東海大學數學系研究所

碩　士　論　文

指導教授:　葉芳栢　博士
　　　　　黃皇男　博士

Numerical Pricing of European Call Options in Black-Scholes and Jump-Diffusion Models

BSM 及 JDM 模型下之數值評價

研究生: 林大鈞

中　華　民　國　一　百　年　七　月

誌 謝

時間匆匆一轉眼就過了三年，我的研究所生涯結束了，這個階段的旅程讓我歷練與成長，回想當初剛進入研究所，一切都還懵懂無知，想說要在這個階段好好努力一窺數學的奧妙，所以跟培瑛經過一番討論後，最後決定跟隨葉芳栢老師研究財金方面的知識，葉老師同時也另外幫我們安排黃皇男老師指導我們關於程式的問題。

這段時間裡幫助過我的人太多了，首先，感謝葉芳栢老師，所謂萬事起頭難，對於剛接觸財金方面的我，剛起頭根本是無所適從，所幸葉老師開的幾門課程使我迅速了解一些財數的基本知識，使我在往後閱讀論文時有基本的概念，才不會感覺像大海撈針無法起步。再來，葉老師在我們 meeting 時，嚴格的要求與犀利的問法，使我了解對於任何細節都要仔細求證。

另外，感謝黃皇男老師，每當我有論文中程式的問題，黃老師都能準確的指出我的錯誤，或著是適時的提供解決的方向，而且在討論的過程中，有時一時沒辦法了解所講解的觀念，黃老師都非常詳盡的一再解釋，直到使我清楚了解，也因爲這樣，才使我在寫程式能力有系統且不易出錯，在老師的磨練之下使我在寫程式方面的訓練非常的扎實。

再來，感謝胡馨云老師，因爲胡老師所開的 matlab 課程，使我對科學計算有了興趣，而且胡老師也很熱心的指導我寫程式的能力，經由這個機會讓我學習到許多的數值計算方法，因此，我論文裡的模擬方法都是跟胡老師所學的，另外，也感謝老師平常的幫忙與照顧，每當我遇到任何疑難雜症找胡老師商量，都會盡心的爲我解說，感覺就像媽媽一樣的關懷與照顧。

最後，感謝培瑛、金龍、口試委員們、學長們和兼任助教們，一起奮鬥打拼的日子感覺眞好，研究所的階段使我成長，增廣見聞，東海數學系給我的感覺就像家一樣，讓我無後顧之憂，有你們眞好。

林大鈞 謹致於東海大學數學系碩士班

中華民國一百年七月

.

東　海　大　學

數　學　系

碩士學位口試委員審定書

本系碩士班 林大鈞 君

所提論文 Numerical Pricing of European Call Options in Black-Scholes and Jump-Diffusion Models
(BSM 及 JDM 模型下之數值評價)

合於碩士班資格水準，業經本委員會評審通過，特此證明。

口試委員：葉芳栢

朱香蕙

胡馨云

曾旭亮

指導教授：葉芳栢 黃皇男

所　　長：陳文豪

中　華　民　國　一〇〇　年　七　月　十四　日

摘 要

由於選擇權契約的多樣化和複雜化加深了評價上的困難，所以大部份都以來討論，但這個模型有個缺點，變動中的選擇權或股票都會突然有個劇烈的漲跌，這時 Black-Scholes 模型就不能描述這個現象，所以本文又更近一步討論跳躍擴散模型的選擇權，因爲它的偏微分方程式中多加入了一項 Poisson 積分項，這使方程式具有描述跳躍的特性，更能貼近選擇權的特性。

在選擇權評價方面，大多數的人都用蒙地卡羅法或是有限差分法來做評價，國內較少使用半徑基底函數 (Radial Basic Function, RBF) 內插來進行評價，另外，在 RBF 法上基底的又有 TPS 、MQ 、Cubic 、Gaussian 等種類可以選擇，本文使用有限差分法和上述四種基底的 RBF 法來對 BSM 進行評價，發現 MQ 、Cubic 較爲準確；之後再對跳躍擴散模型作評價，文中並討論有限差分法與 RBF 兩種方法之計算優缺點。

關鍵字: 有限差分法、無網格法、半徑基底函數、Black-Scholes 模型、跳躍擴散模型

Abstract

Due to the variaties and complexity of Options in the current market, pricing the Option comes up to be a very difficult task. The common tactic is to discuss the pricing problem under the framework of the phamous Black-Scholes Model (BSM). The drawback of this model is quite evident that it does not count the possiblility on the sudden changes in the option or stock. In this study, we also consider he Jump-Diffusion Model (JDM) right after the discussion of the BSM model. The PDE representation of JDM is the same as BSM but with an additional Poisson integral term to model the jump feature, which make this model more close to the features in pricing the options.

Unlike the popular studies using the Monte-Carlo simulation method or applying the finite difference method to solve the PDE in Black-Scholes Model, we use finite difference method and meshfree method with four different types of radial basis functions (RBF), i.e., TPS, MQ, Cubic, Gaussian, to compute the numerical solution of the associated PDE for BSM and PIDE for JDM, resepctively. Based on the numeircal experiment, the MQ and Cubic RBFs are found to provide more accurate result. We also compare the computational efficiency between using the finite difference method and meshfree method.

Keywords: Finite difference method, meshfree method, radial basis function, Black-Scholes Model, Jump-Diffusion Model

Contents

List of Figures

List of symbols

Symbol	Meaning
$a(t)$	- the coefficient of the RBF approximation
c	- the parameter of the radial basis function
f_c	- European call option price
f_i^j	- the numerical solution of European call option price
$\mathscr{F}_t$	- the filtration
I	- the integral part
K	- the strike price
$N(x)$	- normal distribution
p, q	- risk free probability
r	- risk-free interest rate
S_K	- the centers
$W(t)$	- Brownian motion
σ	- the volatility
θ_i	- parameters
λ_i	- the intensity of the i-th Poisson process
ξ_i	- the data points
ϕ	- radial basis function
$(\Omega, \mathscr{F}, P)$	- probability space

CHAPTER 1

Introduction

Black and Scholes found a close form for evaluating European call options in 1973[**2**]. They assumed the asset price is risk-neutral and showed that the European call options value satisfies a lognormal diffusion partial differential equation which is now known as celebrated *Black-Scholes equation*.

The meshfree method based on the radial basis function (RBF) is widely used in many fields within this decade. It is important to choose the basis functions which are multiquadric (MQ), Gaussian, thin-plate spline (TPS), and cubic. It can approximate well not only high dimensional scattered datas but derivative values. Franke showed that the MQ function is better on Accuracy, stability and efficiency in 1982[**9**]. In 1997 Hon and Mao solved the initial problem with MQ function and showed that the RBF unlike the finite difference method to construct the grids[**10**]. |In 1998 they compared the numerical methods for the finite element method, the finite difference method and RBF-MQ on burger's equations[**11**]. And they showed that the numerical result of RBF-MQ is better than others. Generally the numerical methods of pricing option are binomial tree model, the finite difference method and Monte Carlo method. Recnet the RBF get attention on pricing options. In 1999 Hon and Mao used the RBF-MQ to approximate the numerical value of European and American options on Black-Scholes model and had good approximations[**12**]. Moreover they compared a lot of methods of radial basis functions with MQ, Gaussian, TPS and cubic functions in 2000[**13**]. And they chose TPS function in order to selecting parameter.

In this thesis we show how to compute European call option prices in the Black-Scholes model (BSM) and the Jump-Diffusion model (JDM) using Finite Difference Method (FDM) and Radial Basis Function (RBF) interpolation techniques. Although there are many studies on solving the BSM model since 1980 but the JDM related equations such as the Merton

Model[**18**]) and the Kou Model[**14, 15**] have mostly been solved by FDM. In FDM, the idea is to simply fully discretize the equations on an equidistant grid and use around the point to be evaluated. In RBF, the idea is to random discretize the equations on a domain and use the coefficients which are generated by the data points of radial basis function to be approach. The evident drawback of using FDM method to solve JDM is coming up in evaluating the integral term, which needs to use the call option prices for the asset that are located outside the descritization domain of the asset. When using RBF methods, this drawback is automatically solved.

The organization of this thesis is described below. In the next chapter, we introduce the European call option exact solution of Black-Scholes model and using FDM and RBF method to price. In Chapter 3, we show that a brief description of the PIDE of this Jump-diffusion equation. In Chapter 4 we show our computational results of JDM by FDM and RBF method. Finally, we give our conclusions.

CHAPTER 2

The Numerical Solution of the BSM Equations

2.1. Derivation of the Exact Solution for Black-Scholes Equations

Suppose $f_c(t, S_t)$ be the European option price, it staisfies the Black-Scholes partial differential equation in one space dimension with terminal and boundary conditions:

$$\begin{aligned} &\frac{\partial f_c}{\partial t} + rS\frac{\partial f_c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_c}{\partial S^2} = rf_c, \quad f_c = f_c(S,t), \\ &f_c(0,t) = 0 \\ &\lim_{S\to\infty} \frac{\partial^2 f_c}{\partial S^2} = 0 \\ &f_c(S,T) = \max\{S(T) - K, 0\} \end{aligned} \tag{2.1.1}$$

where r is the risk-free interest rate, σ is the volatility, and over the rectangle $0 \leq t \leq T$, $S_L \leq S \leq S_U$, with various boundary conditions on the top, bottom, and right sides of the rectangle. The parameters r, $\sigma > 0$ are arbitrary constants. The interested rectangle is shown in Figure 2.1.

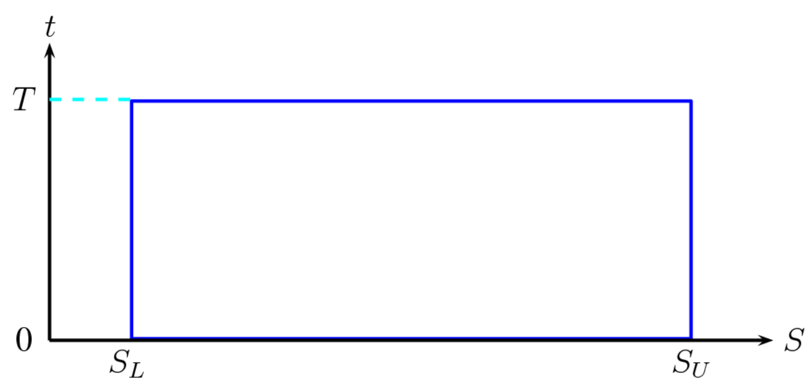


FIGURE 2.1. Computation at domain in (S, t) plane

We assume that X_t is affected by time of the derivatives. And the strike price K at terminal time T is derived as following. The payoff of the derivative X at time t under the probability P can be described by

$$X_t = e^{-r(T-t)} E_p(X_T | \mathscr{F}_t).$$

From the relation

$$\frac{dS_t}{S_t} = rdt + \sigma d\tilde{w}_t \tag{2.1.2}$$

and by Itô's formula

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma\tilde{w}_t}$$

or equivalently, we arrive at

$$\begin{aligned} S_T &= S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\tilde{w}_{T-t}} \\ &= S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\mathcal{N}(0,1)} \end{aligned} \tag{2.1.3}$$

Hence

$$\begin{aligned} f_c(S_t,t) &= e^{-r(T-t)} E_p(f_c(S_T,T) | \mathscr{F}_t) \\ &= e^{-r(T-t)} E_p\Big(f_c(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}N(0,1)}, T)\Big| \mathscr{F}_t) \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} f_c(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}x}, T) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx. \end{aligned}$$

At terminal time the payoff is given by

$$X_T = f_c(S_T,T) = \max\{S_T - K, 0\} = (S_T - K)^+ \tag{2.1.4}$$

and hence

$$f_c(S_t,t) = e^{-r(T-t)} \int_{-\infty}^{\infty} (S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}x} - K)^+ \varphi(x) dx$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. Firstly, compute the payoff when $S_T = K$:

$$S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}x} = K$$

that is

$$(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}x = \ln\frac{K}{S_t}$$

then

$$\begin{aligned} x &= \frac{\ln\frac{K}{S_t} - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= -\frac{\ln\frac{S_t}{K} - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &\triangleq -d_2. \end{aligned} \tag{2.1.5}$$

Thus

$$f_c(S_t,t) = e^{-r(T-t)}\left(\int_{-\infty}^{-d_2}(S_T-K)^+\varphi(x)dx + \int_{-d_2}^{\infty}(S_T-K)^+\varphi(x)dx\right).$$

Secondly, by definition

$$(S_T-K)^+ = \begin{cases} S_T-K, & S_T \geq K, \\ 0, & S_T < K, \end{cases}$$

we know that when $x \leq -d_2$, $(S_T-K)^+ = 0$. Therefore

$$\begin{aligned} f_c(S_t,t) &= e^{-r(T-t)}\int_{-d_2}^{\infty}(S_T-K)\varphi(x)dx \\ &= e^{-r(T-t)}\int_{-d_2}^{\infty}(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}x} - K)\varphi(x)dx \\ &= S_t\int_{-d_2}^{\infty}e^{-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}x}\varphi(x)dx - Ke^{-r(T-t)}\int_{-d_2}^{\infty}\varphi(x)dx. \end{aligned}$$

We can rewrite the integral term as

$$\int_{-d_2}^{\infty}e^{-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx = \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty}e^{-\frac{1}{2}(x-\sigma\sqrt{T-t})^2}dx$$

and by substituting the variable $u = x - \sigma\sqrt{T-t}$ with $du = dx$, then the above integral becomes

$$\int_{-d_2}^{\infty}e^{-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx = \frac{1}{\sqrt{2\pi}}\int_{-d_1}^{\infty}e^{-\frac{1}{2}u^2}du$$

where $d_1 = d_2 + \sigma\sqrt{T-t}$. Recapsulate these procedures give us

$$f_c(S_t,t) = S_t \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du - Ke^{-r(T-t)} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx. \tag{2.1.6}$$

Denote

$$N(d) \triangleq \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \int_{-d}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \tag{2.1.7}$$

whose meaning is shown Figure 2.2. Therefore the payoff of the call European option is given by

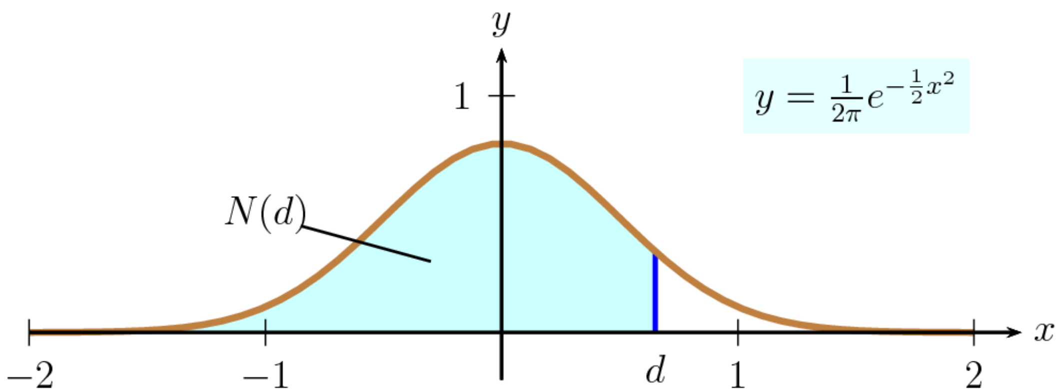


FIGURE 2.2. Normal distribution $N(d)$

$$f_C(t,S_t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2) \tag{2.1.8}$$

and the corresponding payoff for the put European option is then becomes

$$f_P(t,S_t) = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1) \tag{2.1.9}$$

Consider the following parameters:

$$r = 0.10, \quad \sigma = 0.40, \quad T = 0.5, \quad K = 50.00,$$

with $S_L = 0$ and $S_U = 100.00$. Let M, N be the numbers of partitions in price and time over the rectangle $0 \leq t \leq T$, $S_L \leq S \leq S_U$. The exact solution is plotted in Figure 2.3.

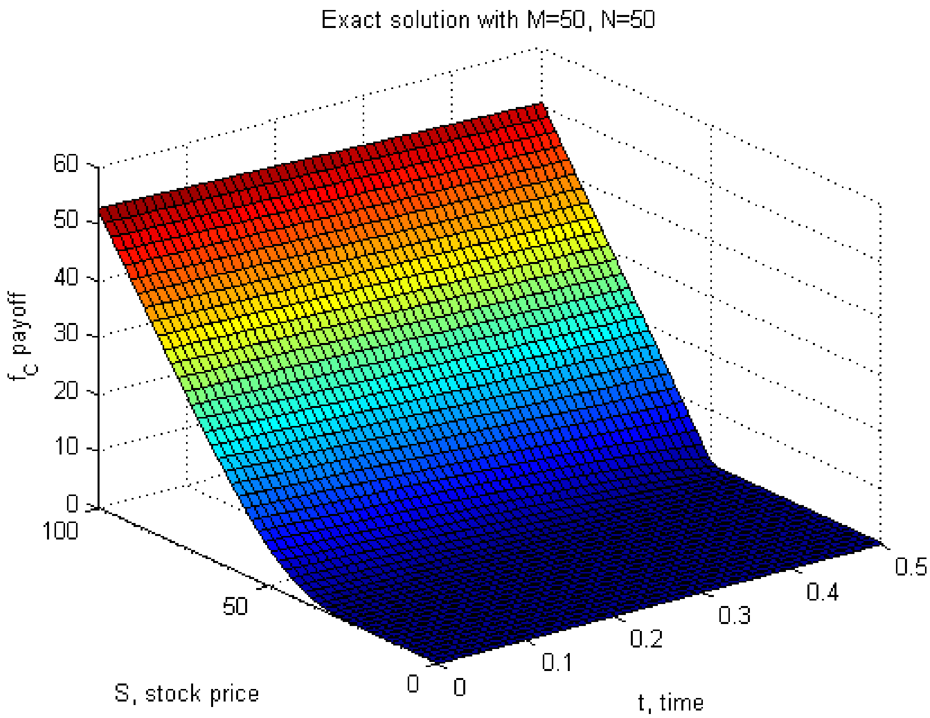


FIGURE 2.3. The exact solution of BSM

2.2. Finite Difference Method

This section presents the finite difference idea of solving the Black-Scholes partial differential equation in one space dimension with the boundary and terminal conditions given in the Equation (2.1.1):

$$\frac{\partial f_c}{\partial t} + rS\frac{\partial f_c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_c}{\partial S^2} = rf_c, \quad f_c = f_c(t, S),$$

Let $t\epsilon[0, T]$, $S_t\epsilon[S_L, S_U]$, we summarize the equations for the finite differences as following:

$$\begin{aligned} \Delta S &= \frac{S_U - S_L}{M}, \quad S_i = S_L + i\Delta S, \quad i = 0, 1, \cdots, M \\ \Delta t &= \frac{T - 0}{N}, \qquad t_j = j\Delta t, \qquad\qquad j = 0, 1, \cdots, N \end{aligned}$$

