $2, \pm 1$ on F_n such that each vertex sum is 2. By multiplying all labels on each edge by a, we may have labels $2a, \pm a$ such that each vertex sum is $2a$. Then we are done in this case.

$\square$

Lemma 4.2.8 *$I_k(F_n) = \mathbb{Z}_k$, for all $n \geq 5$ odd, and $k \geq 5$.*

Proof. At first, we construct four types of labeling L_1, L_2, L_3, and L_4 such that vertex sum is r for F_5, as in the Figure 4.15.

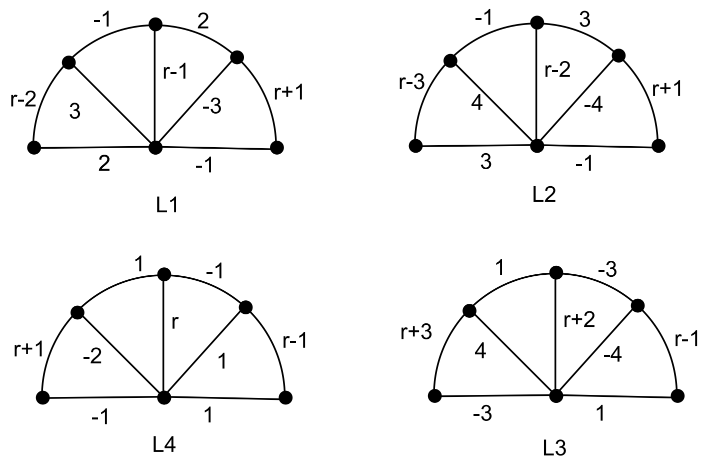

Figure 4.15: L_1, L_2, L_3, and L_4

We use the subdivision method in the following to extend L_1, L_2, L_3, and L_4 to the labeling of F_n, for $n \geq 5$ and n odd. Note that via subdividing edges labeled by -1 and $r+1$ on L_1, L_2 respectively, we have the following sets of possible labels $S_1 = \{r \pm 1, r-2, \pm(r+2)\}$ and $S_2 = \{r+1, r-3, \pm(r+2), r-2\}$ respectively. Subdividing edges labeled by 1 and $r-1$ on L_3 and L_4 respectively, we have the following sets of possible labels $S_3 = \{r-1, r+3, \pm(r-2\}, r+2\}$ and $S_4 = \{r, r \pm 1, \pm(r-2)\}$ respectively. Subdividing edges labeled by -1 and $r+1$ in L_4 again, we then have the sets of possible labels $S_5 = \{r, r \pm 1, \pm(r+2)\}$.

It remains to show that the above sets of possible labels contain no common zero elements in $\mathbb{Z}_k$. It is not hard to see that S_1 contains zero elements only for $r = \pm 1, \pm 2$, then instead we may use the following labeling: $S_2 = \{-1, 2, -2, \pm 3\}$ for $r = 1$, $S_3 = \{1, -2, 2, \pm 3\}$ for $r = -1$, $S_4 = \{-1, -3, -2, \pm 4\}$ for $r = -2$, and $S_5 = \{2, 1, 3, \pm 4\}$ for $r = 2$. Hence we are done. □

To summarize from the above lemmas, we have the following:

Theorem 4.2.2 *The index sets of fans F_n, $n \geq 3$, are as follows.*

when $k = 3$:

$$I_3(F_n) = \begin{cases} \emptyset, & n = 3. \\ \{0\}, & n = 4. \\ \emptyset, & n = 5. \\ \mathbb{Z}_3, & \text{for all } n \geq 6 \text{ and } n \equiv 1 \ (mod\ 3). \\ \mathbb{Z}_3 \backslash \{0\}, & \text{for all } n \geq 6 \text{ and } n \equiv 0, 2 \ (mod\ 3). \end{cases}$$

when $k = 4$:

$$I_4(F_n) = \begin{cases} \{0, 2\}, & n = 3. \\ 2\mathbb{Z}_4 = \{0, 2\}, & n \text{ even}. \\ \mathbb{Z}_4, & n \geq 5 \text{ odd}. \end{cases}$$

when $k \geq 5$:

$$I_k(F_n) = \begin{cases} \mathbb{Z}_k, & \text{for all } n \geq 5 \text{ odd , and } k \geq 5. \\ 2\mathbb{Z}_k, & \text{for all } n \geq 4 \text{ even , and } k \geq 5. \end{cases}$$

Remark. Note that $2\mathbb{Z}_k = \mathbb{Z}_k$ when k is odd, and $2\mathbb{Z}_k = \{0, 2, \cdots, \frac{k}{2}\}$ when k is even.

4.2.3 Index Sets of Wheels W_n

To obtain the index set of the wheel graphs W_n, $n \geq 3$, we look at the magic sum index 0. As mentioned earlier, the magic zero sum has been done in [23], and we still include the proofs here for completeness. We consider the $\mathbb{Z}_3$-magic case first of all.

