東 海 大 學 數 學 系 研 究 所

碩 士 論 文

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Isoperimetric Inequality and Its Applications

等周不等式及其應用

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碩士學位口試委員審定書

本系碩士班 張雅淳 君

所提論文　等周不等式及其應用
(Isoperimetric Inequality and Its Applications)

合於碩士班資格水準，業經本委員會評審通過，特此證明。

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中　華　民　國　一○○　年　七　月　一　日

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Contents

摘 要

等周不等式主要是說在平面上任意的封閉區域，由固定周長所圍成的面積，其中以圓圍出來的面積爲最大。在此論文中我們介紹傳統微分幾何的等周不等式及多邊形的等周問題，接著從Brunn-Minkowski不等式與超平面的概念將等周不等式推廣到d維度。另一方面，介紹Steiner不等式及平行集合的概念來推導相關的等周不等式。

Abstract

The Isoperimetric inequality says that the area of any region in the plane bounded by a curve of a fixed length can never exceed the area of a circle whose boundary has that length. In this paper, we present isoperimetric inequality of classical differential geometry and polygonal isoperimetric problem. Next, we consider isoperimetric problem in $\mathbb{R}^d$ using Brunn-Minkowski inequality and the concept of hyperplanes. On the other hand, we consider isoperimetric inequality of Hadiwger using Steiner's Inequality and the concept of the out t-parallel sets.

Introduction

Isoperimetric Problem can trace back to Ancient Greece time. It was said that Princess Dido was forced to leave her house and move to north Africa which was near the coast of the Mediterranean Sea. Over there, she desired to get a piece of land, and she agreed to pay an amount of money to exchange the land which could be fenced by a male cowhide. The clever Dido cut the male cowhide into very thin line, and then she mounted every tip between line and line. After that, she used the long line to circle a piece of land. The diameter of the land was just equal to the length of the line. According to the tale, Princess Dido decided to use these lines to circle a half circle along the coast which was the right shape of the biggest area. Therefore, isoperimetric problem is called Dido's problem or isoperimetric inequality.

In chapter 1, we introduce what is a simple closed curve and recall isoperimetric inequality in $\mathbb{R}^2$.The isoperimetric inequality is a geometric inequality involving the square of the circumference of a closed curve in the plane and the area of a plane region it encloses, as well as its various generalizations. Isoperimetric literally means "having the same perimeter". Let γ be a simple closed curve of length L and area A, we have

$$L^2 - 4\pi A \geq 0. \tag{0.0.1}$$

Equality holds if and only if γ is a circle. There are many inequality which implies (0.0.1), we used Wirtinger's inequality and Green's Theorem to prove it. In section1.2, we discuss an elementary proof of the isoperimetric inequality for polygons.

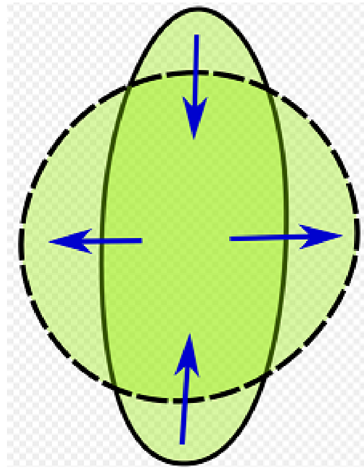

Figure 0.0.1. An elongated shape can be made more round while keeping its perimeter fixed and increasing its area

In chapter 2,we discuss two parts. First, if the set is optimal, it minimizes the perimeter given the area and the Minkowski Sum of any two sets. And we introduce the characteristic of Minkowski sum. Next, we prove the Brunn-Minkowski inequality and used it to prove isoperimetric inequality in $\mathbb{R}^d$. By the way, in the chapter2, we should known what is the hyperplane and definitions of convex set and convex function.

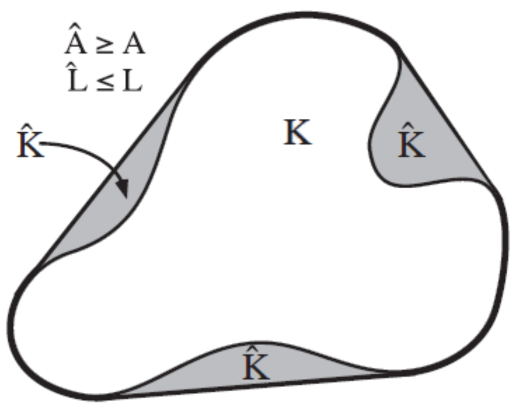

Figure 0.0.2. The region K and its convex hull $\widehat{K}$.

By the figure 0.0.2, we see the convex hull of K, denoted by $\widehat{K}$, is the smallest convex set such that $K \subset \widehat{K}$. And we have $\widehat{A} \geq A, \widehat{L} \leq L$. Thus the isoperimetric inequality for convex sets implies

$$\widehat{L}^2 - 4\pi\widehat{A} \leq L^2 - 4\pi A. \tag{0.0.2}$$

If the equality holds, $\widehat{K} = K$ is convex.

In chapter 3, we use Steiner's Inequality and convexity property of equality (0.0.2) to prove isoperimetric inequality of Hadwiger.

CHAPTER 1

The classical Isoperimetric Inequality

First, we introduce the classical case. We recall isoperimetric inequality in $\mathbb{R}^2$. Let γ be a simple closed curve, and we will used Wirtinger's inequality and Green's Theorem to prove isoperimetric inequality.

1.1. Isoperimetric Inequality for simple closed curve in the plane

Definition 1.1.1. The curve γ is simple (or call a Jordan arc) if for every x, y in the interval I, we have

$$\gamma(x) = \gamma(y) \Rightarrow x = y.$$

Definition 1.1.2. A curve γ is closed (or called a loop) if for the interval $I = [a, b]$ and

$$\gamma(a) = \gamma(b).$$

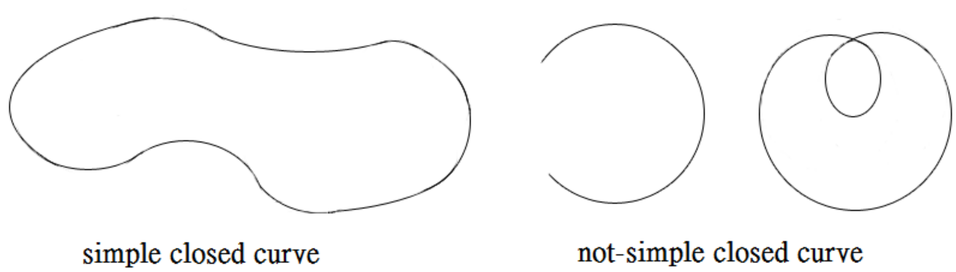

Figure 1.1.1

Remark 1.1.3. A simple closed curve is also called a Jordan curve.

Theorem 1.1.4. (Jordan curve Theorem)
Any simple closed curve γ in the plane $\mathbb{R}^2$ has an interior and an exterior, denoted by int(γ)

and ext(γ) with the following properties:

1.int(γ) is bounded;

2.ext(γ) is unbounded;

3.int(γ) and ext(γ) are connected.

Theorem 1.1.5. (Wirtinger's Inequality)

Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise $C^1(\mathbb{R})$ function with period 2π. Let $\overline{f}$ denoted the mean value of f

$$\overline{f} = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx.$$

Then

$$\int_0^{2\pi} (f(x) - \overline{f})^2 dx \leq \int_0^{2\pi} (f'(x))^2 dx,$$

with equality holds if and only if

$$f(x) = \overline{f} + a\cos x + b\sin x$$

for some $a, b \in \mathbb{R}$.

Proof. Since f is bounded and f is continuous, we expressed by Fourier series in $[0, 2\pi]$. Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

and

$$\begin{aligned} a_m &= \tfrac{1}{\pi} \textstyle\int_0^{2\pi} f(x) \cos mx dx \\ b_m &= \tfrac{1}{\pi} \textstyle\int_0^{2\pi} f(x) \sin mx dx \\ \overline{f} &= 2a_0 \end{aligned}$$

Since the sines and cosines are complete and orthogonal, by Parseval's identity,

$$\int_0^{2\pi} \left(f - \overline{f}\right)^2 = \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

The Fourier series for the derivative f'

$$f'(x) = -\sum_{n=1}^{\infty} n a_n \sin nx + \sum_{n=1}^{\infty} n b_n \cos nx.$$

Since f' is square integrable, by Bessel's inequality

$$\pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) \leq \int_0^{2\pi} (f')^2.$$

Then

$$\int_0^{2\pi} (f')^2 - \int_0^{2\pi} \left(f - \overline{f}\right)^2 \geq \pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) - \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \pi \sum_{n=2}^{\infty} (n^2 - 1)(a_n^2 + b_n^2) \geq 0.$$

If the equality holds, that for $n \geq 2$

$$(n^2 - 1)(a_n^2 + b_n^2) \geq 0,$$

then $a_n = b_n = 0$.