Let f_i^j denote the approximation to $f_c(S_i, t_j)$, $i = 1, 2, \cdots, n-1$. At the terminal condition T, i.e. $j = N$, f_i^N is described by

$$f_i^N = \max\{S(T) - K, 0\} \tag{2.2.1}$$

Applying the boundary condition at $S = S_L$ $(i = 0)$ and $S = S_U$ $(i = M)$ to f_i^j leads to

$$\begin{aligned} f_c(S, t) &= 0 \\ \Rightarrow f_0^j &= 0 \end{aligned} \tag{2.2.2}$$

$$\begin{aligned} \lim_{S\to\infty} \frac{\partial^2 f_c}{\partial S^2} &= 0 \\ \Rightarrow \frac{f_{M-2}^j - 2f_{M-1}^j + f_M^j}{\Delta S^2} &= 0 \\ \Rightarrow f_M^j &= 2f_{M-1}^j - f_{M-2}^j \end{aligned} \tag{2.2.3}$$

for $j = 0, 1, \cdots, N-1$

We can rewrite the PDE to be

$$
\begin{aligned}
&-rf_c + \tfrac{\partial f_c}{\partial t} + rS\tfrac{\partial f_c}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \tfrac{\partial^2 f_c}{\partial S^2} = 0 \\
&\Rightarrow -r[\theta f_i^j + (1-\theta) f_i^{j+1}] + \frac{f_i^{j+1} - f_i^j}{\Delta t} \\
&\quad + r(i\Delta S)[\theta \frac{f_{i+1}^j - f_{i-1}^j}{2\Delta S} + (1-\theta)\frac{f_{i+1}^{j+1} - f_{i-1}^{j+1}}{2\Delta S}] \\
&\quad + \frac{1}{2}\sigma^2 i^2 [\theta \frac{f_{i-1}^j - 2f_i^j + f_{i+1}^j}{\Delta S^2} + (1-\theta)\frac{f_{i-1}^{j+1} - 2f_i^{j+1} + f_{i+1}^{j+1}}{\Delta S^2}] = 0
\end{aligned}
$$

and so arrive at

$$
\begin{aligned}
&[\theta\triangle t(\tfrac{1}{2}ri - \tfrac{1}{2}\sigma^2 i^2) f_{i-1}^j + (1 + \theta\triangle t(\sigma^2 i^2 + r)) f_i^j + \theta\triangle t(-\tfrac{1}{2}ri - \tfrac{1}{2}\sigma^2 i^2) f_{i+1}^j] \\
&= [(1-\theta)\triangle t(-\tfrac{1}{2}ri + \tfrac{1}{2}\sigma^2 i^2) f_{i-1}^{j+1} + (1 + (1-\theta)\triangle t(-\sigma^2 i^2 - r)) f_i^{j+1} + (1-\theta)\triangle t(\tfrac{1}{2}ri + \tfrac{1}{2}\sigma^2 i^2) f_{i+1}^{j+1}]
\end{aligned}
\tag{2.2.4}
$$

where $i = 1, 2, 3, \cdots, M-1$ and $j = 0, 1, \cdots, N$,

Let

$$
\begin{aligned}
&H_{1,i}^+ = (\tfrac{1}{2}ri - \tfrac{1}{2}\sigma^2 i^2), \\
&H_{2,i}^+ = (\sigma^2 i^2 + r) \\
&H_{3,i}^+ = (-\tfrac{1}{2}ri - \tfrac{1}{2}\sigma^2 i^2) \\
&H_{1,i}^- = -H_{1,i}^+, H_{2,i}^- = -H_{2,i}^+, H_{3,i}^- = -H_{3,i}^+,
\end{aligned}
\tag{2.2.5}
$$

The equation 2.2.4 can be expressed as

$$
\begin{aligned}
&[\theta\triangle t H_{1,i}^+ f_{i-1}^j + (1 + \theta\triangle t H_{2,i}^+) f_i^j + \theta\triangle t H_{3,i}^+ f_{i+1}^j] \\
&= [(1-\theta)\triangle t H_{1,i}^- f_{i-1}^{j+1} + (1 + (1-\theta)\triangle t H_{2,i}^-) f_i^{j+1} + (1-\theta)\triangle t H_{3,i}^- f_{i+1}^{j+1}].
\end{aligned}
\tag{2.2.6}
$$

But when $i = 1$

$$
\begin{aligned}
&[\theta\triangle t(\frac{1}{2}r - \frac{1}{2}\sigma^2) f_0^j + (1 + \theta\triangle t(\sigma^2 + r)) f_1^j + \theta\triangle t(-\frac{1}{2}r - \frac{1}{2}\sigma^2) f_2^j] \\
&= [(1-\theta)\triangle t(-\tfrac{1}{2}r + \tfrac{1}{2}\sigma^2) f_0^{j+1} + (1 + (1-\theta)\triangle t(-\sigma^2 - r) f_1^{j+1} + (1-\theta)\triangle t(\tfrac{1}{2}r + \tfrac{1}{2}\sigma^2) f_2^{j+1}]
\end{aligned}
$$

i.e.,

$$\begin{aligned}
&[\theta\triangle tH^{+}_{1,1}f^{j}_{0}+(1+\theta\triangle tH^{+}_{2,1})f^{j}_{1}+\theta\triangle tH^{+}_{3,1}f^{j}_{2}]\\
&=[(1-\theta)\triangle tH^{-}_{1,1}f^{j+1}_{0}+(1+(1-\theta)\triangle tH^{-}_{2,1}f^{j+1}_{1}+(1-\theta)\triangle tH^{-}_{3,1}f^{j+1}_{2}]
\end{aligned}\tag{2.2.7}$$

and for $i=M-1$

$$\begin{aligned}
&[\theta\triangle t(\frac{1}{2}r(M-1)-\frac{1}{2}\sigma^{2}(M-1)^{2})f^{j}_{M-2}+(1+\theta\triangle t(\sigma^{2}(M-1)^{2}+r))f^{j}_{M-1}+\theta\triangle t(-\frac{1}{2}r(M-1)-\frac{1}{2}\sigma^{2}(M-1)^{2})f^{j}_{M}]\\
&=[(1-\theta)\triangle t(-\tfrac{1}{2}r(M-1)+\tfrac{1}{2}\sigma^{2}(M-1)^{2})f^{j+1}_{M-2}+(1+(1-\theta)\triangle t(-\sigma^{2}(M-1)^{2}-r))f^{j+1}_{M-1}+(1-\theta)\triangle t(\tfrac{1}{2}r(M-1)+\tfrac{1}{2}\sigma^{2}(M-1)^{2})f^{j+1}_{M}]
\end{aligned}$$

so it becomes

$$\begin{aligned}
&[\theta\triangle tH^{+}_{1,M-1}f^{j}_{M-2}+(1+\theta\triangle tH^{+}_{2,M-1})f^{j}_{M-1}+\theta\triangle tH^{+}_{3,M-1}f^{j}_{M}]\\
&=[(1-\theta)\triangle tH^{-}_{1,M-1}f^{j+1}_{M-2}+(1+(1-\theta)\triangle tH^{-}_{2,M-1})f^{j+1}_{M-1}+(1-\theta)\triangle tH^{-}_{3,M-1}f^{j+1}_{M}]
\end{aligned}\tag{2.2.8}$$

Putting into the matrix form

$$\begin{aligned}
&A_{\theta}\mathbf{f}^{(j)}=A_{1-\theta}\mathbf{f}^{(j+1)}+\mathbf{b}\\
&\Rightarrow\mathbf{f}^{(j)}=A_{\theta}^{-1}(A_{1-\theta}\mathbf{f}^{(j+1)}+\mathbf{b})
\end{aligned}\tag{2.2.9}$$

where

$$A_{\theta}=\begin{bmatrix}
\theta\triangle tH^{+}_{1,1} & (1+\theta\triangle tH^{+}_{2,1}) & \theta\triangle tH^{+}_{3,1} & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & \theta\triangle tH^{+}_{1,2} & (1+\theta\triangle tH^{+}_{2,2}) & \theta\triangle tH^{+}_{3,2} & 0 & 0 & & 0 & 0\\
0 & 0 & \ddots & \ddots & \ddots & 0 & & \vdots & \vdots\\
\vdots & \vdots & & \theta\triangle tH^{+}_{1,i} & (1+\theta\triangle tH^{+}_{2,i}) & \theta\triangle tH^{+}_{3,i} & & 0 & 0\\
0 & 0 & & \ddots & \ddots & \ddots & & & 0\\
0 & 0 & \cdots & & & & \theta\triangle tH^{+}_{1,M-1} & (1+\theta\triangle tH^{+}_{2,M-1}) & \theta\triangle tH^{+}_{3,M-1}
\end{bmatrix}\tag{2.2.10}$$

$$A_{1-\theta} = \begin{bmatrix} (1-\theta)\triangle tH^{-}_{1,1} & (1+(1-\theta)\triangle tH^{-}_{21}) & (1-\theta)\triangle tH^{-}_{3,1} & 0 & 0 \\ 0 & (1-\theta)\triangle tH^{-}_{1,2} & (1+(1-\theta)\triangle tH^{-}_{2,2}) & (1-\theta)\triangle tH^{-}_{3,2} & 0 \\ 0 & & \ddots & \ddots & \vdots \\ \vdots & (1-\theta)\triangle tH^{-}_{1,i} & (1+(1-\theta)\triangle tH^{-}_{2,i}) & (1-\theta)\triangle tH^{-}_{3,i} & \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & (1-\theta)\triangle tH1^{-}_{M-1} & (1+(1-\theta)\triangle tH2^{-}_{M-1}) & (1-\theta)\triangle tH3^{-}_{M-1} \end{bmatrix} \tag{2.2.11}$$

$$\mathbf{b} = \begin{bmatrix} (\frac{1}{2}r - \frac{1}{2}\sigma^2)(\theta f_0^j + (1-\theta)f_0^{j+1}) \\ \vdots \\ \vdots \\ (\frac{1}{2}r(M-1) - \frac{1}{2}\sigma^2(M-1)^2)(\theta f_0^j + (1-\theta)f_0^{j+1}) \end{bmatrix} \tag{2.2.12}$$

Consider the same parameters for the exact one in previous section,

$$r = 0.10, \quad \sigma = 0.40, \quad T = 0.5, \quad K = 50,$$

and select $S_L = 0$ and $S_U = 100$. When $M = 50$ i.e., $\Delta S = 2$, then the computation ae result of the European call option price for various time steps is described below

For explicit method i.e., $\theta = 0$, the selction of N must be greater or equal to 164 in order to satisfy the CFL condition for obtain stable solution. The corresponding solutions for various N are plotted in Figure 2.4. For the implicit method $\theta = 1$ and Crank-Nicolson method i.e., $\theta = 0.5$ with $N = 160$ the computational results are given in Figure 2.5 with comparison to the exact solution and explicit method for $N = 160$.

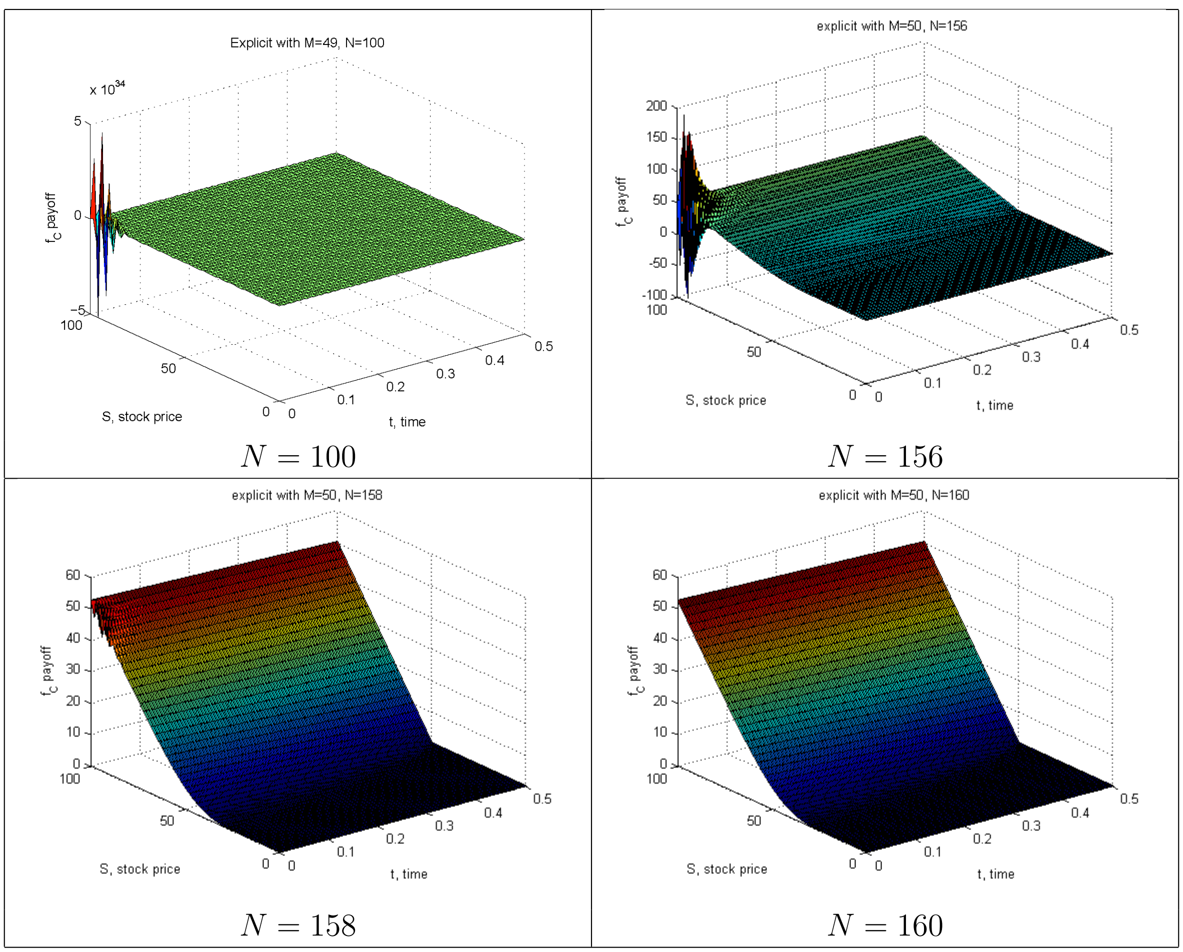


FIGURE 2.4. Computation result for the explicit method

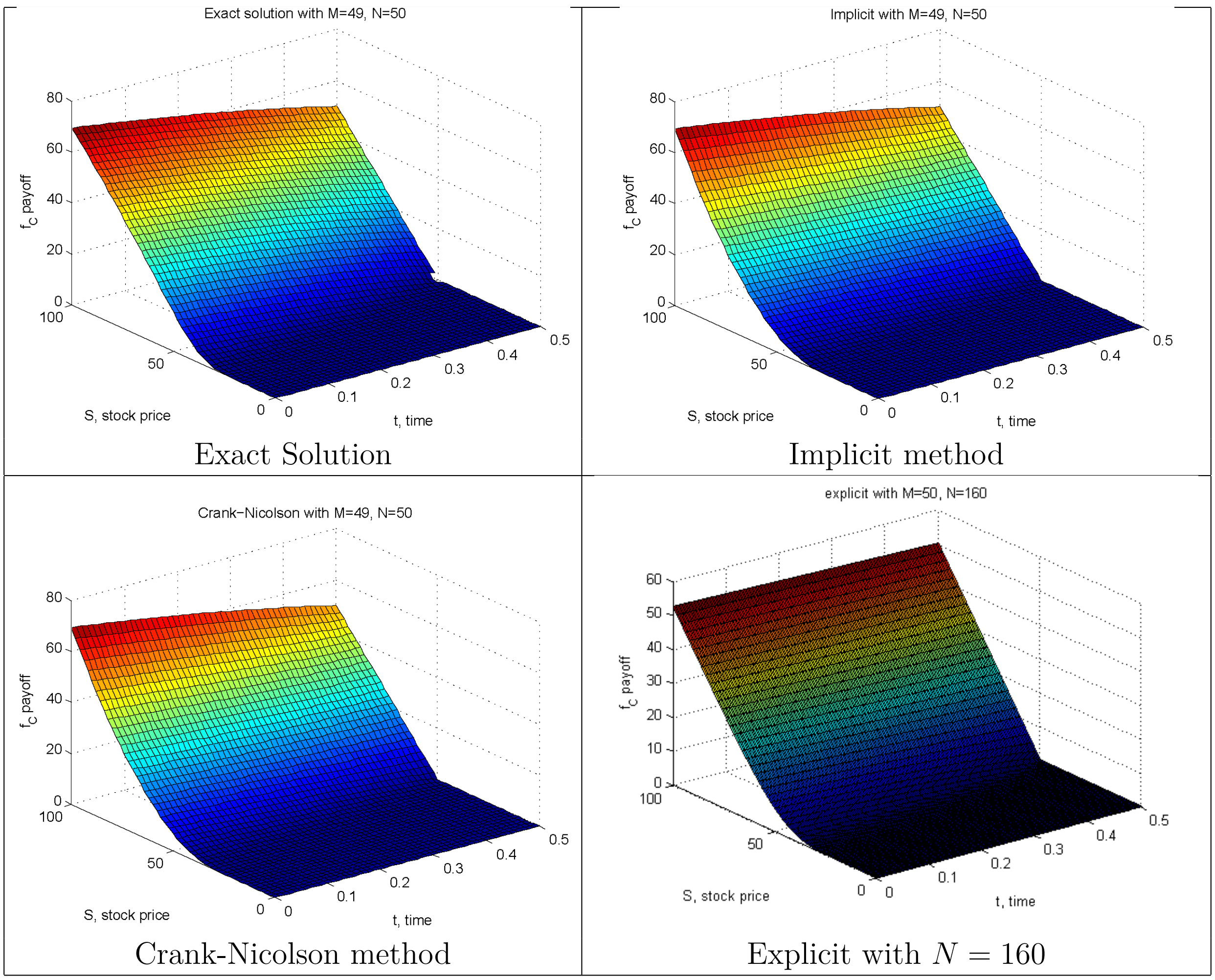


FIGURE 2.5. The result comparison for various finite difference methods, $N = 160$ for implicit and Crank-Nicolson method.

Figure 2.6 shows the level curve of absolute error which is absolute difference between exact solution and FDM for Crank-Nicolson method with $M = 5, 10, 25, 50, 75, 100$, and we can see the selction of M must be greater or equal to 25 in order to obtain the numerical solution with accuracy 4×10^{-3}.

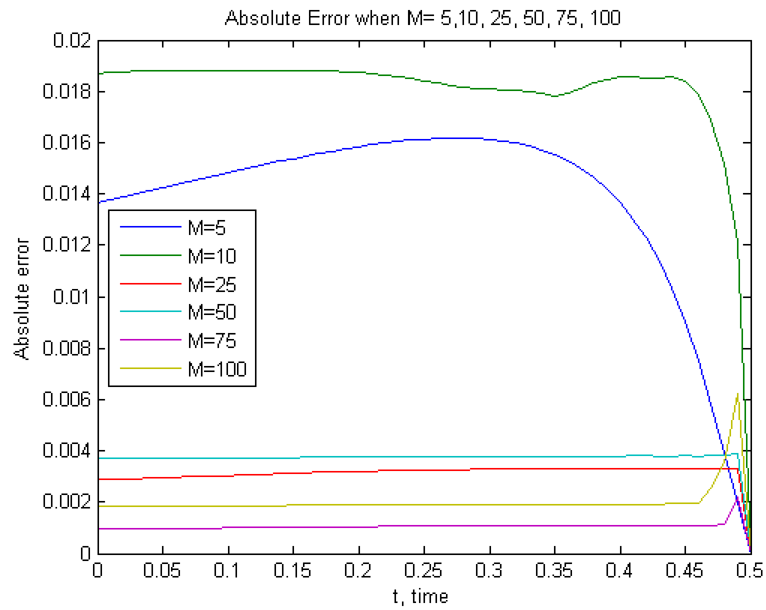


FIGURE 2.6. The absolute error of Crank-Nicolson method for various M

And we discuss the root mean square error between the exact solution and FDM with $N = 25, 50, 75, 100$ and $M = 5, 50, 100, 150, 200, 250, 300, 350, 400, 450, 500$. Obviously in Figure 2.7, the time steps increase, the error decreases.