Lemma 4.2.9 *For $n \geq 3$, $0 \in I_3(W_n)$ if and only if $n \equiv 0$ (mod 3).*

Proof. Assume $0 \in I_3(W_n)$. First observe that the edge labels on the cycle of the W_n have to be all the same, either all $1's$ or all $(-1)'s$. If not, then the there will be 0-edges over the spokes. By symmetry, suppose the edge labels on the cycle are all $1's$. Thus, the labels on the spokes are all $(-1)'s$. Then by calculating the vertex sum of the center, we have $(-1) \cdot n \equiv 0$ $(mod\ 3)$, hence $n \equiv 0$ $(mod\ 3)$. The converse follows from the given labeling.

□

Then we have the $\mathbb{Z}_k$-magic zero sum of the wheels in the following for $k \geq 4$.

Lemma 4.2.10 *Let $n \geq 3$. Then $0 \in I_k(W_n)$ for all $k \geq 4$.*

Proof. We deal with the problem by using induction on $n \geq 6$ from n to $n+2$, and for the cases W_n for $n \leq 5$, we give the labeling in the Figures. Note that W_3 is $\mathbb{Z}_k$-magic with zero sum for $k \geq 3$, please see the Figure 4.16. In case of W_4, by Lemma 4.2.9 and see the Figure 4.16, we see that $0 \in I_k(W_4)$ for all $k \geq 4$. For the case W_5, see the Figure 4.16. The case on the left of the Figure 4.16 is for $\mathbb{Z}_k$-magicness with zero sum for W_5, for all $k \geq 5$, and the case on the right of the Figure 4.16 is for $\mathbb{Z}_4$-magicness with zero sum for W_5.

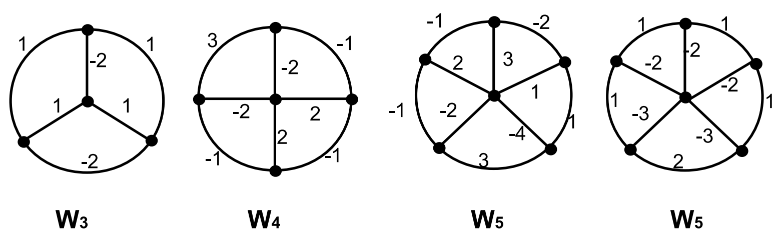

Figure 4.16: W_3, W_4, and W_5

The cases for W_6 and W_7, see the Figure 4.17. Combined with the Lemma 4.2.9, we see $0 \in I_k(W_6)$ for all $k \geq 3$ and $0 \in I_k(W_7)$ for all $k \geq 4$. Assume the result is true for $n \geq 6$. The induction step is the construction from W_n to W_{n+2}, for all $n \geq 6$, using the subdivision method mentioned above. Note that in the labeling given in the Figure 4.17, W_6 and W_7 have both 1 and -1 over the cycles of the wheels, respectively. Therefore we may complete the index sets of W_n, for all $n \geq 6$, by subdividing the 1-edge and (-1)-edge.

□

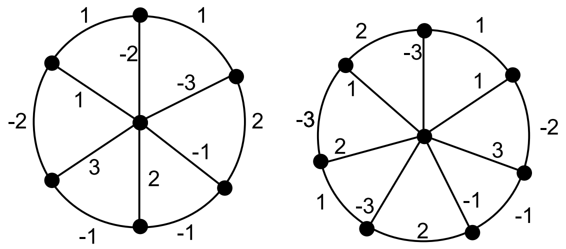

Figure 4.17: W_6 and W_7

Lemma 4.2.11 $\pm 1 \in I_3(W_n)$ *for* $n \geq 3$.

Proof. Suppose $n+1 \equiv r \bmod 3$, and $r = 0, 1$, or 2. Label -1 on r edges of the outer cycle of W_n non-consecutively, and 1 on other edges. Please see the Figure 4.18. In order to get index 1, labeling $2r$ of 1's and $n-2r$ of (-1)'s on the spokes, hence the vertex sum of the center is $2r-(n-2r) \equiv 4r-n \equiv 1$ $(mod\, 3)$.
□

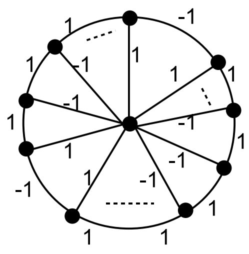

Figure 4.18: $\mathbb{Z}_3$-Magic Sum of 1 of W_n

Lemma 4.2.12 *For n odd,* $I_k(W_n) = \mathbb{Z}_k$.

Proof. Let $n = 2t+1$. In W_{2t+1}, each vertex is of odd degree, and it admits a perfect matching P such that $W_{2t+1} \backslash P$ is an Euler graph of even size. Label

± 1 on $W_{2t+1}\backslash P$ such that each partial vertex sum is 0, then label nonzero element r on the matching P, we get all possible labeling of index $r \neq 0$. The case for zero index is done. □

By the Proposition 3.1.4, we have $I_k(W_n) \subseteq 2\mathbb{Z}_k$ for n even. The following lemma shows the index set is in fact the whole $2\mathbb{Z}_k$.