Therefore, $f = a_0 + a_1 \cos x + b_1 \sin x$. □

Theorem 1.1.6. (Green's Theorem)

Let E be a two -dimensional region whose topological boundary ∂E is a piecewise smooth C^1 curve oriented positive. If $P, Q : E \to \mathbb{R}$ are $C^1$and $\overrightarrow{F} = (P, Q)$, then

$$\int_{\partial E} P\left(x, y\right) dx + Q\left(x, y\right) dy = \iint_E \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Proof. Let

$$\begin{aligned} E &= \left\{ (x, y) \,\middle|\, \begin{array}{c} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{array} , \ g_i \, are \, continuous \, functions \right\} \\ &= \left\{ (x, y) \,\middle|\, \begin{array}{c} h_1(y) \leq x \leq h_2(y) \\ c \leq y \leq d \end{array} , \ h_i \, are \, continuous \, functions \right\} \end{aligned}$$

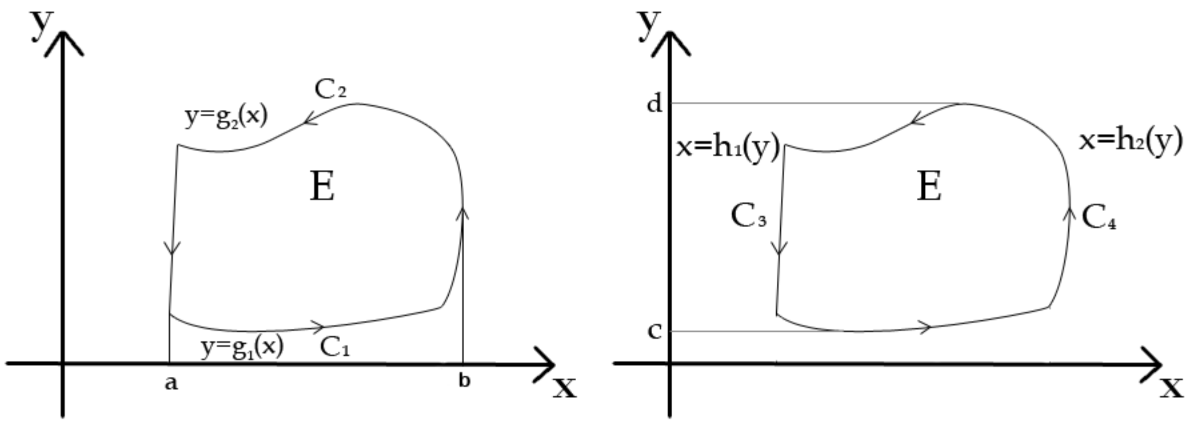

Figure 1.1.2. $\partial E = C_1 \cup C_2 = C_3 \cup C_4 = C$

We know

$$\begin{aligned}
\int_C P(x,y)dx &= \int_{C_1} P(x,y)dx + \int_{C_2} P(x,y)dx \\
&= \int_a^b P(x,g_1(x))dx - \int_a^b P(x,g_2(x))dx \\
&= -\int_a^b [P(x,g_2(x)) - P(x,g_1(x))]dx \\
&= -\int_a^b (\int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x,y)dy)dx \\
&= -\iint_E \frac{\partial P}{\partial y}(x,y)dA.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_C Q(x,y)dy &= \int_{C_3} Q(x,y)dy + \int_{C_4} Q(x,y)dy \\
&= \int_c^d Q(h_2(y),y)dy - \int_c^d Q(h_1(y),y)dy \\
&= \int_c^d [Q(h_2(y),y) - Q(h_1(y),y)]dy \\
&= \int_c^d (\int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial x}(x,y)dx)dy \\
&= \iint_E \frac{\partial Q}{\partial x}(x,y)dA.
\end{aligned}$$

So we get

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \iint_E (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA.$$

$\square$

Remark 1.1.7. Recall Green's theorem. If P and Q are differential functions on the plane and ∂E is a simple closed C^1curve bounding the region E then

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \iint_E (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy.$$

If we take $Q = x$ and $P = -y$ then Green's theorem says

$$\int_C xdy - ydx = 2Area\,(E)\,. \tag{1.1.1}$$

Theorem 1.1.8. Let γ be a simple closed curve, let L be its length and let A be the area of its interior. Then

$$A \leq \frac{L^2}{4\pi}$$

with the equality holding if and only if γ is a circle.

Proof. Let the curve $\gamma : I \to \mathbb{R}^2$ is defined as $\gamma(s) = (x(s), y(s))$ that satisfy

$$|x'(s)|^2 + |y'(s)|^2 = 1$$

with totally length L.

We change to 2π periodic function

$$f(\theta) = x(\frac{L\theta}{2\pi}), g(\theta) = y(\frac{L\theta}{2\pi}).$$

Then

$$(f')^2 + (g')^2 = (\frac{L}{2\pi})^2 = \frac{L^2}{4\pi^2.}$$

We know

$$\int_0^{2\pi} g'd\theta = g(2\pi) - g(0) = 0.$$

By (1.1.1) and Wirtinger's Inequality,

$$\begin{aligned}
2A &= \int_0^{2\pi} fg' - gf' d\theta \\
&= \int_0^{2\pi} fg' + gf' d\theta - 2\int_0^{2\pi} gf' d\theta \\
&= 2\int_0^{2\pi} fg' d\theta \\
&= 2\int_0^{2\pi} fg' d\theta - 2\overline{f}\int_0^{2\pi} g' d\theta \\
&= 2\int_0^{2\pi} (f-\overline{f})g' d\theta \\
&= \int_0^{2\pi} (f-\overline{f})^2 + (g')^2 - (f-\overline{f}-g')^2 d\theta \\
&\leq \int_0^{2\pi} (f')^2 + (g')^2 d\theta = \int_0^{2\pi} \tfrac{L^2}{4\pi^2} d\theta = \tfrac{L^2}{2\pi}.
\end{aligned}$$

Then

$$A \leq \frac{L^2}{4\pi}.$$

If the equality holds, then

$$f(\theta) = \overline{f} + a\cos\theta + b\sin\theta$$

for some $a, b \epsilon \mathbb{R}$, $\overline{f}$ is constant and

$$\int_0^{2\pi} (f - \overline{f} - g')^2 d\theta = 0.$$

So that

$$g' = f - \overline{f} = a\cos\theta + b\sin\theta.$$

Hence

$$g(\theta) = \overline{g} + a\sin\theta - b\cos\theta$$

for $\overline{g}$ is constant, $a, b\epsilon\mathbb{R}$.

Then

$$\begin{aligned}
(f')^2 + (g')^2 &= \left(a^2\sin^2\theta + b^2\cos^2\theta\right) + \left(a^2\cos^2\theta + b^2\sin^2\theta\right) \\
&= a^2 + b^2 \\
&= \tfrac{L^2}{4\pi^2}.
\end{aligned}$$

It means γ is a circle of radius $\frac{L}{2\pi}$. $\square$

1.2. Polygonal Isoperimetric Problem

The proof of the isoperimetric inequality for general polygons.