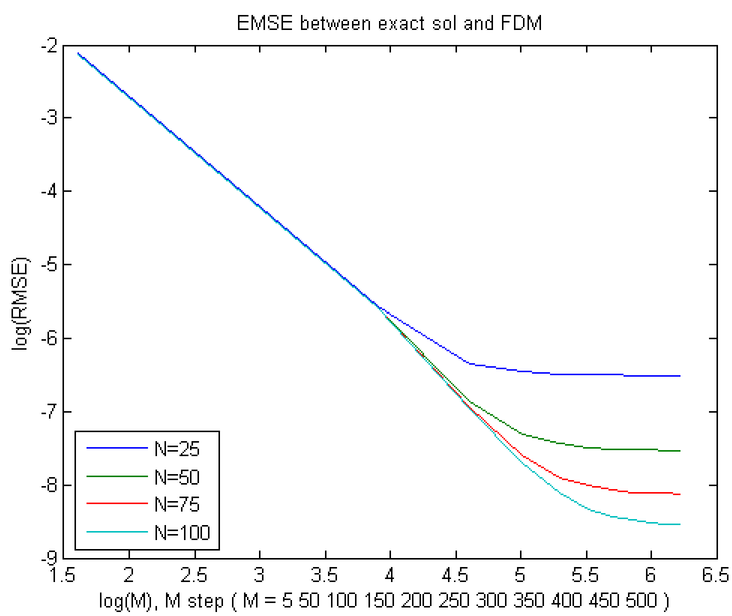


FIGURE 2.7. The root mean square error between the exact solution and the Crank-Nicolson method

2.3. The Radial Basis Function Method

The Black scholes equation for the evaluation of an call option price equation (2.1.1) is restated here:

$$\frac{\partial f_c}{\partial t} + rS\frac{\partial f_c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_c}{\partial S^2} = rf_c, \quad f_c = f_c(S,t),$$

Four radial basis functions are used:

$$\begin{aligned}
&TPS : \phi_k(S) = \|S - S_k\|^4 \log(\|S - S_k\|) \\
&MQ : \phi_k(S) = \sqrt{C^2 + \|S - S_k\|^2} \\
&Cubic : \phi_k(S) = \|S - S_k\|^3 \\
&Gaussian : \phi_k(S) = e^{-\frac{\|S-S_k\|^2}{C^2}}
\end{aligned} \tag{2.3.1}$$

where$\|S - S_k\|$is the Euclidean norm and C is an arbitrary constant.

Figure 2.8 and 2.9 show the graphs of these four RBF's centered at $S_k = 0, 20, 40, 60, 80, 100$, respectively.

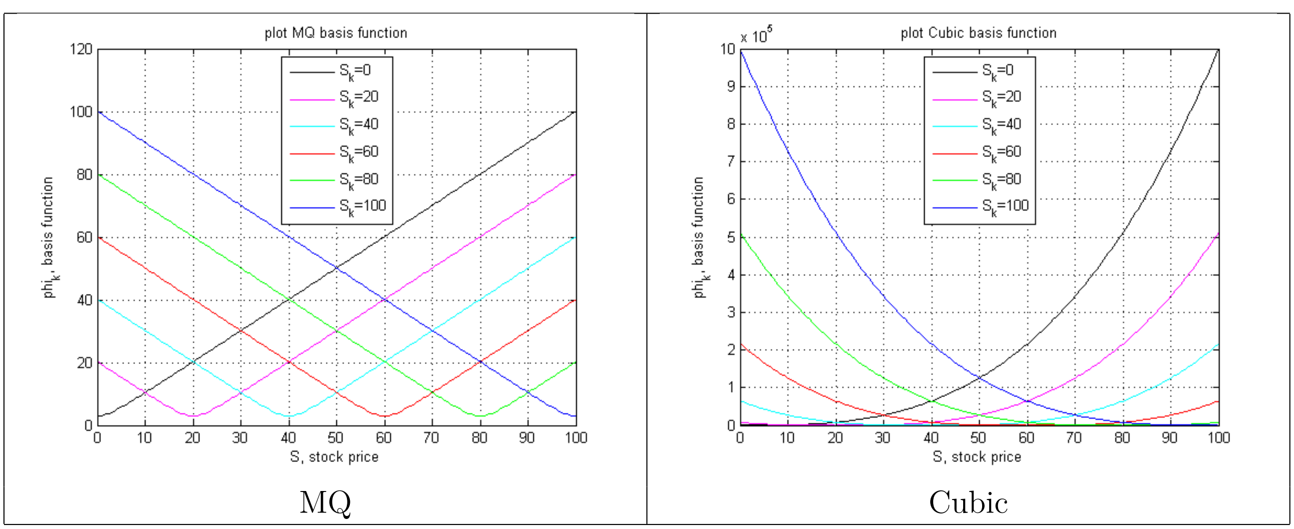


FIGURE 2.8. The graph of MQ-RBF, Cubic-RBF centered at various S_k

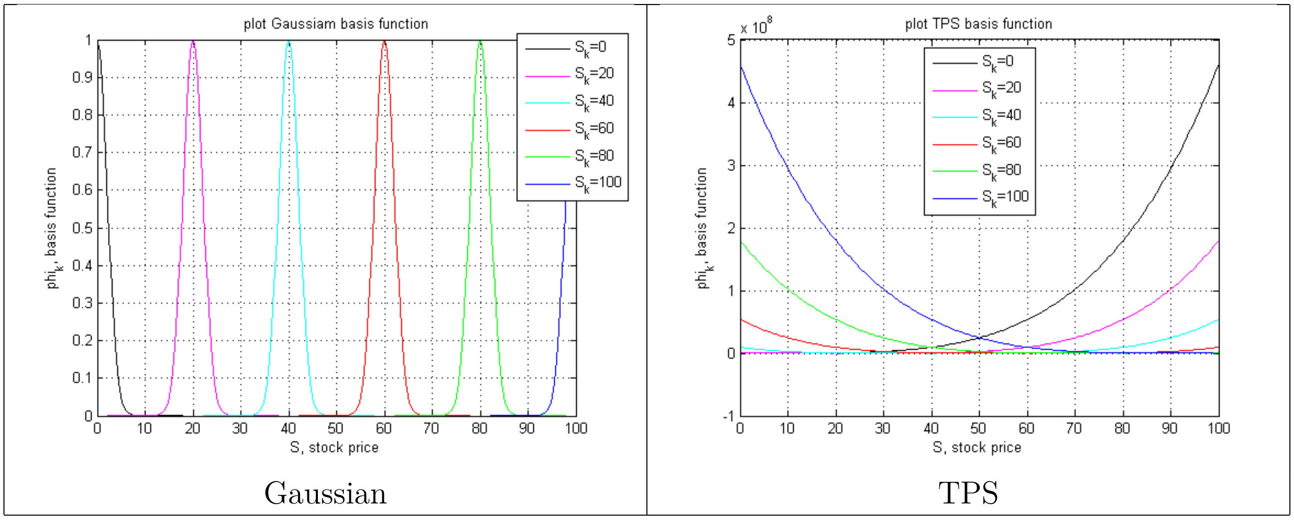


FIGURE 2.9. The graph of Gaussian-RBF, TPS-RBF centered at various S_k

We choose data points $\xi = \{\xi_0, \xi_1, \xi_2, \xi_3, \cdots, \xi_M\}$, $i = 0, 1, \cdots, M$, and centers $S = \{S_0, S_1 \cdots, S_L\}$, and $\Delta t = \frac{T-0}{N}$, $t_j = j \cdot \Delta t$, $j = 0, 1, \cdots, N$. Then the call price, f_c, can be expressed by

$$f_c(t, S) = \sum_{k=0}^{L} a_k(t)\phi_k(S) \tag{2.3.2}$$

and the derivatives of f_c are computed by

$$\begin{aligned} \frac{\partial f_c}{\partial S} &= \sum_{k=0}^{L} a_k(t)\phi'_k(S) \\ \frac{\partial^2 f_c}{\partial S^2} &= \sum_{k=0}^{L} a_k(t)\phi''_k(S) \end{aligned} \tag{2.3.3}$$

To satisfy the terminal condition

$$\begin{aligned} f_c(S_i, T) &= \sum_{k=0}^{L} a_k(T)\phi_k(\xi_i) \\ &= \max\{S(T) - K, 0\} \\ &= f_c(S_i) \end{aligned} \tag{2.3.4}$$

and the boundary conditions S are then given by:

1.when $t = T$ and $S = \xi_0$:

$$
\begin{aligned}
&\frac{\partial f_c}{\partial S}(\xi_0, T) - r f_c(\xi_0, T) = 0 \\
&\Rightarrow \sum_{k=0}^{L} a_k(T)\phi_k'(\xi_0) - r\sum_{k=0}^{l} a_k(T)\phi_k(\xi_0) \quad = 0 \\
&\Rightarrow \sum_{k=0}^{L} (\phi_k'(\xi_0) - r\phi_k(\xi_0))a_k(T) = 0
\end{aligned} \tag{2.3.5}
$$

2.when $t = T$ and $S = \xi_M$:

$$
\begin{aligned}
&\frac{\partial^2 f_c}{\partial S^2}(\xi_M, T) = 0 \\
&\Rightarrow \sum_{k=0}^{L} a_k(T)\phi_k''(\xi_M) = 0
\end{aligned} \tag{2.3.6}
$$

we can finally obtain

$$
\begin{bmatrix}
\phi_0(\xi_0) & \phi_1(\xi_0) & \cdots & \phi_L(\xi_0) \\
\phi_0(\xi_1) & \phi_1(\xi_1) & \cdots & \phi_L(\xi_1) \\
\vdots & \vdots & \vdots & \vdots \\
\phi_0(\xi_{M-1}) & \phi_1(\xi_{M-1}) & \cdots & \phi_L(\xi_{M-1}) \\
\phi_0(\xi_M) & \phi_1(\xi_M) & \cdots & \phi_L(\xi_M)
\end{bmatrix}
\begin{bmatrix}
a_0^N \\ a_1^N \\ \vdots \\ a_{L-1}^N \\ a_L^N
\end{bmatrix}
=
\begin{bmatrix}
f_c(S_0) \\ f_c(S_1) \\ \vdots \\ f_c(S_{M-1}) \\ f_c(S_M)
\end{bmatrix} \tag{2.3.7}
$$

Denote

$$\mathbf{a}^{(N)} = \left\{a_0^N, a_1^N, \cdots, a_{L-1}^N, a_L^N\right\}$$

and

$$
\begin{aligned}
M &= \begin{bmatrix}
\phi_0(\xi_0) & \phi_1(\xi_0) & \cdots & \phi_L(\xi_0) \\
\phi_0(\xi_1) & \phi_1(\xi_1) & \cdots & \phi_L(\xi_1) \\
\vdots & \vdots & \vdots & \vdots \\
\phi_0(\xi_{M-1}) & \phi_1(\xi_{M-1}) & \cdots & \phi_L(\xi_{M-1}) \\
\phi_0(\xi_M) & \phi_1(\xi_M) & \cdots & \phi_L(\xi_M)
\end{bmatrix} \\
\mathbf{b} &= \begin{bmatrix} f_c(S_0), & \cdots & , f_c(S_M) \end{bmatrix}^T
\end{aligned}
$$

then (2.3.7) becomes

$$M \cdot \mathbf{a}^{(N)} = \mathbf{b}$$

thus

$$\mathbf{a}^{(N)} = M^{+} \cdot \mathbf{b} \tag{2.3.8}$$

where M^{+} is the pseudo inverse of M.

The descritization of the PDE is described below. Substituting the derivatives into PDE,

$$\begin{aligned}
&-r\Delta t[(1-\theta)\sum_{k=0}^{L} a_k(j+1)\phi_k(\xi_i) + \theta \sum_{k=0}^{L} a_k(j)\phi_k(\xi_i)] \\
&\quad +[\sum_{k=0}^{L} a_k(j+1)\phi_k(\xi_i) - \sum_{k=0}^{L} a_k(j)\phi_k(\xi_i)] \\
&\quad +r\xi_i\Delta t[(1-\theta)\sum_{k=0}^{L} a_k(j+1)\phi_k'(\xi_i) + \theta \sum_{k=0}^{L} a_k(j)\phi_k'(\xi_i)] \\
&\quad +\frac{1}{2}\sigma^2\xi_i^2[(1-\theta)\sum_{k=0}^{L} a_k(j+1)\phi_k''(\xi_i) + \theta \sum_{k=0}^{L} a_k(j)\phi_k''(\xi_i)] = 0 \\
&\Rightarrow \sum_{k=0}^{L} a_k(j) \left\{ \phi_k(\xi_i) - \theta\Delta t[r\xi_i\phi_k'(\xi_i) + \frac{1}{2}\sigma^2\xi_i^2\phi_k''(\xi_i) - r\phi(\xi_i)] \right\} \\
&\quad = \sum_{k=0}^{L} a_k(j+1) \left\{ \phi_k(\xi_i) + (1-\theta)\Delta t[r\xi_i\phi_k'(\xi_i) + \tfrac{1}{2}\sigma^2\xi_i^2\phi_k''(\xi_i) - r\phi(\xi_i)] \right\}
\end{aligned}$$

we get

$$\begin{aligned}
&[1-\theta\Delta t(r\xi_i\tfrac{\partial}{\partial S} + \tfrac{1}{2}\sigma^2\xi_i^2\tfrac{\partial^2}{\partial S^2} - r)] \textstyle\sum_{k=0}^{L} a_k(j)\phi_k(\xi_i) \\
&= [1+(1-\theta)\Delta t(r\xi_i\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2\xi_i^2\frac{\partial^2}{\partial S^2} - r)] \sum_{k=0}^{L} a_k(j+1)\phi_k(\xi_i)
\end{aligned} \tag{2.3.9}$$

where $i = 1, 2, 3, \cdots, M-1$, $0 \le \theta \le 1$, and we define two new operators D and E by

$$\begin{aligned}
D &= [1+(1-\theta)\Delta t(r\xi_i\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2\xi_i^2\frac{\partial^2}{\partial S^2} - r)] \\
E &= [1-\theta\Delta t(r\xi_i\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2\xi_i^2\frac{\partial^2}{\partial S^2} - r)]
\end{aligned} \tag{2.3.10}$$

The operator H_+ and H_- are applied to the approximation (2.3.9) to yield:

$$D\sum_{k=0}^{L} a_k(j)\phi_k(\xi_i) = E\sum_{k=0}^{L} a_k(j+1)\phi_k(\xi_i)$$

Expressed in matrix form

$$
\begin{bmatrix}
d_0(\xi_0) & d_1(\xi_0) & \cdots & d_L(\xi_0) \\
d_0(\xi_1) & d_1(\xi_1) & \cdots & d_L(\xi_1) \\
\vdots & & & \\
d_0(\xi_{M-1}) & d_1(\xi_{M-1}) & \cdots & d_L(\xi_{M-1}) \\
d_0(\xi_M) & d_1(\xi_M) & \cdots & d_L(\xi_M)
\end{bmatrix}
\begin{bmatrix}
a_0^j \\ a_1^j \\ \vdots \\ a_{L-1}^j \\ a_L^j
\end{bmatrix}
=
\begin{bmatrix}
e_0(\xi_0) & e_1(\xi_0) & \cdots & e_L(\xi_0) \\
e_0(\xi_1) & e_1(\xi_1) & \cdots & e_L(\xi_1) \\
\vdots & \vdots & \ddots & \vdots \\
e_0(\xi_{M-1}) & e_1(\xi_{M-1}) & \cdots & e_L(\xi_{M-1}) \\
e_0(\xi_M) & e_1(\xi_M) & \cdots & e_L(M_{M-1})
\end{bmatrix}
\begin{bmatrix}
a_0^{j+1} \\ a_1^{j+1} \\ \vdots \\ a_{L-1}^{j+1} \\ a_L^{j+1}
\end{bmatrix}
\tag{2.3.11}
$$

four radial basis functionswhere

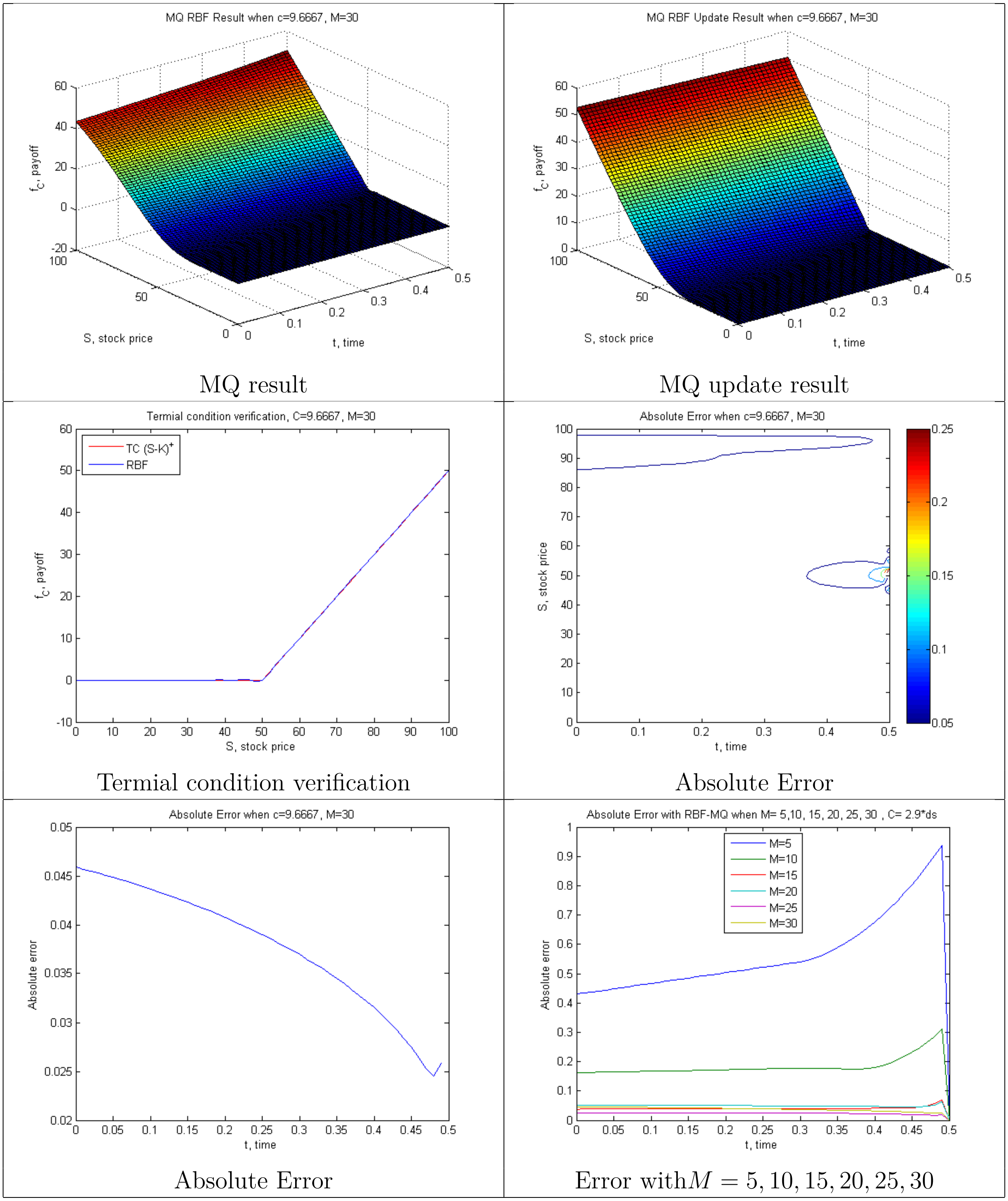
$$
\begin{aligned}
d_k(\xi_i) &= \phi_k(\xi_i) - (1-\theta)\Delta t[r\phi_k'(\xi_i) + \tfrac{1}{2}\sigma^2\xi_i^2\phi_k''(\xi_i) - r\phi(\xi_i)] \\
e_k(\xi_i) &= \phi_k(\xi_i) - \theta\Delta t[r\phi_k'(\xi_i) + \tfrac{1}{2}\sigma^2\xi_i^2\phi_k''(\xi_i) - r\phi(\xi_i)]
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
D \cdot \mathbf{a}^{(j)} &= E \cdot \mathbf{a}^{(j+1)} \\
\Rightarrow \mathbf{a}^{(j)} &= D^{-1}E \cdot \mathbf{a}^{(j+1)}
\end{aligned}
\tag{2.3.12}
$$

for $j = 0, 1, 2, \cdots, N$.

