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碩 士 論 文

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Gauss-Bonnet Theorem and Its Applications

Gauss-Bonnet 定理及其應用

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本系碩士班　曾柏耕　　君

所提論文　高斯-柏涅定理及其應用
(Gauss-Bonnet Theorem and Its Applications)

合於碩士班資格水準，業經本委員會評審通過，特此證明。

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中　華　民　國　一〇〇　年　七　月　一　日

- 摘　要 -

Gauss-Bonnet 定理是一個美麗的定理， 它把曲面上的曲率和曲面的尤拉特徵數做一個連結。 換句話說， Gauss-Bonnet 定理是幾何和拓樸之間的橋樑。 在本論文中，我們提出 Gauss-Bonnet 定理的發展及証明， 並討論它的一些應用。 例如，龐加萊-霍普夫指標定理，毛球定理，和代數基本定理。 除此之外，我們還討論 $\mathbb{R}^3$ 空間中多面體的離散型 Gauss-Bonnet 定理。

Abstract

Gauss Bonnet theorem is beautiful because it relates the curvature of a surface with its Euler characteristic. It links differential geometry with topology. In this paper, we present some developments on the proof and some applications of Gauss-Bonnet theorem. For example, the Poincar**é**-Hopf index theorem, the hairy ball theorem, and the fundamental theorem of algebra. Moreover, we discuss the discrete Gauss-Bonnet theorem about a convex polyhedron in $\mathbb{R}^3$.

- 致 謝 辭 -

時間過得很快，在這幾年的求學生涯中，體會很多，也學到了很多。 感謝我的指導教授陳文豪老師這些日子的教導與照顧，而在擔任老師的小助教這段時間，不管是微積分、高等微積分、或幾何學，都讓我有了更深的體悟。 另外也要感謝系上其他老師的教誨，謝謝黃皇男老師 TeX 的教導以及上課分享的人生閱歷，讓我學到很多， 曹景懿老師、陳淑珍老師、胡馨云老師、和楊智烜老師的鼓勵讓人充滿鬥志。 感謝口試委員曹景懿老師、陳昇國老師、林盈甄老師，除了細心地審查論文內容之外，並提出了不少寶貴的意見，使本論文更加完善。 還要我的實習學校的老師們，林孟儒老師、李浚標主任、林詩舫老師、徐大欽老師、江幸紋老師，和宇秀姐等等的幫忙及鼓勵。

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最後要感謝我的家人們在我從小到大的求學路上，總是無怨尤的一直支持著，讓我能夠全心全力的朝我的目標前進，謝謝你們，沒有你們，就沒有現在的我。

Contents

Chapter 1

Introduction

In this paper, we investigate the Gauss-Bonnet theorem in differential geometry, the discrete Gauss-Bonnet theorem in discrete differential geometry and its applications.

The Gauss-Bonnet theorem is probably the most beautiful and deepest theorem in the differential geometry of surfaces. The simplest version of the Gauss-Bonnet theorem states that the sum of the interior angles of a triangle in the Euclidean plane equals π. The local Gauss-Bonnet theorem states as follows:

Theorem 1.0.1. *[18] Suppose R is a simply connected and regular region with simple, closed, piecewise regular, and positively oriented boundary in an oriented regular surface M. If $\gamma = \partial R$ with length $\ell(\gamma)$ has exterior angles ϵ_i at the vertices $\gamma(s_i)$, $i = 1, 2, \ldots, n$, then*

$$\int_{\partial R} k_g ds + \iint_R K dA + \sum_{i=1}^{n} \epsilon_i = 2\pi. \tag{1.0.1}$$

Gauss published his formula in 1827 and deal with geodesic triangles on surfaces, and then Bonnet generalized it in 1848 to any simple connected region enclosed by arbitrary curves. A few years later, the equation (1.0.1) became:

$$\iint_M K dA = 2\pi\chi(M), \tag{1.0.2}$$

where M is an oriented compact surface and $\chi(M)$ is the Euler characteristic of M.

The equation (1.0.2) provides a remarkable relation between the topology of a compact surface and the integral of its curvature. It is striking that the total curvature does not change as we deform the surface, for example, as shown in Figure 1.0.1.

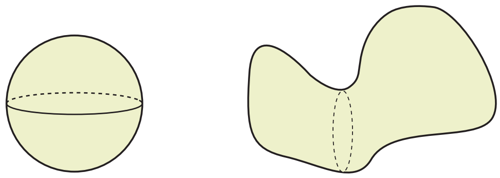


Figure 1.0.1: The total Gaussian curvature is unchanged.

H. Hopf found an application of Gauss-Bonnet theorem in 1885.

Theorem 1.0.2. *[12] The sum of the indices of a differentiable vector field* $\boldsymbol{V}$ *with isolated singular points on a compact surface* M *is equal to the Euler characteristic of* M*.*

We apply the theorem to find a interesting statement in meteorology. The statement is the hairy ball theorem.

Theorem 1.0.3. *[9] There does not exist a non-vanishing continuous tangent vector filed* $\boldsymbol{V}$ *on a sphere* S^2*.*

Moreover, we introduce the fundamental theorem of algebra.

Theorem 1.0.4. *[5] Every nonconstant polynomial* $P(z) \in \mathbb{C}[z]$ *has at least one complex root.*

In chapter 2, we review some basic definitions and theorems in differential geometry. The content of this chapter contains the first fundamental form, tangent spaces, the Gauss map, the second fundamental form, normal curvatures, geodesic curvatures, principal curvatures and the Gaussian curvature.

In chapter 3, we present a topical subject that is Gauss-Bonnet theorem. In section 3.1 and 3.2, we need some topological definitions and prove carefully the local and global Gauss-Bonnet theorem. Besides, in section 3.3, we first discuss a regular complex with a triangulation on a sphere S^2. Then we define the discrete Gaussian curvature of it, and find the discrete Gauss-Bonnet theorem related to the total curvature.

In chapter 4, we investigate the applications of Gauss-Bonnet theorem: the Poincaré-Hopf index theorem, the hairy ball theorem, and the fundamental theorem of algebra. In section 4.1, we list some singular points and their indices with a tangent vector field, and then we prove the Poincaré-Hopf index theorem. In section 4.2, we construct a tangent vector field on a sphere S^2 to recheck the hairy ball theorem is true. In addition, we find two tangent vector fields on a torus T. One is a nowhere zero tangent vector fields on a torus T, and the other is not. In section 4.3, we use Gauss-Bonnet theorem to prove the fundamental theorem of algebra.

In chapter 5, we find the generalizations of Gauss-Bonnet theorem and its application. Besides, we have more topics about discrete differentiable geometry in other papers.

Chapter 2

Preliminaries

In this chapter, we introduce the Gaussian curvature in differential geometry.

Definition 2.0.1. [2] A subset S of $\mathbb{R}^3$ is a surface if for every point $p \in S$, there exists an open set U in $\mathbb{R}^2$ and an open set W in $\mathbb{R}^3$ containing p such that $S \cap W$ is homeomorphic to U. That is,

$$\exists\, \sigma : U \to W \cap S \quad \text{s.t.} \quad \sigma \text{ is a homeomorphism.}$$

We call σ surface patches or parametrizations. A surface S should mean a smooth surface, and a surface patch σ should mean a regular surface patch, i.e., $\sigma_u \times \sigma_v \neq \mathbf{0}$.

We look at the first fundamental form. It allows us to make measurements on the surface, for instance, lengths of curves, angles of tangent vectors, areas of regions, without referring back to the ambient space $\mathbb{R}^3$ where the surface lies.

Definition 2.0.2. [2] Let $\gamma(t) = \sigma(u(t), v(t))$ be a curve in a surface patch σ. Then the first fundamental form of σ is

$$ds^2 = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2,$$

where

$$g_{11} :=< \sigma_u, \sigma_u >,$$

$$g_{12} :=< \sigma_u, \sigma_v >,$$

$$g_{22} :=< \sigma_v, \sigma_v >,$$

and $< \, , \, >$ is an inner product. It is obvious that $g_{12} = g_{21} :=< \sigma_v, \sigma_u >$.

Definition 2.0.3. [2] The tangent space T_pS at a point p of a surface S is the set of tangent vectors at p of all curves in S passing through p.

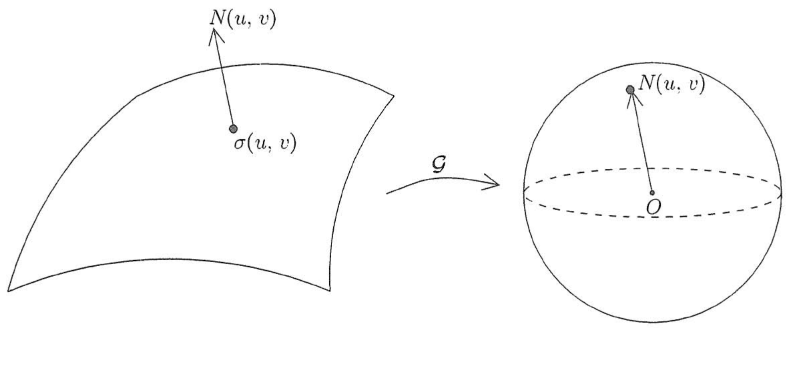
Proposition 2.0.4. *[2] Let $\sigma : U \to \mathbb{R}^3$ be a patch of a surface S containing a point $p \in S$, and let (u, v) be coordinates in U. The tangent space to S at p is the vector subspace of $\mathbb{R}^3$ spanned by the vector σ_u and σ_v.*

Proposition 2.0.4 shows that a surface patch $\sigma : U \to \mathbb{R}^3$ containing $p = \sigma(u_0, v_0)$ leads to a choice,

$$\mathbf{N}(u_0, v_0) = \frac{\sigma_u(u_0, v_0) \times \sigma_v(u_0, v_0)}{||\sigma_u(u_0, v_0) \times \sigma_v(u_0, v_0)||}.$$

We can think of $\mathbf{N}$ to be a map $\mathbf{N} : \sigma(U) \to \mathbb{R}^3$. Thus, each point $q \in \sigma(U)$ has a normal vector associated to it. We say that $\mathbf{N}$ is a differential field of unit normal vectors on U. A regular surface is oriented if it has a differentiable field of unit normal vectors defined on the whole surface.

The values of $\mathbf{N}$ at the points of S are recorded by its Gauss map $\mathcal{G}$, as pictured below.



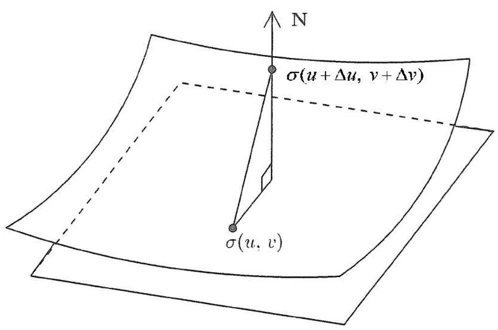
Definition 2.0.5. [2] Let $S \subset \mathbb{R}^3$ be a surface with an orientation $\mathbf{N}$ and $S^2 \subset \mathbb{R}^3$ be the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

The map $\mathcal{G} : S \to S^2$ is called the Gauss map.