Theorem 1.2.1. *In every polygon with perimeter L and area A we have $L^2 \geq 4\pi A$.*

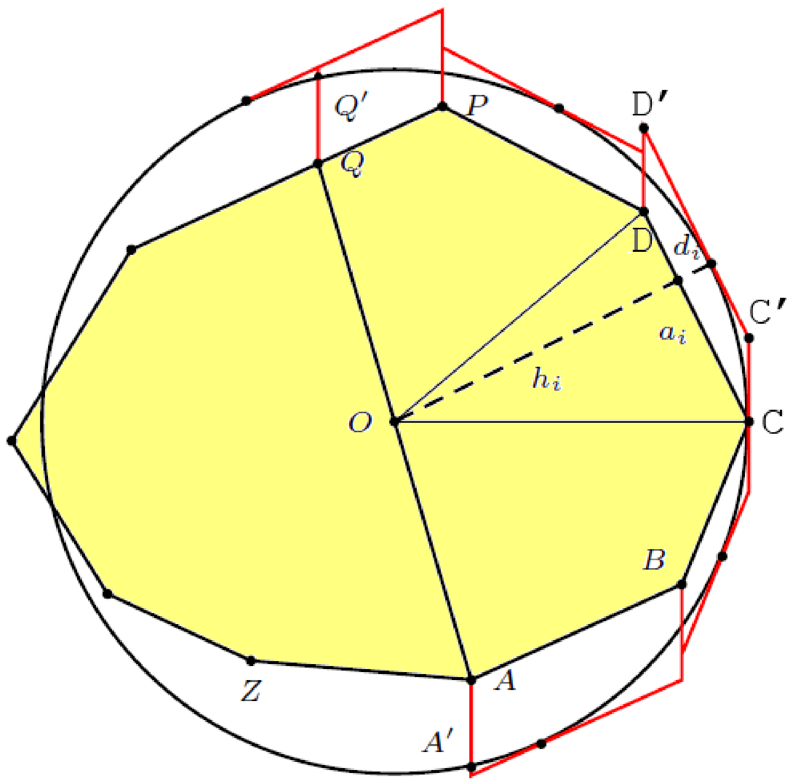

Figure 1.2.1. Draw a circle with center O and radius R, which $R = \overline{OC}$, C is a vertex of polygon farthest from O

Proof. Consider any convex polygon $ABC \cdots Z$. From the vertex A of the polygon, we draw the segment $\overline{AQ}$ which dividing the polygon in two polygons and satisfied follows:
(1) $\overline{AB} + \overline{BC} + \cdots + \overline{PQ} = \frac{L}{2}$;
(2) the area of $ABC \cdots PQA$ is A_1 and $A_1 \geq \frac{A}{2}$.
Let O be the mid-point of A and Q. Let C be the vertex of polygon farthest from O and let $\overline{OC} = R$. Draw a circle with center O and radius R. We find the points A' and Q' in the circle respectively, such that $\overleftrightarrow{AA'} \perp \overleftrightarrow{OC}$ and $\overleftrightarrow{QQ'} \perp \overleftrightarrow{OC}$. By the symmetry, the area

of the part of circle $AA'CQ'QA$ has equal to the area of hemicycle i.e.

$$S = \frac{1}{2}\pi R^2.$$

And outside the polygon $ABC \cdots PQ$, we can draw parallelograms touching the circle. Let $\overline{AB} = a_1$, h_1 is the altitude of $\triangle OAB$ and d_1 is the height of the parallelogram $AA'B'B$. So let $\overline{CD} = a_i$, h_i is the altitude of $\triangle OCD$ and d_i is the height of the parallelogram $CC'D'D$, and then $h_i + d_i = R$. Therefore,

$$\begin{aligned} A_1 &= \triangle OAB + \cdots + \triangle OPQ \\ &= \tfrac{1}{2}\sum_i a_i h_i. \end{aligned}$$

If we denote by A_2 is the sum of the areas of parallelograms. We have,

$$\begin{aligned} A_2 &= parallelogram AA'B'B + \cdots + parallelogram PP'Q'Q \\ &= \sum_i a_i d_i \\ &= \sum_i a_i (R - h_i) \\ &= \sum_i a_i R - \sum_i a_i h_i \\ &= \tfrac{L}{2} \cdot R - 2A_1. \end{aligned}$$

Since

$$A_1 + A_2 \geq S,$$

we have

$$\frac{L}{2}R - A_1 \geq \frac{1}{2}\pi R^2.$$

So

$$\begin{aligned} & \pi R^2 - LR + 2A_1 \leq 0 \\ \Rightarrow\ & \pi\left[R^2 - \tfrac{L}{\pi}R + \left(\tfrac{L}{2\pi}\right)^2\right] - \tfrac{L^2}{4\pi} + 2A_1 \leq 0 \\ \Rightarrow\ & \pi\left(R - \tfrac{L}{2\pi}\right)^2 - \left(\tfrac{L^2}{4\pi} - 2A_1\right) \leq 0 \\ \Rightarrow\ & \tfrac{L^2}{4\pi} - 2A_1 \geq \pi\left(R - \tfrac{L^2}{2\pi}\right)^2. \end{aligned}$$

Since

$$\pi\left(R-\frac{L^2}{2\pi}\right)^2 \geq 0$$

and

$$A_1 \geq \frac{A}{2},$$

then

$$\frac{L^2}{4\pi} \geq 2A_1.$$

We conclude that

$$L^2 \geq 2A_1 \cdot 4\pi \geq 4\pi A,$$

as desired. $\square$

CHAPTER 2

Isoperimetric Problem in $\mathbb{R}^d$

It is well known that among all regions in the plane with the same area, the circle has smallest perimeter. In this chapter we always assume that our sets are convex.

2.1. Isoperimetric problem in the plane

Definition 2.1.1. A region $\Omega \subset \mathbb{R}^2$ is convex if for every $x, y \subset \Omega$ the line segment $\overline{xy} \subset \Omega$.

Definition 2.1.2. The set S is optimal if it minimizes the perimeter with a given area.

Theorem 2.1.3. Among all convex sets in the plane with a given area, the circle has smallest perimeter.

Before we proof Isoperimetric Inequality in the plane, we have to known what is Steiner symmetrization.

Steiner symmetrization. Take a convex set $X \subset \mathbb{R}^2$ and any line ℓ. Divide X into interval $[a, b]$, $a, b \in \partial X$, orthogonal to ℓ. Move each interval $[a, b]$ along the line (a, b), so that it is symmetric with respect to ℓ. Denoted by $Y = (X, \ell)$ the resulting set. The resulting map $f : X \to Y$ is called the Steiner symmetrization.

Remark 2.1.4. Suppose the sets is not convex. We can find two points x, y from the figure such that connecting segment $\overline{xy}$ lie inside our shape. Then reflecting a region between the segment $\overline{xy}$, it would be to increasing its area with the same perimeter.

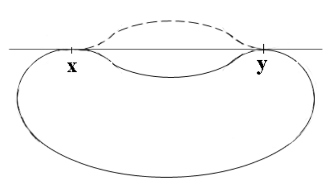

Figure 2.1.1. It would be possible to increase its area without modifying its perimeter.

Proof. We can split the proof into two steps. Let $X \subset \mathbb{R}^2$be a compact set in the plane, and X is optimal.

Step1. Let two opposite points x and y on the boundary $P = \partial X$, and they divide P in half. Such the interval $[xy]$ is called diameter. We claim $[xy]$ divides the area into two equal parts. If not, use the bigger part and its reflection to a set Y of bigger area and equal perimeter. Then X is not optimal.

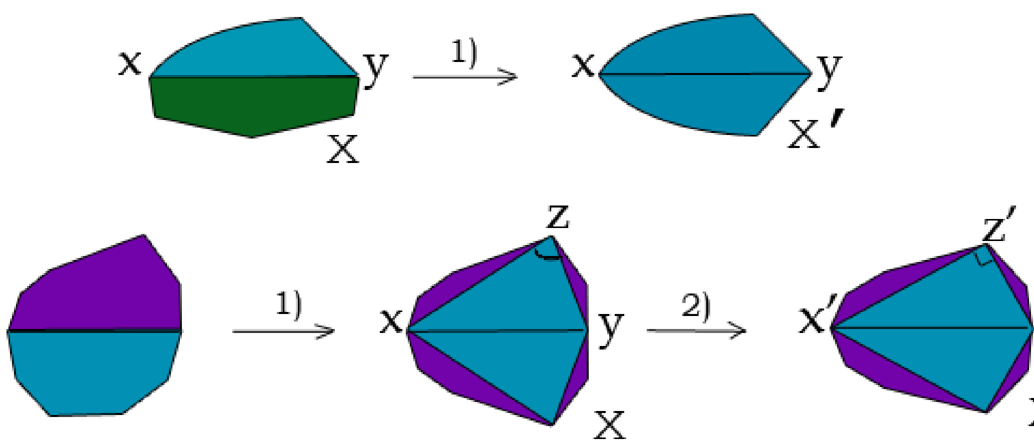

Figure 2.1.2. Increasing the area with the same perimeter.