Given the following and parameters: $r = 0.10$, $\sigma = 0.40$, $T = 0.5$, $K = 50.00$, and selecting $S_L = 0$ and $S_U = 100.00$ with appropriate initial and boundary conditions. We adapte the collocated method i.e. $\xi_k = S_k$, $k = 0, \cdots, L$, in our computation. Let $M = 25$, i.e., $\Delta t = 0.005$ and $\Delta S = 0.4975$, $c = 2.9 \cdot \triangle S$ then the computation result of the basic stock call option price and the error at the terminal condition are described in Figure 2.10-2.12. For MQ-RBF and Cubic-RBF with $M = 5, 10, 15, 20, 25, 30$, we can know the selction of M must be greater or equal to 15 in order to obtain accurate solution with max error ≤ 0.1. For Gaussian-RBF with $M = 5, 10, 15, 20, 25, 30$, we can know the solution is worse than MQ and Cubic. The reason is the matrix D for Gaussian by collocation method is nearly singular. Because we are using the collcation method, the TPS-RBF can not form a matrix.



MQ result

MQ update result

Termial condition verification

Absolute Error

Absolute Error

Error with $M = 5, 10, 15, 20, 25, 30$

FIGURE 2.10. Computation ae result and error of using MQ-RBF

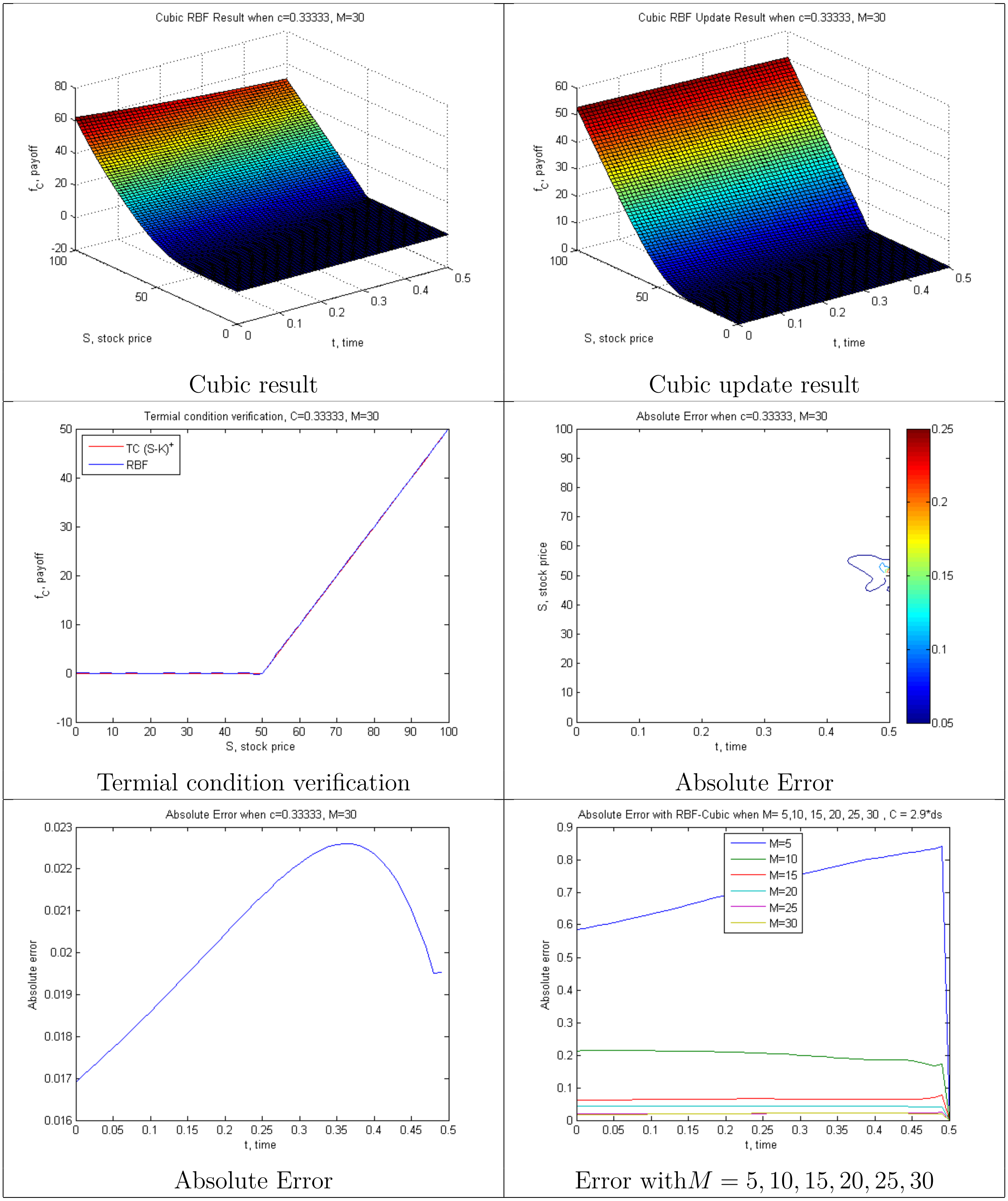
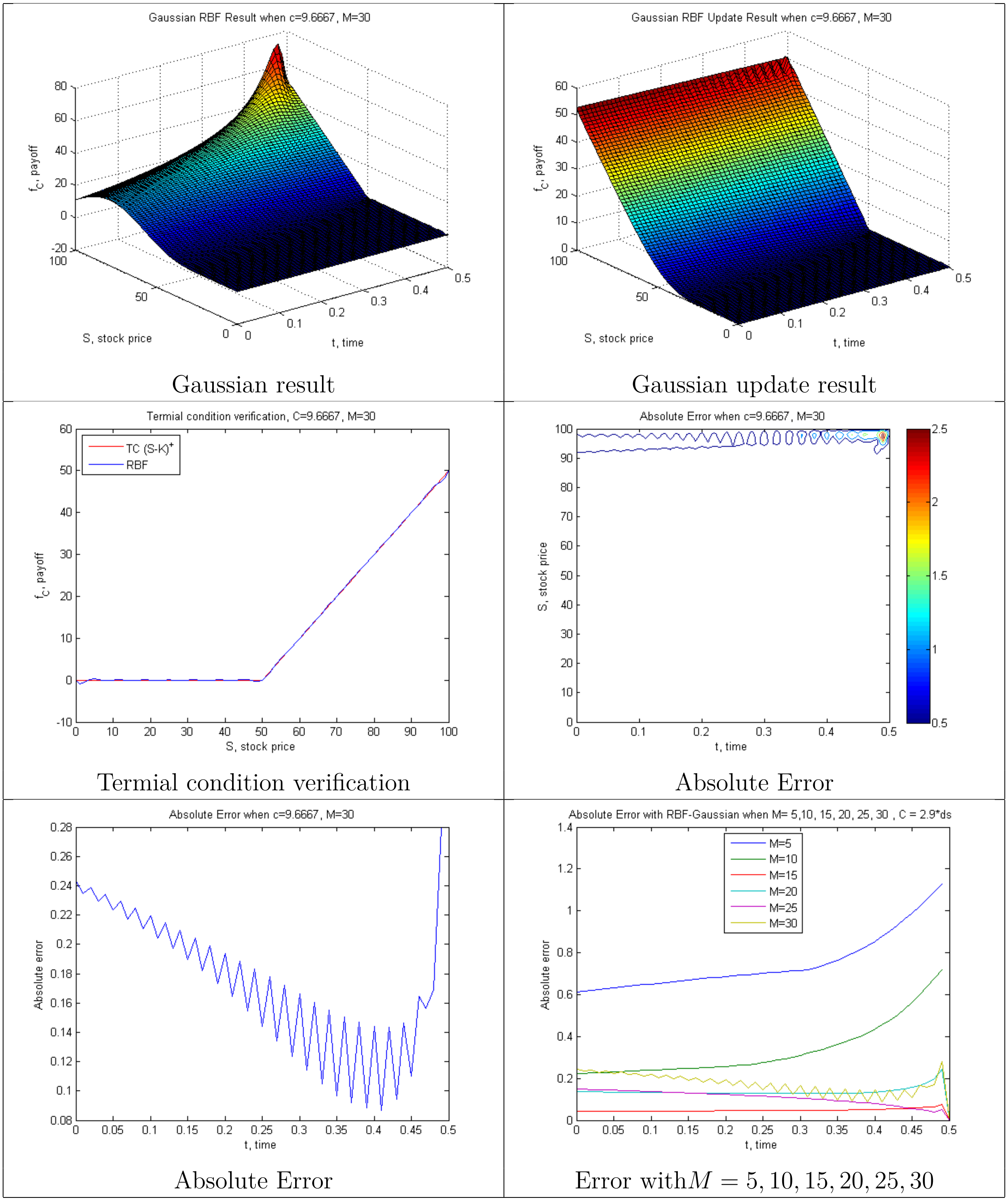


FIGURE 2.11. Computation ae result and errors of using Cubic-RBF



Gaussian result

Gaussian update result

Termial condition verification

Absolute Error

Absolute Error

Error with$M = 5, 10, 15, 20, 25, 30$

FIGURE 2.12. Computation ae result and errors of using Gaussian-RBF

The conparison errors of MQ-RBF when $M = 15, 17, 19$ are described on Figure 2.13. We can know that the max error are reduced.

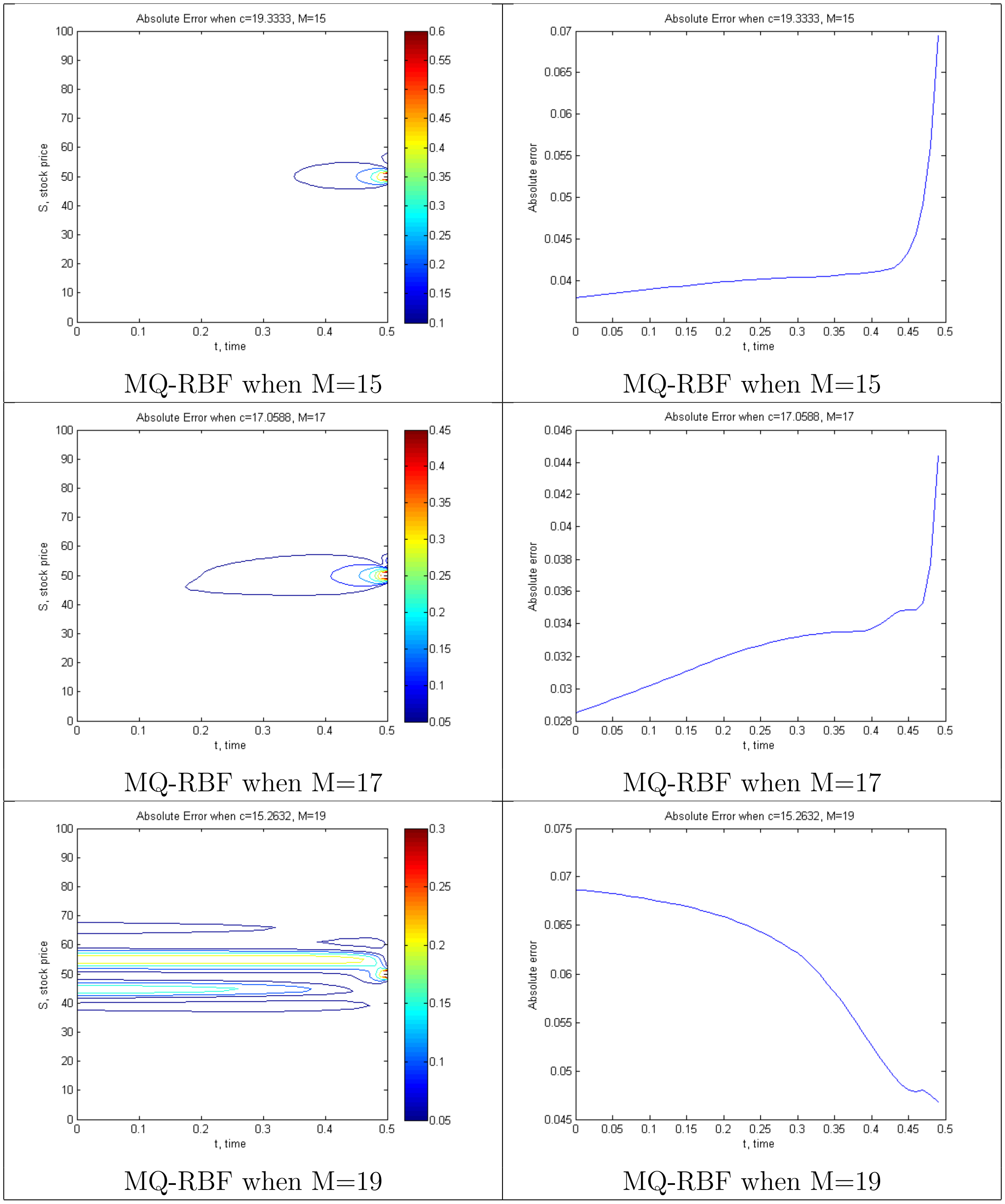


FIGURE 2.13. The absolute error of MQ-RBF for various stock price partition $M = 15, 17, 19$

Figure 2.14 shows the European call options for exact solution, FDM and RBF methods at $t = 0$ with $N = 200$ and $M = 50$. We can see only the explicit method lower than the exact solution at the price of 30. At the price of 50 all of finite difference methods relatively low than the exact solution. Finally, after the price of 70 just Gaussian and MQ functions are greater than the exact solution.

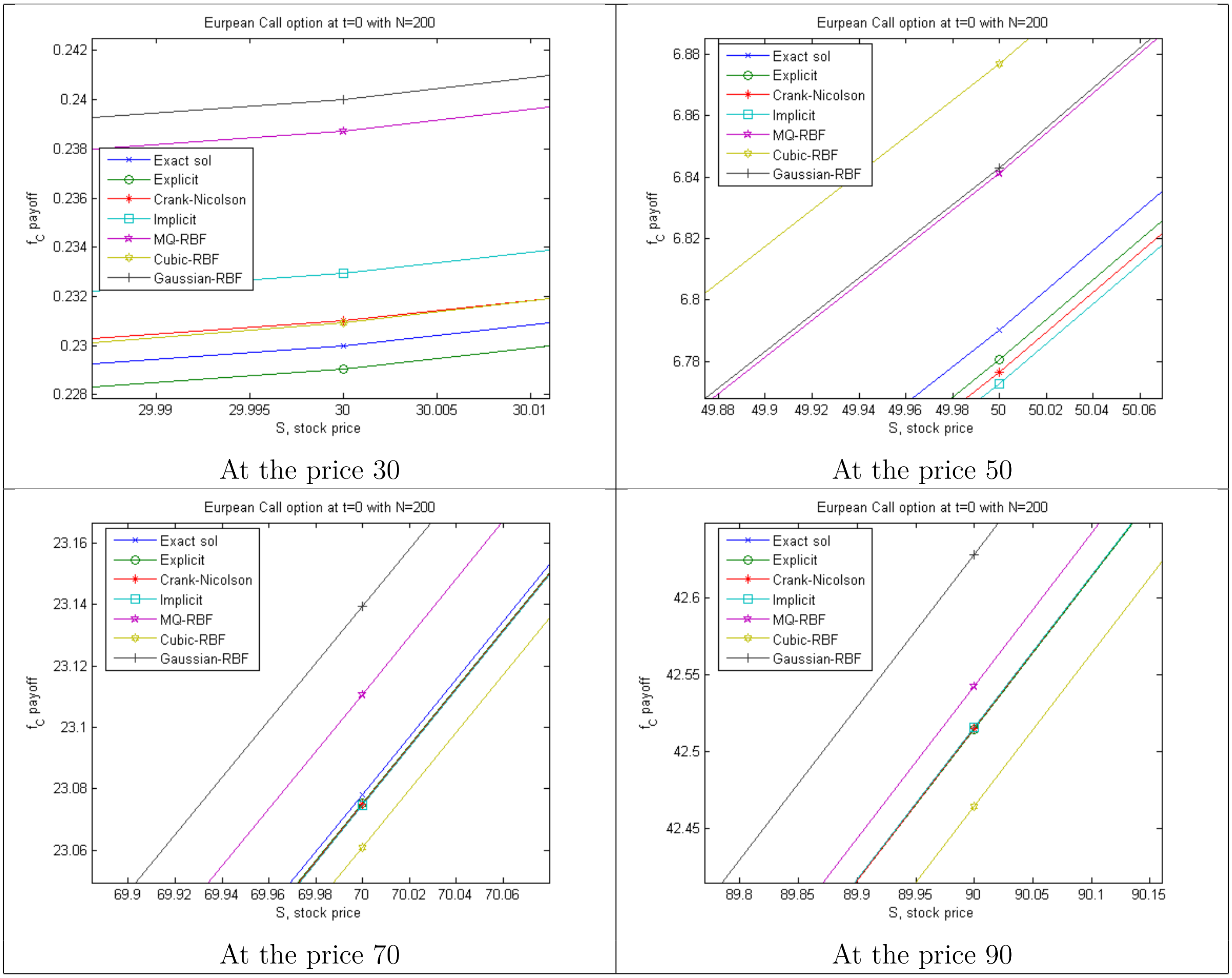


FIGURE 2.14. The European Call Options at $t = 0$

We discuss the root mean square error between the exact solution and FDM with $N = 25, 50, 75, 100$ and $M = 5, 10, 15, 25$. Obviously, the partition number of times is bigger, the error is smaller. And we can know from the Figure 2.15 that

$$RMSE \sim M^{-2} \sim (\Delta S)^2$$

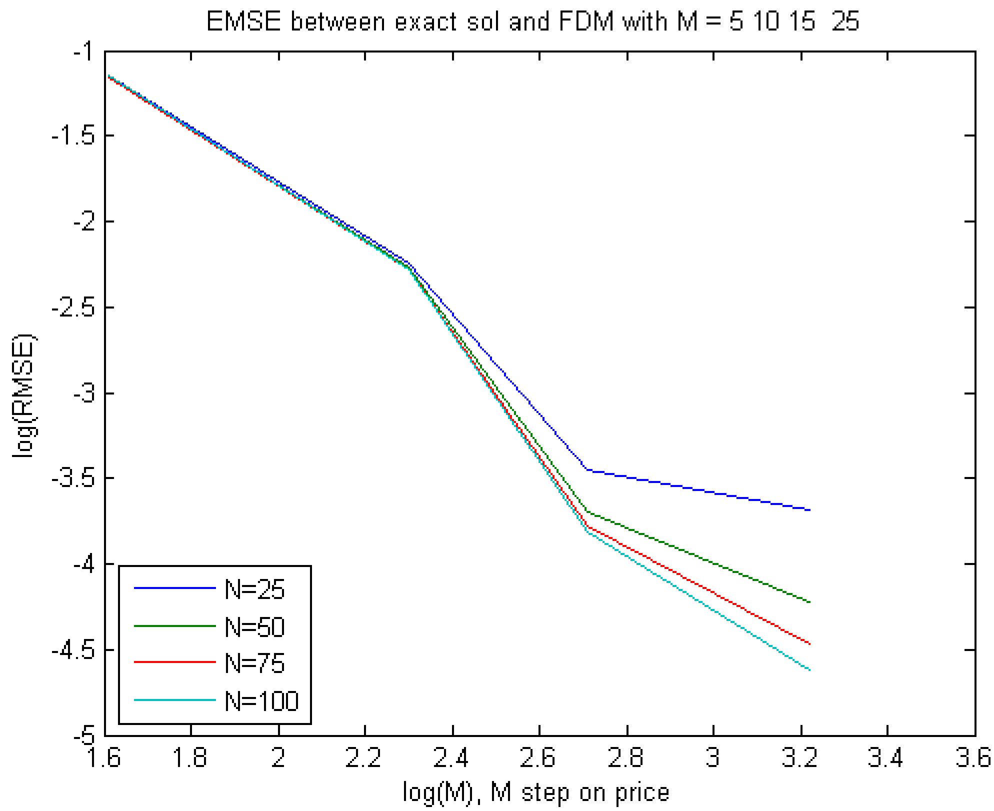


FIGURE 2.15. The root mean square error between the exact solution and RBF

CHAPTER 3

Pricing a European Call in a Jump Diffusion Model

In this chapter we show the exact solution of a double exponential jump-diffusion model.