The rate of change of the tangent line to a curve γ is the curvature of γ. We may try to measure how rapidly a surface S curves from the tangent plane T_pS in a neighborhood of a point $p \in S$. This is equivalent to mearsuring the rate of change at p of a unit normal vector $\mathbf{N}$ on a neighborhood of p.

From Taylor's theorem, we induce the second fundamental form of a surface patch.



Definition 2.0.6. [2] Let $\gamma(t) = \sigma(u(t), v(t))$ be a unit-speed curve in a surface patch σ. Then the second fundamental form of σ is

$$L_{11}du^2 + 2L_{12}dudv + L_{22}dv^2,$$

where

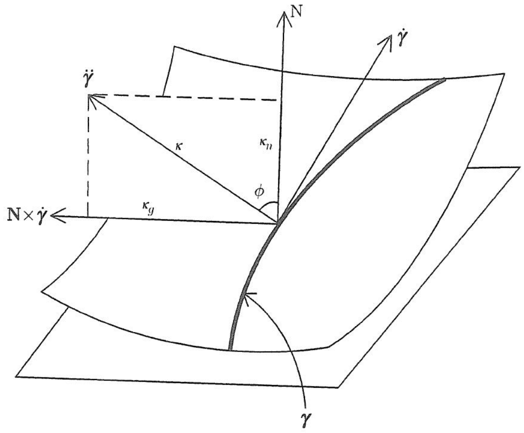
$$L_{11} :=< \sigma_{uu}, \mathbf{N} >,$$

$$L_{12} :=< \sigma_{uv}, \mathbf{N} >,$$

$$L_{22} :=< \sigma_{vv}, \mathbf{N} > .$$

It is obvious that $L_{12} = L_{21} :=< \sigma_{vu}, \mathbf{N} >$.

Another way to investigate how much a surface curves is to look at the curvature of various curves on the surface.



Definition 2.0.7. [2] If $\gamma(t) = \sigma(u(t), v(t))$ is a unit-speed curve in a surface patch σ and $\mathbf{N}$ is the unit normal vector of σ, then

$$\gamma'' = k_n \mathbf{N} + k_g(\mathbf{N} \times \gamma').$$

The scalars k_n and k_g are called the normal curvature and the geodesic curvature of γ.

Definition 2.0.8. [2] The principal curvatures of a surface patch are the roots of the equations:

$$\det\left(\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} - k\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\right) = 0.$$

We use some notations to simplify the equation. Let

$$\Psi = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}.$$

Since we discuss a regular surface patch, by Lagrange's identity, we know $g_{11}g_{22} - g_{12}^2 \neq 0$. Therefore we can solve the equations to find the principal curvatures.

We introduce the Gaussian curvature of a surface.

Definition 2.0.9. [2] Let k_1 and k_2 be the principal curvature of a surface patch. Then the Gaussian curvature of the surface patch is

$$K = k_1 k_2.$$

It is easy to get explicit formulas for the Gaussian curvature K.

Proposition 2.0.10. *[2] Let $\sigma(u, v)$ be a surface patch with the first and the second fundamental forms*

$$g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2 \quad \textit{and} \quad L_{11}du^2 + 2L_{12}dudv + L_{22}dv^2,$$

respectively. Then

$$K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}$$

Proof. By Definition 2.0.8, we have

$$(g_{11}g_{22} - g_{12}^2)k^2 - (L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11})k + (L_{11}L_{22} - L_{12}^2) = 0.$$

And the Gaussian curvature is the product of roots, we are done. □

Chapter 3

Gauss-Bonnet Theorem

In this chapter, we introduce Gauss-Bonnet theorem in differential geometry. The Gauss-Bonnet theorem is the most beautiful and profound result in theory of surfaces. It connects the intrinsic differential geometry of a surface with its topology.

As we discuss spaces curves, we have an orthonormal basis $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ of $\mathbb{R}^3$. However, when we discuss the curves on a surface, we should make use of a smooth basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of the tangent plane at each point of the surface patch, where "smooth" means that $\{\mathbf{e}_1, \mathbf{e}_2\}$ are smooth functions of the surface parameters (u, v). Moreover, we construct $\{\mathbf{e}_1, \mathbf{e}_2\}$ by applying the Gram-Schmidt process to the basis $\{\sigma_u, \sigma_v\}$ of the tangent plane. Then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{N}\}$ is a right-handed orthonormal basis on a surface, and $\mathbf{N}$ is the standard unit normal of the surface patch σ.

3.1 The Local Gauss-Bonnet Theorem

The simpler version of the Gauss-Bonnet theorem involves simple closed curves on a surface. To prove the theorem, we may need some lemmas and propositions.

Lemma 3.1.1. *[2] For every vector* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, *and* $\boldsymbol{d} \in \mathbb{R}^3$

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}).$$

Proof. We can easily check the result. □

Lemma 3.1.2. *[2] $\boldsymbol{N}$ is a unit normal vector of a surface patch, then*

$$(\boldsymbol{N}_u \times \boldsymbol{N}_v) \cdot (\boldsymbol{e}_1 \times \boldsymbol{e}_2) = (\boldsymbol{N}_u \cdot \boldsymbol{e}_1)(\boldsymbol{N}_v \cdot \boldsymbol{e}_2) - (\boldsymbol{N}_u \cdot \boldsymbol{e}_2)(\boldsymbol{N}_v \cdot \boldsymbol{e}_1).$$

Proof. From Lemma 3.1.1, we quickly get the result. □

Proposition 3.1.3. *[2] Let $\boldsymbol{N}$ be a unit normal vector of a surface patch σ. Then*

$$\boldsymbol{N}_u = a\sigma_u + b\sigma_v \quad \text{and} \quad \boldsymbol{N}_v = c\sigma_u + d\sigma_v, \tag{3.1.1}$$

where

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = -\Psi^{-1}\Phi. \tag{3.1.2}$$

Proof. Since $\mathbf{N}$ is a unit vector, $\mathbf{N}_u$ and $\mathbf{N}_v$ are perpendicular to $\mathbf{N}$. Therefore, $\mathbf{N}_u$ and $\mathbf{N}_v$ are linear combinations of σ_u and σ_v. So, we can suppose

$$\mathbf{N}_u = a\sigma_u + b\sigma_v \quad \text{and} \quad \mathbf{N}_v = c\sigma_u + d\sigma_v.$$

Moreover, σ_u and σ_v are tangent vectors to the surface patch, then

$$\mathbf{N} \cdot \sigma_u = 0 \quad \text{and} \quad \mathbf{N} \cdot \sigma_v = 0.$$

Differentiating the equation with respect to u and v gives

$$(\mathbf{N} \cdot \sigma_u)_u \Rightarrow \mathbf{N}_u \cdot \sigma_u = -\mathbf{N} \cdot \sigma_{uu} = -L_{11},$$

$$(\mathbf{N} \cdot \sigma_u)_v \Rightarrow \mathbf{N}_v \cdot \sigma_u = -\mathbf{N} \cdot \sigma_{uv} = -L_{12},$$

$$(\mathbf{N} \cdot \sigma_v)_u \Rightarrow \mathbf{N}_u \cdot \sigma_v = -\mathbf{N} \cdot \sigma_{vu} = -L_{12},$$

$$(\mathbf{N} \cdot \sigma_v)_v \Rightarrow \mathbf{N}_v \cdot \sigma_v = -\mathbf{N} \cdot \sigma_{vv} = -L_{22}.$$

Taking the dot product of each of the equations in (3.1.1) with σ_u and σ_v, we have

$$-L_{11} = ag_{11} + bg_{12},$$
$$-L_{12} = ag_{12} + bg_{22},$$
$$-L_{12} = cg_{11} + dg_{12},$$
$$-L_{22} = cg_{12} + dg_{22}.$$

Thus

$$-\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

and then the equation (3.1.2) is proved. □

In the natation of the proof of Proposition 3.1.3, we have

Lemma 3.1.4. *[2] Let $\boldsymbol{N}$ be a unit normal vector of a surface patch σ of the surface S and K be the Gaussian curvature of σ. Then*

$$\boldsymbol{N}_u \times \boldsymbol{N}_v = K\sigma_u \times \sigma_v.$$

Proof. The Gauss map $\mathcal{G} : S \to S^2$ is defined by

$$\mathcal{G}(\sigma(u,v)) = \mathbf{N}(u,v).$$

Form the equations (3.1.1) and (3.1.2), we see that

$$\begin{aligned} \mathbf{N}_u \times \mathbf{N}_v &= (ad - bc)\,\sigma_u \times \sigma_v \\ &= \det(-\Psi^{-1}\Phi)\,\sigma_u \times \sigma_v \\ &= K\sigma_u \times \sigma_v. \end{aligned}$$

This proves the lemma. □

Lemma 3.1.5. *[2] $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{N}\}$ is an orthonormal basis of $\mathbb{R}^3$ and K is the Gaussian curvature of a surface patch σ. Then*

$$(\boldsymbol{e}_1)_u \cdot (\boldsymbol{e}_2)_v - (\boldsymbol{e}_1)_v \cdot (\boldsymbol{e}_2)_u = K\|\sigma_u \times \sigma_v\|.$$

Proof. Since $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors, $(\mathbf{e}_1)_u$ and $(\mathbf{e}_1)_v$ are perpendicular to $\mathbf{e}_1$. Similarly, $(\mathbf{e}_2)_u$ and $(\mathbf{e}_2)_v$ are perpendicular to $\mathbf{e}_2$. Thus,

$$\begin{aligned}
(\mathbf{e}_1)_u &= 0\mathbf{e}_1 + a\mathbf{e}_2 + b\mathbf{N},\\
(\mathbf{e}_1)_v &= 0\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{N},\\
(\mathbf{e}_2)_u &= a'\mathbf{e}_1 + 0\mathbf{e}_2 + b'\mathbf{N},\\
(\mathbf{e}_2)_v &= c'\mathbf{e}_1 + 0\mathbf{e}_2 + d'\mathbf{N},
\end{aligned}$$

for some scalars $a, b, c, d, a', b', c', d'$ which depend on u and v. Moreover, by differentiating the equation $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ with respect to u and v, we see that

$$\begin{aligned}
(\mathbf{e}_1)_u \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot (\mathbf{e}_2)_u &= 0,\\
(\mathbf{e}_1)_v \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot (\mathbf{e}_2)_v &= 0.
\end{aligned}$$

It implies that $a = -a'$ and $c = -c'$, thus

$$\begin{aligned}
(\mathbf{e}_1)_u &= 0\mathbf{e}_1 + a\mathbf{e}_2 + b\mathbf{N},\\
(\mathbf{e}_1)_v &= 0\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{N},\\
(\mathbf{e}_2)_u &= -a\mathbf{e}_1 + 0\mathbf{e}_2 + b'\mathbf{N},\\
(\mathbf{e}_2)_v &= -c\mathbf{e}_1 + 0\mathbf{e}_2 + d'\mathbf{N}.
\end{aligned}$$