Step2. Show that only a circle can be optimal. Let $[xy]$ be a diameter of an optimal set X and let $P = \partial X$. We claim every point $z \in P$, such that $\angle xzy = \frac{\pi}{2}$. By step1, we know $[xy]$ splits the area into equal parts. If $\angle xzy \neq \frac{\pi}{2}$, take segments $[xy]$ and $[yz]$ and attach them to $\triangle x'y'z'$ with

$$|x'z'| = |xz| \, , |z'y'| = |zy| \, ,$$

and

$$\angle x'z'y' = \frac{\pi}{2}.$$

We know the area of triangle

$$area(\triangle xyz) = \frac{1}{2} \cdot |xz| \cdot |zy| \cdot \sin(\angle xzy).$$

When $\angle xzy = \frac{\pi}{2}$, the area is maximum.

So

$$area(\triangle xyz) < area(\triangle x'y'z').$$

Copy the construction symmetrically on the other side of the diameter. Then X' has the same perimeter and bigger area. Thus, an optimal the angle at all points of the boundary to a fixed diameter must be $\frac{\pi}{2}$. And we know the center of circumcircle of right-triangle lies in the mid-point of adjacent side. Therefore, X is a circle. □

Remark 2.1.5. The above proof due to Steiner symmetrization is beautiful, so there are several ways to get around the problem.

(1) We want to used a general compactness argument to show that the optimal must exist. Consider all convex sets with area π, and ask which of them has the smallest perimeter. We assume the convex sets contain the origin $O \in \mathbb{R}$. Let Λ be a circle with radius 4. If a set X contains O and point $x \notin \Lambda$. Since X is a convex set, the perimeter of X is at least

$$2\,|Ox| \geq 2 \cdot 4 = 8 > 2\pi.$$

So the perimeter of X is bigger than perimeter of a unit circle. If the optimal set X exists, then $X \subset \Lambda$. i.e. X is a convex subset of Λ. By compactness argument, a set X of convex subset of a compact set is a compact, and a least one minimum must exist. Therefore, the unit circle is the desired minimum.

(2) Another way to prove the result convergent to a circle. Consider a convex set X, let $P = \partial X$. By step1, we find a diameter $[xy]$ divide the area into two equal parts. And by step2, for any points $z \in P$ to a fixed diameter must be $\frac{\pi}{2}$. Apply step1 and repeatedly apply step2 to points $z \in P$ chosen to split the perimeter into 2^n pieces of equal length. Every piece under the transformation, we can stay the same perimeter but the area increase. Every piece region increase the area, and totally region converges to a circle. Finally, the region converges to a circle. That implies the area of convex sets with given perimeter was smaller than circles.

2.2. Minkowski Content

Definition 2.2.1. The Minkowski Sum of any two sets $A, B \subset \mathbb{R}^d$, defined as follows:

$$A + B := \{a + b \,|a \in A, b \in B\} .$$

Definition 2.2.2. For every $\lambda \in \mathbb{R}$, let $\lambda A = \{\lambda \cdot a \,|a \in A\}$. If there exists $\lambda \neq 0$, such that $B = \lambda A$. Then we call B is an expansion of A. This implies $A = \frac{1}{\lambda}B, \lambda \neq 0$, A is also called an expansion of B.

(1) If A and B are two intervals starting at the origin O, then $(A + B)$ is a parallelogram spanned by the intervals.
(2) If $A \subset \mathbb{R}^d$ is a convex set and B is a closed ball with radius $\varepsilon > 0$ centered at the origin, then every point $z \in (A + B)$ such that $dist\,(z, A) \leq \varepsilon$.
(3) The Minkowski sum of the rectangles is a rectangle.

$$e.g. \left([0, a] \times [0, b]\right) + \left([0, c] \times [0, d]\right) = [0, a + c] \times [0, b + d] .$$

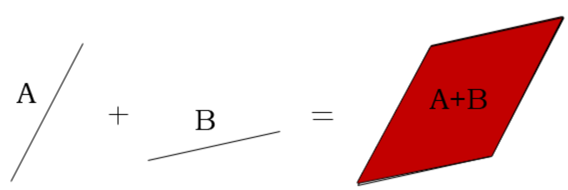

Figure 2.2.1. A and B are two intervals, then (A+B) is a parallelogram spanned by the intervals.

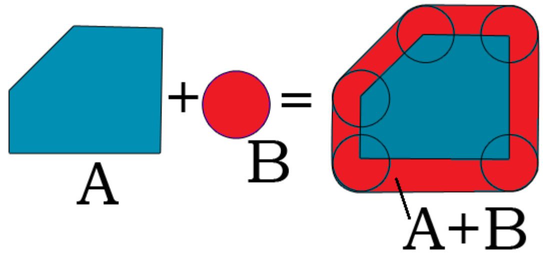

Figure 2.2.2. Every points $z \in (A + B)$ such that $dist(z, A) \leq \varepsilon$

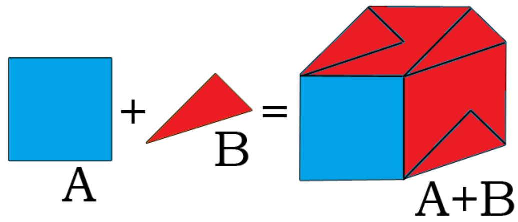

Figure 2.2.3. Minkowski addition tends to ''round out'' the figures being added

Characteristic of Minkowski sum.

(1) The sum is symmetric.
(2) Minkowski addition tends to ''round out'' the figures being added, the area of the figure exceeds the area of the summands.
(3) The sum respects translation.
(4) The sum does not respect rotation.

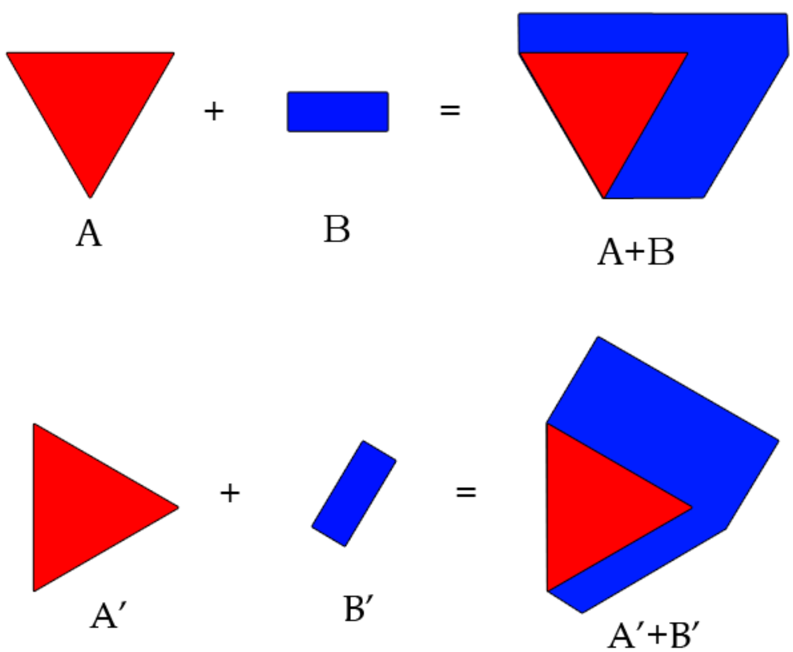

Figure 2.2.4. If A' and B' are (possible different) rotation of A and B, then (A'+B') does not have to be a rotation of (A+B).

2.3. Brunn-Minkowski Inequality

Before we discuss the subject of Brunn-Minkowski Inequality, we introduce the concept of hyperplane and define the area of convex sets in $\mathbb{R}^d$.

What is a hyperplane? An n-dimensional generalization of a plane; an affine subspace of dimension n-1 that splits an n-dimensional space. In one-dimensional space, it is a point; In two-dimensional space, it is a line; In three-dimensional space, it is an ordinary plane.

Definition 2.3.1. If H_1 and H_2 are two parallel hyperplanes. Then

$$H = \lambda_1 H_1 + \lambda_2 H_2, \lambda_1, \lambda_2 \in \mathbb{R}$$

is yet another parallel hyperplane.

