東海大學數學系研究所

碩　士　論　文

指導教授:　葉芳栢　博士
黃皇男　博士

Maximum Likelihood Estimation in Vasicek, Black-Scholes and Jump-Diffusion Models

最大概似法估計利率與股價跳躍模型之參數

研究生: 吳培瑛

中　華　民　國　一　百　年　七　月

誌謝

一轉眼研究所生涯就要結束了，回想當初進研究所時還不能確定自己未來的研究方向，直到上了葉芳栢老師的課後，加上當時美國發生次貸風暴，才對這塊領域有了興趣，之後便決定朝這個方向研究，並選擇葉芳栢老師和黃皇男老師為指導教授。

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本系碩士班 吳培瑛 君

所提論文 Maximum Likelihood Estimation in Vasicek, Black-Scholes and Jump-Diffusion Models
（最大概似法估計利率與股價跳躍模型之參數）

合於碩士班資格水準，業經本委員會評審通過，特此證明。

口試委員：黃皇男

杜淑儀

朱香蕙

李孟育

指導教授：葉芳栢

所　　長：陳文豪

中　華　民　國　一〇〇　年　七　月　二十八　日

Abstract

There are two main topics in the present study. The first topic focuses on determining the parameters of the Vasicek model. Since if we use the zero coupon bond as a price model to pricing the option value, the simulation of term structure is very important. Before the bond price is obtained, we have to simulate the short interest rate using Vasicek model. The likelihood function is used to estimate the parameters of Vasicek model with U.S. Treasury one year bond rate data.

The asset price in the current market is modeled with Black-Sholes, and Merton construct a Poisson process into their model to describe if there are some extreme jumps in the asset price. The second part of present study is to estimate the parameters of Black-Scholes model and the jump-diffusion mode by using the maximum likelihood approach.

Keywords: Maximum likelihood, Black-Scholes model, jump-diffusion model

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摘 要

本篇論文所要探討的主題有兩個部份：第一部份在評價歐式債券選擇權時，通常利用零息債券作爲評價的工具，並得到選擇權的確解，在這之前必須使用利率模型來模擬零息債券的價格，若能以實際市場資料來對利率模型的參數做估計，便可帶入選擇權的確解來模擬價格，本文是以 Vasicek 模型做爲參數估計的對象。

第二部份是以 Blcak-Scholes 模型作爲評價歐式選擇權的工具，但股價有時會有劇烈的大漲或大跌，Merton 在原本的 Blcak-Scholes 模型中加入了 Poisson 過程，來描述上述現象，稱爲跳躍擴散模型，本文利用最大概似函數估計以上模型的參數。

關鍵字: 最大概似法、Vasicek 模型、Black-Scholes 模型、跳躍擴散模型

List of symbols

Symbol	Meaning
$\mathscr{F}$	-filtration
$\ln L(\theta)$	-the maximum likelihood estimator
$N(t)$	-Poisson process
$\mathcal{N}(\mu, \sigma^2)$	-normal distribution
$\mathbb{P}$	-risk free probability
$W(t)$	-Brownian motion
r	-Risk free rate
$(\Omega, \mathscr{F}, \mathbb{P})$	-probability space
λ	-the intesity of Poisson process
σ	-volatility

CHAPTER 1

Introduction

In the last twenty years, the option pricing have become increasingly popular, and the values of European options can be calculated by some pricing models. The estimation of the associated parameters for interest rate and pricing models for the current market is very important in pricing the option. Hence this study discuss the parameter estimation of those models by using the maximal likelihood method.

The first part of the present study is to estimate the parameters of Vasicek model. If we use the zero coupon bond as a price model to pricing the option value, the simulation of term structure is very important. Vasicek (1976) was the first to give an explicit characterization with mean reverting of the term structure. Before we get the bond price, we have to simulate the short interest rate using Vasicek model. So the focus of our study on determining the parameters of the Vasicek model. Now we use the likelihood function to estimate the parameters of Vasicek model with U.S. Treasury one year bond rate data.

The second part is to estimate the parameters of BSM and JDM models. Black and Scholes (1973) models the asset price, and Merton (1976) construct a Poisson process in the BSM to describe if there are some extreme jumps in the asset price which is called jump-diffusion model (JDM). We also use the likelihood to estimate the two models and compare which model more fits the data.

The organization of this thesis is as following. In the next chapter we introduce the three models and their explicit solutions. In Chapter 3, we estimate parameters of Vasicek model. In Chapter 4, we estimate the parameters of BSM and JDM. Finally, some conclusions are given.

CHAPTER 2

Mathematical Preliminary

2.1. Brownian motion

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and $\xi = \{\xi(t) \mid t \in [0,T]\}, \forall\, t \in [0,T]$, where $\xi(t)$ is a random variable and ξ is called the stochastic process. Let $\{\mathscr{F}(t)\}_{t\in[0,T]}$ be the $\sigma - algebra$ set $\mathscr{F}(t) \subset \mathscr{F}\ \forall\, t$ and $\forall s \leq t$, $\mathscr{F}_s \subseteq \mathscr{F}_t$ is called the filtration.

DEFINITION 2.1. $W = \{W(t) | t \in [0,T]\}$ is called a Brownian motion if

- $W(0) = 0$ a.s..
- $t \mapsto W(t)$ is continuous a.s..
- $\forall\ 0 < t_1 < t_2 < \ldots < t_n$

  $W(t_1) - W(0) < W(t_2) - W(t_1) < ... < W(t_n) - W(t_{n-1})$ are all independent.
- $\forall\ 0 < s < t$, $W(t) - W(s) = W(t-s) \sim \mathcal{N}(0, t-s)$

  $\forall\ A \in \mathscr{B}(\mathbb{R})$

$$\mathbb{P}(W(t) - W(s) \quad \in \quad A) = \int_A \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}} dx$$

The following figure shows a typical path of the Brownian motion with $t - s = 1/250$.

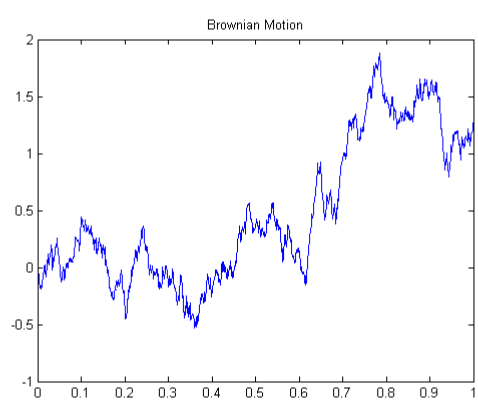

FIGURE 2.1. Brownian Motion path with $t - s = 1/250$

2.2. Vasicek model

For a suitable choice of the market price of risk, this is equivalent to assume that r follows an Ornstein-Uhlenbeck process (Vasicek 1976) with constant coefficients under the risk-neutral probability as well, that is,

$$dr(t) = a(b - r(t))\,dt + \sigma dW(t),$$

where a is the speed of reversion, and b is long term mean level, σ is the instantaneous volatility, $\frac{\sigma^2}{2a}$ is long term variance.

Integrating equation $dr(t)$ for each $s \leq t$, then,

$$r(t) = r(s)\,e^{-a(t-s)} + b\left(1 - e^{-a(t-s)}\right) + \sigma \int_s^t e^{-a(t-u)} dW(u).$$

2.3. Black-Scholes model

The Black-Scholes model (Black, Scholes 1973) is

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW(t)$$

under the risk-neutral probability.