We compute

$$\begin{aligned}
(\mathbf{e}_1)_u \cdot (\mathbf{e}_2)_v - (\mathbf{e}_1)_v \cdot (\mathbf{e}_2)_u &= bd' - b'd\\
&= [(\mathbf{e}_1)_u \cdot \mathbf{N}][(\mathbf{e}_2)_v \cdot \mathbf{N}] - [(\mathbf{e}_2)_u \cdot \mathbf{N}][(\mathbf{e}_1)_v \cdot \mathbf{N}].
\end{aligned}$$

By differentiating the equations $\mathbf{e}_1 \cdot \mathbf{N} = 0$ and $\mathbf{e}_2 \cdot \mathbf{N} = 0$ with respect to u and v, we know that

$$\begin{aligned}
(\mathbf{e}_1)_u \cdot \mathbf{N} &= -\mathbf{e}_1 \cdot \mathbf{N}_u, \\
(\mathbf{e}_1)_v \cdot \mathbf{N} &= -\mathbf{e}_1 \cdot \mathbf{N}_v, \\
(\mathbf{e}_2)_u \cdot \mathbf{N} &= -\mathbf{e}_2 \cdot \mathbf{N}_u, \\
(\mathbf{e}_2)_v \cdot \mathbf{N} &= -\mathbf{e}_2 \cdot \mathbf{N}_v.
\end{aligned}$$

Thus

$$(\mathbf{e}_1)_u \cdot (\mathbf{e}_2)_v - (\mathbf{e}_1)_v \cdot (\mathbf{e}_2)_u = (-\mathbf{e}_1 \cdot \mathbf{N}_u)(-\mathbf{e}_2 \cdot \mathbf{N}_v) - (-\mathbf{e}_2 \cdot \mathbf{N}_u)(-\mathbf{e}_1 \cdot \mathbf{N}_v).$$

By Lemma 3.1.2 and Lemma 3.1.4, we get

$$\begin{aligned}
(\mathbf{e}_1)_u \cdot (\mathbf{e}_2)_v - (\mathbf{e}_1)_v \cdot (\mathbf{e}_2)_u &= (K\sigma_u \times \sigma_v) \cdot (\mathbf{e}_1 \times \mathbf{e}_2) \\
&= (K\sigma_u \times \sigma_v) \cdot \mathbf{N} \\
&= K\|\sigma_u \times \sigma_v\|.
\end{aligned}$$

This completes the proof. □

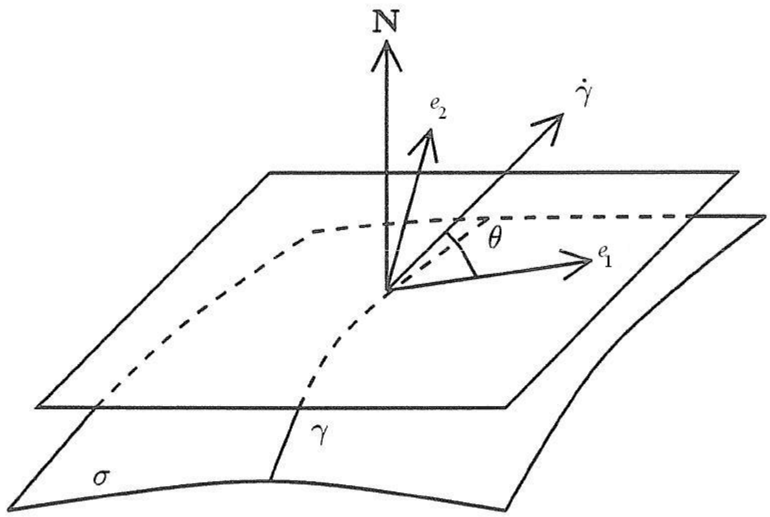
We can now state the first version of the Gauss-Bonnet Theorem.

Theorem 3.1.6. *[2]* (The Gauss-Bonnet Theorem for Simple Closed Curve). *Let S be an oriented regular surface in $\mathbb{R}^3$. Let $\sigma : U \subset \mathbb{R}^2 \to S$ be a surface patch of S such that $\sigma(U)$ is simply connected. Let $\gamma : \mathbb{R} \to S$ be a unit-speed, simple, closed and positively oriented curve on S with length $\ell(\gamma)$. Then*

$$\int_0^{\ell(\gamma)} k_g ds = 2\pi - \iint_{int(\gamma)} K dA.$$

Proof. Choose a right-handed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{N}\}$ at each point of the surface patch σ which we obtained by applying the Gram-Schmidt process on the basis $\{\sigma_u, \sigma_v\}$. Along the curve $\gamma : \mathbb{R} \to \sigma(U)$, let $\theta : \mathbb{R} \to \mathbb{R}$ be the angle between the unit vector $\mathbf{e}_1(s)$

and $\gamma'(s)$ at the same point. That is,



$$\gamma'(s) = \cos\theta(s)\mathbf{e}_1(s) + \sin\theta(s)\mathbf{e}_2(s).$$

Then,

$$\mathbf{N}\times\gamma' = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix} = \begin{pmatrix} -\sin\theta(s) \\ \cos\theta(s) \end{pmatrix}$$
$$= -\sin\theta(s)\mathbf{e}_1(s) + \cos\theta(s)\mathbf{e}_2(s).$$

And for the second derivative γ'' we have

$$\gamma'' = -\theta'\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_1' + \theta'\cos\theta\mathbf{e}_2 + \sin\theta\mathbf{e}_2'.$$

So the geodesic curvature satisfies

$$\begin{aligned} k_g &= \gamma''\cdot(\mathbf{N}\times\gamma) \\ &= \theta' - (\mathbf{e}_1\cdot\mathbf{e}_2')\sin^2\theta + (\mathbf{e}_1'\cdot\mathbf{e}_2)\cos^2\theta \\ &= \theta' - \mathbf{e}_1\cdot\mathbf{e}_2'. \end{aligned} \tag{3.1.3}$$

we integrate the geodesic curvature k_g over one period,

$$\int_0^{\ell(\gamma)} k_g ds = \int_0^{\ell(\gamma)} \theta' ds - \int_0^{\ell(\gamma)} \mathbf{e}_1 \cdot \mathbf{e}_2' \, ds.$$

Let $\pi = \sigma^{-1} \circ \gamma : \mathbb{R} \to U$ be entirely contained in the simply connected region U. The curve π is simple, closed and positively oriented. Then

$$\begin{aligned} \int_0^{\ell(\gamma)} \mathbf{e}_1(s) \cdot \mathbf{e}_2'(s) \, ds &= \int_0^{\ell(\gamma)} \mathbf{e}_1 \cdot [u'(\mathbf{e}_2)_u + v'(\mathbf{e}_2)_v] ds \\ &= \int_\pi \mathbf{e}_1 \cdot (\mathbf{e}_2)_u \, du + \mathbf{e}_1 \cdot (\mathbf{e}_2)_v \, dv. \end{aligned}$$

By Green's theorem and Lemma 3.1.5

$$\begin{aligned} \int_0^{\ell(\gamma)} \mathbf{e}_1(s) \cdot \mathbf{e}_2'(s) \, ds &= \iint_{int(\pi)} \{[\mathbf{e}_1 \cdot (\mathbf{e}_2)_v]_u - [\mathbf{e}_1 \cdot (\mathbf{e}_2)_u]_v\} dudv \\ &= \iint_{int(\pi)} \{(\mathbf{e}_1)_u \cdot (\mathbf{e}_2)_v - (\mathbf{e}_1)_v \cdot (\mathbf{e}_2)_u\} dudv \\ &= \iint_{int(\pi)} K \|\sigma_u \times \sigma_v\| dudv \\ &= \iint_{int(\gamma)} K dA. \end{aligned}$$

Consider

$$\int_0^{\ell(\gamma)} \theta' ds = \int_0^{\ell(\gamma)} d\theta = \theta(\ell(\gamma)) - \theta(0).$$

When $\sigma(U)$ is simple connected, i.e., $\sigma(U)$ can be continuously deformed to a point. As we shrink the curve to a point, $\mathbf{e}_1$ becomes almost constant along the curve, but tangent vector must make one full rotation. Therefore,

$$\int_0^{\ell(\gamma)} \theta' ds = 2\pi.$$

Hence

$$\int_0^{\ell(\gamma)} k_g ds = 2\pi - \iint_{int(\gamma)} K dA.$$

This proves the statement. □

We should begin the details of a local version of the Gauss Bonnet theorem. We need a few definitions. Let S be a regular surface in $\mathbb{R}^3$, and $\gamma : [0,l] \to S$ be a simple, closed, and piecewise regular curve. That is,

1. $\gamma(0) = \gamma(l)$.

2. For all $t_1, t_2 \in [0,l]$, if $t_1 \neq t_2$, then $\gamma(t_1) \neq \gamma(t_2)$.

3. There exists a subdivision

$$0 = t_0 < t_1 < \cdots < t_n = l$$

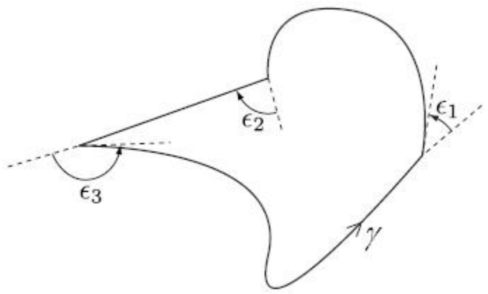
of $[0,l]$ such that γ is differentiable and regular on each of the intervals (t_{i-1}, t_i) for $i = 1, 2, \ldots, n$.

Moreover, the one-sided derivatives of γ at the endpoints of each subinterval

$$\dot{\gamma}^-(t_i) = \lim_{t \to t_i^-} \frac{\gamma(t) - \gamma(t_i)}{t - t_i},$$
$$\dot{\gamma}^+(t_i) = \lim_{t \to t_i^+} \frac{\gamma(t) - \gamma(t_i)}{t - t_i}$$

exist, are non-zero and not parallel. The points $\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_n)$ are called the vertices of the curve γ.

Let $\theta_i^\pm$ be the angle between $\dot{\gamma}^\pm(t_i)$ and $\mathbf{e}_1$. Typically, $\epsilon_i = \theta_i^+ - \theta_i^-$ (see the picture below) is called the exterior angle at the vertex $\gamma(t_i)$ from $\dot{\gamma}^-(t_i)$ to $\dot{\gamma}^+(t_i)$, and $\alpha_i = \pi - \epsilon_i$ is called the interior angle at the vertex $\gamma(t_i)$.



The next result generalizes Theorem 3.1.6.