Definition 2.3.2. The set $B := \frac{1}{2}(A_1 + A_2)$ is the average of sets A_1 and A_2 which $A_1 \subset H_1$ and $A_2 \subset H_2$. Then the set $B = \left\{\frac{1}{2}(a_1 + a_2) \mid a_1 \in A_1, a_2 \in A_2\right\}$ is consisting of midpoints of the intervals between the sets, and $B \subset H_3 = \frac{1}{2}(H_1 + H_2)$ is in the middle between hyperplanes H_1 and H_2.

Definition 2.3.3. (Convex sets) Let C be a convex set, if for all $x, y \in C$ and $\forall t \in [0, 1]$. Then the point $(1-t)x + y$ is in C.

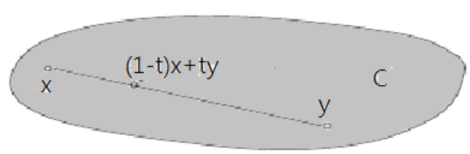

Figure 2.3.1. Convex set

Definition 2.3.4. Let $X \subset \mathbb{R}^d$ be a convex set with the surface $S = \partial X$, and let B be a unit ball. Then

$$area(S) = \frac{d}{dt} vol(X + tB)\mid_{t=0}.$$

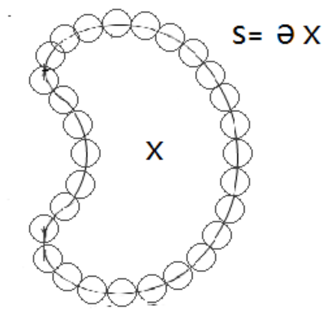

Figure 2.3.2. $\lim\limits_{t\to 0}\frac{vol(X+tB)-vol(X)}{t} = area(S) = \frac{d}{dt}vol(X+tB)\mid_{t=0}$

Theorem 2.3.5. (Brunn-Minkowski inequality)

Let $A, B \subset \mathbb{R}^d$ be two convex bodies. Then

$$vol(A+B)^{\frac{1}{d}} \geq vol(A)^{\frac{1}{d}} + vol(B)^{\frac{1}{d}}.$$

The equality holds, if and only if A is an expansion of B.

The main result in the following inequality used to prove Brunn-Minkowski inequality. As the reader shall see this is really a disguised form of the arithmetic mean vs. geometric mean inequality.

Theorem 2.3.6. (The Minkowski inequality)

For every $x_1, \cdots\cdots, x_n, y_1, \cdots\cdots, y_n > 0$.Then

$$\left[\prod_{i=1}^{n}(x_i+y_i)\right]^{\frac{1}{n}} \geq \left[\prod_{i=1}^{n}x_i\right]^{\frac{1}{n}} + \left[\prod_{i=1}^{n}y_i\right]^{\frac{1}{n}}.$$

Then equality holds, if and only if $x_i = cy_i$ for all $i = 1, \cdots\cdots, n$, and some $c > 0$.

Proof. We use the Arithmetic-Geometric Mean inequality:

$$\frac{a_1 + \cdots\cdots + a_n}{n} \geq (a_1 \cdots\cdots a_n)^{\frac{1}{n}}$$

for all $a_1, \cdots\cdots, a_n > 0$ and if the equality holds, if and only if $a_1 = \cdots\cdots = a_n$.

Then

$$\begin{aligned}\frac{\left(\prod\limits_{i=1}^{n} x_i\right)^{\frac{1}{n}}+\left(\prod\limits_{i=1}^{n} y_i\right)^{\frac{1}{n}}}{\prod\limits_{i=1}^{n}(x_i+y_i)^{\frac{1}{n}}} &= \prod_{i=1}^{n}\left(\frac{x_i}{x_i+y_x}\right)^{\frac{1}{n}}+\prod_{i=1}^{n}\left(\frac{y_i}{x_i+y_i}\right)^{\frac{1}{n}} \\ &\leq \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_i+y_x}+\frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{x_i+y_x} \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{x_i+y_i}{x_i+y_x}=1,\end{aligned}$$

and if the equality holds, if and only if $\frac{x_i}{y_i}=c, c>0$ for all $i=1,\cdots\cdots,n$. $\square$

Brick-by-Brick proof of Brunn-Minkowski inequality. The proof of Brunn-Minkowski inequality will proceed by induction, we prove the inequality for brick regions. Defined as disjoint unions of bricks with edges parallel to the axes. Denote by B_d is the set of all brick regions.

Proof. (Proof of Brunn-Minkowski inequality.)

Let A and B be two brick regions. We use induction on the total number k of bricks in A and B. Suppose $k=2$, assume

$$\begin{aligned} A &= \left\{(x_1,x_2,\cdots\cdots,x_d)\in\mathbb{R}^d \mid x_i\in[0,a_i], i=1,\cdots\cdots,d\right\} \\ B &= \left\{(x_1,x_2,\cdots\cdots,x_d)\in\mathbb{R}^d \mid x_i\in[0,b_i], i=1,\cdots\cdots,d\right\}\end{aligned}$$

for $a_i, b_i > 0 \in \mathbb{R}$.

Then

$$\begin{aligned} A+B &= \{a+b \mid a\in A, b\in B\} \\ &= \left\{(x_1,x_2,\cdots\cdots,x_d)\in\mathbb{R}^d \mid x_i\in[0,a_i+b_i], i=1,\cdots\cdots,d\right\}.\end{aligned}$$

By Minkowski inequality, then

$$vol\,(A+B)^{\frac{1}{d}} = \left(\prod_{i=1}^{d} a_i+b_i\right)^{\frac{1}{d}} \geq \left(\prod_{i=1}^{d} a_i\right)^{\frac{1}{d}}+\left(\prod_{i=1}^{d} b_i\right)^{\frac{1}{d}} = vol\,(A)^{\frac{1}{d}}+vol\,(B)^{\frac{1}{d}}.$$

Suppose the result holds for brick regions with k bricks, $k\geq 2$. Now take two brick regions $A, B\in B_d$ with $(k+1)$ bricks. Suppose A contains at least two bricks. Let P, Q are two of them, $P, Q\subset A$. Since they are disjoint, there exists a hyperplane $H\subset\mathbb{R}^d$ with equation $x_i=c$, c is a constant, which separates bricks P and Q. Denoted by A_1

and A_2 are the regions on the two sides of H.

Define

$$A_1 = A \cap \left\{x \in \mathbb{R}^d \mid x_i \geq c\right\}$$
$$A_2 = A \cap \left\{x \in \mathbb{R}^d \mid x_i \leq c\right\}.$$

Let $\theta = \frac{vol(A_1)}{vol(A)}$. Since the Minkowski sum is independent of translation, we can assume that hyperplane H' with equation $x_i = c'$, such that

$$B_1 = B \cap \left\{x \in \mathbb{R}^d \mid x_i \geq c'\right\}$$
$$B_2 = B \cap \left\{x \in \mathbb{R}^d \mid x_i \leq c'\right\}.$$

The hyperplane H' divided B into two sets B_1 and B_2, with the same ratio $\theta = \frac{vol(B_1)}{vol(B)}$. Moreover,

$$A_1 + B_1 \subset \left\{x \in \mathbb{R}^d \mid x_i \geq c + c'\right\}$$

and

$$A_2 + B_2 \subset \left\{x \in \mathbb{R}^d \mid x_i \leq c + c'\right\}$$

$(A_1 + B_1)$ and $(A_2 + B_2)$ lie on different sides of $\overline{H}$ with the equation $x_i = c + c'$ and they are not intersect. We have

$$\begin{aligned}
vol\,(A+B) &\geq vol\,(A_1+B_1) + vol\,(A_2+B_2) \\
&\geq \left[vol\,(A_1)^{\frac{1}{d}} + vol\,(B_1)^{\frac{1}{d}}\right]^d + \left[vol\,(A_2)^{\frac{1}{d}} + vol\,(B_2)^{\frac{1}{d}}\right]^d \\
&\geq \left[(\theta vol\,(A))^{\frac{1}{d}} + (\theta vol\,(B))^{\frac{1}{d}}\right]^d + \left[((1-\theta)\,vol\,(A))^{\frac{1}{d}} + ((1-\theta)\,vol\,(B))^{\frac{1}{d}}\right]^d \\
&\geq [\theta + (1-\theta)] \cdot \left[vol\,(A)^{\frac{1}{d}} + vol\,(B)^{\frac{1}{d}}\right]^d \\
&\geq \left[vol\,(A)^{\frac{1}{d}} + vol\,(B)^{\frac{1}{d}}\right]^d.
\end{aligned}$$

Thus

$$vol\,(A+B)^{\frac{1}{d}} \geq vol\,(A)^{\frac{1}{d}} + vol\,(B)^{\frac{1}{d}},$$

and we are done.