Lemma 4.2.13 *For n even, $I_k(W_n) = 2\mathbb{Z}_k$.*

Proof. Let $n = 2t$. Since the zero index case is done, let $2a \neq 0$ in $2\mathbb{Z}_k$. Let $W_n = \{v\}+\{u_1u_2\cdots u_{2t-1}u_{2t}u_1\}$, we delete the edges $u_{2i}u_{2i+1}, i = 1, 2, \cdots, t-1$ and $u_{2t}u_1$ alternatively, and obtain a windmill M, see the Figure 4.19 please.

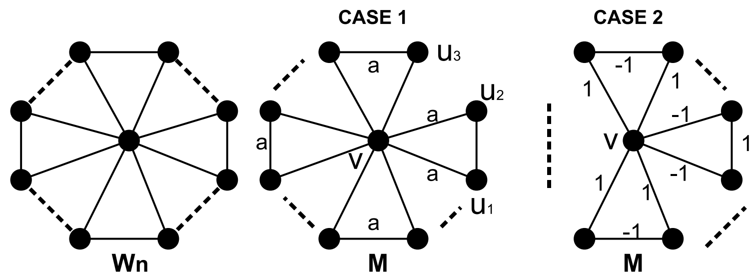

Figure 4.19: Index Sets of Wheels W_n, n even

Case 1: M contains even number of triangles.

Since M is of even size, we may labels ± 1 on one Eulerian tour of M such that the partial vertex sum is 0 on each vertex. Add value a to labels on edges $vu_1, vu_2, u_{2j-1}u_{2j}, j = 2, \cdots, t$, then we have labels $a \pm 1$ and ± 1 on M such that partial vertex sums are $2a$ on v, and a on other vertices. Now put back the edges $u_{2i}u_{2i+1}, i = 1, 2, \cdots, t-1$ and $u_{2t}u_1$, and then label a on them, we have labels $a, a \pm 1$ on W_n such that each vertex sum is $2a$.

To remedy the case when $a \pm 1 = 0$, similarly we put labels ± 2 on the Eulerian tour of M, we then have labels $a, a \pm 2$ and ± 2 on W_n such that each vertex sum is $2a$. Note that at least one of the two labeling on W_n contains no zero element since $a \neq 0$ and $k \geq 3$ is under consideration.

Case 2: M contains odd number of triangles.

We have an Eulerian tour of odd size starting and ending at v and label consecutively the edges $1, -1, \cdots, 1, -1, 1$, then we have labels ± 1 on M such that the partial vertex sum is 2 on v and is 0 on other vertices. Put the $u_{2i}u_{2i+1}, i = 1, 2, \cdots, t-1$ and $u_{2t}u_1$ back as in previous case and label 2 on them, we have then the labels $2, \pm 1$ on W_n such that each vertex sum is 2.

Now adjust the labeling through multiplying the labels on each edge by a, we may have the labeling with $2a, \pm a$ such that each vertex sum is $2a$. Then we are done in this case.

□

To summarize from the above lemmas, we have the following:

Theorem 4.2.3 *The index sets of wheels W_n, $n \geq 3$, are as follows.*

when $k = 3$:

$$I_3(W_n) = \begin{cases} \mathbb{Z}_3, & n \equiv 0 \ (mod\ 3). \\ \mathbb{Z}_3 \backslash \{0\}, & n \equiv 1, 2 \ (mod\ 3). \end{cases}$$

when $k \geq 4$:

$$I_k(W_n) = \begin{cases} \mathbb{Z}_k, & \text{for all } n \text{ even} . \\ 2\mathbb{Z}_k, & \text{for all } n \text{ odd} . \end{cases}$$

Remark. Again note that $2\mathbb{Z}_k = \mathbb{Z}_k$ when k is odd, and $2\mathbb{Z}_k = \{0, 2, \cdots, \frac{k}{2}\}$ when k is even.

4.3 Open Problems of Magic Spectrum

There are more open problems left as follows from our work in this thesis:

- Characterize the graphs which is $\mathbb{Z}_k$-magic, for $k \geq 3$.
- Identify classes of graphs other than $K_{m,n}$ with the index sets to be the cyclic subgroup $\mathbb{Z}_d$ of $\mathbb{Z}_k$.
- When is the index set $I_k(G)$ a subgroup of $\mathbb{Z}_k$? Or, when do one have $r + r' \in I_k(G)$ for $r, r' \in I_k(G)$?
- Characterize the graphs G having the index sets $I_k(G) = \mathbb{Z}_d$, where d is a divisor of k. Note that this open problem includes cases $d = 1$, $1 < d < k$, and $d = k$, respectively. The case $d = 1$ corresponds to characterizing the graphs G having the index sets $I_k(G) = \{0\}$, for all $k \geq 3$. The case $d = k$ corresponds to characterizing the graphs G having the index sets $I_k(G) = \mathbb{Z}_k$, for all $k \geq 3$.

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