Theorem 3.1.7. *[2]* (The Local Gauss-Bonnet Theorem). *Suppose R is a simply connected region with simple, closed, piecewise regular, and positively oriented boundary in an oriented regular surface S. If $\gamma = \partial R$ with length $\ell(\gamma)$ has exterior angles ϵ_i at the vertices $\gamma(s_i)$, $i = 1, 2, \ldots, n$, then*

$$\int_{\partial R} k_g ds + \iint_R K dA + \sum_{i=1}^{n} \epsilon_i = 2\pi.$$

Note, as pictured above, that we measure exterior angles so that $|\epsilon_i| < \pi$ for all i.

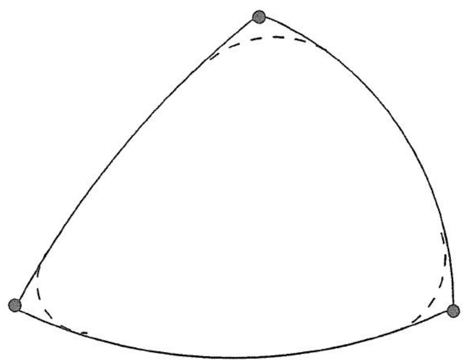
Proof. If ∂R is smooth, then from our easier discussion we infer that

$$\int_0^{\ell(\gamma)} k_g ds + \iint_{int(\gamma)} K dA = 2\pi = \int_{\partial R} \theta' ds.$$

But ∂R is piecewise smooth, then our goal is to prove that

$$\int_0^{\ell(\gamma)} \theta' ds = 2\pi - \sum_{i=1}^{n} \epsilon_i. \tag{3.1.4}$$

To establish the equation (3.1.4), we image "smoothing" each vertex of γ as shown in the following diagram.



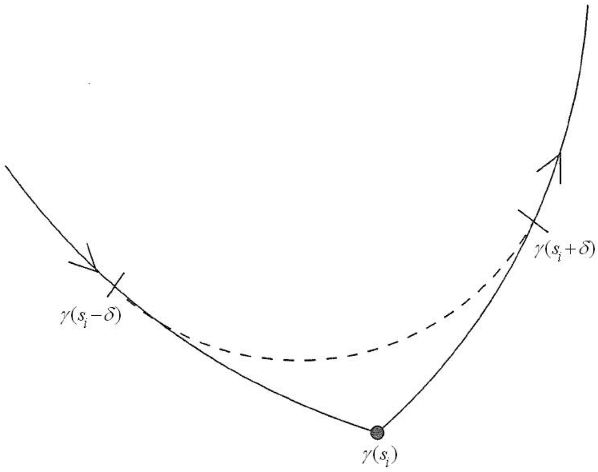
If the "smoothed" curve $\tilde{\gamma}$ is smooth, then

$$\int_0^{\ell(\tilde{\gamma})} \tilde{\theta}' ds = 2\pi,$$

where $\tilde{\theta}$ is the angle between the unit tangent vector $\mathbf{e}_1$ and $\tilde{\gamma}'(s)$ at the same point. Since γ and $\tilde{\gamma}$ are the same except near the vertices of γ, the difference is

$$\int_0^{\ell(\tilde{\gamma})} \tilde{\theta}' ds - \int_0^{\ell(\gamma)} \theta' ds = \sum_{i=1}^{n} \left(\int_{s_{i-1}}^{s_i} \tilde{\theta}' ds - \int_{s_{i-1}}^{s_i} \theta' ds \right).$$

Consider the situation, near $\gamma(s_i)$, the picture is



i.e. γ and $\tilde{\gamma}$ agree except when s belongs to a small interval $(s_i - \delta, s_i + \delta)$, for some $\delta > 0$, so the contribution from the ith vertex is

$$\begin{aligned} & \int_{s_i-\delta}^{s_i+\delta} \tilde{\theta}' ds - \left(\int_{s_i-\delta}^{s_i} \theta' ds + \int_{s_i}^{s_i+\delta} \theta' ds \right) \\ = \ & \tilde{\theta}(s_i + \delta) - \tilde{\theta}(s_i - \delta) - (\theta(s_i) - \theta(s_i - \delta) + \theta(s_i + \delta) - \theta(s_i)). \end{aligned}$$

As $\delta \downarrow 0$, the first integral is the angle between $\dot{\gamma}^+(s_i)$ and $\dot{\gamma}^-(s_i)$, i.e., ϵ_i. On the other hand, since $\gamma(s)$ is smooth on each interval $(s_i - \delta, s_i)$ and $(s_i, s_i + \delta)$, the last two integrals become zero when $\delta \downarrow 0$. So

$$\int_0^{\ell(\tilde{\gamma})} \tilde{\theta}' ds - \int_0^{\ell(\gamma)} \theta' ds = \sum_{i=1}^{n} \epsilon_i.$$

This completes the proof. □

3.2 The Global Gauss-Bonnet Theorem

To globalize the Gauss-Bonnet theorem, we need further topological preliminaries.

Let S be a regular surface. A region $R \subset S$ is said to be regular if R is compact and its boundary ∂R is the union of a finite number of simple closed piecewise regular curves that do not intersect to each other. The region in Figure 3.2.1 is not regular.

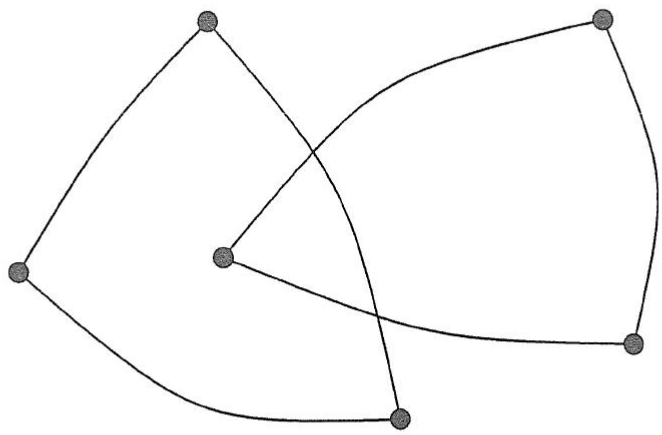


Figure 3.2.1: The situation is not allowed.

For convenience, we should consider a compact surface as a regular region, and its boundary is empty.

A simple region which has only three vertices with external angles $\alpha_i \neq 0$, $i = 1, 2, 3$, is called a triangle.

Definition 3.2.1. [12] A triangulation of a regular region $R \subset S$ is a finite family $\mathfrak{T}$ of triangles T_i, $i = 1, 2, \ldots, n$, such that

1. $\displaystyle\bigcup_{i=1}^{n} T_i = R$.
2. If $T_i \cap T_j \neq \emptyset$, then $T_i \cap T_j$ is either a common edge of T_i and T_j or a common vertex of T_i and T_j.

Can every regular region of a regular surface be triangulated? The next theorem tells us the answer.

Theorem 3.2.2. *[12] Every regular region of a regular surface admits a triangulation.*

This proof was first proven in 1924 by Tibor Radó. His proof is provided rigorously in Chapter 1 of the text [10] by Ahlfors and Sario, and a relatively shorter proof is given in Doyle and Moran [14]. We will continue to use the fact to prove the global Gauss-Bonnet Theorem.

Given a triangulation $\mathfrak{T}$ of a regular region $R \subset S$ of a surface S, we should consider the relation of the number of vertices, the number of edges, and the number of faces of the triangulation.

Definition 3.2.3. [12] The Euler characteristic $\chi(S)$ of a triangulation of a compact surface S is

$$\chi(S) = V - E + F,$$

where V, E, and F are the number of vertices, edges, and faces respectively of the triangulation.

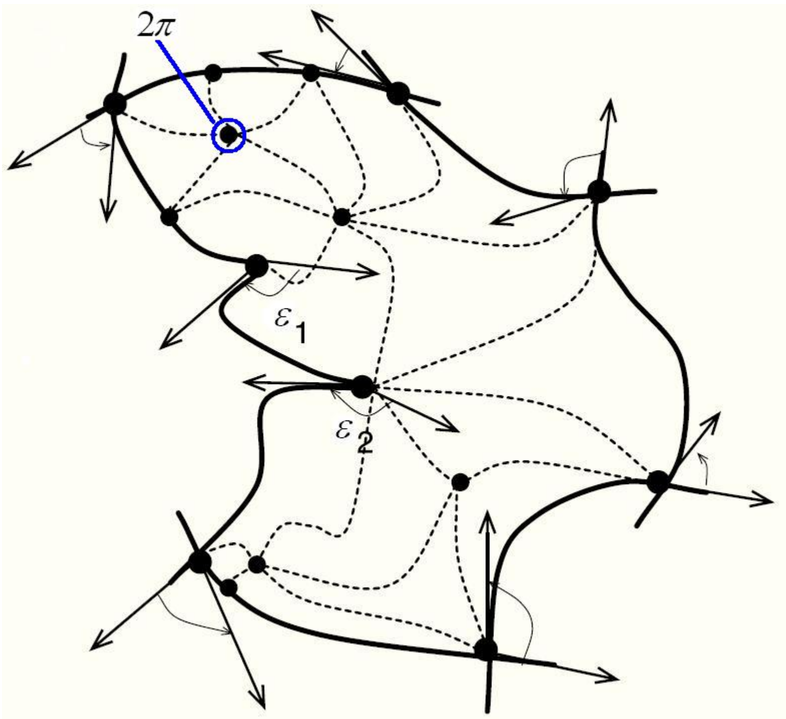


Figure 3.2.2: A triangulation on a regular region R.

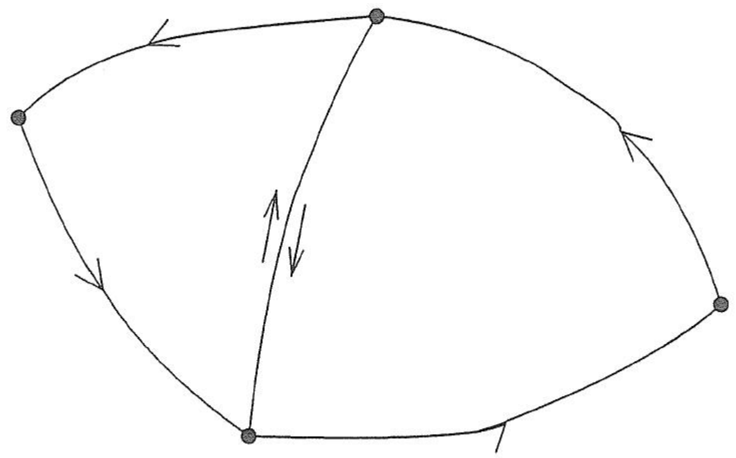
We have proven the local case of this theorem, and the global theorem tells us similar information. We prove this generalization by using the local theorem in each triangular region of our triangulation for the given surface. The beautiful result to which we have been headed is now the following.