And the exact solution:

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma W(t)}$$

we can check this by Itô lemma.

2.4. Jump-Diffusion Model

Definition 2.2. $N = \{N(t)|t \geq 0\}$ is a Poisson process if

- $N(0) = 0$, a.s..
- $\forall\ t \mapsto N(t)$ is increasing indicator function, and the jump size is 1.
- $\forall\ 0 < t_1 < t_2 < \ldots < t_n$,
  $N(t_1) - N(0) < N(t_2) - N(t_1) < ... < N(t_n) - N(t_{n-1})$ are all independent.

- $\forall\ 0 < s < t$, $N(t) - N(s) \sim \mathcal{P}(\lambda(t-s))$

$$\mathbb{P}(N(t) - N(s) = k) = \frac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{k!},$$

λ is the intesity of Poisson process.

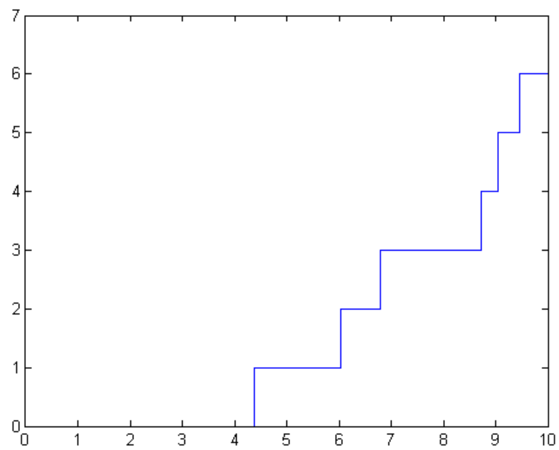

FIGURE 2.2. Poisson Process with $\lambda = 1$

The Jump-Diffusion model can be expressed as the following :

$$\frac{dS(t)}{S(t^-)} = rdt + \sigma dW(t) + d(\sum_{i=1}^{N(t)}(y_i - 1)).$$

The asset price jumps from $S(t)$ to $y_t S(t)$, so the relative price jump size is

$$\frac{dS(t)}{S(t^-)} = \frac{y_t S(t) - S(t)}{S(t^-)} = y_t - 1,$$

and the $dW(t)$ is a Brownian motion with mean 0 and variance dt, and $\sum_{i=1}^{N(t)}(y_i - 1)$ is a compound poisson process, $y_i \sim log\mathcal{N}(e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1, e^{2\mu_J + \sigma_J^2}(e^{\sigma_J^2} - 1))$.

The exact solution:

$$S(t) = S(0)e^{(r - \frac{1}{2}\sigma^2)t + \sigma W(t)} \prod_{i=0}^{N(t)} Y_i$$

, where $Y_i = \ln y_i \sim \mathcal{N}(\mu_J, \sigma_J^2)$.

the asset price jumps from $S(t)$ to $y_t S(t)$, so the relative price jump size is

$$\frac{dS(t)}{S(t^-)} = \frac{y_t S(t) - S(t)}{S(t^-)} = y_t - 1,$$

if the number of jump time $N(t) = 1$, the relative price jump size is $y_t - 1$, we suppose $y_t = 0.8$, that means the asset price falls by 20%.

Now we check the exact solution, Cont and Tankov(2004) gave the Itô lemma for the JDM in differential notation,

$$\begin{aligned} df(X(t),t) &= \frac{\partial f(X(t),t)}{\partial t}dt + b(t)\frac{\partial f(X(t),t)}{\partial x}dt + \frac{\sigma^2(t)}{2}\frac{\partial^2 f(X(t),t)}{\partial x^2}dt \\ &\quad + \frac{\partial f(X(t),t)}{\partial x}\sigma(t)dW(t) + [f(X(t^-) + \Delta X(t)) - f(X(t^-))], \end{aligned}$$

where the $b(t)$ is the drift and $\sigma(t)$ is the volatility of a JDM,

$$X(t) = X(0) + \textstyle\int_0^t \; b(s)ds + \displaystyle\int_0^t \sigma(s)dW(s) + \sum_{i=1}^{N(t)} \Delta X(i),$$

by applying this:

$$\begin{aligned} dS(t) &= S(t^-)(rdt + \sigma dW(t) + d(\sum_{i=1}^{N(t)}(y_i - 1))) \\ d\ln S(t) &= \frac{\partial \ln S(t)}{\partial t}dt + rS(t^-)\frac{\partial \ln S(t)}{\partial S(t)}dt + \frac{\sigma^2}{2}S^2(t^-)\frac{\partial^2 \ln S(t)}{\partial S^2(t)}dt \\ &\quad + \frac{\partial \ln S(t)}{\partial S(t)}\sigma S(t^-)dW(t) + [\ln y_t S(t) - \ln S(t)] \\ &= rS(t^-)\frac{1}{S(t)}dt - \frac{\sigma^2}{2}S^2(t^-)\frac{1}{S^2(t)}dt + \sigma S(t^-)\frac{1}{S(t)}dW(t) + \ln y_t \end{aligned}$$

$$
\begin{aligned}
d\ln S(t) &= (r-\frac{\sigma^2}{2})dt+\sigma dW(t)+\ln y_t \\
\ln S(t)-\ln S(0) &= y_t(r-\frac{\sigma^2}{2})t+\sigma W(t)+\sum_{i=1}^{N(t)}\ln y_t \\
\ln S(t) &= \ln S(0)+(r-\frac{\sigma^2}{2})t+\sigma W(t)+\sum_{i=1}^{N(t)}Y_i \\
S(t) &= S(0)e^{(r-\frac{\sigma^2}{2})t+\sigma W(t)}\prod_{i=0}^{N(t)}Y_i.
\end{aligned}
$$

2.5. Maximum Likelihood Estimation

This estimation method assumes that the distribution of an oberved phenomenon is known, except for a finite number of unknown parameters. Then, the unknown parameters will be estimated by looking at the sample values and then choosing our extimates of the unknown parameters the values for which the probability of getting the sample values is a maximum.

Let the probability density function of a random variable $x=(x_{1,\ldots,}x_n)$ with parameter θ, denote by $f(x\mid\theta)=f(x_{1,\ldots,}x_n\mid\theta)$. If the observations are independent and identically distributed then the joint density is the product of the individual densities:

$$f(x_{1,\ldots,}x_n\mid\theta)=\prod_{i=1}^{n}f(x_i\mid\theta),$$

The likelihood function for the sample data is given by

$$L(\theta)=\prod_{i=1}^{n}f(x_i\mid\theta).$$

We take the ln of L on both sides first,

$$\ln L(\theta)=\sum_{i=1}^{n}\ln f(x_i\mid\theta),$$

since logarithm is a monotonic function then the value of θ that maximizes the log-likelihood function must also maximize the likelihood function.

To find the maximum of log-likelihood function, the necessary condition is

$$\frac{\partial \ln L\left(\theta\right)}{\partial \theta} = 0$$

the parameter θ is the root of the likelihood equation.

In order to get the standard error, first we have to caculate the variance-covariance matrix of the maximum likelihood estimator θ

$$Var(\theta) = (-E(\frac{\partial^2 \ln L}{\partial \theta^2}(\theta)))^{-1},$$

the $-E(\frac{\partial^2 \ln L}{\partial \theta^2}(\theta))$ is called the expected Fisher information matrix and the standard errors of the estimator θ which is the square root of the diagonal terms in the variance-covariance matrix.