$\square$

2.4. Isoperimetric Inequality for $\mathbb{R}^d$

We need slicing polytopes with hyperplane to solve the isoperimetric problem in $\mathbb{R}^d$.

Definition 2.4.1. Let $A, B \subset \mathbb{R}^d$ be two convex bodies and let $X_t = (1-t)\,A + tB$ where $t \in [0,1]$.

Theorem 2.4.2. Let X be a convex body and let H_1, H_2 and H_3 be three parallel hyperplanes in $\mathbb{R}^d$ intersecting X. Suppose $A_i = X \cap H_i, i = 1,2,3$. Then

$$vol(A_2) \geq min\left\{vol\left(A_1\right); vol\left(A_3\right)\right\}.$$

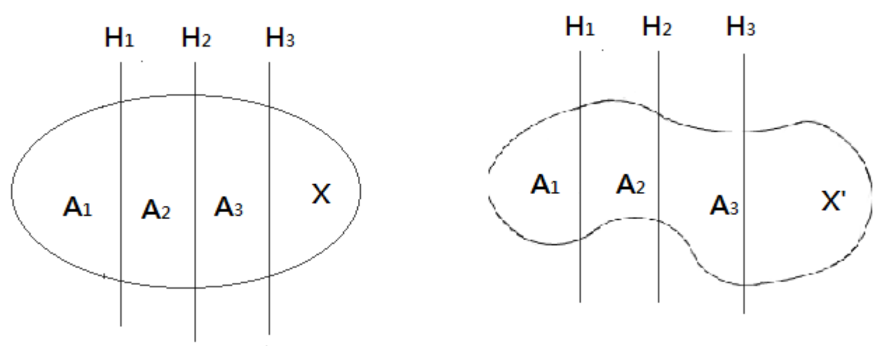

Figure 2.4.1. X is a convex body but X' is not

Proof. Let $D = \lambda A_1 + (1-\lambda)\,A_3$ where $\lambda = \frac{dist(H_1,H_2)}{dist(H_1,H_3)}$ is the ratio of the distance between hyperplanes. Then

$$\begin{aligned} D &= \tfrac{dist(H_1,H_2)}{dist(H_1,H_3)}A_1 + \left(1 - \tfrac{dist(H_1,H_2)}{dist(H_1,H_3)}\right)A_3 \\ &= \tfrac{dist(H_1,H_2)}{dist(H_1,H_3)}A_1 + \tfrac{dist(H_2,H_3)}{dist(H_1,H_3)}A_3. \end{aligned}$$

So D lies in H_2. By the Minkowski sum, we have

$$D \subset conv\left\{A_1, A_3\right\} \subset X.$$

Thus $D \subset A_2$ (By the Brunn Theorem, $A_2 = X \cap H_2$). And by the definition

$$\begin{aligned} vol\left(A_2\right) \geq vol\left(D\right) &= vol(\lambda A_1 + (1-\lambda)\, A_3) \\ &= \lambda vol\left(A_1\right) + (1-\lambda)\, vol\left(A_3\right) \\ &= \lambda \left[vol\left(A_1\right) - vol\left(A_3\right)\right] + vol\left(A_3\right). \end{aligned}$$

Since $\lambda \in [0,1]$. Then

$$vol\left(D\right) \geq vol\left(A_1\right)$$

or

$$vol\left(D\right) \geq vol\left(A_3\right).$$

Therefore,

$$vol\left(A_2\right) \geq vol\left(D\right) \geq min\left\{vol\left(A_1\right); vol\left(A_3\right)\right\},$$

and the proof is complete. □

Proposition 2.4.3. Suppose $A, B \subset \mathbb{R}^d$ be two convex bodies and let $X_t = (1-t)A + tB$, where $t \in [0,1]$. Then the function $\varphi\left(t\right) = vol\left(X_t\right)^{\frac{1}{d}}$ is concave downward on $[0,1]$. Moreover,

$$\varphi'\left(0\right) \geq vol\left(B\right)^{\frac{1}{d}} - vol\left(A\right)^{\frac{1}{d}},$$

and the equality holds, if and only if A is an expansion of B.

Proof. First, we claim the function $\varphi\left(t\right)$ is concave downward for $t \in [0,1]$.

For every three values $0 \leq a < t < b \leq 1$, consider $A = X_a$, $B = X_b$ and $C = X_t$.

We have $C = (1-t)\, A + tB$, where $0 \leq t \leq 1$.

We Know

$$\begin{aligned} \varphi\left(0\right) &= vol\left(X_0\right)^{\frac{1}{d}} = vol\left(A\right)^{\frac{1}{d}} \\ \varphi\left(1\right) &= vol\left(X_1\right)^{\frac{1}{d}} = vol\left(B\right)^{\frac{1}{d}} \end{aligned}$$

And

$$\begin{aligned} vol\,(C)^{\frac{1}{d}} &= vol\,((1-t)\,A + tB)^{\frac{1}{d}} \\ &\geq (1-t)\,vol\,(A)^{\frac{1}{d}} + t \cdot vol\,(B)^{\frac{1}{d}} = (1-t)\,\varphi\,(0) + t\varphi\,(1)\,, \end{aligned}$$

therefore $vol\,(A)^{\frac{1}{d}} \leq vol\,(C)^{\frac{1}{d}}$ and $vol\,(B)^{\frac{1}{d}} \leq vol(C)^{\frac{1}{d}}$ for $t \in [0,1]$.

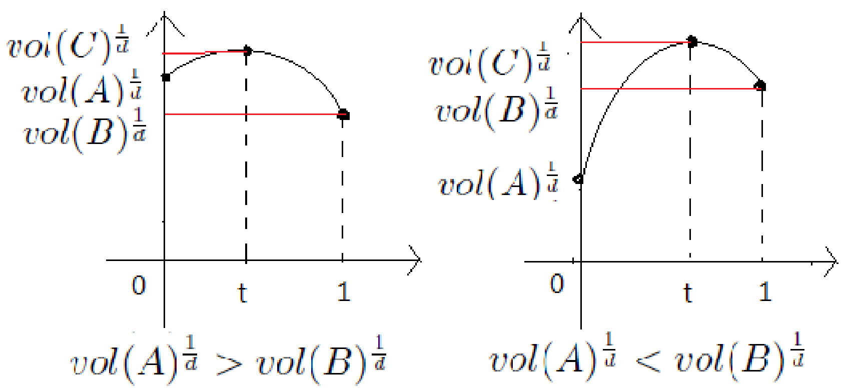

Figure 2.4.2. $\varphi\,(t)$ is concave downward for $t \in [0,1]$

Without loss of generality, $\varphi\,(t)$ is concave downward for $t \in [0,1]$.