Theorem 3.2.4. *[1]* (The Global Gauss-Bonnet Theorem). *Let $R \subset S$ be a regular region of an oriented regular surface S with simple, closed, piecewise regular and positively oriented boundary ∂R. If ϵ_k, $k = 1, 2, \ldots, p$ are exterior angles of ∂R, then*

$$\int_{\partial R} k_g ds + \iint_R K dA + \sum_{k=1}^{p} \epsilon_k = 2\pi\chi(R).$$

Proof. Since every regular region of a regular surface admits a triangulation. Let $\mathfrak{T}$ be a triangulation of R such that every triangular region T_i in $\mathfrak{T}$ is contained in a coordinate neighborhood of one of a family of patches (see Figure 3.2.2).

We give the position orientation to each triangular region $T_i \in \mathfrak{T}$. In this way, adjacent triangular region give opposite orientations to their common edge, as pictured below.



Suppose $\mathfrak{T} = \{T_1, T_2, \ldots, T_F\}$ is a triangulation of the region region R. We obverse

$$\iint_R K dA = \sum_{i=1}^{F} \iint_{T_i} K dA$$

and

$$\int_{\partial R} k_g dA = \sum_{i=1}^{F} \int_{\partial T_i} k_g ds.$$

Let ϵ_{ij}, $j = 1, 2, 3$, denote the exterior angles of the triangle T_i. Then applying the local Gauss-Bonnet theorem to T_i, we have

$$\int_{\partial T_i} k_g ds + \iint_{T_i} K dA + \sum_{j=1}^{3} \epsilon_{ij} = 2\pi. \qquad (3.2.1)$$

The equation (3.2.1) can be written as

$$\int_{\partial T_i} k_g ds + \iint_{T_i} K dA = -\pi + \sum_{j=1}^{3} \alpha_{ij},$$

where α_{ij}, $j = 1, 2, 3$, is the interior angles of T_i.

We go on the process to every triangular region in $\mathfrak{T}$ and add up the result. Then

$$\int_{\partial R} k_g ds + \iint_{R} K dA = -\pi F + \sum_{i,j=1}^{F,3} \alpha_{ij}, \qquad (3.2.2)$$

where $\sum_{i,j=1}^{F,3} \alpha_{ij}$ is the sum of all interior angles.

Suppose

$$V_e = \text{number of exterior vertices of } \mathfrak{T},$$

$$V_i = \text{number of internal vertices of } \mathfrak{T},$$

$$E_e = \text{number of exterior vertices of } \mathfrak{T},$$

$$E_i = \text{number of internal vertices of } \mathfrak{T}.$$

Moreover, the angles around each internal vertex add up to 2π (see Figure 3.2.2); hence the sum of all internal interior angles is $2\pi V_i$. A similar calculation computes the sum of the external interior angles, and so

$$\sum_{i,j=1}^{F,3} \alpha_{ij} = 2\pi V_i + \pi V_e - \sum_{k=1}^{p} \epsilon_k.$$

Thus the equation (3.2.2) can be written as

$$\int_{\partial R} k_g ds + \iint_R K dA + \sum_{k=1}^{p} \epsilon_k = -\pi F + 2\pi V_i + \pi V_e. \tag{3.2.3}$$

We still need some relations

$$V_e = E_e, \quad V = V_e + V_i, \quad E = E_e + E_i.$$

Furthermore, by mathematical induction,

$$3F = 2E_i + E_e.$$

Thus the equation (3.2.3) can be written as

$$\begin{aligned} \int_{\partial R} k_g ds + \iint_R K dA + \sum_{k=1}^{p} \epsilon_k &= 2\pi F - 3\pi F + 2\pi V_i + \pi V_e \\ &= 2\pi F - \pi(2E_i + E_e) + 2\pi V_i + \pi V_e \\ &= 2\pi F - 2\pi E_i - \pi E_e + 2\pi V_i + 2\pi V_e - \pi V_e \\ &= 2\pi V - 2\pi E + 2\pi F \\ &= 2\pi \chi(R) \end{aligned}$$

which is exactly what we wanted to prove. □

We now derive some conclusions:

Corollary 3.2.5. *[18] The Euler characteristic $\chi(R)$ does not depend on the triangulation $\mathfrak{T}$ of a regular region R of an oriented surface S.*

Proof. The left-hand side of the equality in Theorem 3.2.4 has nothing whatsoever to do with the triangulation. □

It is therefore legitimate that the Euler characteristic has no reference to the triangulation. It is proved in a course in algebraic topology that the Euler characteristic is a "topological invariant". In other words, if we deform the surface R in a bijective, continuous manner (so as to obtain a homeomorphic surface), the Euler characteristic does not change. We therefore deduce:

Corollary 3.2.6. *[18] The quantity*

$$\int_{\partial R} k_g ds + \iint_R K dA + \sum_{k=1}^{p} \epsilon_k$$

is a topological invariant, i.e., does not change as we deform the surface S.

By taking into account that a closed surface can be considered as a region without boundary, we obtain

Corollary 3.2.7. *[12] Let S be an oriented compact surface, then*

$$\iint_S K dA = 2\pi\chi(S).$$

3.3 The Discrete Gauss-Bonnet Theorem

In this section, we are interested in the discrete curvatures related to the integral Gaussian curvature. Before discussing it, we have some definitions.

The Euler characteristic is obtained by counting vertices, edges, and faces, but it applies to structure much more general than polyhedra. We will focus on the notation of a cell complex.

Definition 3.3.1. [17] Cells are defined to be those topological objects whose interiors are homeomorphic to disks of some dimension. Looking at the first few dimensions we obtain:

0-cell (vertex)	point
1-cell (edge)	interior homeomorphic to an open interval
2-cell (face)	interior homeomorphic to an open disk in $\mathbb{R}^2$

A cell complex is a union of finitely many 0-cells, 1-cells, and 2-cells so that the interior of the cells are pairwise disjoint and the boundary of each cell is the union of other lower-dimensional cells.

Definition 3.3.2. [17] If the cell complex is homeomorphic to a surface S, we say it is a cell decomposition of S.

We continue our study of some special types of cell complexes: regular and b-valent. The valency of a vertex is simply the number of edges emanating from it.

Definition 3.3.3. [17] A regular complex on a surface S is a cell decomposition of S where

1. Each face has the same number of edges a with $a \geq 3$.
2. Each vertex has the same valency b with $b \geq 3$.
3. Two faces meet along a single edge, at a single vertex, or not at all.
4. No face meets itself.

We denote a regular complex on S that has all faces a-sided polygons and all vertices of valency b by $(a,b)S$.

Our task is to determine all possible regular complexes of a given surface S. The key is the following:

$$2E = aF \quad \text{and} \quad 2E = bV.$$

Since each face has a edges, this gives aF. We note that each edge bounds precisely two faces, so we have counted each edge exactly twice. Hence, $aF = 2E$.

The second equation is similar. Since each vertex has b edges, this counting gives a total of bV edges. Noticing that each edge has exactly two vertex ends, we have $bV = 2E$.

Therefore, the Euler characteristic $\chi(S)$ can be denoted by

$$\chi(S) = 2E\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{2}\right),$$

where $a, b \in \mathbb{N}$ and $a, b \geq 3$.

Theorem 3.3.4. *[10] All Platonic solids—the tetrahedron, cube, octahedron, dodecahedron, and icosahedron—are the only regular complexes of S^2.*

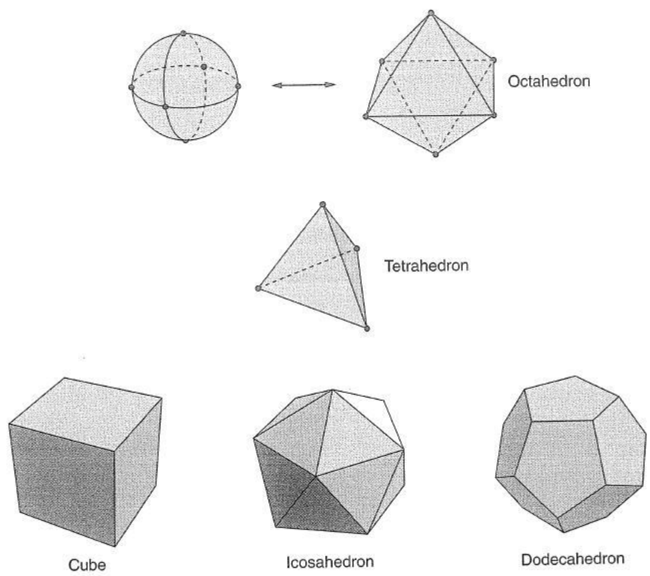


Figure 3.3.1: The Platonic solids give regular complexes of the sphere.

Proof. Since $\chi(S^2) = 2$, then

$$2 = 2E\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{2}\right), \quad a, b \in \mathbb{N}, a, b \geq 3.$$

It implies that

$$\frac{1}{a} + \frac{1}{b} > \frac{1}{2}, \quad a, b \in \mathbb{N}, a, b \geq 3. \tag{3.3.1}$$

We will analyze each possible case separately:

Case1 $a = 3$ (faces are triangles).

From the equation (3.3.1), we know

$$\frac{1}{6} < \frac{1}{b} \leq \frac{1}{3}.$$

The only possibilities are $b = 3, 4$, or 5.

(1) If $b = 3$, then $2 = 2E(\frac{1}{3} + \frac{1}{3} - \frac{1}{2})$. So $E = 6$, $F = \frac{2E}{a} = 4$, and $V = \frac{2E}{b} = 4$. This is the tetrahedron $(3,3)S^2$.

(2) If $b = 4$, then $2 = 2E(\frac{1}{3} + \frac{1}{4} - \frac{1}{2})$. So $E = 12$, $F = \frac{2E}{a} = 8$, and $V = \frac{2E}{b} = 6$. This is the octahedron $(3,4)S^2$.

(3) If $b = 5$, then $2 = 2E(\frac{1}{3} + \frac{1}{5} - \frac{1}{2})$. So $E = 30$, $F = \frac{2E}{a} = 20$, and $V = \frac{2E}{b} = 12$. This is the icosahedron $(3,5)S^2$.

Case2 $a = 4$ (faces are squares).

From the equation (3.3.1), we know

$$\frac{1}{4} < \frac{1}{b} \leq \frac{1}{3}.$$

The only possibility is $b = 3$. If $b = 3$, then $2 = 2E(\frac{1}{4} + \frac{1}{3} - \frac{1}{2})$. So $E = 12$, $F = \frac{2E}{a} = 6$, and $V = \frac{2E}{b} = 8$. This is the cube $(4,3)S^2$.

Case3 $a = 5$ (faces are pentagons).

From the equation (3.3.1), we know

$$\frac{3}{10} < \frac{1}{b} \leq \frac{1}{3}.$$

The only possibility is $b = 3$. If $b = 3$, then $2 = 2E(\frac{1}{5} + \frac{1}{3} - \frac{1}{2})$. So $E = 30$, $F = \frac{2E}{a} = 12$, and $V = \frac{2E}{b} = 20$. This is the dodecahedron $(5,3)S^2$.

Case4 $a = 6$ (faces are hexagons or bigger).