2.6. Bayesian Information Criterion

To compare the different kth models we defined the Bayesian information criterion(BIC) as following:

$$BIC^k = -2 \ln L(\theta^k) + d^k \ln n,$$

where n is the number of observe datas, d is the number of parameters. We can choose the smallest value of BIC for each model which is the best fit model.

CHAPTER 3

Parameter Estimation in the Vasicek Interest Rate Model

In this chapter, we want to estimate the parameters of the Vasicek model to represent the following U.S. T-bill daily data 1962/10/02 to 2010/12/31 with its size $N = 12000$. Figure 3.1 shows the time variation of this data set and some statistical information are listed in Table 3.1.

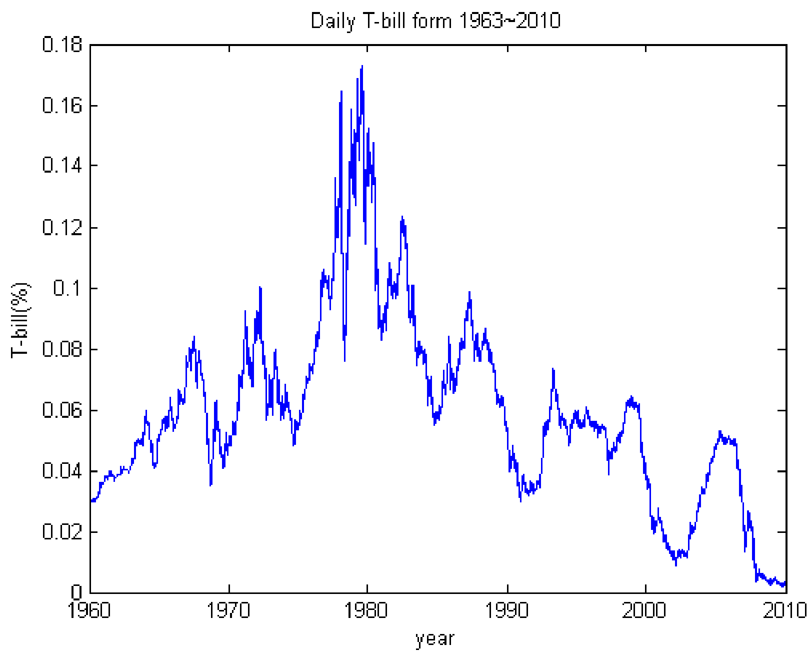

Figure 3.1. The time variation of U.S. T-bill daily data from 1962/10/02 to 2010/12/31.

Mean	Variance	Rate max	Rate min
0.0588	9.52×10^{-4}	0.1731	0.0021

Table 3.1. Some statistics of the U.S. T-bill daily data from 1962/10/02 to 2010/12/31.

The exact solution of Vasicek model is given by

$$r\left(t\right) = r\left(s\right) e^{-a(t-s)} + b\left(1 - e^{-a(t-s)}\right) + \sigma \int_s^t e^{-a(t-u)} dW\left(u\right),$$

with risk-neutral probability, so that $r(t)$ conditional on $\mathscr{F}_s$ is normally distributed with mean and variance

$$\begin{aligned}
E\left[r\left(t\right)\mid r(s)\right] &= r\left(s\right)e^{-a(t-s)}+b\left(1-e^{-a(t-s)}\right)\\
Var\left[r\left(t\right)\mid r(s)\right] &= \frac{\sigma^2}{2a}\left[1-e^{-2a(t-s)}\right]\\
r\left(t\right) &\sim N\left(r\left(s\right)e^{-a(t-s)}+b\left(1-e^{-a(t-s)}\right),\frac{\sigma^2}{2a}\left[1-e^{-2a(t-s)}\right]\right)
\end{aligned}$$

The interest can be discretized as

$$r_{i+1} = r_i e^{-adt}+b\left(1-e^{-adt}\right)+\sigma\int_t^{t+dt}e^{-a(t-u)}dW\left(u\right).$$

where $dt=\frac{1}{250}$ such that we can obtain a discrete set $\{r_i\}$ corresponding to the U.S. T-bill daily data. Denote $\theta=(a,b,\sigma^2)$ and let $\alpha=e^{-adt}$, $\beta=b\left(1-e^{-adt}\right)$, and $V^2=\frac{\sigma^2}{2a}\left[1-e^{-2adt}\right]$. The log-likelihood fuction for the discrete data set $\{r_i\}$ is defined by

$$\begin{aligned}
\ln L(\theta) &= \sum_{i=1}^{n}\ln\left[(\frac{1}{\sqrt{2\pi V^2}}e^{-\frac{(r_{i+1}-\alpha r_i-\beta)^2}{2V^2}}\right]\\
&= -\frac{1}{2}\sum_{i=1}^{n}\left[(\ln 2\pi+\ln V^2)+\frac{(r_{i+1}-\alpha r_i-\beta)^2}{V^2}\right].
\end{aligned}$$

To determine the maximum of the log-likelihood function, we have to find the parameter θ such that the zero gradient of $\ln L(\theta)$ is achieved. Since

$$\begin{aligned}
\frac{\partial \ln L(\theta)}{\partial a} &= \frac{\partial \ln L}{\partial \alpha}\frac{d\alpha}{da}+\frac{\partial \ln L}{\partial \beta}\frac{d\beta}{da}+\frac{\partial \ln L}{\partial V^2}\frac{dV^2}{da}\\
&= (-e^{-adt}dt)\sum_{i=1}^{n}\frac{r_{i+1}r_i-\alpha r_i^2-\beta r_i}{V^2}+(be^{-adt}dt)\sum_{i=1}^{n}\frac{r_{i+1}-\alpha r_i-\beta}{V^2}+\\
&\quad\left\{\left[-\frac{n}{2V^2}+\frac{1}{2}\sum_{i=1}^{n}\frac{(r_{i+1}-\alpha r_i-\beta)^2}{(V^2)^2}\right]\left[-\frac{\sigma^2}{2a^2}(1-e^{-2adt})+\frac{\sigma^2}{a}(e^{-2adt}dt)\right]\right\}\\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \ln L(\theta)}{\partial b} &= \frac{\partial \ln L}{\partial \beta}\frac{d\beta}{db} \\
&= \sum_{i=1}^{n} \frac{r_{i+1} - \alpha r_i - \beta}{V^2}(1 - e^{-adt}) \\
&= 0 \\
\frac{\partial \ln L(\theta)}{\partial \sigma^2} &= \frac{\partial \ln L}{\partial V^2}\frac{dV^2}{d\sigma^2} \\
&= \left[\frac{-n}{2V^2} + \frac{1}{2}\sum_{i=1}^{n} \frac{(r_{i+1} - \alpha r_i - \beta)^2}{(V^2)^2}\right]\frac{1}{2a}\left(1 - e^{-2adt}\right) = 0 \\
&= 0
\end{aligned}
$$

and $a \neq 0$, we arrive at the system of equations

$$
\begin{cases}
\displaystyle\sum_{i=1}^{n} \frac{r_{i+1}r_i - \alpha r_i^2 - \beta r_i}{V^2} & = 0, \\
\displaystyle\sum_{i=1}^{n} \frac{r_{i+1} - \alpha r_i - \beta}{V^2} & = 0, \\
\displaystyle\frac{-n}{2V^2} + \frac{1}{2}\sum_{i=1}^{n} \frac{(r_{i+1} - \alpha r_i - \beta)^2}{(V^2)^2} & = 0.
\end{cases}
$$