3.1. A Jump Diffusion Model for European Call options pricing

In this section we present the Kou's analytical option price formula [**15**]. The following dynamics is proposed to model the asset price, S_t , under the physical probability measure P:

$$\frac{dS(t)}{S(t^-)} = rdt + \sigma dW(t) + d(\sum_{i=1}^{N(t)}(V_i - 1)) \tag{3.1.1}$$

where $W(t)$ is a standard Brownian motion, $N(t)$ is a Poisson process with rate λ, and $\{V_i\}$ is a sequence of independent identically distributed (i.i.d.) nonnegative random variables such that $Y = \log V$ has an asymmetric double exponential distribution with the density

$$f_Y(y) \sim p\eta_1 e^{-\eta_1 y} I_{\{y \geqq 0\}} + q\eta_2 e^{\eta_2 y} I_{\{y<0\}} \tag{3.1.2}$$

where $p, q \geqq 0$ and $p + q = 1$, represent the probabilities of upward and downward jumps.

Solving the stochastic differential equation gives the dynamics of the asset price:

$$S(T) = S(t)e^{\left\{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma W(T-t)\right\}} \prod_{i=1}^{N(t)} V_i \tag{3.1.3}$$

Hence

$$\begin{aligned} f_c(t, S_t) &= e^{-r(T-t)} E_p(f_c(T, S_T)| \mathscr{F}_t) \\ &= S(t)\Upsilon(r + \tfrac{1}{2}\sigma^2 - \lambda\varsigma, \sigma, \tilde{\lambda}, \tilde{P}, \tilde{\eta}_1, \tilde{\eta}_2; \log(\tfrac{K}{S(t)}), T - t) \\ &\quad -Ke^{-r(T-t)}\Upsilon(r - \tfrac{1}{2}\sigma^2 - \lambda\varsigma, \sigma, \lambda P, \eta_1, \eta_2; \log(\tfrac{K}{S(t)}), T - t) \end{aligned} \tag{3.1.4}$$

where $\tilde{P} = \frac{p}{1+\varsigma} \cdot \frac{\eta_1}{\eta_1 - 1}$,$\tilde{\eta}_1 = \eta_1 - 1$,$\tilde{\eta}_2 = \eta_2 + 1$,$\tilde{\lambda} = \lambda(\varsigma + 1)$,$\varsigma = p\dfrac{\eta_1}{\tilde{\eta}_1} + q\dfrac{\eta_2}{\tilde{\eta}_2}$, and for any given probability P, the notation Υ define:

$$\Upsilon(\mu, \sigma, \lambda P, \eta_1, \eta_2; a, T) := P\{Z(t) \geqq a\} \tag{3.1.5}$$

where $Z(t) = \mu t + \sigma W(t) + \sum_{i=1}^{N(t)} V_i$, V has a double exponential distribution with density $f_Y(y) \sim p\eta_1 e^{-\eta_1 y} 1_{\{y \geqq 0\}} + q\eta_2 e^{\eta_2 y} 1_{\{y<0\}}$, and $N(t)$ is a Poisson process with rate λ.

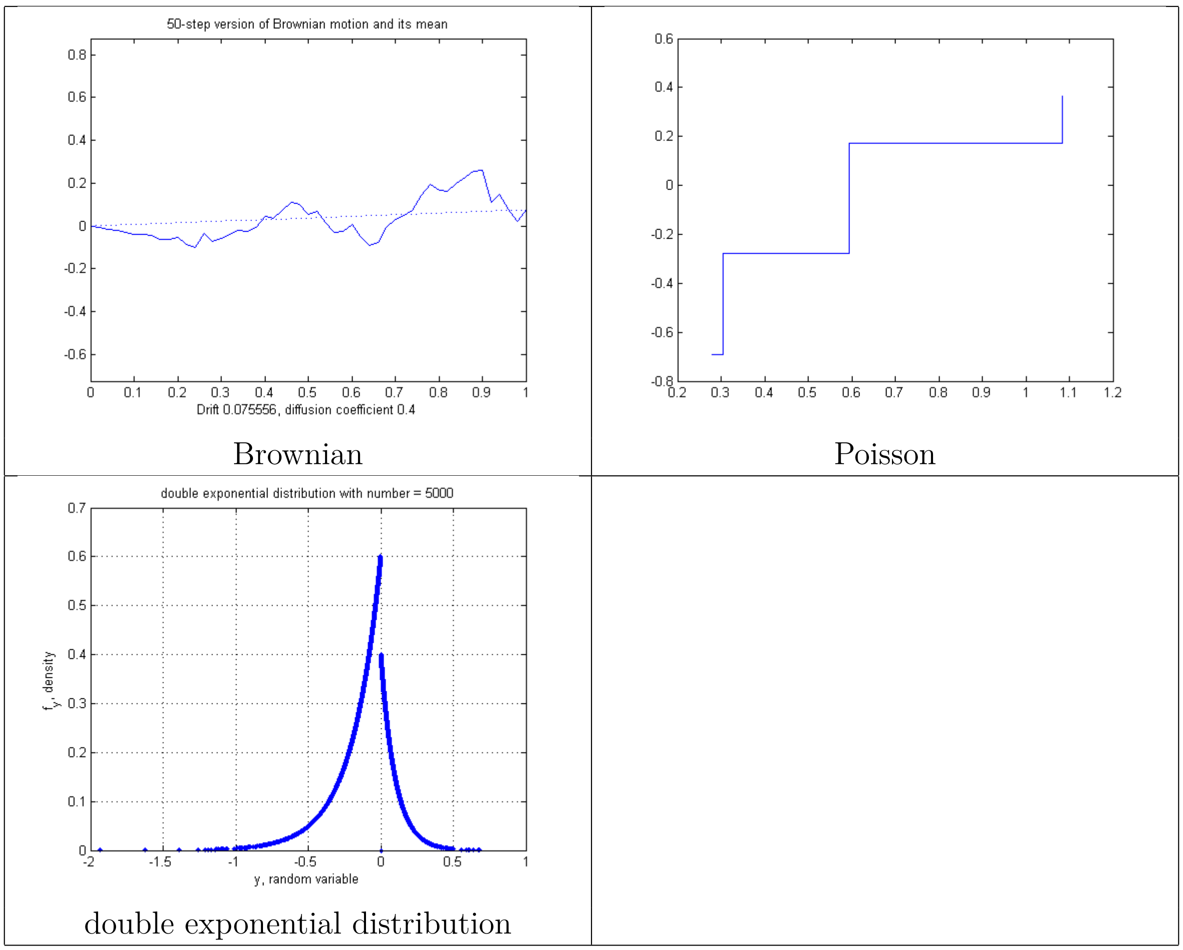


FIGURE 3.1. Brownian, Poisson and double exponential distribution

Consider the following and parameters:

$$r = 0.10, \quad \sigma = 0.40, \quad T = 1, \quad K = 50.00, \quad \lambda = 1, \quad p = 0.4 \quad , \eta_1 = 10, \quad \eta_2 = 5,$$

and selecting $S_L = 0$ and $S_U = 100.00$ with initial and boundary conditions. We apply various numbers of Brownian motion $W(t)$ to evaluate the exact solution by the equation (3.1.4) whose result is shown in Figure 3.2. It is evident that there exists no diffusion effect when the time is decreasing from 0.5 to 0 which means that our computation of the exact solution in this way is not correct and we will not refer to this result for verification in the numerical solution of PIDE.

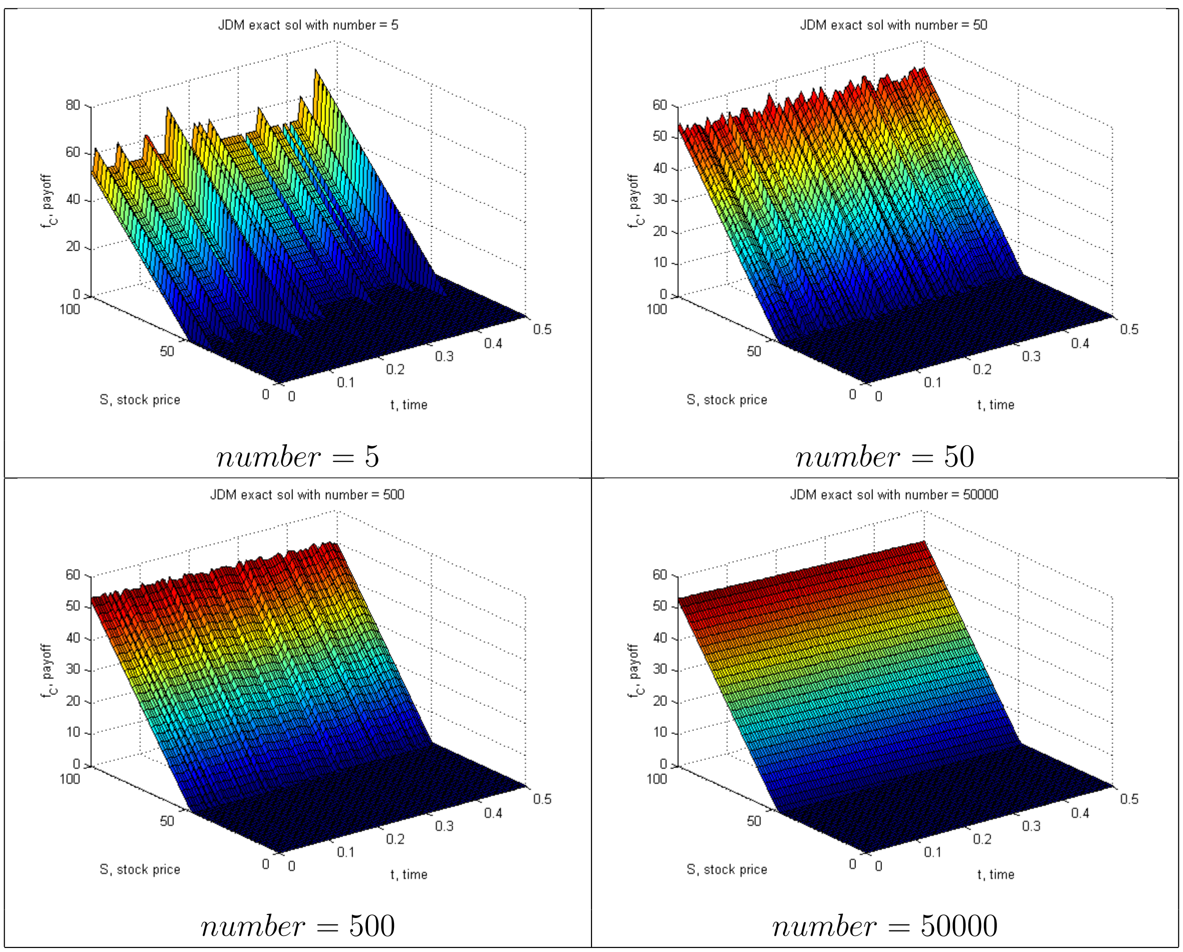


FIGURE 3.2. The exact solution of JDM for various numbers of Brownian motion $W(t)$ to be $5, 50, 500, 50000$

3.2. A Jump Diffusion Model in PIDE Form

In this section we present the Shreve's approach[**22**] to set up the PIDE for the JDM model. The exact solution of the Jump-Diffusion partial differential equation in one space dimension with terminal and boundary conditions:

$$\begin{aligned}
&-(\lambda + r) f_c + \tfrac{\partial f_c}{\partial t} + (r-\lambda) S \frac{\partial C}{\partial S} + \frac{1}{2} S^2 \frac{\partial f_c}{\partial S} + \lambda \textstyle\int_0^\infty P(\eta) C(t, \eta S(t)) d\eta = 0, \quad f_c = f_c(S,t) \\
&\frac{\partial f_c}{\partial S} - r f_c = 0, S \to 0 \\
&\frac{\partial^2 f_c}{\partial S^2} = 0,\ S \to \infty \\
&f_c(S,T) = \max\{S(T) - K, 0\}
\end{aligned} \tag{3.2.1}$$

On the probability space $(\Omega, \mathscr{F}, P)$, there are M independent Poisson Processes

$$N_1(t),\ N_2(t),\ \cdots,\ N_M(t), \quad 0 \le t \le T$$

and $\lambda_m > 0$ denotes the intensity of the m-th Poisson process. Define $W(t)$ to be a Brownian motion, and $\mathscr{F}(t)$ to be the filtration generated by the Brownian motion and M Poisson processes. Let $0 < \eta_1 < \eta_2 < \cdots < \eta_M$ be constant, and

$$\begin{aligned}
N(t) &= \sum_{m=1}^{M} N_m(t), \\
Q(t) &= \sum_{m=1}^{M} \eta_m N_m(t)
\end{aligned}$$

where N is a Poisson process with instensity $\lambda = \sum_{m=1}^{M} \lambda_m$ and Q is a compound Poisson process consisting from M independent Poisson Processes $N_1(t), N_2(t), \cdots, N_M(t)$ .Let Y_i denote the size of the i-th jump of Q . The random variables V_i , $1 \le i \le m$ take values in the set $\{\eta_1, \eta_2, \cdots, \eta_M\}$ and $Q(t)$ can be written as

$$Q(t) = \sum_{i=1}^{N(t)} V_i \tag{3.2.2}$$

with the probability of intensity of the m-th Poisson process given by $P(\eta_m) = \dfrac{\lambda_m}{\lambda}$. The random variables $V_1, V_2, \cdots, V_m$ are indenpendent and identically distributed, with

$$V_i \underrightarrow{i.i.d.} P\{V_i = \eta_m\} = P(\eta_m).$$

Set β to denote the expectation of the random variables, i.e.,

$$\beta = E_p(V_i) = \sum_{m=1}^{M} \eta_m P(\eta_m) = \frac{1}{\lambda}\sum_{m=1}^{M}\lambda_m\eta_m \tag{3.2.3}$$

then

$$Q(t) - \beta\lambda t = Q(t) - \sum_{m=1}^{M}\lambda_m\eta_m t \tag{3.2.4}$$

is a martingale.

In the following, the stock will be modeled by the stochastic differential equation

$$\begin{aligned}\frac{dS(t)}{S(t^-)} &= \mu dt + \sigma dW(t) + d(Q(t) - \beta\lambda t) \\ &= (\mu - \beta\lambda)dt + \sigma dW(t) + dQ(t)\end{aligned} \tag{3.2.5}$$

and we let $\beta = 1$, then it's asset price.

$$\begin{aligned}\frac{dS(t)}{S(t^-)} &= \mu dt + \sigma dW(t) + d(Q(t) - \lambda t) \\ &= (\mu - \lambda)dt + \sigma dW(t) + d(\sum_{i=1}^{N(t)} V_i)\end{aligned} \tag{3.2.6}$$

under the original probability measure P where the mean rate on the stock is μ and t^- denotes the moment just before the time t. The assumption that $\eta_i > 0$ for $i = 1, 2, \cdots, M$ guarantees that although the stock price can jump down, it cannot jump frome a positive to a negative value or to zero. Denote the set $S = \{S_t \mid t \in [0, T]\}$ to be the collection of the stock satisfying

$$\frac{dS(t)}{S(t^-)} = (\mu - \lambda)dt + \sigma dW(t) + d(\sum_{i=1}^{N(t)} V_i) \tag{3.2.7}$$

Because the mean rate of return of the stock under P is μ which is not equal to the interset rate r, we must chang it by now constructing a risk-neutral measure.

Let θ be a constant and $\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_M$ be positive constants. Define

$$Z_0(t) = e^{-\theta W(t) - \frac{1}{2}\theta^2 t},$$

$$Z_m(t) = e^{(\lambda_m - \widetilde{\lambda}_m)t}(\frac{\widetilde{\lambda}_m}{\lambda_m})^{N_m(t)}, m = 1, 2, 3 \cdots M,$$

and

$$Z(t) = Z_0(t) \prod_{m=1}^{M} Z_m(t)$$

$$\widetilde{P}(A) = \int_A Z(T)dP$$

for all $A \in \mathscr{F}$. Under the probability measure $\tilde{P}$, the following facts hold:

- the process $\tilde{W}(t) = W(t) + \theta t$ is a brownian motion,
- each N_m is a Poisson process with intensity $\tilde{\lambda}$,
- $\tilde{W}$ and $N_1, N_2, \cdots, N_m$ are independent of each other.

Denoting

$$\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m, \qquad \tilde{P}(v_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}$$

and under the measure $\tilde{P}$, the process $\tilde{N}(t) = \sum_{m=1}^{M} \tilde{N}_m(t)$ is Poisson with intensity $\tilde{\lambda}$, the jump-size random variables $V_1, V_2, \cdots$ are independent and identically distribution with

$$V_i \underrightarrow{i.i.d.} \tilde{P}\{V_i = \eta_m\} = \tilde{P}(\eta).$$

The corresponding expectation of jump-size becomes

$$\tilde{\beta} = E_{\tilde{p}}(V_i) = \sum_{m=1}^{M} \eta_m \tilde{P}(\eta_m) = \frac{1}{\tilde{\lambda}} \sum_{m=1}^{M} \tilde{\lambda}_m \eta_m = 1$$

Since the probability measure $\tilde{P}$ is risk-neutral if and only if the mean rate of return of the stock under $\tilde{P}$ is the interest rate r, we need to examine the stock model under new measure which gives us

$$\begin{aligned} \frac{dS(t)}{S(t^-)} &= (\mu - \lambda)dt + \sigma dW(t) + d(\sum_{i=1}^{\tilde{N}(t)} V_i) \\ &= (r - \tilde{\lambda})dt + \sigma d\tilde{W}(t) + d(\sum_{i=1}^{\tilde{N}(t)} V_i) \end{aligned} \tag{3.2.8}$$

with its solution given by

$$S(t) = S(0)e^{\{\sigma \tilde{W}(t) + (r - \tilde{\lambda} - \frac{1}{2}\sigma^2)t\}} \prod_{i=1}^{\tilde{N}(t)} V_i. \tag{3.2.9}$$

We can verify that $S(t)$ satisfy the stochastic differential equation.