From the equation (3.3.1), we know

$$\frac{1}{b} > \frac{1}{3}.$$

This can not happen.

Hence there are only five solutions. □

What are the regular complexes on the torus T ? We know $\chi(T) = 0$. A similar discuss is the following:

$$\chi(T) = 0 = 2E\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{2}\right),$$
$$\Rightarrow \quad (a-2)(b-2) = 4,$$

where $a, b \in \mathbb{N}$, $a, b \geq 3$.

There are only three possibilities, they are $(3,6)T$, $(4,4)T$, and $(6,3)T$.

We begin the study of polyhedral surfaces with one of the most important results: the Gauss-Bonnet theorem. Here is a lemma which will prove useful.

Lemma 3.3.5. *[15] Every simplicial polyhedron with n vertices has $3n-6$ edges and $2n-4$ faces. More generally, this holds for every triangulated surface homeomorphic to a sphere S^2.*

Proof. By Theorem 3.2.2, we suppose the simplicial polyhedron has F faces. Since

$$2E = 3F.$$

Then the number of edges is $\frac{3}{2}F$, and by Euler's formula

$$\chi(S^2) = n - \frac{3}{2}F + F = 2.$$

It implies

$$F = 2n - 4 \quad \text{and} \quad E = \frac{3}{2}F = 3n - 6.$$

This lemma is proved. □

We now define the discrete Gaussian curvature of a convex polyhedron P.

Definition 3.3.6. [15] Let $P \subset \mathbb{R}^3$ be a convex polyhedron with the set of vertices $X = \{v_1, v_2, \ldots, v_n\}$. Denote by $\alpha_i = \alpha(v_i)$ the sum of the face angles around v_i and let

$$w_i = 2\pi - \alpha_i$$

be the Gaussian curvature of v_i. The sum of Gaussian curvatures at all vertices is called the total curvature of the polyhedron.

Theorem 3.3.7. *[15]* (The Discrete Gauss-Bonnet Theorem). *Let $w_1, w_2, \ldots, w_n$ be the Gaussian curvatures of vertices of a convex polyhedron $P \subset \mathbb{R}^3$. Then*

$$w_1 + w_2 + \cdots + w_n = 4\pi.$$

Proof. Suppose the polyhedron P has n vertices, then we triangulate the faces of P. By Lemma 3.3.5, the resulting triangulation has $2n - 4$ faces. Then, the total sum of face angle $A = (2n - 4)\pi$. We conclude:

$$w_1 + w_2 + \cdots + w_n = 2\pi n - A = 4\pi$$

which finishes the proof of the discrete Gauss-Bonnet theorem. □

We try to generalize the Theorem 3.3.7. Consider the situation that we triangulate on different compact surfaces.

surface S	$\chi(S)$	vertex	edge	face	total Gaussian curvature
S^2	2	n	$3n-6$	$2n-4$	4π
T	0	n	$3n$	$2n$	0
T_2	-2	n	$3n+6$	$2n+4$	-4π
T_g	$2-2g$	n	$3n+6(g-1)$	$2n+4(g-1)$	$-4\pi(g-1)$

We define the genus of an oriented surface to be the number of handles we add to the sphere to get the surface. So if the surface $S = T_g$, the genus is g. Moreover, the table above tells us the discrete total curvature on a different compact surface is the same as the smooth total curvature on it.

Chapter 4

The Applications of Gauss-Bonnet Theorem

In this chapter, we will present some applications of the Gauss-Bonnet theorem below: the Poincaré-Hopf index theorem, the hairy ball theorem and the fundamental theorem of algebra.

4.1 The Poincaré-Hopf Index Theorem

Before introducing the statement of the Poincaré-Hopf index theorem, we need some definitions related to vector fields are provided.

Definition 4.1.1. [12] A tangent vector field $\mathbf{V}$ on a surface S is a correspondence which assigns to each $p \in S$ a vector $\mathbf{V}(p) \in T_pS$. The tangent vector field $\mathbf{V}$ is differentiable at $p \in S$ if, for some parametrization $\sigma(u, v)$ at p, the functions $\alpha(u, v)$ and $\beta(u, v)$ given by

$$\mathbf{V}(\sigma(u, v)) = \alpha(u, v)\sigma_u(u, v) + \beta(u, v)\sigma_v(u, v)$$

are differentiable functions at p.

Definition 4.1.2. [12] If $\mathbf{V}$ is a differentiable tangent vector field on a surface S, a point $p \in S$ at which $\mathbf{V} = \mathbf{0}$ is called a singular point of $\mathbf{V}$.

By the existence and uniqueness theorem of differentiable equations, if $p \in S$, there is a unique $\gamma(t)$ on S such that $\gamma' = \mathbf{V}$ and $\gamma(0) = p$; γ is called an integral curve on $\mathbf{V}$.

Definition 4.1.3. [12] A singular point p is said to be isolated if there exists an open set containing the point p that contains no other singular points.

Then we define the index of $\mathbf{V}$ at p.

Definition 4.1.4. [2] The index of the singular point p of the tangent vector field $\mathbf{V}$ is

$$\mu(p) = \frac{1}{2\pi} \int_0^{\ell(\gamma)} \frac{d\psi}{ds} ds,$$

where $\gamma(s)$ is any unit-speed, simple, closed, and positively oriented curve of length $\ell(\gamma)$ in S with $p \in \text{int}(\gamma)$, and $\psi(s)$ is the angle between a nowhere vanishing differentiable tangent vector field ξ on S and $\mathbf{V}$ at the point $\gamma(s)$.

From Definition 4.1.4, it is clear that $\mu(p)$ is an integer.

Example 4.1.5. We show some examples of indices of the tangent vector fields in the xy plane which have $(0,0)$ as singular point. The curves that appear in the drawings are the trajectories of the tangent vector fields.

(1) $\mathbf{V}(x,y) = (x,y)$; $\mu = +1$

(2) $\mathbf{V}(x,y) = (-x,-y)$; $\mu = +1$

(3) $\mathbf{V}(x,y) = (y,-x)$; $\mu = +1$

(4) $\mathbf{V}(x,y) = (x,-y)$; $\mu = -1$

The singular point in examples (1), (2), (3), and (4) is called a course, sink, center, and saddle, respectively.

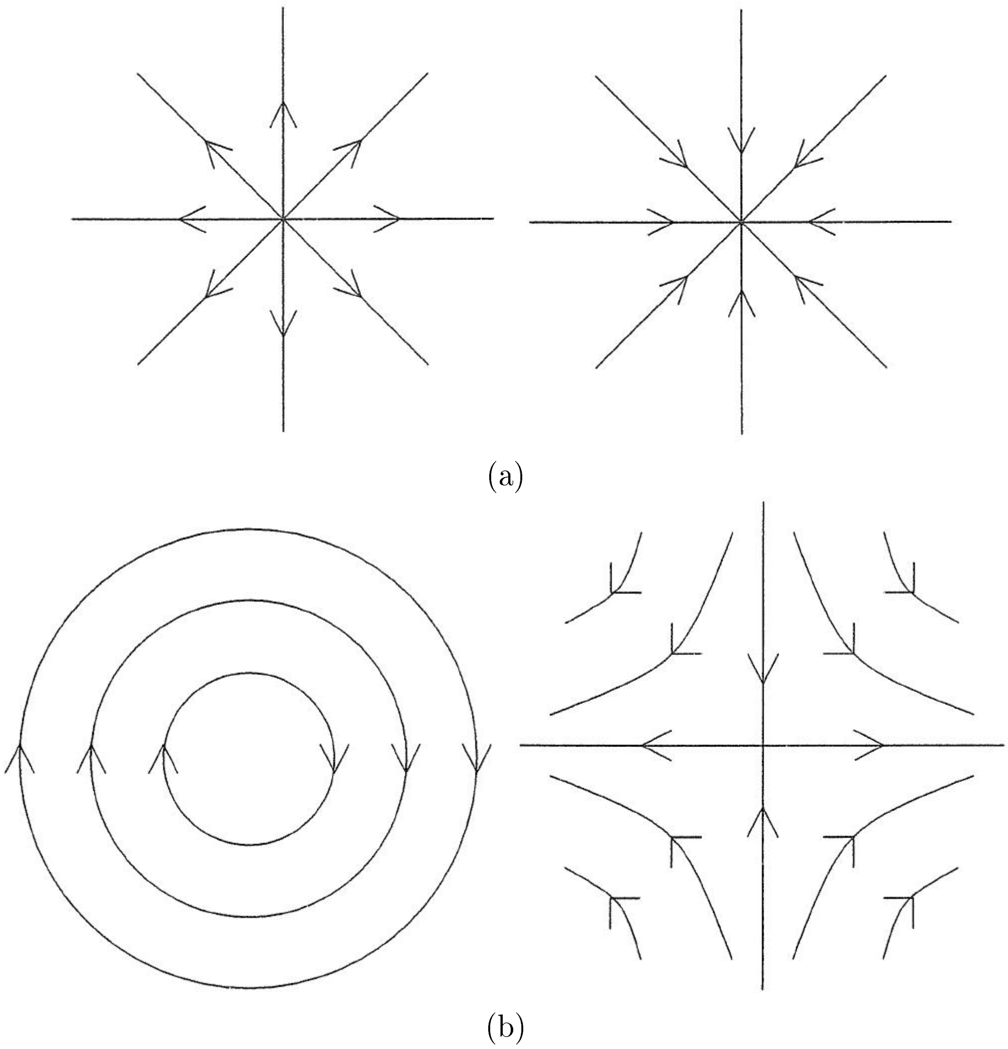


Figure 4.1.1: (a) Course, Sink (b) Center, Saddle

Let us verify the index in case (4). Choose the "reference" tangent vector field to be the constant vector field $\xi = (1, 0)$. Then, the angle ψ is given by

$$(\cos\psi, \sin\psi) = \frac{\mathbf{V}}{\|\mathbf{V}\|} = \left(\frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}} \right).$$

Taking $\gamma(s) = (\cos s, \sin s)$ to be the unit circle, at $\gamma(s)$ the angle ψ satisfies

$$(\cos\psi, \sin\psi) = (\cos s, -\sin s).$$

So $\psi = 2\pi - s$, and hence

$$\mu(0,0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d}{ds}(2\pi - s)ds = -1.$$

Similarly, in case (1), (2), and (3), $\psi = \pi$, $\psi = \pi + s$, and $\psi = \frac{3\pi}{2} + s$ respectively. Hence, we get the results that we want.

Now, let $S \subset \mathbb{R}^3$ be an oriented, compact surface and $\mathbf{V}$ is a differentiable tangent vector field with only isolated singular points. We remark that they are finite in number. Otherwise, if there are infinitely many singular points, then the Bolzano-Weierstrass theorem implies that they have a limit point in S. By continuity of the tangent vector field, this point is also a singular point, and hence the singular point is not isolated.