And the parameters α, β and V^2 are then given by

$$
\begin{aligned}
\alpha &= \frac{n\displaystyle\sum_{i=1}^{n} r_{i+1}r_i - \sum_{i=1}^{n} r_{i+1}\sum_{i=1}^{n} r_i}{n\displaystyle\sum_{i=1}^{n} r_i^2 - (\sum_{i=1}^{n} r_i)^2}, \\
\beta &= \frac{\alpha\displaystyle\sum_{i=1}^{n} r_i - \sum_{i=1}^{n} r_{i+1}}{n}, \\
V^2 &= \frac{\displaystyle\sum_{i=1}^{n}(r_{i+1} - \alpha r_i - \beta)^2}{n}.
\end{aligned}
$$

For the Vasicek model, the expected Fisher information matrix is described by

$$-E(\frac{\partial^2 \ln L}{\partial \theta^2}(\theta)) = -E\left[\left[\begin{array}{ccc} \frac{\partial^2 \ln L}{\partial a^2} & \frac{\partial^2 \ln L}{\partial a \partial b} & \frac{\partial^2 \ln L}{\partial a \partial \sigma^2} \\ \frac{\partial^2 \ln L}{\partial b \partial a} & \frac{\partial^2 \ln L}{\partial b^2} & \frac{\partial^2 \ln L}{\partial b \partial \sigma^2} \\ \frac{\partial^2 \ln L}{\partial \sigma^2 \partial a} & \frac{\partial^2 \ln L}{\partial \sigma^2 \partial b} & \frac{\partial^2 \ln L}{\partial (\sigma^2)^2} \end{array}\right]\right]$$

where

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial a^2} &= \frac{\partial^2 \ln L}{\partial \alpha^2}\left(\frac{\partial \alpha}{\partial a}\right)^2 + \frac{\partial^2 \ln L}{\partial \beta^2}\left(\frac{\partial \beta}{\partial a}\right)^2 + \frac{\partial^2 \ln L}{(\partial V^2)^2}\left(\frac{\partial V^2}{\partial a}\right)^2 + \\
&\quad 2\frac{\partial^2 \ln L}{\partial \alpha \partial \beta}\frac{\partial \alpha}{\partial a}\frac{\partial \beta}{\partial a} + 2\frac{\partial^2 \ln L}{\partial \alpha \partial V^2}\frac{\partial \alpha}{\partial a}\frac{\partial V^2}{\partial a} + 2\frac{\partial^2 \ln L}{\partial \beta \partial V^2}\frac{\partial \beta}{\partial a}\frac{\partial V^2}{\partial a}
\end{aligned}$$

$$\frac{\partial^2 \ln L}{\partial b^2} = \frac{\partial^2 \ln L}{\partial \beta^2}\left(\frac{\partial \beta}{\partial b}\right)^2$$

$$\frac{\partial^2 \ln L}{\partial (\sigma^2)^2} = \frac{\partial^2 \ln L}{\partial (V^2)^2}\left(\frac{\partial V^2}{\partial \sigma^2}\right)^2$$

$$\frac{\partial^2 \ln L}{\partial a \partial b} = \left(\frac{\partial^2 \ln L}{\partial \alpha \partial \beta}\frac{\partial \alpha}{\partial a} + \frac{\partial^2 \ln L}{\partial \beta^2}\frac{\partial \beta}{\partial a} + \frac{\partial^2 \ln L}{\partial \beta \partial V^2}\frac{\partial V^2}{\partial a}\right)\frac{\partial \beta}{\partial b}$$

$$\frac{\partial^2 \ln L}{\partial a \partial \sigma^2} = \left(\frac{\partial^2 \ln L}{\partial \alpha \partial V^2}\frac{\partial \alpha}{\partial \sigma^2} + \frac{\partial^2 \ln L}{\partial \beta \partial V^2}\frac{\partial \beta}{\partial \sigma^2} + \frac{\partial^2 \ln L}{\partial (V^2)^2}\frac{\partial V^2}{\partial \sigma^2}\right)\frac{\partial V^2}{\partial \sigma^2}$$

$$\frac{\partial^2 \ln L}{\partial \sigma^2 \partial b} = \frac{\partial^2 \ln L}{\partial \beta \partial V^2}\frac{\partial V^2}{\partial \sigma^2}\frac{\partial \beta}{\partial b}.$$

The estimate parameters and its standard errors (computed by using Fisher matrix) corresponding to the Vasicek model are listed in Table 3.2. And the plot of the orginal data versus the simulated Vasicek behavior is shown in Figure 3.2. The trend in the figure represents the long term average of the given data set but it does not fit the entire data curve very well and hence we can also see that the simulated behavior of the Vasicek model is not consistent to the time trajectory of the data set.

	a	b	σ	$\ln L$
Parameters	0.07837	0.05219	0.01407	67264.11
Standard error	0.06573	0.02660	2.56×10^{-6}	

Table 3.2. Corresponding arameters for the Vasicek model

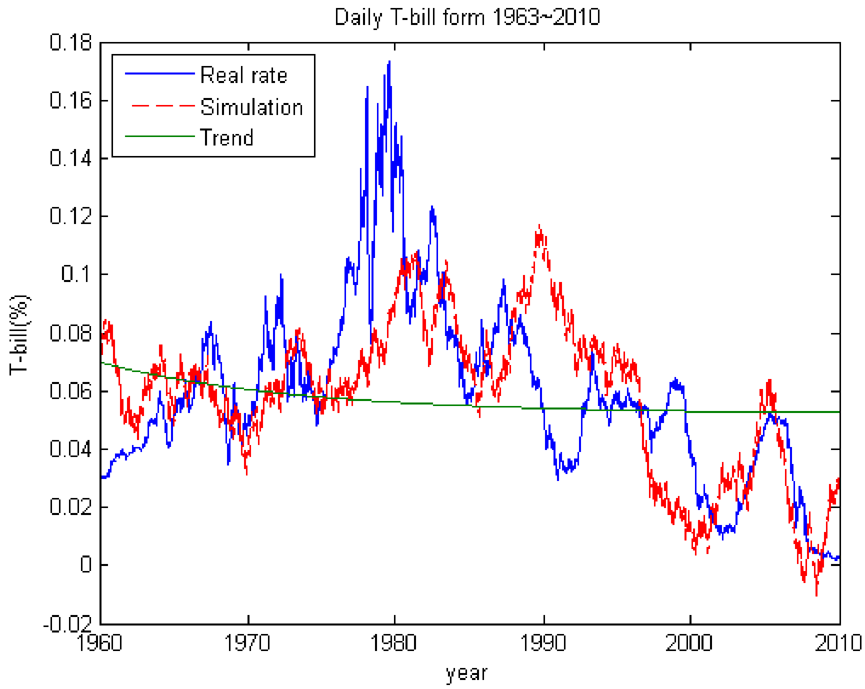

Figure 3.2. The Vasicek behavior to simulate the data

Figure 3.3 shows the annaul inflation rate of USA from 1963 to 2010. In comparing the time series shown in Figures 3.1 and 3.3, it is observed that increasing or decreasing of the interest rate concides with the change of the inflation rate, i.e., when the inflaction rate is increasing then the corresponding interest rate is also increasing, and vice verse. Thus we divide the US T-bill daily data into several intervals according to the increasing/decreasing behavior of the annual inflation rate.