Define the continuous stochastic process

$$X(t) = S(0)e^{\sigma \tilde{W}(t) + (r - \tilde{\lambda} - \frac{1}{2}\sigma^2)t}$$

and the pure jump process

$$J(t) = \prod_{i=1}^{\tilde{N}(t)} V_i$$

then $S(t) = X(t)J(t)$. The Itô formula gives us the differential forms of the continuous process

$$dX(t) = (r - \tilde{\lambda})X(t)dt + \sigma X(t)d\tilde{W}(t)$$

and the pure jump process

$$dJ(t) = J(t^-)d\tilde{Q}(t)$$

Itô product rule for jump process implies that

$$S(t) = X(t)J(t) = S(0) + \int_0^t X(u^-)dJ(u) + \int_0^t J(u)dX(u) + [X, J](t)$$

Since $J(t)$ is a pure process and $X(t)$ is continuous with $[X, J] = 0$, thus

$$\begin{aligned} S(t) &= X(t)J(t) \\ &= S(0) + \int_0^t X(u^-)J(u^-)d\tilde{Q}(u) + (r - \tilde{\lambda}) \int_0^t J(u)X(u)du + \sigma \int_0^t J(u)X(u)d\tilde{W}(u) \end{aligned}$$

in which the differential form is

$$\begin{aligned} dS(t) = d(X(t)J(t)) &= X(t^-)J(t^-)d\tilde{Q}(t) + (r - \tilde{\lambda})J(t)X(t)dt + \sigma J(t)X(t)d\tilde{W}(t) \\ &= S(t^-)d\tilde{Q}(t) + (r - \tilde{\lambda})S(t)dt + \sigma S(t)d\tilde{W}(t) \end{aligned}$$

For $0 \le t \le T$, the risk-neutral price of a call

$$f_c(S, t) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^+ \mid \mathscr{F}(t)] \tag{3.2.10}$$

where $S(T) = S(t)e^{\{\sigma \tilde{W}(T-t) + (r - \tilde{\lambda} - \frac{1}{2}\sigma^2)(T-t)\}} \prod_{i=N(t)+1}^{\tilde{N}(T)} V_i$. The call price satisfies the following differential-difference equation. From the differential form of stock price $\frac{dS(t)}{S(t^-)} = (r - \tilde{\lambda})Sdt + \sigma d\tilde{W}(t) + d(\sum_{i=1}^{\tilde{N}(t)} V_i)$, the Itô formula implies

$$\begin{aligned} e^{-rt} f_c(S(t), t) &= f_c(S(0), 0) + \int_0^t e^{-ru} \Big|_0^t [-r f_c(S(u), u) + \frac{\partial f_c}{\partial t}(S(u), u) + (r - \tilde{\lambda})S(u)\frac{\partial f_c}{\partial S}(S(u), u) \\ &\quad + \frac{1}{2}\sigma^2 S^2(u)\frac{\partial^2 f_c}{\partial S^2}(S(u), u)]du + \textstyle\int_0^t e^{-ru}\sigma S(u)\frac{\partial f_c}{\partial S}(S(u), u)d\tilde{W}(u) \\ &\quad + \sum_{0 \le u \le t} e^{-ru}[f_c(S(u), u) - f_c(S(u^-), u^-)] \end{aligned}$$

We examine the last term. If u is the time where a jump occurs in the m-th Poisson process N_m, the stock price satisfies $S(u) = \eta_m S(u^-)$. Therefore,

$$
\begin{aligned}
&\sum_{0\le u\le t} e^{-ru}[f_c(S(u),u) - f_c(S(u^-),u^-)] \\
&= \sum_{m=1}^{M}\sum_{0\le u\le t} e^{-ru}[f_c(\eta_m S(u^-),u^-) - f_c(S(u^-),u^-)]\Delta\tilde{N}_m(u) \\
&= \sum_{m=1}^{M}\int_0^t e^{-ru}[f_c(\eta_m S(u^-),u^-) - f_c(S(u^-),u^-)]d(\tilde{N}_m(u) - \tilde{\lambda}u) \\
&\quad + \int_0^t e^{-ru}[\sum_{m=1}^{M}\tfrac{\tilde{\lambda}_m}{\tilde{\lambda}} f_c(\eta_m S(u),u) - f_c(S(u),u)]\tilde{\lambda}du \\
&= \sum_{m=1}^{M}\int_0^t e^{-ru}[f_c(\eta_m S(u^-),u^-) - f_c(S(u^-),u^-)]d(\tilde{N}_m(u) - \tilde{\lambda}u) \\
&\quad + \int_0^t e^{-ru}\tilde{\lambda}[\sum_{m=1}^{M}\tilde{P}(\eta) f_c(\eta_m S(u),u) - f_c(S(u),u)]du.
\end{aligned}
$$

Substituting and taking derivatives, we arrive at

$$
\begin{aligned}
&d(e^{-rt}f_c(S(t),t)) \\
&= e^{-rt}\{-rf_c(S(t),t) + \frac{\partial f_c}{\partial t}(S(t),t) + (r-\tilde{\lambda})S(t)\frac{\partial f_c}{\partial S}(S(t),t) + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 f_c}{\partial S^2}(S(t),t) \\
&\quad + \tilde{\lambda}[\sum_{m=1}^{M}\tilde{P}(\eta_m)f_c(\eta_m S(u),u) - f_c(S(u),u)]dt + e^{-ru}\sigma S(t)\frac{\partial f_c}{\partial S}(S(t),t)d\tilde{W}(t) \\
&\quad + \sum_{m=1}^{M} e^{-ru}[f_c(\eta_m S(t^-),t^-) - f_c(S(t^-),t^-)]d(\tilde{N}_m(t) - \tilde{\lambda}t)
\end{aligned}
$$

The integrators $\tilde{N}_m(u) - \tilde{\lambda}u$ and $d\tilde{W}(t)$ are martingales under $\tilde{P}$ which implies the coefficient in dt term must be equal to zero. Thus, the call price $f_c(S(t),t)$ of the stock

$$
\frac{dS(t)}{S(t^-)} = (r-\tilde{\lambda})dt + \sigma d\tilde{W}(t) + d(\sum_{i=1}^{\tilde{N}(t)} V_i)
$$

must satisfy the equation

$$
\begin{aligned}
&-rf_c(S(t),t) + \frac{\partial f_c}{\partial t}(S(t),t) + (r-\tilde{\lambda})S(t)\frac{\partial f_c}{\partial S}(S(t),t) \\
&+\frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 f_c}{\partial S^2}(S(t),t) + \tilde{\lambda}[\sum_{m=1}^{M}\tilde{P}(\eta_m)f_c(\eta_m S(t),t) - f_c(S(t),t) \quad = \quad 0
\end{aligned}
\tag{3.2.11}
$$

where $0 \le t \le T, S(t) \ge 0$, and the terminal condition

$$f_c(S,T) = (S(T) - K)^+, S(T) \ge 0.$$

Consider the last term of the equation (3.2.11) by taking the jump size $M \to \infty$, and then we can obtain

$$\tilde{\lambda}\sum_{m=1}^{M}\tilde{P}(\eta_m)C(\eta_m S(u), u) \to \tilde{\lambda}\int_0^\infty \tilde{P}(\eta) f_c(\eta S(t), t) d\eta$$

therefore the associated PIDE can be written as

$$-(\tilde{\lambda} + r) f_c + \frac{\partial f_c}{\partial t} + (r - \tilde{\lambda}) S \frac{\partial f_c}{\partial S} + \frac{1}{2} S^2 \frac{\partial f_c}{\partial S} + \tilde{\lambda} \int_0^\infty \tilde{P}(\eta) f_c(\eta S(t), t) d\eta = 0 \qquad (3.2.12)$$

where $0 \le t \le T, S(t) \ge 0$, and the terminal condition

$$f_c(S,T) = (S(T) - K)^+, S(T) \ge 0.$$

On the probability space $(\Omega, \mathscr{F}, \tilde{P})$, we can write the equation (3.2.12) to be

$$-(\lambda + r) f_c + \frac{\partial f_c}{\partial t} + (r - \lambda) S \frac{\partial C}{\partial S} + \frac{1}{2} S^2 \frac{\partial f_c}{\partial S} + \lambda \int_0^\infty P(\eta) C(\eta S(t), t) d\eta = 0 \qquad (3.2.13)$$

for convenience.

CHAPTER 4

Th Numerical Solution of the Jump Diffusion Model

4.1. Truncation of Integration Domain

We rewrite the integral term in equation (3.2.13) so that it is formulated form for computational purpose. Consider the change of variables:

$$y = \log \eta, \qquad x = \log S$$

then the integral term becomes

$$\lambda \int_0^\infty P(\eta) f_c(\eta S(t), t) d\eta = \lambda \int_{-\infty}^\infty P(e^y) e^y f_c(e^y e^x, t) dy \tag{4.1.1}$$

and where $P(e^y)e^y = \frac{1}{\sqrt{2\pi}\delta} e^{(-\frac{y^2}{2\delta^2})}$. It is refered to Zemanian [**23**] for more detail disussion.

The integral term in (4.1.1) is defined on an infinite interval and we must truncate this to a finite interval such that we can compute its approximation value. In general, the procedure is to choose two finite values A and B such that the difference between the infinite and truncated integrals is less than a given tolerance ϵ:

$$\left| \int_{-\infty}^\infty P(e^y) e^y f_c(e^y e^x, t) dy - \int_A^B P(e^y) e^y f_c(e^y e^x, t) dy \right| < \epsilon$$

In the our problem we have a specific type of integrand (kerne), namely a probability density function, of the form

$$P(e^y)e^y = \frac{1}{\sqrt{2\pi}\delta} e^{(-\frac{y^2}{2\delta^2})}$$

This function goes to zero very quickly and we only look at this when it values are greater than a given tolerance ε. Following the suggestion by Chioma [?] then we have the inequalities:

$$
\begin{aligned}
&P(e^y)e^y \geq \varepsilon \\
&\Longleftrightarrow -\sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})} \leq y \leq \sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})}
\end{aligned}
\tag{4.1.2}
$$

Therefore

$$
\begin{aligned}
&\lambda \int_{-\infty}^{\infty} P(e^y)e^y f_c(e^y e^x, t)dy \\
&= \lambda \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\delta} e^{(-\dfrac{y^2}{2\delta^2})} f_c(e^y e^x, t)dy \\
&\approx \lambda \int_{-\sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})}}^{\sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})}} \frac{1}{\sqrt{2\pi}\delta} e^{(-\frac{y^2}{2\delta^2})} f_c(e^y e^x, t)dy
\end{aligned}
$$

Transform back to original (S, η) domain, where $A = e^{-\sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})}}$, $\qquad B = e^{\sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})}}$.

$$
\begin{aligned}
&\lambda \int_{0}^{\infty} P(\eta) f_c(\eta S(t), t)d\eta \\
&= \lambda \int_{e^{-\sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})}}}^{e^{\sqrt{-2\delta^2 \log(\varepsilon\delta\sqrt{2\pi})}}} \frac{1}{\sqrt{2\pi}\delta} e^{(-\frac{(\log(\eta))^2}{2\delta^2})} f_c(\eta S, t)\frac{1}{\eta}d\eta \\
&= \lambda \int_{A}^{B} \frac{1}{\sqrt{2\pi}\delta\eta} e^{(-\frac{(\log(\eta))^2}{2\delta^2})} f_c(\eta S, t)d\eta
\end{aligned}
$$

Thus the PIDE is approximately the following PIDE:

$$
-(\lambda+r)f_c + \frac{\partial f_c}{\partial t} + (r-\lambda)S\frac{\partial f_c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_c}{\partial S^2} + \lambda \int_{A}^{B} \frac{1}{\sqrt{2\pi}\delta\eta} e^{(-\frac{(\log(\eta))^2}{2\delta^2})} f_c(\eta S, t)d\eta = 0 \tag{4.1.3}
$$

Together with the boundary condition

$$
\begin{aligned}
&\frac{\partial f_c}{\partial S} - r f_c = 0, \, S \to 0 \\
&\frac{\partial^2 f_c}{\partial S^2} = 0, \; S \to \infty
\end{aligned}
\tag{4.1.4}
$$

and the terminal condition

$$
f_c(T, S_T) = \max\{S_T - K, 0\} \tag{4.1.5}
$$

4.2. Finite difference Method

Fixed $t\epsilon[0,T]$, $S_t\epsilon[S_L,S_U]$, and $\eta\epsilon[A,B]$. Define $w_i^j = f_c(S_i,t_j)$ with

$$\begin{aligned}
\Delta S &= \frac{SU-SL}{M}, & S_i &= SL + i\Delta S, & i &= 0,1,\cdots,M\\
\Delta t &= \frac{T-0}{N}, & t_j &= j\Delta t, & j &= 0,1,\cdots,N\\
\Delta \eta &= \frac{B-A}{L}, & \eta_i &= A + k\Delta\eta, & k &= 0,1,\cdots,L
\end{aligned}$$

and the terminal condition is modeled at $j=N$, for $i=1,2,\cdots,M-1$ are described by

$$f_c^N = \max\{S(T)-K,0\}$$

The boundary condition$i=0$and$i=M$,$j=0,1,\cdots,N$

- at $i=0$

$$\begin{aligned}
&\lim_{S\to 0}\frac{\partial f_c}{\partial S} - rf_c = 0\\
&\Rightarrow \frac{f_1^j - w_0^j}{\Delta S} - rf_0^j = 0\\
&\Rightarrow (1+r\Delta S)f_0^j = f_1^j\\
&\Rightarrow f_0^j = \frac{1}{(1+r\Delta S)}f_1^j
\end{aligned}$$

- at $i=M$

$$\begin{aligned}
&\lim_{S\to 0}\frac{\partial^2 f_c}{\partial S^2} = 0\\
&\Rightarrow \frac{f_{M-2}^j - 2f_{M-1}^j + f_M^j}{\partial S^2} = 0\\
&\Rightarrow f_M^j = 2f_{M-1}^j - f_{M-2}^j
\end{aligned}$$

The integral part is computed by Simpson's rule:

$$\begin{aligned}
&\lambda\int_A^B \frac{1}{\sqrt{2\pi}\delta\eta}e^{(-\frac{(\log(\eta))^2}{2\delta^2})}f_c(\eta S,t)d\eta\\
&= \lambda\frac{\Delta\eta}{3}\frac{1}{\sqrt{2\pi}\delta}[\frac{1}{A}e^{-\frac{(\log A)^2}{2\delta^2}}f_c(AS_i,t_j) + 4\frac{1}{\eta_1}e^{-\frac{(\log\eta_1)^2}{2\delta^2}}f_c(\eta_1 S_i,t_j) + 2\frac{1}{\eta_2}e^{-\frac{(\log\eta_2)^2}{2\delta^2}}f_c(\eta_2 S_i,t_j)\\
&\quad +4\frac{1}{\eta_3}e^{-\frac{(\log\eta_3)^2}{2\delta^2}}f_c(\eta_3 S_i,t_j) + \cdots + 4\frac{1}{\eta_{n-1}}e^{-\frac{(\log\eta_{n-1})^2}{2\delta^2}}f_c(\eta_{n-1}S_i,t_j) + \frac{1}{B}e^{-\frac{(\log B)^2}{2\delta^2}}f_c(BS_i,t_j)]
\end{aligned}$$

We let

$$\mathbf{I}^j = \frac{\lambda\Delta\eta}{3\sqrt{2\pi}\delta}[\tfrac{1}{A}e^{-\frac{(\log A)^2}{2\delta^2}}f_c(AS,t_j) + 2\sum_{k=1}^{\frac{L}{2}-1}\frac{1}{\eta_{2k}}e^{-\frac{(\log\eta_{2k})^2}{2\delta^2}}f_c(\eta_{2k}S,t_j)$$
$$+4\sum_{k=1}^{\frac{L}{2}}\frac{1}{\eta_{2k-1}}e^{-\frac{(\log\eta_{2k-1})^2}{2\delta^2}}f_c(\eta_{2k-1}S,t_j) + \tfrac{1}{B}e^{-\frac{(\log B)^2}{2\delta^2}}f_c(BS,t_j)] \quad (4.2.1)$$

and $f_c(S_i,t_j) = f_i^j$, so the PIDE becomes

$$-(\lambda+r)f_c + \frac{\partial f_c}{\partial t} + (r-\lambda)S\frac{\partial f_c}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 f_c}{\partial S^2} + \lambda\int_A^B \frac{1}{\sqrt{2\pi}\delta\eta}e^{(-\frac{(\log(\eta))^2}{2\delta^2})}f_c(\eta S,t)d\eta = 0$$

i.e.,

$$\Rightarrow -(\lambda+r)[\theta_1 f_i^j + (1-\theta_1)f_i^{j+1}] + \frac{f_i^{j+1}-f_i^j}{\Delta t}$$
$$+(r-\lambda)(i\Delta S)[\theta_1\frac{f_{i+1}^j - f_{i-1}^j}{2\Delta S} + (1-\theta_1)\frac{f_{i+1}^{j+1}-f_{i-1}^{j+1}}{2\Delta S}]$$
$$+\frac{1}{2}\sigma^2 i^2[\theta_1\frac{f_{i-1}^j - 2f_i^j + f_{i+1}^j}{\Delta S^2} + (1-\theta_1)\frac{f_{i-1}^{j+1} - 2f_i^{j+1} + f_{i+1}^{j+1}}{\Delta S^2}]$$
$$+\Delta t[\theta_2\mathbf{I}^j + (1-\theta_2)\mathbf{I}^{j+1}] = 0$$

Since we don't know the value f^j in advance, we choose $\theta_2 = 0$ for simplicity:

$$[\theta\triangle t(\frac{1}{2}(r-\lambda)i - \frac{1}{2}\sigma^2i^2)f_{i-1}^j + (1+\theta\triangle t(\sigma^2i^2+r+\lambda))f_i^j\ldots$$
$$+\theta\triangle t(-\tfrac{1}{2}(r-\lambda)i - \tfrac{1}{2}\sigma^2i^2)f_{i+1}^j]$$
$$= [(1-\theta)\triangle t(-\tfrac{1}{2}(r-\lambda)i + \tfrac{1}{2}\sigma^2i^2)f_{i-1}^{j+1} + (1+(1-\theta)\triangle t(-\sigma^2i^2-r-\lambda))f_i^{j+1}\ldots$$
$$+(1-\theta)\triangle t(\tfrac{1}{2}(r-\lambda)i + \tfrac{1}{2}\sigma^2i^2)f_{i+1}^{j+1}] + \Delta t\mathbf{I}^{j+1} \quad (4.2.2)$$

where $i = 1,2,3,\cdots,M-1$and $j = 0,1,\cdots,N$,

Define $H_{1,i}^+ = \frac{1}{2}(r-\lambda)i - \frac{1}{2}\sigma^2i^2$, $H_{2,i}^+ = \sigma^2i^2+r+\lambda$, $H_{3,i}^+ = -\frac{1}{2}(r-\lambda)i - \frac{1}{2}\sigma^2i^2$, and $H_{1,i}^- = -H_{1,i}^+$, $H_{2,i}^- = -H_{2,i}^+$, $H_{3,i}^- = -H_{3,i}^+$ then (4.2.2) becomes

$$[\theta\triangle tH_{1,i}^+f_{i-1}^jf_i^j + (1+\theta\triangle tH_{2,i}^+) + \theta\triangle tH_{3,i}^+f_{i+1}^j]$$
$$= [(1-\theta)\triangle tH_{1,i}^-f_{i-1}^{j+1} + (1+(1-\theta)\triangle tH_{2,i}^-)f_i^{j+1} + (1-\theta)\triangle tH_{3,i}^-f_{i+1}^{j+1}] + \Delta t\mathbf{I}^{j+1} \quad (4.2.3)$$

The boundary condition at $i = 1$

$$\begin{aligned}&[\theta\triangle tH^+_{1,1}f^j_0 + (1+\theta\triangle tH^+_{2,1})f^j_1 + \theta\triangle tH^+_{3,1}f^j_2]\\&= [(1-\theta)\triangle tH^-_{1,1}f^{j+1}_0 + (1+(1-\theta)\triangle tH^-_{2,1})f^{j+1}_1 + (1-\theta)\triangle tH^-_{3,1}f^{j+1}_2] + \Delta t\mathbf{I}^{j+1}\end{aligned} \tag{4.2.4}$$

and the boundary condition at $i = M-1$ gives

$$\begin{aligned}&[\theta\triangle tH^+_{1,M-1}f^j_{M-2} + (1+\theta\triangle tH^+_{2,M-1})f^j_{M-1} + \theta\triangle tH^+_{3,M-1}f^j_M]\\&= [(1-\theta)\triangle tH^-_{1,M-1}f^{j+1}_{M-2} + (1+(1-\theta)\triangle tH^-_{2,M-1})f^{j+1}_{M-1} + (1-\theta)\triangle tH^-_{3,M-1}f^{j+1}_M] + \Delta t\mathbf{I}^{j+1}\end{aligned} \tag{4.2.5}$$