Theorem 4.1.6. *[2]* (The Poincaré-Hopf Index Theorem). *Let $\boldsymbol{V}$ be a differentiable tangent vector field on a compact surface S which has only finitely many isolated singular points $p_1, p_2, \ldots, p_n$. Then*

$$\sum_{i=1}^{n} \mu(p_i) = \chi(S),$$

where $\chi(S)$ is the Euler characteristic of S.

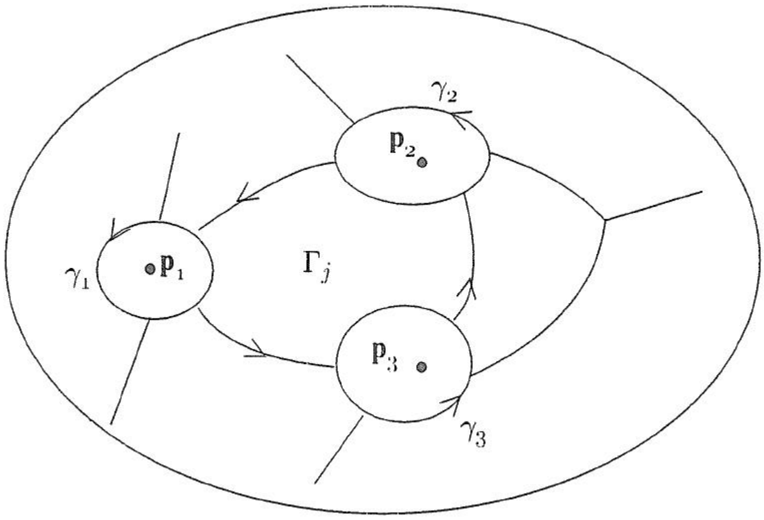
Proof. Let γ_i be a unit-speed, simple, closed, and positively oriented curve contained in a patch σ_i of S with $p_i \in \operatorname{int}(\gamma_i)$. Since p_i is isolated, then we assume that the γ_i are chosen so small that their interiors are disjoint. Let

$$S_1 = \operatorname{int}(\gamma_1) \cup \operatorname{int}(\gamma_2) \cup \cdots \cup \operatorname{int}(\gamma_n) \quad \text{and} \quad S_2 = S \setminus S_1.$$

By Theorem 3.2.2, we can choose a triangulation $\mathfrak{T}$ of S_2. Note that the edges of some of these triangular region $\Gamma_j \in \mathfrak{T}$ will be segments of the curve γ_i (see the picture below, in which the arrows indicate the sense of the positive orientation). Moreover, when these triangular regions are positively oriented, we obverse that

1. the induced orientation of the γ_i is opposite to their positive orientation.

2. any common edge of the two triangular regions Γ_j appears twice with opposite orientations.



From Corollary 3.2.7, we know

$$\iint_S KdA = \sum_{i=1}^{n} \iint_{\text{int}(\gamma_i)} KdA + \iint_{S_2} KdA = \chi(S). \tag{4.1.1}$$

On S_2, because of $\mathbf{V} \neq \mathbf{0}$, we define

$$\mathbf{u}_1 = \frac{\mathbf{V}}{\|\mathbf{V}\|}, \quad \mathbf{u}_1 \perp \mathbf{u}_2, \quad \text{and} \quad \|\mathbf{u}_2\| = 1,$$

then $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis of the tangent plane of S at each point. Arguing as in the proof of Theorem 3.1.6, we get

$$\iint_{S_2} KdA = \sum_j \int_0^{\ell(\Gamma_j)} \mathbf{u}_1 \cdot \mathbf{u}_2' \, ds = \sum_{i=1}^{n} - \int_0^{\ell(\gamma_i)} \mathbf{u}_1 \cdot \mathbf{u}_2' \, ds \tag{4.1.2}$$

On S_1, for the sake of $\mathbf{V} = \mathbf{0}$ at p_i, we choose an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ of the tangent plane of S on each patch σ_i. By the proof of Theorem 3.1.6, we get

$$\iint_{S_1} KdA = \sum_{i=1}^{n} \int_0^{\ell(\gamma_i)} \mathbf{w}_1 \cdot \mathbf{w}_2' \, ds \tag{4.1.3}$$

Combining the equations (4.1.1), (4.1.2), and (4.1.3), we see that

$$\sum_{i=1}^{n}\int_{0}^{\ell(\gamma_i)}(\mathbf{w}_1\cdot\mathbf{w}_2'-\mathbf{u}_1\cdot\mathbf{u}_2')ds=2\pi\chi(S). \tag{4.1.4}$$

However, from the proof of Theorem 3.1.6,

$$\mathbf{w}_1\cdot\mathbf{w}_2'=\varphi'-k_g \quad \text{and} \quad \mathbf{u}_1\cdot\mathbf{u}_2'=\theta'-k_g,$$

where k_g is the geodesic curvature of γ_i, φ is the angles between $\mathbf{w}_1$ and γ' at the same point, and θ is the angles between $\mathbf{u}_1$ and γ' at the same point. Then $\psi\equiv\varphi-\theta$ is the angle between $\mathbf{w}_1$ and $\mathbf{u}_1$. That is, the angle is between the "reference" tangent vector field ξ on σ_i and $\mathbf{V}$. Hence the equation (4.1.4) can be written as

$$\sum_{i=1}^{n}\int_{0}^{\ell(\gamma_i)}\frac{d\psi}{ds}ds=2\pi\chi(S),$$

as we want. □

This is a remarkable result. It implies that $\sum_{i=1}^{n}\mu(p_i)$ does not depend on $\mathbf{V}$ but only on the topology of S. For instance, in any surface homeomorphic to a sphere S^2, all tangent vector fields with isolated singular points must have the sum of their indices equal to 2.

4.2 The Hairy Ball Theorem

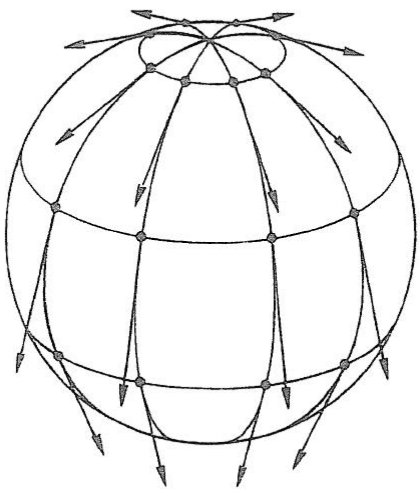
An interesting application of the Poincaré-Hopf index theorem which finds application in meteorology is the hairy ball theorem.

Theorem 4.2.1. *[10]* (The Hairy Ball Theorem). *There does not exist a non-vanishing continuous tangent vector filed* $\boldsymbol{V}$ *on a sphere* S^2. *Or, you can not comb a hairy ball straight.*

Proof. Goal: There is at least one isolated singular point p such that $\mathbf{V}(p) = \mathbf{0}$.
Suppose to the contrary! Then there exists a tangent vector field $\mathbf{F}$ on S^2 such that $\mathbf{F}$ has no isolated singular points, the sum of indices is zero. However, by the Poincaré-Hopf index theorem, we know the sum of the indices on S^2 equal to 2. This is a contradiction. Hence, we can not comb a hairy ball straight. □

We now give some examples of vector fields on surfaces.

Example 4.2.2. A tangent vector field on a sphere S^2 with 1 source and 1 sink: $\chi = 2$



A tangent vector field on the sphere S^2 is obtained by parametrizing the meridian of S^2 and defining $\mathbf{V}(p)$ as the velocity vector of the meridian through p. We know the parametrization of S^2 cam be denoted by

$$\sigma(\theta, \varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta),$$

where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \varphi \leq 2\pi$.

The meridians of S^2 is that φ is fixed, therefore

$$\sigma_\theta(\theta, \varphi) = (-\sin\theta\cos\varphi, -\sin\theta\sin\varphi, \cos\theta).$$

Let

$$\begin{aligned} x &= \cos\theta\cos\varphi \Rightarrow \cos\varphi = \frac{x}{\cos\theta}, \\ y &= \cos\theta\sin\varphi \Rightarrow \sin\varphi = \frac{y}{\cos\theta}, \\ z &= \sin\theta, \end{aligned}$$

where $(x, y, z) \in S^2$, and we have

$$\begin{aligned} \cos\theta &= \sqrt{1-z^2} \quad (\because -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}), \\ -\sin\theta\cos\varphi &= \frac{-xz}{\sqrt{1-z^2}}, \\ -\sin\theta\sin\varphi &= \frac{-yz}{\sqrt{1-z^2}}. \end{aligned}$$

So the velocity of the meridian is

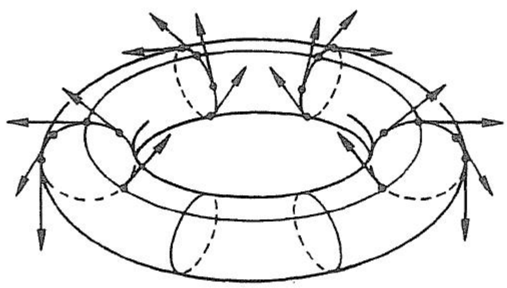
$$\mathbf{F}(x, y, z) = \left(\frac{-xz}{\sqrt{1-z^2}}, \frac{-yz}{\sqrt{1-z^2}}, \sqrt{1-z^2}\right), \quad \forall (x, y, z) \in S^2 \setminus (0, 0, \pm 1).$$

In order to obtain a tangent vector field defined in the whole sphere, we define

$$\mathbf{V}(x, y, z) = \begin{cases} \mathbf{0} & , (x, y, z) = (0, 0, 1) \\ \mathbf{F}(x, y, z) & , (x, y, z) \in S^2 \setminus (0, 0, \pm 1) \\ \mathbf{0} & , (x, y, z) = (0, 0, -1) \end{cases}$$

Indeed, there exists two isolated singular point with the tangent vector field on S^2.

Example 4.2.3. We construct a nowhere zero tangent vector filed $\mathbf{V}$ on a torus T: $\chi = 0$



A similar procedure with Example 4.2.2, the nowhere zero tangent vector field $\mathbf{V}$ can be constructed. The parametrization of T can be denoted by

$$\sigma(\theta, \varphi) = ((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta),$$

where $a > b$ and $0 \leq \theta, \varphi \leq 2\pi$.