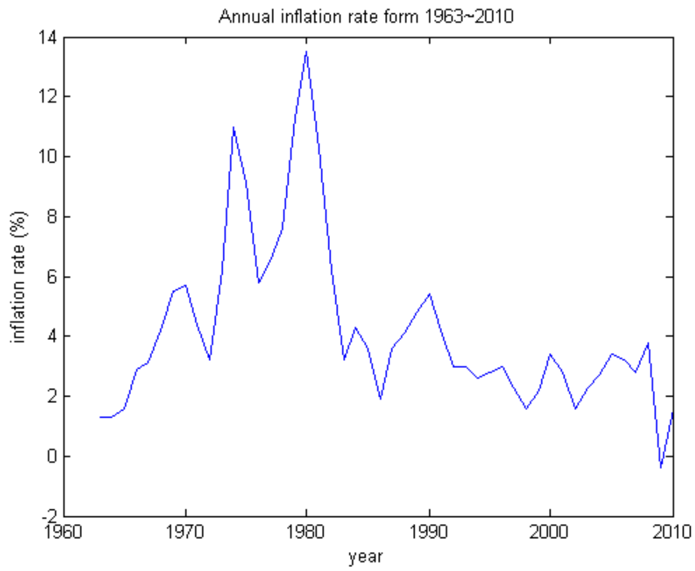

FIGURE 3.3. The annual inflation rate of USA from 1963 to 2010

Therefore we divide the data into 4 intervals, from 1962 to 1972, from 1973 to 1981, from 1982 to 1990, and from 1991 to 2010, because of the annual inflaction rate rises up or falls down rapidly. The estimated parameters with their standard errors are listed in Table 3.3 and Figure 3.4 shows that the trend of each interval now captures the data trend more well than the trend given in Figure 3.2. Also the simulated behavior of the Vasicek model shows in Figure 3.5 is more consistent with the given data. From this example, a proper segmentation of the time history is a very important step in the parameter estimation.

TABLE 3.3. The estimated parameters for 4 intervals

N	a	b	σ	Mean	Variance	$\frac{\sigma^2}{2a}$
$1962 \sim 1972$	0.26506	0.06056	0.00707	0.05055	0.00016	0.9×10^{-5}
Std error	0.17229	0.01060	1.40×10^{-6}			
$1973 \sim 1981$	0.25162	0.12294	0.02420	0.08894	0.00095	0.00116
Std error	0.26304	0.04793	1.75×10^{-5}			
$1982 \sim 1990$	0.59362	0.07653	0.01741	0.08853	0.00045	0.00025
Std error	0.29075	0.01198	9.66×10^{-5}			
$1991 \sim 2010$	0.07790	-0.00560	0.00805	0.04004	0.00042	0.00042
Std error	0.08517	0.05476	1.26×10^{-6}			

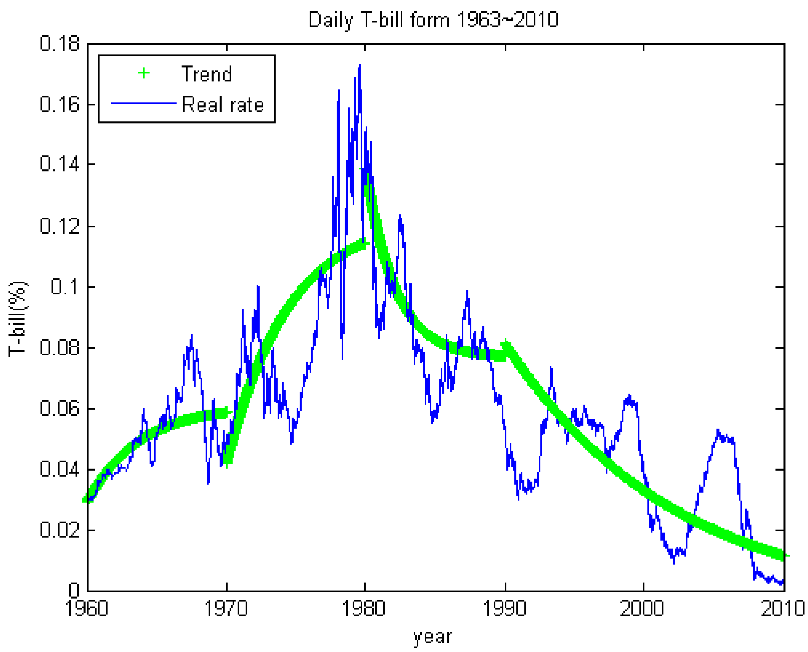

FIGURE 3.4. The trend of each interval of the US T-bill daily data.

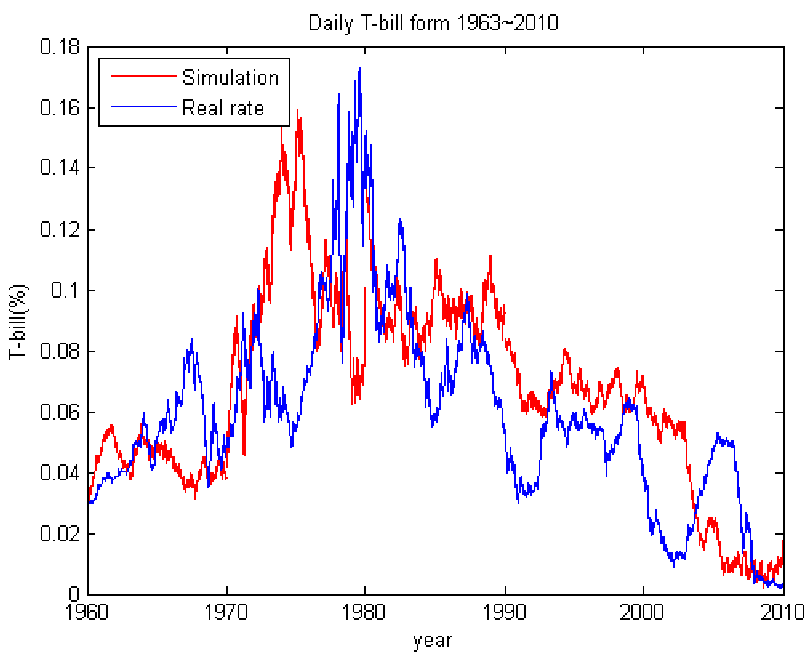

FIGURE 3.5. The simulated behavior of the Vasicek model consisting of four intervals.

CHAPTER 4

Parameter Estimation in Black-Scholes and Jump Diffusion Models

It is interesting to use the asset price model to estimate the stock price variation in the current market. We choose the HTC and Acer stock-price daily data and estimate parameters which are used in BSM and JDM for European option pricing. The statisctical properties of the stock-price data for two companies are listed in Table 4.1. And the time variation of two companies are shown in Figure 4.1.