Thus we can recapulate into the matrix form:

$$\begin{aligned}A_\theta\mathbf{f}^{(j)} &= A_{1-\theta}\mathbf{f}^{(j+1)} + \triangle t\mathbf{I}^{j+1} + \mathbf{b}\\ \Rightarrow \mathbf{f}^{(j)} &= A_\theta^{-1}(A_{1-\theta}\mathbf{f}^{(j+1)} + \triangle t\mathbf{I}^{j+1} + \mathbf{b})\end{aligned} \tag{4.2.6}$$

where

$$\mathbf{f}^j = \begin{bmatrix} f^j_1 & f^j_2 & \cdots & f^j_{M-2} & f^j_{M-1}\end{bmatrix}^T \tag{4.2.7}$$

$$A_\theta = \begin{bmatrix} 1+\theta\triangle tH^+_{2,1} & \theta\triangle tH^+_{3,1} & 0 & 0 & \cdots & 0\\ \theta\triangle tH^+_{1,2} & 1+\theta\triangle tH^+_{2,2} & \theta\triangle tH^+_{3,2} & & & 0\\ 0 & \ddots & \ddots & \ddots & & \vdots\\ 0 & & \theta\triangle tH^+_{1,i} & 1+\theta\triangle tH^+_{2,i} & \theta\triangle tH^+_{3,i} & 0\\ \vdots & & \ddots & \ddots & \ddots & 0\\ 0 & & & \theta\triangle tH^+_{1,M-2} & 1+\theta\triangle tH^+_{2,M-2} & \theta\triangle tH^+_{3,M-2}\\ 0 & 0 & \cdots & 0 & \theta\triangle tH^+_{1,M-1} & 1+\theta\triangle tH^+_{2,M-1}\end{bmatrix} \tag{4.2.8}$$

$$A_{1-\theta} = \begin{bmatrix} 1+\theta\triangle tH^{-}_{2,1} & \theta\triangle tH^{-}_{3,1} & 0 & 0 & \cdots & 0 \\ \theta\triangle tH^{-}_{1,2} & 1+\theta\triangle tH^{-}_{2,2} & \theta\triangle tH^{-}_{3,2} & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ 0 & & \theta\triangle tH^{-}_{1,i} & 1+\theta\triangle tH^{-}_{2,i} & \theta\triangle tH^{-}_{3,i} & 0 \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & \theta\triangle tH^{-}_{1,M-2} & 1+\theta\triangle tH^{-}_{2,M-2} & \theta\triangle tH^{-}_{3,M-2} \\ 0 & 0 & \cdots & 0 & \theta\triangle tH^{-}_{1,M-1} & 1+\theta\triangle tH^{-}_{2,M-1} \end{bmatrix} \tag{4.2.9}$$

$$\mathbf{b} = \begin{bmatrix} -\theta\triangle tH^{+}_{1,1}f^{j}_{0} + (1-\theta)\triangle tH^{-}_{1,1}f^{j+1}_{0} \\ 0 \\ \vdots \\ 0 \\ -\theta\triangle tH^{+}_{3,M-1}f^{j}_{M} + (1-\theta)\triangle tH^{-}_{3,M-1}f^{j+1}_{M} \end{bmatrix}$$

$$\Rightarrow \mathbf{b} = \begin{bmatrix} \triangle tH^{-}_{1,1}(\theta f^{j}_{0} + (1-\theta)f^{j+1}_{0}) \\ 0 \\ \vdots \\ 0 \\ \triangle tH^{-}_{3,M-1}(\theta f^{j}_{M} + (1-\theta)f^{j+1}_{M}) \end{bmatrix} \tag{4.2.10}$$

$$\mathbf{I}^{j} = \begin{bmatrix} I^{j}_{1} \\ I^{j}_{2} \\ \vdots \\ I^{j}_{M-2} \\ I^{j}_{M-1} \end{bmatrix} \tag{4.2.11}$$

with the component I_i^j given by

$$
\begin{aligned}
I_i^j = & \tfrac{\lambda \Delta \eta}{3\sqrt{2\pi}\delta}[\tfrac{1}{A} e^{-\frac{(\log A)^2}{2\delta^2}} f_c(AS_i, t_j) + 2\sum_{k=1}^{\frac{L}{2}-1} \frac{1}{\eta_{2k}} e^{-\frac{(\log \eta_{2k})^2}{2\delta^2}} f_c(\eta_{2k} S_i, t_j) \\
& +4\sum_{k=1}^{\frac{L}{2}} \frac{1}{\eta_{2k-1}} e^{-\frac{(\log \eta_{2k-1})^2}{2\delta^2}} f_c(\eta_{2k-1} S_i, t_j) + \tfrac{1}{B} e^{-\frac{(\log B)^2}{2\delta^2}} f_c(BS_i, t_j)
\end{aligned}
\tag{4.2.12}
$$

Since substitute $\lambda = 0$

$$
-(\lambda + r) f_c + \frac{\partial f_c}{\partial t} + (r - \lambda) S \frac{\partial f_c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_c}{\partial S^2} + \lambda \int_A^B \frac{1}{\sqrt{2\pi}\delta\eta} e^{(-\frac{(\log(\eta))^2}{2\delta^2})} f_c(t, \eta S) d\eta = 0
$$

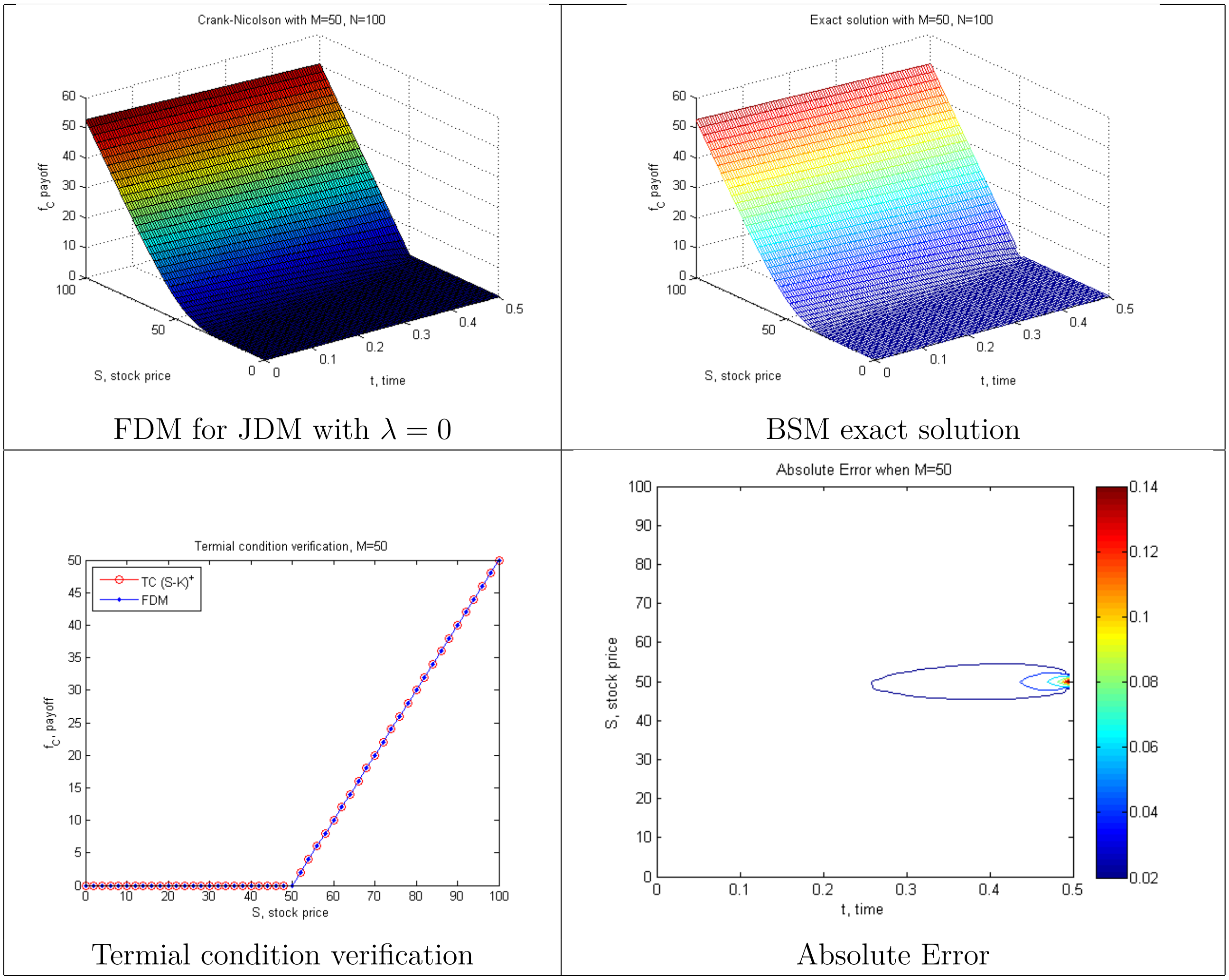
becomes

$$
-r f_c + \frac{\partial f_c}{\partial t} + rS \frac{\partial f_c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_c}{\partial S^2} = 0
$$

i.e., the PDE of the BSM model, we can use this to verify the implementation of our code for PIDE and is gualified after.

$$
r = 0.10, \quad \sigma = 0.40, \quad T = 0.5, \quad K = 50.00
$$

Given select $S_L = 0$ and $S_U = 100.00$ with appropriate initial and boundary conditions compaired with the BSM exact one. The computation results with $M = 100$ for JDM with $\lambda = 0$ is shown in Figure 2.12.



FDM for JDM with $\lambda = 0$

BSM exact solution

Termial condition verification

Absolute Error

FIGURE 4.1. Computate the JDM result with $\lambda = 0$

After setting the following parameters for jump behavior:

$$\lambda = 1, \quad p = 0.4, \quad q = 0.6, \quad \eta_1 = 10, \quad \eta_2 = 5, \quad \delta = \frac{1}{2}\sigma^2$$

the numerical solution of FDM is depicted in Figure 4.2

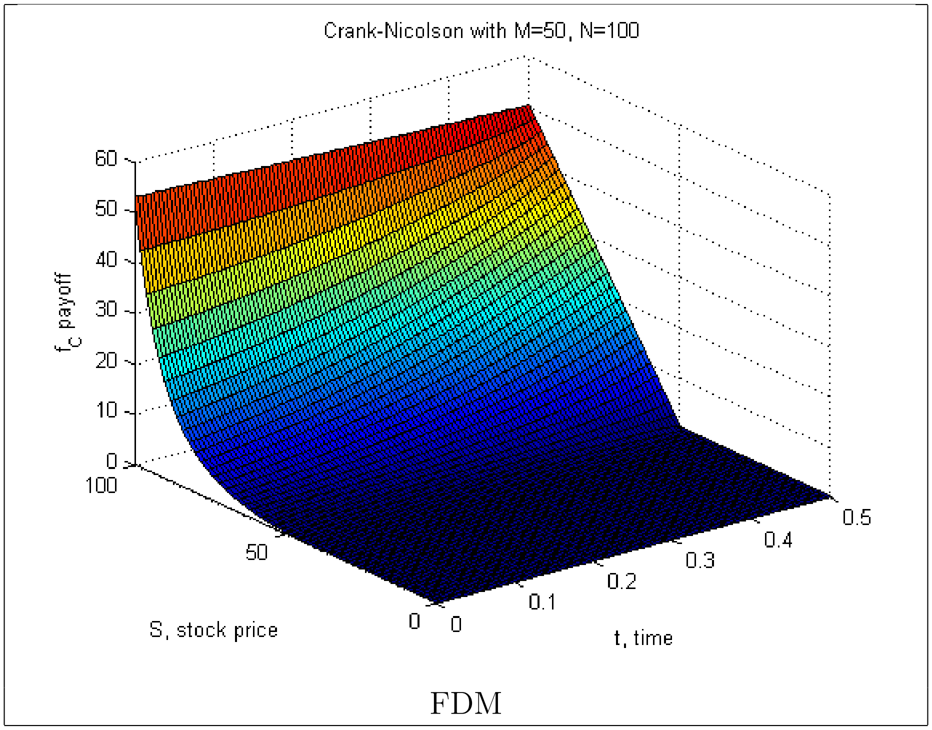


FIGURE 4.2. Numerical solution of the JDM via FDM

4.3. Meshfree RBF Method

We choose data points $\xi = \{\xi_1, \xi_2, \xi_3, \cdots, \xi_M\}$, $i = 0, 1, \cdots, M$ and basis functions $\phi_k(S) = e^{-\frac{x_k^2}{C^2}} = e^{-\frac{|S-S_k|^2}{C^2}}$ where centers $S = \{S_0, S_1 \cdots, S_L\}$,$k = 0, 1, \cdots, L$ and $\Delta t = \frac{T-0}{N}$, $t_j = j\Delta t$, $j = 0, 1, \cdots, N$ and $\Delta\eta = \frac{B-A}{P}$, $\eta_h = h\Delta\eta$, $h = 0, 1, \cdots, P$

The derivatives of the call function are approximated the same in Chapter 2. Since $f_c(t, S) = \sum_{k=0}^{L} a_k(t)\phi_k(S)$, the integral part in PIDE is computed by using Simpson rule:

$$\begin{aligned}
I_i^j &= \lambda \int_A^B \frac{1}{\sqrt{2\pi}\delta\eta} e^{(-\frac{(\log(\eta))^2}{2\delta^2})} f_c(\eta\xi_i, t_j) d\eta \\
&= \lambda \int_A^B \frac{1}{\sqrt{2\pi}\delta\eta} e^{(-\frac{(\log(\eta))^2}{2\delta^2})} \sum_{k=0}^{L} a_k(t_j)\phi_k(\eta\xi_i) d\eta \\
&= \sum_{k=0}^{L} a_k(t_j) \frac{\lambda}{\sqrt{2\pi}\delta} \int_A^B \frac{1}{\eta} e^{(-\frac{(\log(\eta))^2}{2\delta^2})} \phi_k(\eta\xi_i) d\eta \\
&\approx \sum_{k=0}^{L} a_k(t_j) \frac{\lambda}{\sqrt{2\pi}\delta} \frac{\Delta\eta}{3} [\frac{1}{A} e^{(-\frac{(\log(A))^2}{2\delta^2})} \phi_k(A\xi_i) + 2\sum_{h=1}^{\frac{P}{2}-1} \frac{1}{\eta_{2h}} e^{(-\frac{(\log(\eta_{2h}))^2}{2\delta^2})} \phi_k(\eta_{2h}\xi_i) \\
&\quad + 4\sum_{h=1}^{\frac{P}{2}-1} \frac{1}{\eta_{2h-1}} e^{(-\frac{(\log(\eta_{2h-1}))^2}{2\delta^2})} \phi_k(\eta_{2h-1}\xi_i) + \frac{1}{B} e^{(-\frac{(\log(B))^2}{2\delta^2})} \phi_k(B\xi_i)].
\end{aligned} \tag{4.3.1}$$

To satisfy the terminal condition

$$\begin{aligned}
f_c(S_i, T) &= \sum_{k=0}^{L} a_k(T)\phi_k(\xi_i) \\
&= \max\{S(T) - K, 0\} \\
&= f_c(S_i)
\end{aligned} \tag{4.3.2}$$

and the boundary conditions S are then given by:

(1) when $t = T$and $S = \xi_0$:

$$\begin{aligned}
&\frac{\partial f_c}{\partial S}(\xi_0, T) - r f_c(\xi_0, T) = 0 \\
&\Rightarrow \sum_{k=0}^{L} a_k(T)\phi_k'(\xi_0) - r\sum_{k=0}^{l} a_k(T)\phi_k(\xi_0) \quad = 0 \\
&\Rightarrow \sum_{k=0}^{L} (\phi_k'(\xi_0) - r\phi_k(\xi_0)) a_k(T) = 0
\end{aligned} \tag{4.3.3}$$

(2) when $t = T$and $S = \xi_M$:

$$\frac{\partial^2 f_c}{\partial S^2}(\xi_M, T) = 0$$
$$\Rightarrow \sum_{k=0}^{L} a_k(T)\phi_k''(\xi_M) = 0 \tag{4.3.4}$$

we can finally obtain

$$\begin{bmatrix} \phi_0(\xi_0) & \phi_1(\xi_0) & \cdots & \phi_L(\xi_0) \\ \phi_0(\xi_1) & \phi_1(\xi_1) & \cdots & \phi_L(\xi_1) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_0(\xi_{M-1}) & \phi_1(\xi_{M-1}) & \cdots & \phi_L(\xi_{M-1}) \\ \phi_0(\xi_M) & \phi_1(\xi_M) & \cdots & \phi_L(\xi_M) \end{bmatrix} \begin{bmatrix} a_0^N \\ a_1^N \\ \vdots \\ a_{L-1}^N \\ a_L^N \end{bmatrix} = \begin{bmatrix} f_c(S_0) \\ f_c(S_1) \\ \vdots \\ f_c(S_{M-1}) \\ f_c(S_M) \end{bmatrix} \tag{4.3.5}$$

Denote

$$\mathbf{a}^{(N)} = \left\{a_0^N, a_1^N, \cdots, a_{L-1}^N, a_L^N\right\}$$

and

$$M = \begin{bmatrix} \phi_0(\xi_0) & \phi_1(\xi_0) & \cdots & \phi_L(\xi_0) \\ \phi_0(\xi_1) & \phi_1(\xi_1) & \cdots & \phi_L(\xi_1) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_0(\xi_{M-1}) & \phi_1(\xi_{M-1}) & \cdots & \phi_L(\xi_{M-1}) \\ \phi_0(\xi_M) & \phi_1(\xi_M) & \cdots & \phi_L(\xi_M) \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} f_c(S_0), & \cdots & , f_c(S_M) \end{bmatrix}^T,$$

then (4.3.5) becomes

$$M \cdot \mathbf{a}^{(N)} = \mathbf{b}$$

thus

$$\mathbf{a}^{(N)} = M^+ \cdot \mathbf{b} \tag{4.3.6}$$

where M^+ is the pseudo inverse of M.