The meridians of T is that φ is fixed, therefore

$$\sigma_\theta(\theta, \varphi) = (-b\sin\theta\cos\varphi, -b\sin\theta\sin\varphi, b\cos\theta).$$

Let

$$\begin{aligned}
x &= (a + b\cos\theta)\cos\varphi \Rightarrow \cos\varphi = \frac{x}{(a + b\cos\theta)}, \\
y &= (a + b\cos\theta)\sin\varphi \Rightarrow \sin\varphi = \frac{y}{(a + b\cos\theta)}, \\
z &= b\sin\theta \Rightarrow \sin\theta = \frac{z}{b},
\end{aligned}$$

where $(x, y, z) \in T$, and we have

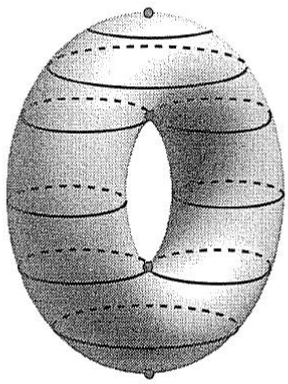
$$\begin{aligned} b\cos\theta &= \pm\sqrt{b^2 - z^2}, \\ -b\sin\theta\cos\varphi = -z\cos\varphi &= \frac{-xz}{a \pm \sqrt{b^2 - z^2}}, \\ -b\sin\theta\sin\varphi = -z\sin\varphi &= \frac{-yz}{a \pm \sqrt{b^2 - z^2}}. \end{aligned}$$

Since $a > b$, $a - \sqrt{b^2 - z^2} > 0$. Then we define the tangent vector field

$$\mathbf{V}(x, y, z) = \left(\frac{-xz}{a \pm \sqrt{b^2 - z^2}}, \frac{-yz}{a \pm \sqrt{b^2 - z^2}}, \pm\sqrt{b^2 - z^2} \right).$$

Indeed, there exists no isolated singular point with tangent vector field on T.

Example 4.2.4. Here is another tangent vector field on a torus T: $\chi = 0$



From Example 4.1.5, the level sets of the height function on this upright torus give a tangent vector field with two saddles and two centers. Indeed, the sum of the indices of a differentiable vector field $\mathbf{V}$ with isolated singular points on a torus T is equal to the Euler characteristic of T.

4.3 The Fundamental Theorem of Algebra

In this section, we need some definitions and theorems in Riemannian geometry and complex analysis to prove the fundamental theorem of algebra.

Definition 4.3.1. [13] A differentiable manifold of dimension n is a set M and a family of injective mappings $\mathbf{x}_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ of open sets U_α of $\mathbb{R}^n$ into M such that:

1. $\bigcup\limits_{\alpha} \mathbf{x}_\alpha(U_\alpha) = M$.
2. For any pair α and β with $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W \neq \emptyset$, the sets $\mathbf{x}_\alpha^{-1}(W)$ and $\mathbf{x}_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mappings $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ are differentiable.
3. The family $\{(U_\alpha, \mathbf{x}_\alpha)\}$ is maximal relative to the conditions 1 and 2.

The pair $(U_\alpha, \mathbf{x}_\alpha)$ (or the mapping $\mathbf{x}_\alpha$) with $p \in \mathbf{x}_\alpha(U_\alpha)$ is called a parametrization of M at p. A 2-manifold is often called a surface. We would like to be able to measure the lengths of and the angles between tangent vectors. In a vector space such a notion of measurement is usually given by a scalar product. We thus define

Definition 4.3.2. [13] A Riemannian metric (or Riemannian structure) on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $<\,,\,>_p$ on the tangent space T_pM, which varies differentiablely in the following sense : If $\mathbf{x} : U \subset \mathbb{R}^n \to M$ is a system of coordinates around p with $\mathbf{x}(x_1, x_2, \ldots, x_n) = q \in \mathbf{x}(U)$ and $\frac{\partial}{\partial x_i}(q) = d\mathbf{x}_q(0, \ldots, 1, \ldots, 0)$, then

$$< \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) >_q = g_{ij}(x_1, \ldots, x_n)$$

is a differentiable function on U.

We still need a little definitions and theorems in complex analysis, and will not prove he following theorems in this pater. The details of the proof can be found in the books [4] and [8].

Definition 4.3.3. [4] The complex function f is analytic at the point z_0, provided there exists $\varepsilon > 0$ such that $f'(z)$ exists for all $z \in B_\varepsilon(z_0)$, where $B_\varepsilon(z_0)$ is a neighborhood of

z_0. In other words, f must be differentiable not only at z_0, but also at all points in some ε neighborhood of z_0.

If f is analytic at each point in the region R, then we say that f is analytic on R. Is there some criterion that we can use to determine whether f is differentiable? Theorem 4.3.5 helps us obtain an important result.

Theorem 4.3.4. *[4]* (Cauchy-Riemann Equations). *Suppose that*

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

is differentiable at the point $z_0 = x_0 + iy_0$. Then the partial derivatives of u and v exist at the point $x_0 + iy_0 = (x_0, y_0)$, and

$$f'(z) = u_x(x_0, y_0) + iv_x(x_0, y_0) \tag{4.3.1}$$

$$f'(z) = v_y(x_0, y_0) - iu_y(x_0, y_0). \tag{4.3.2}$$

Equating the real and imaginary parts of the equations (4.3.1) and (4.3.2) gives

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \textit{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0). \tag{4.3.3}$$

The equations (4.3.3) are the Cauchy-Riemann equations.

However, the mere satisfaction of the Cauchy-Riemann equations is not sufficient to guarantee the differentiability of a function. The following theorem gives the conditions that guarantee the differentiability of f at z_0, so we can use the equation (4.3.1) and (4.3.2) to compute $f'(z_0)$.

Theorem 4.3.5. *[4]* (Cauchy-Riemann Conditions for Differentiability). *Let $f(z) = u(x,y) + iv(x,y)$ be a continuous function that is defined in some neighborhood of the point $z_0 = x_0 + iy_0$. If all the partial derivatives u_x, u_y, v_x, and v_y are continuous at the point (x_0, y_0) and if the Cauchy-Riemann equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$ hold at (x_0, y_0), then f is differentiable at z_0, and the derivative $f'(z_0)$ can be computed with either the equation (4.3.1) or (4.3.2).*

Using $z = re^{i\theta}$ in the expression of a complex function f may be convenient. It gives us the polar representation

$$f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta).$$

We can now restate the Theorem 4.3.5 using polar coordinates.

Theorem 4.3.6. *[4] Let $f(z) = u(r, \theta) + iv(r, \theta)$ be a continuous function that is defined in some neighborhood of the point $z_0 = r_0 e^{i\theta_0}$. If all the partial derivatives u_r, u_θ, v_r, and v_θ are continuous at the point (r_0, θ_0) and if the polar form of the Cauchy-Riemann equations,*

$$r_0 u_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \quad \text{and} \quad u_\theta(r_0, \theta_0) = -r_0 v_r(r_0, \theta_0). \tag{4.3.4}$$

holds, then f is differentiable at z_0.

We now introduce harmonic functions. Let $\phi(x, y)$ be a real-valued function of the two real variables x and y defined on a connected open region D. The partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

is called as Laplace's equation. If ϕ, ϕ_x, ϕ_y, ϕ_{xx}, ϕ_{xy}, ϕ_{yx}, and ϕ_{yy} are all continuous, and if $\phi(x, y)$ satisfy Laplace's equation, then $\phi(x, y)$ is harmonic on D. We begin with an important theorem relating analytic and harmonic functions.

Theorem 4.3.7. *[4] Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function on a connect open region D. Then both u and v are harmonic functions on D. In other words, the real and imaginary parts of an analytic function are harmonic.*

Now, we return to our topics: the fundamental theorem of algebra. Here we have stronger condition about it.

Theorem 4.3.8. *[5]* (The Fundamental Theorem of Algebra). *Every nonconstant polynomial $P(z) \in \mathbb{C}[z]$ has at least one complex root.*

Proof. As it is well known, the stereographic projection is a mapping $\pi : S^2 \to \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let us take

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

with a_0, $a_n \neq 0$ and suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$. In addition, we set

$$p_*(z) = a_n + a_{n-1} z + \cdots + a_0 z^n,$$

and we observe that

1. $p_*(z) = z^n p\left(\frac{1}{z}\right)$.
2. $p_*\left(\frac{1}{z}\right) = \frac{1}{z^n}\, p(z)$.
3. $z_0 \neq 0$ is a root of $p(z) \Leftrightarrow \dfrac{1}{z_0}$ is a root of $p_*(z)$.

Put

$$f(z) = p(z) p_*(z),$$

for all $z \in \mathbb{C}$. Then f satisfies the functional equation

$$\left| f\left(\frac{1}{z}\right) \right| = \frac{1}{|z|^{2n}}\, |f(z)|,$$

for all $z \in \mathbb{C} \setminus \{0\}$. It follows that there exists a Riemannian metric g on $\widehat{\mathbb{C}}$, such that

$$g = \frac{1}{|f(z)|^{\frac{2}{n}}}\, |dz|^2 \quad \text{for } z \in \mathbb{C}$$

and

$$g = \frac{1}{|f(1/z)|^{\frac{2}{n}}}\, |d(1/z)|^2 \quad \text{for } z \in \widehat{\mathbb{C}} \setminus \{0\}.$$

And we have Gaussian curvature (see the book [16] pp. 4)

$$K_g(z) = -\frac{1}{2} |f(z)|^{\frac{2}{n}} \triangle \log\left(\frac{1}{|f(z)|^{\frac{2}{n}}}\right),$$

where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denotes the standard Laplace operator.

Consider if $z = x + iy$, then we have

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}.$$

Use the chain rule, we get

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Moreover, use the chain rule again

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Consider

$$\begin{aligned}\frac{1}{|f(z)|^{\frac{2}{n}}} K_g(z) &= \frac{1}{n}\triangle(\log|f(z)|) \\ &= 4\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}} \log|f(z)|\right) \\ &= 4\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}} \mathbf{Re}\log(f(z))\right) \\ &= \frac{1}{n}\triangle\mathbf{Re}\log(f(z)).\end{aligned}$$

By Theorem 4.3.6, we know $\log(f(z))$ is analytic. And then from Theorem 4.3.7, we get

$$\frac{1}{|f(z)|^{\frac{2}{n}}} K_g(z) = 0.$$

Since $f(z) \neq 0$ for all $z \in \mathbb{C}$, then it follows that $K_g = 0$ over all sphere S^2. However, Gauss-Bonnet theorem, when applied to any Riemannian metric g on the sphere S^2, claims that

$$\iint_{S^2} K_g dA_g = 4\pi.$$

It is a contradiction, and hence the proof is complete. □

Chapter 5

Conclusion

The Gauss-Bonnet theorem is probably the most beautiful and deepest theorem in the differential geometry of surfaces. It connects the intrinsic differential geometry of a surface with its topology.

Higher dimensional versions of the Gauss-Bonnet theorem were given, for submanifolds of higher dimensional Euclidean space, by C. B. Allendoerfer (1940) and W. Fenchel (1940). The first intrinsic proof of the higher dimensional Gauss-Bonnet theorem was given by S.S. Chern (1944).

In section 3.3, we discuss the discrete Gauss-Bonnet theorem. In addition, the paper [6], John M. Sullivan provide more topics about discrete differentiable geometry. In section 4.2, we discuss the hairy ball theorem by the application of Gauss-Bonnet theorem. In the paper [7], John Milnor offer an analytic proof of the hairy ball theorem.

Gauss-Bonnet theorem has many applications in other fields. Moreover, it presents the beauty of mathematics.

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