Table 4.1. Some statistics of log-return data of two companies

	Mean	Variance	Max. Val.	Min Val.	skewness	kurtosis
HTC	0.001251627	0.000835238	-0.1347	0.0677	-0.133728006	3.585435315
Acer	0.000150829	0.000551879	-0.0854	0.0676	-0.0598493832	4.201222636

4.1. Parameter Estimation and Simulation in BSM

In the BSM model, the unknown parameters are r and σ, and denote them by $\theta = (r, \sigma)$. Before we estimate the parameters, we have to take the logarithm of the stock price and calculate the return of the BSM model which is given by

$$\begin{aligned} R^B(t+dt) &\equiv \ln S(t+dt) - \ln S(t) \\ &= (r - \frac{1}{2}\sigma^2)dt + \sigma dW(t), \end{aligned}$$

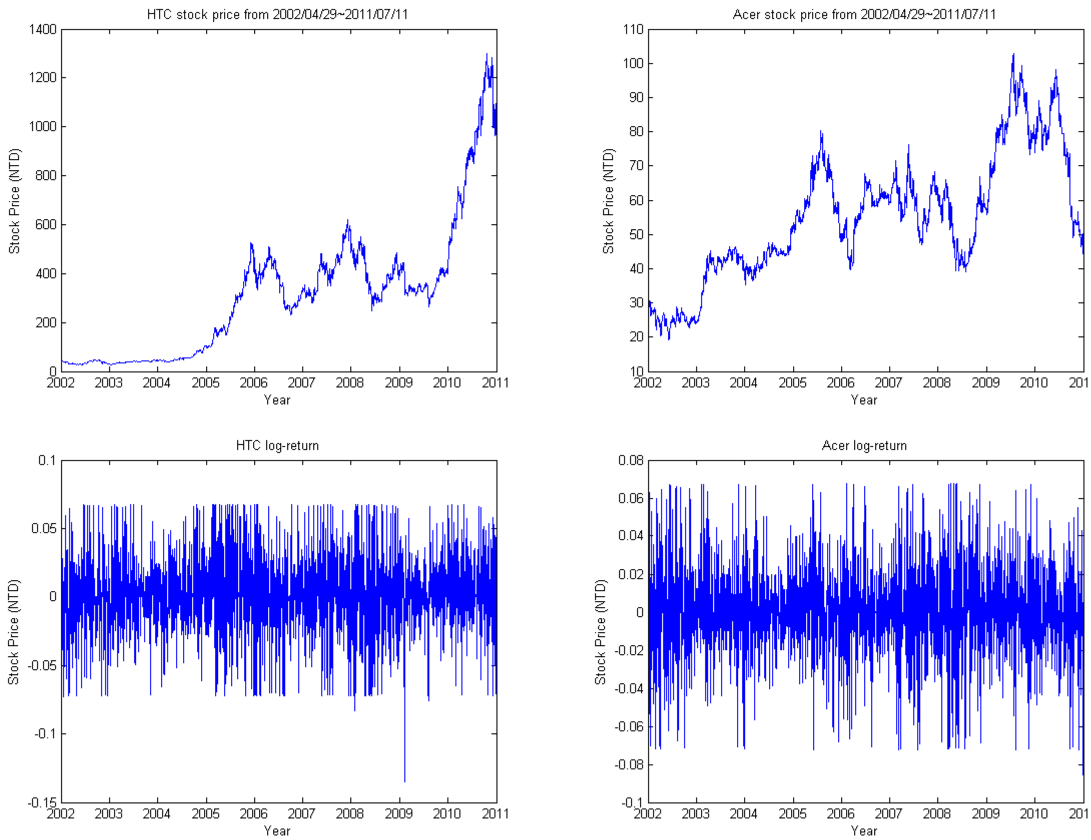

FIGURE 4.1. Time variation of two companies' daily data

where $s = 1$,$\mu = r - \frac{1}{2}\sigma^2$,$dt = 1$, $R^B(t + dt)$ it is alss a Geometric Brownian motion. The conditional mean and conditional variance are:

$$\begin{aligned} E\left[R^B(t+dt) \mid R^B(t)\right] &= \mu \\ Var\left[R^B(t+dt) \mid R^B(t)\right] &= \sigma^2, \end{aligned}$$

and $R^B(t + dt) \sim \mathcal{N}(\mu, \sigma^2)$.

The log-likelihood function is defined by

$$\ln L(\theta) = -\frac{n}{2}\ln 2\pi s - n\ln\sigma - \frac{1}{2}\sum_{t=1}^{n}\frac{(R^B(t+dt) - \mu dt)}{\sigma^2 dt}$$

and the zero derivatives are

$$
\begin{aligned}
\frac{\partial \ln L(\theta)}{\partial r} &= \sum_{t=1}^{n} \frac{(R^B(t+dt) - \mu dt)}{\sigma^2 dt} = 0, \\
\frac{\partial \ln L(\theta)}{\partial \sigma} &= -\frac{n}{\sigma} + \sum_{t=1}^{n} \frac{(R^B(t+sdt) - \mu dt)^2}{\sigma^3 dt} = 0.
\end{aligned}
$$

where the extrema value of $\ln L(\theta)$ occurs. The close-form solutions for μ and σ^2 are then given by

$$
\begin{aligned}
\mu &= \sum_{t=1}^{n} \frac{R^B(t+sdt)}{ndt} \\
\sigma^2 &= \sum_{t=1}^{n} \frac{(R^B(t+dt) - \mu dt)^2}{ndt},
\end{aligned}
$$

and the expected Fish information matrix is

$$
-E(\frac{\partial^2 \ln L}{\partial \theta^2}(\theta)) = E \begin{bmatrix} \frac{\partial^2 \ln L}{\partial r^2} & \frac{\partial^2 \ln L}{\partial r \partial \sigma} \\ \frac{\partial^2 \ln L}{\partial \sigma \partial r} & \frac{\partial^2 \ln L}{\partial \sigma^2} \end{bmatrix}
$$

where

$$
\begin{aligned}
\frac{\partial^2 \ln L}{\partial r^2} &= \sum_{t=1}^{n} \frac{R^B(t+dt) - dt}{\sigma^2 dt}, \\
\frac{\partial^2 \ln L}{\partial \sigma^2} &= \sum_{t=1}^{n} \frac{R^B(t+dt) - dt}{dt} + \frac{n}{\sigma^2} - 3\sum_{t=1}^{n} \frac{(R^B(t+dt) - \mu dt)^2}{\sigma^4 dt}, \\
\frac{\partial^2 \ln L}{\partial r \partial \sigma} &= -\sum_{t=1}^{n} \frac{R^B(t+dt) - dt}{\sigma dt}.
\end{aligned}
$$

Table 4.2 lists all the parameters for use in the BSM model for the data of HTC and Acer and the simulated behavior for each company is shown in Figures 4.2 and 4.3, respectively. From the figures, we can see that the trend and time variation that are represented by the estimated model fit well with the real data.

HTC	μ	σ	$\ln L$
Parameters	0.001669072	0.028894464	5100.400
Standard Error	0.000590298	0.000417056	
Acer	μ	σ	$\ln L$
Parameters	0.00039218	0.023552301	5594.757
Standard Error	0.000480854	0.000339948	

TABLE 4.2. Associated estimated parameters for the BSM model of two companies

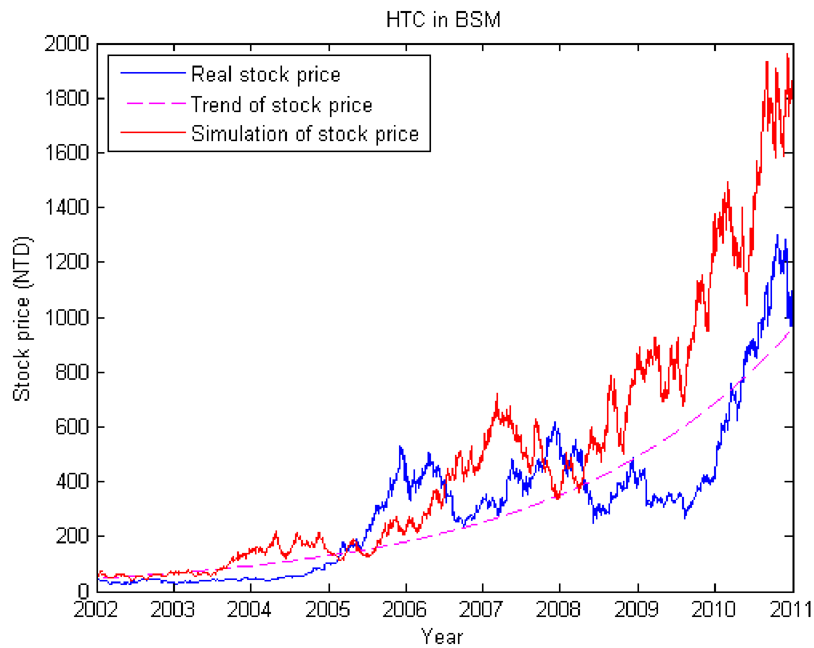

FIGURE 4.2. The simulated variation of the HTC by the BSM model.