Observe that

$$\varphi\left(\frac{1}{2}\right) = vol\left(X_{\frac{1}{2}}\right)^{\frac{1}{d}} = vol\left(\frac{1}{2}A + \frac{1}{2}B\right)^{\frac{1}{d}} = \frac{1}{2}vol\,(A+B)^{\frac{1}{d}}\,.$$

Consider the right-hand derivative of $\varphi'\,(0)$ is well-defined,

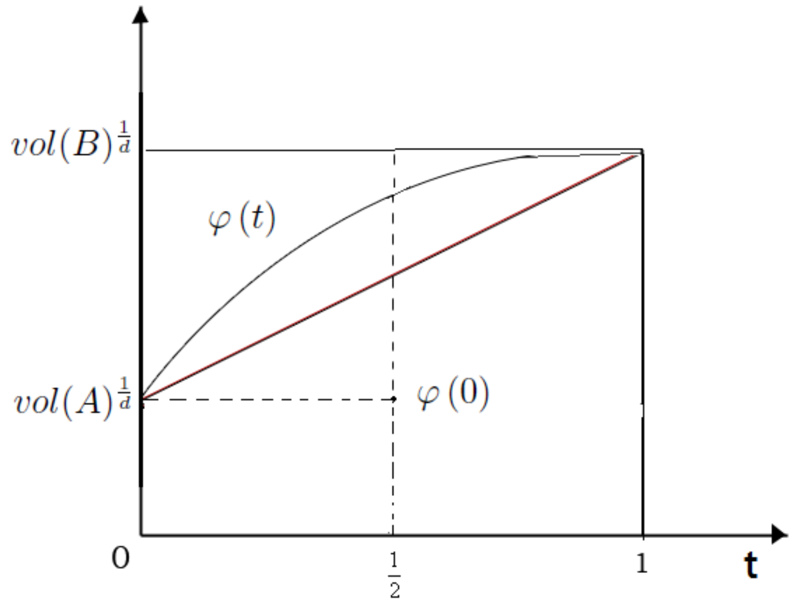

Figure 2.4.3. Convexity of $\varphi(t) = vol(X_t)^{\frac{1}{d}}$

then

$$\varphi\left(\tfrac{1}{2}\right) \leq \varphi(0) + \varphi'(0) \cdot \left(\tfrac{1}{2} - 0\right)$$
$$\varphi'(0) \geq \frac{\varphi\left(\frac{1}{2}\right) - \varphi(0)}{\frac{1}{2}}$$

By the Brunn-Minkowski inequality, if the equality holds, we get

$$vol(A+B)^{\frac{1}{d}} = vol(A)^{\frac{1}{d}} + vol(B)^{\frac{1}{d}},$$

which A is an expansion of B.

Thus

$$\varphi'(0) \geq \frac{\frac{1}{2}\left[vol(A)^{\frac{1}{d}} + vol(B)^{\frac{1}{d}}\right] - vol(A)^{\frac{1}{d}}}{\frac{1}{2}} = vol(B)^{\frac{1}{d}} - vol(A)^{\frac{1}{d}}.$$

Therefore, the equality holds, if and only if A is an expansion of B. $\square$

Theorem 2.4.4. (Isoperimetric inequality in $\mathbb{R}^d$)

Among all convex sets in $\mathbb{R}^d$ with a given volume, the ball has the smallest surface area.

Proof. Let A be a convex set, and let B be a unit ball in $\mathbb{R}^d$. By the proposition2.4.4,

$\varphi(t) = vol((1-t)A + tB)^{\frac{1}{d}}$ is convex on $[0,1]$. Then

$$\begin{aligned} area(A) &= \tfrac{d}{dt} vol(A+tB)\,|_{t=0} \\ &= \tfrac{d}{dt}\left[(1+t)^d vol\left(\tfrac{1}{1+t}A + \tfrac{t}{1+t}B\right)\right]|_{t=0} \\ &= \tfrac{d}{dt}\left[(1+t)\varphi\left(\tfrac{t}{1+t}\right)\right]^d |_{t=0} \\ &= d\left((1+t)\varphi\left(\tfrac{t}{1+t}\right)\right)^{d-1}\cdot\left[1\cdot\varphi\left(\tfrac{t}{1+t}\right) + (1+t)\cdot\varphi'\left(\tfrac{t}{1+t}\right)\cdot\tfrac{1\cdot(1+t)-t\cdot 1}{(1+t)^2}\right]|_{t=0} \\ &= d\varphi(0)^{d-1}\left[\varphi(0)+\varphi'(0)\right]. \end{aligned} \tag{2.4.1}$$

We know $\varphi'(0) \geq vol(B)^{\frac{1}{d}} - vol(A)^{\frac{1}{d}}$, and the equality holds, if and only if A is an expansion of B. In other words, for every $\lambda \in \mathbb{R}$, $A = \lambda B, \lambda \neq 0$.

Then

$$\varphi(0) = vol(A)^{\frac{1}{d}} = vol(\lambda B)^{\frac{1}{d}} = \lambda vol(B)^{\frac{1}{d}}.$$

Therefore, the equality (2.4.1) can be expressed as

$$\begin{aligned} area(A) &\geq d\varphi(0)^{d-1}\left[vol(A)^{\frac{1}{d}} + vol(B)^{\frac{1}{d}} - vol(A)^{\frac{1}{d}}\right] &= d\varphi(0)^{d-1}\left[vol(B)^{\frac{1}{d}}\right] \\ & &= d\left[\lambda\cdot vol(B)^{\frac{1}{d}}\right]^{d-1}\cdot\left[vol(B)^{\frac{1}{d}}\right]. \end{aligned}$$

If A has the smallest area, when the equality holds. Thus, A is an expansion of B implies A is a ball. $\square$

CHAPTER 3

Applications

3.1. Steiner's Inequality

Steiner's Inequality for Parallel Sets. Let $D \subset \mathbb{R}^2$ be a closed and bounded region. For $t \geq 0$ and let B be the closed unit ball, then the out t-parallel set is

$$D_t = D \times tB = \left\{x \in \mathbb{R}^2 \mid dist\,(x, D) \leq t\right\}$$

the set of points whose distance to D is at most t.

Theorem 3.1.1. (Steiner's Inequality)

Let $D \subset \mathbb{R}^2$ be a closed and bounded set whose area is A and whose boundary has length L. For $t \geq 0$, the out t-parallel set satisfies the inequalities

$$\begin{aligned} Area\,(D_t) &\leq A + Lt + \pi t^2, \\ L\,(\partial D_t) &\leq L + 2\pi t. \end{aligned}$$

Remark 3.1.2. If D is convex, then the inequalities are equalities. The inequalities can be written as

$$\begin{aligned} Area\,(D_t) &= A + Lt + \pi t^2, \\ L\,(\partial D_t) &= L + 2\pi t, \end{aligned}$$

which be called Steiner's formulas.

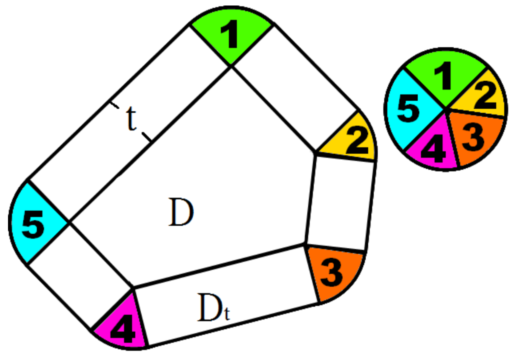

Figure 3.1.1. The out t-parallel set of a convex polygon.

Proof. By the figure3.1.1, we see for each segment of D of length l, derive from an $l \times t$ rectangle of D_t consists of sectors of circles with radius t, and which add up exactly one complete circle. Therefore, the total length of ∂D_t is

$$L(\partial D_t) = Length\,(rectangle) + Length\,(circle) = L + 2\pi t$$

and the total area is

$$Area\,(D_t) = Area\,(D) + Area\,(rectangle) + Area\,(circle) = A + Lt + \pi t^2.$$

$\square$

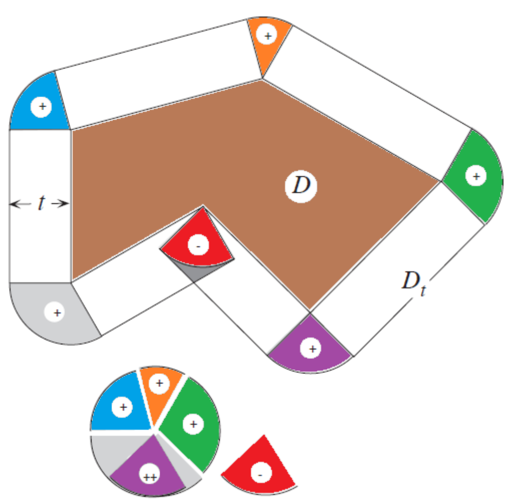

Figure 3.1.2. The out t-parallel set of a nonconvex polygon and Steiner's Inequality.

Proof. (Proof of the Steiner's Inequality)

By the figure3.1.2, we see nonconvex curves is that the out t-parallel set has extra overlap. Therefore, $Area\,(D_t) \leq A + Lt + \pi t^2$ and $L\,(\partial D_t) \leq L + 2\pi t$ still holds. We omitted from the details. (see [2]) $\square$

The proof of Hadwiger depends on Steiner's Inequality. For the purpose, we need to define the circumradius and inradius of a compact set $\Omega \subset \mathbb{R}^2$.