The PIDE to

$$
\begin{aligned}
&-(\lambda+r)f_c+\frac{\partial f_c}{\partial t}+(r-\lambda)S\frac{\partial f_c}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f_c}{\partial S^2}+\lambda\int_A^B\frac{1}{\sqrt{2\pi}\delta\eta}e^{(-\frac{(\log(\eta))^2}{2\delta^2})}f_c(t,\eta S)d\eta=0\\
&\Rightarrow -(\lambda+r)\Delta t[\theta_1\sum_{k=0}^{L}a_k(j+1)\phi_k(\xi_i)+(1-\theta_1)\sum_{k=0}^{L}a_k(j)\phi_k(\xi_i)]\\
&\quad +[\sum_{k=0}^{L}a_k(j+1)\phi_k(\xi_i)+\sum_{k=0}^{L}a_k(j)\phi_k(\xi_i)]\\
&\quad +(r-\lambda)\xi_i\Delta t[\theta_1\sum_{k=0}^{L}a_k(j+1)\phi_k'(\xi_i)+(1-\theta_1)\sum_{k=0}^{L}a_k(j)\phi_k'(\xi_i)]\\
&\quad +\frac{1}{2}\sigma^2\xi_i^2[\theta_1\sum_{k=0}^{L}a_k(j+1)\phi_k''(\xi_i)+(1-\theta_1)\sum_{k=0}^{L}a_k(j)\phi_k''(\xi_i)]+\Delta t[(1-\theta_2)\mathbf{I}^j+\theta_2\mathbf{I}^{j+1}]=0\\
&\Rightarrow \sum_{k=0}^{L}a_k(j)\left\{\phi_k(\xi_i)-(1-\theta)\Delta t[(r-\lambda)\phi_k'(\xi_i)+\frac{1}{2}\sigma^2\xi_i^2\phi_k''(\xi_i)-(r+\lambda)\phi_k(\xi_i)]\right\}+\Delta t(1-\theta)I_i^j\\
&\quad =\sum_{k=0}^{L}a_k(j+1)\left\{\phi_k(\xi_i)-\theta\Delta t[(r-\lambda\beta)\phi_k'(\xi_i)+\tfrac{1}{2}\sigma^2\xi_i^2\phi_k''(\xi_i)-(r+\lambda)\phi_k(\xi_i)]\right\}+\Delta t\theta I_i^{j+1}
\end{aligned}
$$

so the equation can been rewritten

$$
\begin{aligned}
&-(\lambda+r)f_c+\frac{\partial f_c}{\partial t}+(r-\lambda)S\frac{\partial f_c}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 f_c}{\partial S^2}+\lambda\int_A^B\frac{1}{\sqrt{2\pi}\delta\eta}e^{(-\frac{(\log(\eta))^2}{2\delta^2})}f_c(t,\eta S)d\eta=0\\
&\Rightarrow \sum_{k=0}^{L}a_k(j)\left\{\phi_k(\xi_i)-(1-\theta_1)\Delta t[(r-\lambda)\phi_k'(\xi_i)+\frac{1}{2}\sigma^2\xi_i^2\phi_k''(\xi_i)-(r+\lambda)\phi_k(\xi_i)]\right\}+\Delta t(1-\theta_2)\mathbf{I}^j\\
&\quad =\sum_{k=0}^{L}a_k(j+1)\left\{\phi_k(\xi_i)-\theta_1\Delta t[(r-\lambda)\phi_k'(\xi_i)+\tfrac{1}{2}\sigma^2\xi_i^2\phi_k''(\xi_i)-(r+\lambda)\phi_k(\xi_i)]\right\}+\Delta t\theta_2\mathbf{I}^{j+1}
\end{aligned}
\tag{4.3.7}
$$

where $i=1,2,3,\cdots,N-1$ and let

$$
\begin{aligned}
d_k(\xi_i)&=\phi_k(\xi_i)-(1-\theta)\Delta t[(r-\lambda)\phi_k'(\xi_i)+\tfrac{1}{2}\sigma^2\xi_i^2\phi_k''(\xi_i)-(r+\lambda)\phi_k(\xi_i)]\\
e_k(\xi_i)&=\phi_k(\xi_i)-\theta\Delta t[(r-\lambda)\phi_k'(\xi_i)+\tfrac{1}{2}\sigma^2\xi_i^2\phi_k''(\xi_i)-(r+\lambda)\phi_k(\xi_i)]
\end{aligned}
\tag{4.3.8}
$$

Therefore

$$
\begin{bmatrix}
d_0(\xi_0) & d_1(\xi_0) & \cdots & d_L(0_1) \\
d_0(\xi_1) & d_1(\xi_1) & \cdots & d_L(\xi_1) \\
\vdots & \vdots & \ddots & \vdots \\
d_0(\xi_{M-1}) & d_1(\xi_{M-1}) & \cdots & d_L(\xi_{M-1}) \\
d_0(\xi_M) & d_1(\xi_M) & \cdots\cdots & d_L(\xi_M)
\end{bmatrix}
\begin{bmatrix}
a_0^j \\ a_1^j \\ \vdots \\ a_{L-1}^j \\ a_L^j
\end{bmatrix}
+
$$

$$
\Delta t(1-\theta_2)
\begin{bmatrix}
\psi_0(\xi_0) & \psi_1(\xi_0) & \cdots & \psi_L(\xi_0) \\
\psi_0(\xi_1) & \psi_1(\xi_1) & \cdots & \psi_L(\xi_1) \\
\vdots & \vdots & \ddots & \vdots \\
\psi_0(\xi_M) & \psi_1(\xi_M) & \cdots & \psi_L(\xi_M)
\end{bmatrix}
\begin{bmatrix}
a_0^j \\ a_1^j \\ \vdots \\ a_L^j
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
e_0(\xi_0) & e_1(\xi_0) & \cdots & e_L(\xi_0) \\
e_0(\xi_1) & e_1(\xi_1) & \cdots & e_L(\xi_1) \\
\vdots & \vdots & \ddots & \vdots \\
e_0(\xi_{M-1}) & e_1(\xi_{M-1}) & \cdots & e_L(\xi_{M-1}) \\
e_0(\xi_M) & e_1(\xi_M) & \cdots & e_L(\xi_M)
\end{bmatrix}
\begin{bmatrix}
a_0^{j+1} \\ a_1^{j+1} \\ \vdots \\ a_{L-1}^{j+1} \\ a_L^{j+1}
\end{bmatrix}
+
$$

$$
\Delta t\theta_2
\begin{bmatrix}
\psi_0(\xi_0) & \psi_1(\xi_0) & \cdots & \psi_L(\xi_0) \\
\psi_0(\xi_1) & \psi_1(\xi_1) & \cdots & \psi_L(\xi_1) \\
\vdots & \vdots & \ddots & \vdots \\
\psi_0(\xi_M) & \psi_1(\xi_M) & \cdots & \psi_L(\xi_M)
\end{bmatrix}
\begin{bmatrix}
a_0^{j+1} \\ a_1^{j+1} \\ \vdots \\ a_L^{j+1}
\end{bmatrix}
$$

where

$$
\begin{aligned}
\psi_k(\xi_i) = \frac{\lambda}{\sqrt{2\pi}\delta}\frac{\Delta\eta}{3}[\frac{1}{A}e^{(-\frac{(\log(A))^2}{2\delta^2})}\phi_k(A\xi_i) + 2\sum_{h=1}^{\frac{p}{2}-1}\frac{1}{\eta_{2h}}e^{(-\frac{(\log(\eta_{2h}))^2}{2\delta^2})}\phi_k(\eta_{2h}\xi_i) \\
+4\sum_{h=1}^{\frac{p}{2}-1}\frac{1}{\eta_{2h-1}}e^{(-\frac{(\log(\eta_{2h-1}))^2}{2\delta^2})}\phi_k(\eta_{2h-1}\xi_i) + \frac{1}{B}e^{(-\frac{(\log(B))^2}{2\delta^2})}\phi_k(B\xi_i)]
\end{aligned}
\qquad (4.3.9)
$$

$$
D = \begin{bmatrix} d_0(\xi_0) & d_1(\xi_0) & \cdots & d_L(0_1) \\ d_0(\xi_1) & d_1(\xi_1) & \cdots & d_L(\xi_1) \\ \vdots & \vdots & \ddots & \vdots \\ d_0(\xi_{M-1}) & d_1(\xi_{M-1}) & \cdots & d_L(\xi_{M-1}) \\ d_0(\xi_M) & d_1(\xi_M) & \cdots\cdots & d_L(\xi_M) \end{bmatrix} \tag{4.3.10}
$$

$$
E = \begin{bmatrix} e_0(\xi_0) & e_1(\xi_0) & \cdots & e_L(\xi_0) \\ e_0(\xi_1) & e_1(\xi_1) & \cdots & e_L(\xi_1) \\ \vdots & \vdots & \ddots & \vdots \\ e_0(\xi_{M-1}) & e_1(\xi_{M-1}) & \cdots & e_L(\xi_{M-1}) \\ e_0(\xi_M) & e_1(\xi_M) & \cdots & e_L(\xi_M) \end{bmatrix} \tag{4.3.11}
$$

with $i = 0, 1, 2, \cdots, M, \quad k = 0, 1, 2, \cdots, L, \quad h = 0, 1, 2, \cdots, P$

Denote as:

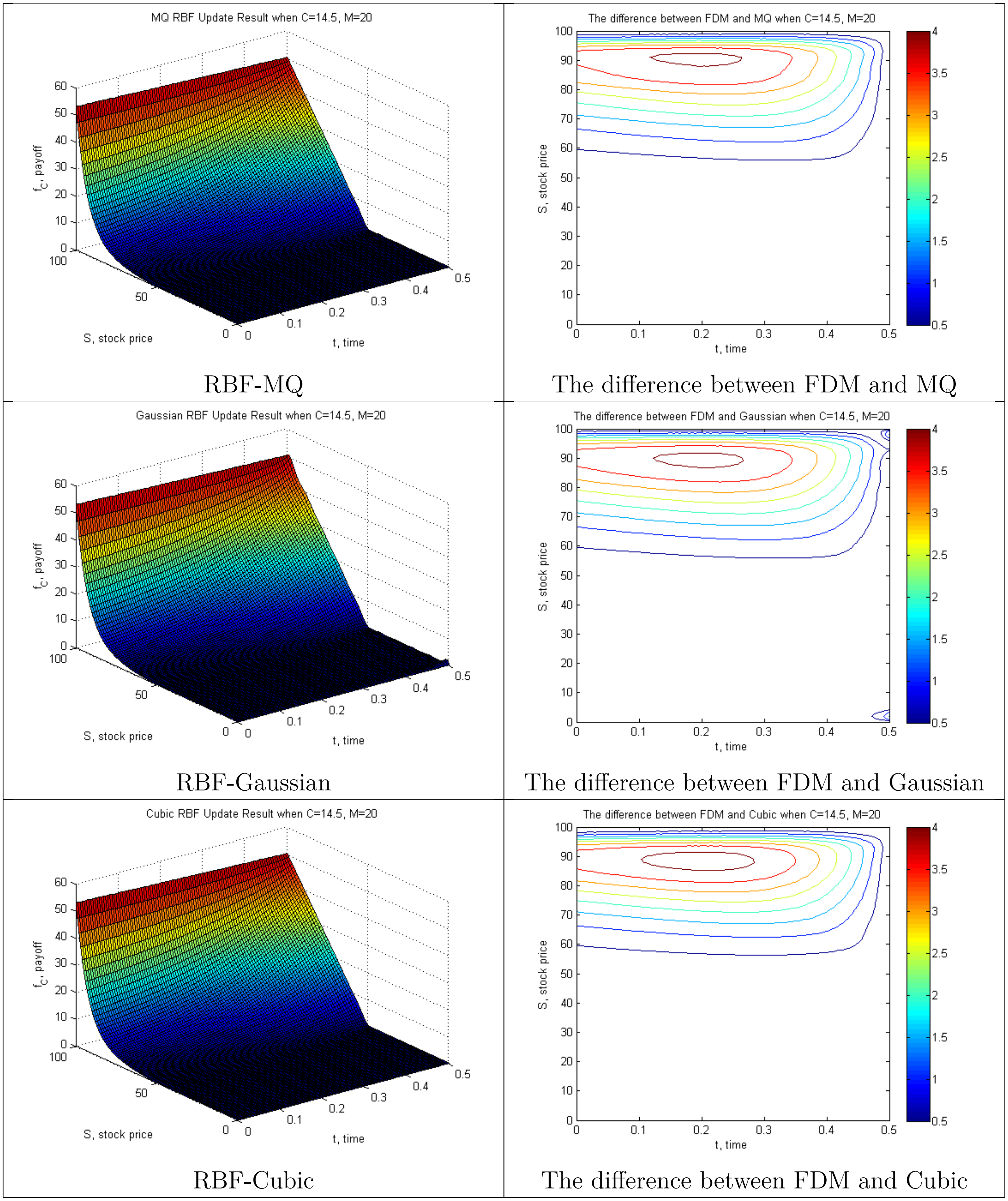
$$
\begin{aligned}
& D \cdot \mathbf{a}^{(j)} + \Delta t \cdot (1 - \theta_2) \cdot I \cdot \mathbf{a}^{(j)} = E \cdot \mathbf{a}^{(j+1)} + \Delta t \cdot \theta_2 \cdot I \cdot \mathbf{a}^{(j+1)} \\
& \Rightarrow [D + \Delta t \cdot (1 - \theta_2) \cdot I] \cdot \mathbf{a}^{(j)} = [E + \Delta t \cdot \theta_2 \cdot I] \cdot \mathbf{a}^{(j+1)} \\
& \Rightarrow \mathbf{a}^{(j)} = [D + \Delta t \cdot (1 - \theta_2) \cdot I]^{-1} \cdot [E + \Delta t \cdot \theta_2 \cdot I] \cdot \mathbf{a}^{(j+1)}
\end{aligned} \tag{4.3.12}
$$

where $j = 0, 1, 2, \cdots, N$

For the parameters are

$r = 0.10, \quad \sigma = 0.40, \quad T = 0.5, \quad K = 50.00, \quad \lambda = 1, \quad p = 0.4, \quad q = 0.6, \quad \eta_1 = 10, \quad \eta_2 = 5,$
$\delta = \frac{1}{2}\sigma^2,$

with $S_L = 0$ and $S_U = 100.00$, the numerical solution computed by RBF's are presented in Figure4.3, and their difference to FDM is given in Figure 4.3. Thus, we know the Cubic and MQ RBF have better performace than TPS and Gaussian RBF. Also the Cubic RBF can avoid the open question of choosing an optimal shape parameter c.



RBF-MQ | The difference between FDM and MQ

RBF-Gaussian | The difference between FDM and Gaussian

RBF-Cubic | The difference between FDM and Cubic

FIGURE 4.3. The numerical result and the difference between FDM of three RBFs

CHAPTER 5

Conclusion

The aim of this paper is to compute that European call option prices in Black-Scholes and Jump-Diffusion models via Finite Difference Method and meshfree methods based on Radial Basis Functions.

In Chapter 2 we present the follow things. First, derive that exact solution of the Black-Scholes partial differential equation in one space dimension. Secondly using explicit, implicit and Crank-Nicolson of Finite difference Method to compute the Black-Scholes European call option prices. Thirdly four radial basis functions are used to pricing compute the Black-Scholes European call option numerically.

In Chapter 3 we disscuss the exact solution on the problem of pricing a European call when the underlying asset is dirven by a Brownian motion and a compound Poisson process. In the model, one should notice that the price jump process V_i is i.i.d so for each $i\epsilon\{1, 2, 3, \cdots\}$, V_i has the same mean and variance. PIDE consists of a 'classical' Black Scholes part and an intergral part. Starting with the jump-diffusion model for the underlying asset we motivate how to find the corresponding PIDE that models a derivative quantity on that asset.

In Chapter 4 we discuss numerical methods that approximate the solution of the PIDE that models contingent claims with jumps. And a partial integral difference equation is obtained to describe the JDM model. We define a finite integral of integration for improper integral and propose simple numerical algorithms for finding a finite computational range of an improper integral term in the PIDE so that the accuracy of approximation of the integral can be improved. And final we use FDM and RBF to propose for numerically solving initial value and free boundary problem for the Jump-Diffusion model.

In summery, two different approaches, i.e., FDM and meshfree method with RBF are use to numerically solve the PDE of BSM and PIDE of JDM with appropriate buondary conditions for pricing European call options. In general, the RBF meshfree method is faster than FDM with the same tolerance, and it is obviously the Cubic and MQ function have better performance than Gaussian and TPS functions. Beside, the Cubic function can avoid the open question of choosing an optimal shape parameter c.

In the future, we can suggest to extend the present method in two assets. Alternatively, the maximal likelihood method can be used to obtain the appropriate parameters from the market informations instead of using historical information. And then we can use these parameters into our numerical model to pricing the options.

Bibliography

[1] Amadori, A. L., The Obstacle Problem for Nonlinear Integro-Differential Equations Arising in Option Pricing, *Ricerche di Matematica* 56, 1-17, 2007.

[2] Black, F. and Scholes, M., The Pricing of Options and Corporate Liabilities, *Journal of Political Economy,* 81, 637, 1973. 17

[3] Briani, M., and Natalini, R., and Russo, G., Implicit-Explicit Numerical Schemes for Jump-Diffusion Processes, *Calcolo* 44, 33-57, 2004.

[4] Chan, R. T. L. and Hubbert, S., A Numerical Study of Radial Basis Function Based Methods for Option Pricing Under One Dimension JumpDiffusion Model, *Applied Numerical Mathematics (Submitted),* 2010.

[5] Chan, R. T. L., Pricing Options under Jump-Diffusion Models by Adaptive Radial Basic Functions, Bath Economics Research Papers No. 06/10, Department of Economics, University of Bath, 2010.

[6] Cont, R. and Voltchkova, E., A Finite Difference Scheme for Option Pricing in Jump Diffusion and Exponential Lèvy Model, *SIAM J. Numer. Anal.,* 43, 1596–1626, 2005.

[7] Duffy, D. J., A Critique of the Crank-Nicolson Scheme Strengths and Weaknesses for Financial Instrument Pricing, Technical Article, *WILMOTT Magazine*, 68-76, 2004.

[8] Duffy, D. J., *Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach*, John Wiley & Sons, Inc., 2005.

[9] Franke, R., Scattered Data Interplation: Test of Some Methods, *Journal Mathematical Computation*, 38, 181-200, 1982. 17

[10] Hon, Y. C. and Mao, X. Z., A Multiquadric Interpolation Method for Solving Initial Value Problem, *Journal of Scientific Computing*, 12(1), 51-55, 1997. 17

[11] Hon, Y. C. and Mao, X. Z., An Efficient Numerical Scheme for Burger'sequation, *Applied Mathematics and Computation*, 95, 37-50, 1998. 17

[12] Hon, Y. C. and Mao, X. Z., A Radial Basis Function Method for Solving Options Pricing Model, *Financial Engineering*, 8(1), 31-50, 1999. 17

[13] Hon, Y. C. and Zhou, X., A Comparison on Using Various Radial Basis Functions for Options Pricing, http://www.cityu.edu.hk/ma/staff/ychon/option-XZhou.ps.gz, 2000. 17

[14] Kou, S.G., A Jump Diffusion Model for Option Pricing with three Properties: Leptokurtic Feature, Volatility Smile, and Analytical Tractability, *Proceedings of the IEEE/IAFE/INFORMS 2000 Conference on Computational Intelligence for Financial Engineering* (CIFEr), 129-131, 2000. 18

[15] Kou, S. G., A Jump Diffusion Model for Option Pricing, *Management Science* 48, 1086-1101, 2002. 18, 43

[16] Kou, S. G. and Wang H., Option Pricing under a Double Exponential Jump Diffusion Model, *Manegement Science* 59, 1178-1192, 2004.

[17] Matsuda, K., Introduction to Merton Jump Diffusion Model, http://www.maxmatsuda.com/Papers/Intro/Intro%20to%20MJD%20Matsuda.pdf, 2004.

[18] Merton, R., Option Pricing When Underlying Stock Returns Are Discontinuous, *Journal of Financial Economics* 3, 125-144, 1976. 18

[19] Randall, C., and Tavella, D., *Pricing Financial Instruments: the Finite Difference Method*, John Wiley & Sons, Inc., 2000.

[20] Pan, J.L., *Mesh-based and Mesh-free Methods: Theory, Implementation and Application.*, Department of Mathematics, Tunghai University, 2011.

[21] Santos, G. T. and Fortes, M., Use of Radial Basis Functions for Meshless Numerical Solutions Applied to Financial Engneering Barrier Options, *Pesquisa Operacional,* 29, 419–437, 2009.

[22] Shreve, S. E., *Stochastic Calculus for Finance II. Continuous-Time Models*, Springer, 2004. 46

[23] Zemanian, A.H., *Distribution Theory and Transform Analysis*, Dover, 1987. 53