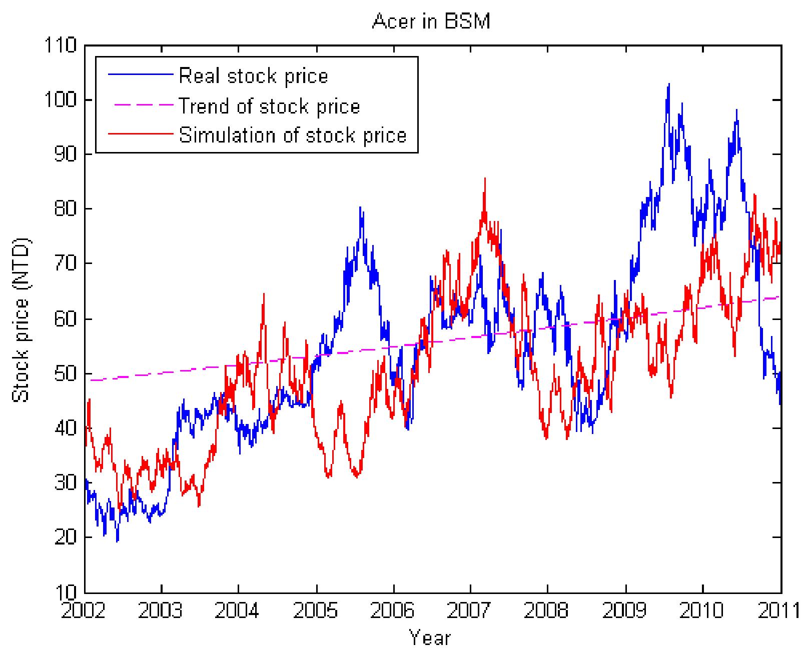

FIGURE 4.3. The simulated variation of theAcerby the BSM model.

4.2. Estimation Parameter and Simulation in JDM

We use the similar procedures in Section 4.1 for estimating the paramters corresponding to the JDM model. The log-return of JDM is calcuated to be

$$\begin{aligned} R^J(t+dt) &= \ln \frac{S(t+dt)}{S(dt)} \\ &= (r - \frac{1}{2}\sigma^2)dt + \sigma dW(t) + d\sum_{i=1}^{N(t)} Y_i \\ &= R^B(t+dt) + d\sum_{i=1}^{N(t)} Y_i, \end{aligned}$$

where X denotes the Geometric Brownian motion $\sim \mathcal{N}(\mu_B, \sigma_B^2)$ with $\mu_B = r - \frac{1}{2}\sigma^2$.

The log-return contains two components

$$R^J(t+dt) = \begin{cases} R^B(t+dt), & N(t) = 0, \\ R^B(t+dt) + Y_1 + Y_2 + ... + Y_k, & N(t) \geq 1, \end{cases}$$

where $Y_i \sim \mathcal{N}(\mu_J, \sigma_J^2)$ and the distribution of $R^J(t+dt)$ is the sum of i.i.d random variable. If the number k is the jump times, the probability density of $R^J(t+dt)$ is $\mathcal{N}(\mu_B + k\mu_J, \sigma_B^2 + k\sigma_J^2)$. Let $\mathbb{P}(N(t) = k) = \frac{(\lambda s)^k e^{-\lambda s}}{k!}$, so the probability density of log-return is described by

$$f(R^J(t+dt)) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{1}{\sqrt{2\pi(\sigma_B^2 + k\sigma_J^2)}} e^{\frac{-(R^J(t+sdt) - \mu_B - k\mu_J)^2}{2(\sigma_B^2 + k\sigma_J^2)}}.$$

The unconditional mean and unconditional variance of the JDM equal to

$$\begin{aligned} E(R^J(t+dt)) &= \mu_B + \lambda\mu_J, \\ Var(R^J(t+dt)) &= \sigma_B^2 + \lambda(\mu_J^2 + \sigma_J^2) \end{aligned}$$

and the skewness and kurtosis are also given by

$$\begin{aligned} skewness &= \frac{\lambda(\mu_J^3 + 3\mu_J\sigma_J^2)}{(\sigma_B^2 + \lambda\sigma_J^2 + \lambda\mu_J^2)^{3/2}}, \\ kurtosis &= \frac{\lambda(3\sigma_J^4 + 6\mu_J^2\sigma_J^2 + \mu_J^4)}{(\sigma_B^2 + \lambda\sigma_J^2 + \lambda\mu_J^2)^2}. \end{aligned} \tag{4.2.1}$$

The maximum likelihood function is defined to be

$$\ln L(\theta) \quad = \quad -n\lambda - \frac{n}{2}\ln(2\pi) + \sum_{t=1}^{n} \ln \sum_{k=0}^{Nt)} \frac{\lambda^j}{j!} \frac{1}{\sqrt{\sigma_B^2 + k\sigma_J^2}} e^{\frac{-(R^J(t+dt) - \mu_B - k\mu_J)^2}{2(\sigma_B^2 + k\sigma_J^2)}} \tag{4.2.2}$$

where the $\theta = (\lambda, \mu_B, \mu_J, \sigma_B, \sigma_J)$.

Tables 4.3and 4.4 give us the estimated parameters for the JDM model of the HTC and Acer data, respectively. The simuated behavior of the stock price variation of the JDM model for HTC is shown in Figure 4.4. Figure 4.5 shows the comparision between the simulated behavior for the BSM and JDM model, and it is obviously that the result for JDM is much closer to the original data than the result of BSM. For the Acer company, the similar result of the simulated behavior is given in Figure 4.6 and its comparison with BSM is given in Figure 4.7. The comparison between different types of model is introduced in Chapter 2 with the BIC criterion. For our computation the BIC criterion is listed in Table 4.5 for the two companies and it indicates that the JDM is more fit than BSM for both companies' stock price variation.