Definition 3.1.3. Let Ω be the region bounded by K. The radius of the smallest circular disk containing Ω is called the circumradius, denoted R_I. The radius of the largest circular disk contained in Ω is called the inradius, denoted r_I.

$$r_I = \sup\{r \geq 0 \mid there\ is\ p \in \mathbb{R}^2\ such\ that\ B_r\,(p) \subset \Omega\}$$
$$R_I = \inf\{r \geq 0 \mid there\ exists\ p \in \mathbb{R}^2\ such\ that\ \Omega \subset B_r\,(p)\}$$

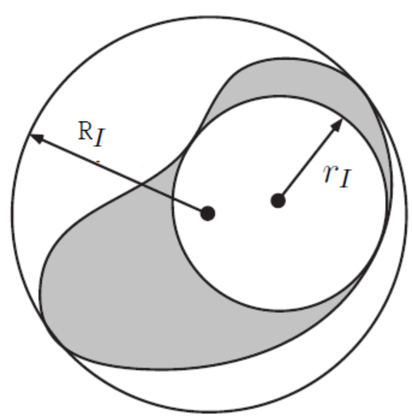

Figure 3.1.3. The disk realizing the circumradius,R_I and inradius, r_I of K.

Theorem 3.1.4. (Isoperimetric Inequality of Hadwiger)

Suppose Ω is a compact set with piecewise C^1 boundary of area A and boundary length L. There is a line M through the incenter X_I of the convex hull K of Ω and $a = L\,(K \cap \Omega)$ the length of chord passing through the center. Then

$$L^2 - 4\pi A \geq \frac{\pi^2}{4}\,(a - 2r_I)^2\,.$$

If the equality holds, all chords through X_I of K agree with the diameter of the incircle implies K is a circle.

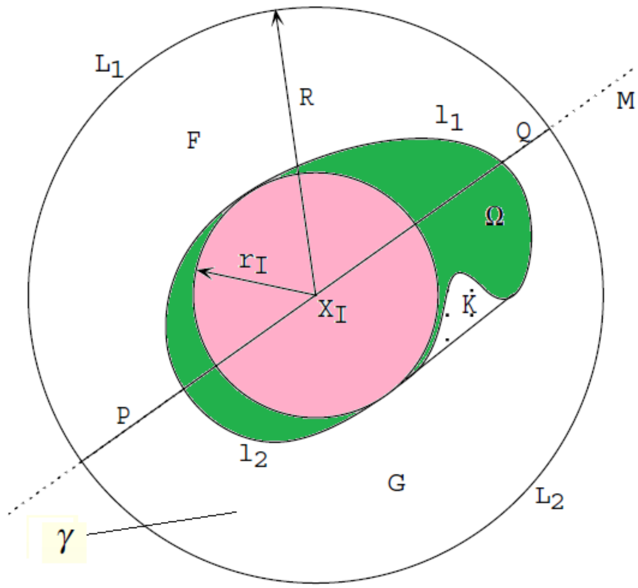

Figure 3.1.4. Diagram for the Hadwiger's proof.

Proof. Let $Area\,(K) = \widehat{A}$ and $Length\,(K) = \widehat{L}$. By convexity property of equality (0.0.2), we know $Area\,(\Omega) \leq Area\,(K)$ implies $\widehat{L}^2 - 4\pi\widehat{A} \leq L^2 - 4\pi A$. Choose $R > 0$ such that $K \subset B_R\,(X_I)$. Let γ be the annular region between the ball $B_R\,(X_I)$ and K, $\gamma = B_R\,(X_I) - K^o$. And the line $\gamma - M$ splits the annulus into two equal parts F and G. Let P and Q be the disjoint line segments $P \cup Q = M \cap \gamma$. Let l_1 and l_2 be the two lengths of ∂K in F and G respectively, and L_1 and L_2 be the two lengths of $\partial F \cap B_R\,(X_I)$ and $\partial G \cap B_R\,(X_I)$ respectively. Now, we know

$$\begin{aligned} \widehat{L} = &\quad l_1 + l_2 \\ \pi R^2 = &\quad Area\,(F) + Area\,(G) + \widehat{A}. \end{aligned}$$

Choose a number r, such that $2r_I \leq 2r \leq a$. Consider the out r-parallel set of the ring domain γ_r. Since $r \geq r_I$ the all region is covered. So

$$Area\,(\gamma_r) = \pi\,(R + r)^2\,. \tag{3.1.1}$$

The length of P and Q denoted by p and q respectively.

By the Steiner's Formula,

$$Area\,(P_r) = 2pr + \pi r^2 \tag{3.1.2}$$

$$Area\,(Q_r) = 2qr + \pi r^2. \tag{3.1.3}$$

Finally, we apply Steiner's Inequality to F_r and G_r

$$Area\,(F_r) \leq Area\,(F) + (p + L_1 + q + \pi R)\,r + \pi r^2, \tag{3.1.4}$$

$$Area\,(G_r) \leq Area\,(G) + (p + L_2 + q + \pi R)\,r + \pi r^2. \tag{3.1.5}$$

For any point $x \in \gamma_r \subset (F_r \cup G_r)$. Also any points $x \in (P_r \cup Q_r) \subset (F_r \cap G_r)$.

If any points $x \in \gamma_r \cup (P_r \cup Q_r) \subset (F_r \cup G_r) \cup (F_r \cap G_r)$. Hence

$$Area\,(\gamma_r) + Area\,(P_r) + Area\,(Q_r) \leq Area\,(F_r) + Area\,(G_r)$$

by equalities (3.1.1), (3.1.2), (3.1.3), (3.1.4), (3.1.5) which implies

$$\pi\,(R + r)^2 + 2\,(p + q)\,r + 2\pi r^2 \leq Area\,(F) + Area\,(G) + (2p + L_1 + 2q + L_2 + 2\pi R)\,r + 2\pi r^2.$$

We can reduced the inequality to

$$\widehat{A} + \pi r^2 \leq (L_1 + L_2)\,r \leq \widehat{L} r$$

which is equivalent to

$$\widehat{L}^2 - 4\pi \widehat{A} \geq \left(\widehat{L} - 2\pi r\right)^2 = \frac{1}{2}\left(\widehat{L} - 2\pi r\right)^2 + \frac{1}{2}\left(\widehat{L} - 2\pi r\right)^2.$$

Choose some r, such that $r_I \leq r \leq \frac{a}{2}$, which implies

$$\widehat{L}^2 - 4\pi\widehat{A} \geq \frac{1}{2}\left(\widehat{L} - 2\pi r_I\right)^2 + \frac{1}{2}\left(\widehat{L} - \pi a\right)^2. \tag{3.1.6}$$

Since $2A^2 + 2B^2 \geq (A-B)^2$ multiples by $\frac{1}{4}$ is

$$\frac{1}{2}A^2 + \frac{1}{2}B^2 \geq \frac{1}{4}(A-B)^2. \tag{3.1.7}$$

Therefore, by the inequality (3.1.7), the inequality (3.1.6) can be written as

$$\widehat{L}^2 - 4\pi\widehat{A} \geq \frac{1}{4}\left[\left(\widehat{L} - 2\pi r_I\right) + \left(\widehat{L} - \pi a\right)\right]^2 = \frac{\pi^2}{4}(a - 2r_I)^2.$$

Then

$$L^2 - 4\pi A \geq \widehat{L}^2 - 4\pi\widehat{A} \geq \frac{\pi^2}{4}(a - 2r_I)^2$$

as desired. □

Conclusions

The isoperimetric problem is to determine a plane figure of the largest possible area whose boundary had a specified length. In other words, given a fixed perimeter L of a closed curve and the area A of the region, satisfies

$$4\pi A \leq L^2$$

and the equality holds if and only if the curve is a circle.

In the three-dimension Euclidean space, among all simple closed surfaces with given surface area, the sphere enclosed a region of maximal volume. An analogous statement holds in Euclidean space of any dimension. We consider Brunn-Minkowski Inequality to prove isoperimetric problem in $\mathbb{R}^d$. Finally, using Steiner's Inequality to prove isoperimetric inequality of Hadwiger.

In addition, there are analogs of isoperimetric inequality for domain on surfaces, for examples: Bonnesen inequalities, minimal surfaces and inequalities depending on Gauss curvature. If you are interested in the topics, you can see the paper [9].

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