Table 4.3. The estimated parameters for the JDM model of the HTC data

HTC	μ_B	σ_B	μ_J	σ_J	λ	$\ln L$
parameter	0.00420	0.02608	-0.05391	3.65×10^{-8}	0.05468	5114.12
Standard Error	0.0008	0.0006	0.0052	0.0057	0.0132	
	Mean	Variance	Skewness	kurtosis		
Data	0.00125	0.00083	-0.13372	3.58544		
Parameter	0.00125	0.00084	-0.35270	3.65654		

Table 4.4. The estimated parameters for the JDM model of the HTC data

Acer	μ_B	σ_B	μ_J	σ_J	λ	$\ln L$
parameter	0.00194	0.02128	-0.05660	0.00001	0.03166	5624.022
Standard Error	0.00050	0.00038	0.00417	0.00469	0.00654	
	Mean	Variance	Skewness	kurtosis		
Data	0.00015	0.00055	-0.05985	4.20122		
Parameter	0.00015	0.00055	-0.43994	4.05762		

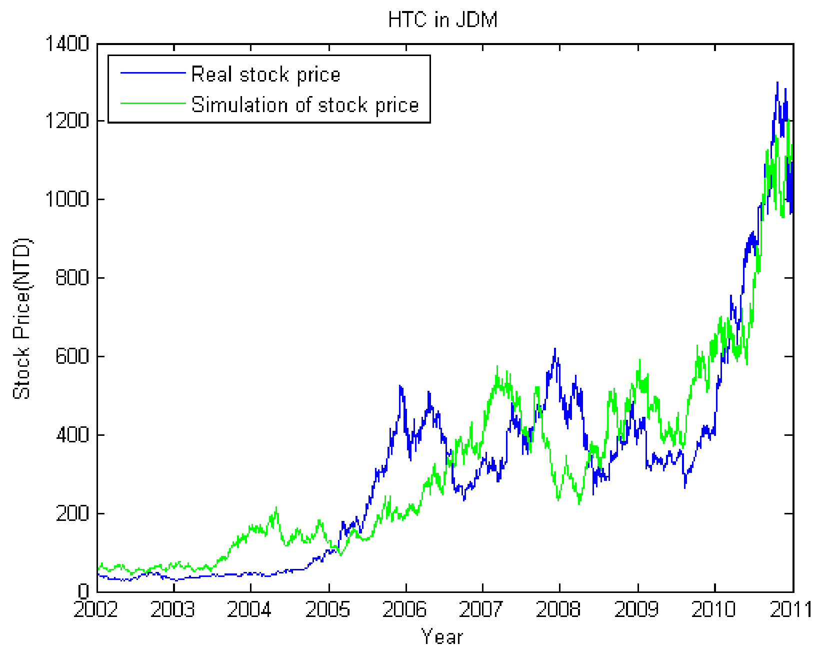

FIGURE 4.4. The simulated stock price variation for the JDM model of the HTC data

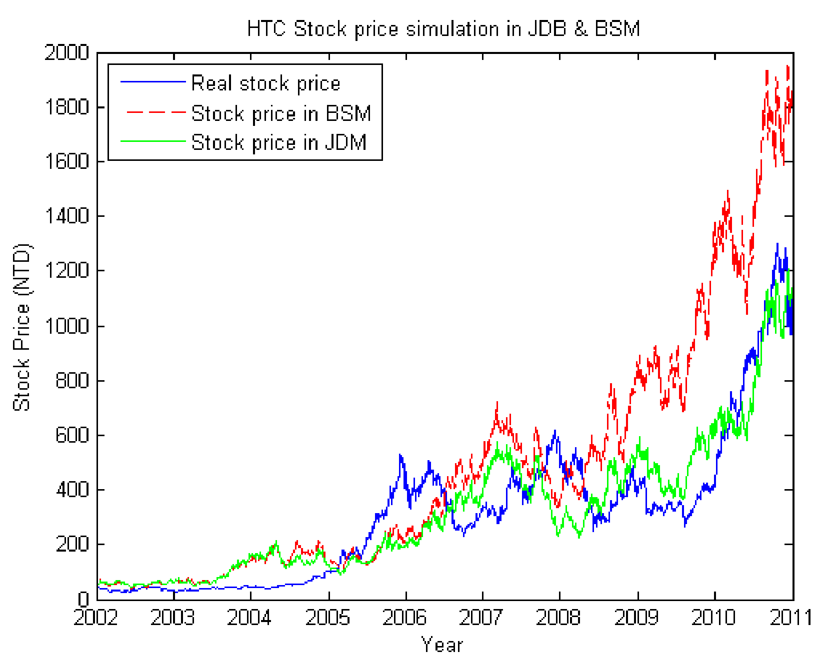

FIGURE 4.5. The simulated stock price variation for the BSM and JDM models of the HTC data

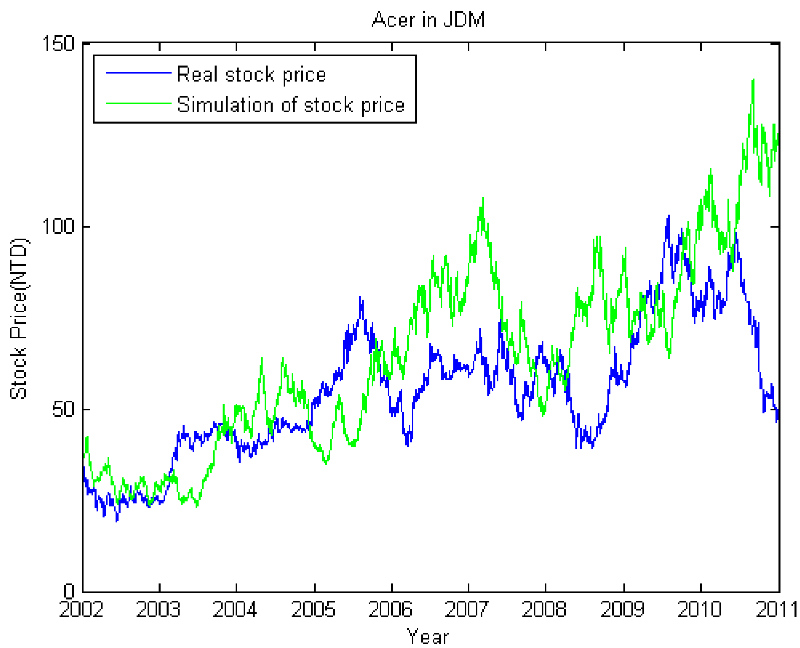

FIGURE 4.6. The simulated stock price variation for the JDM model of the HTC data

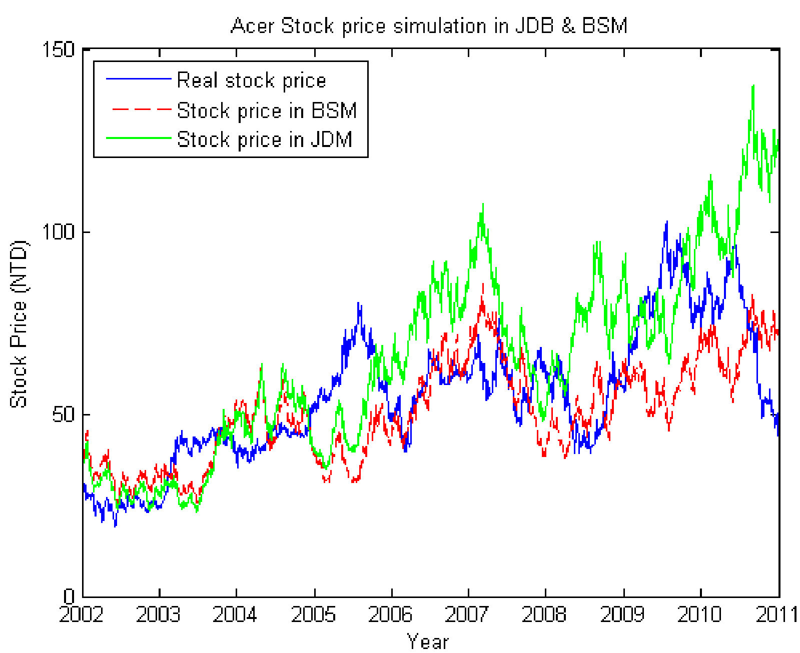

FIGURE 4.7. The simulated stock price variation for the BSM and JDM models of the HTC data

TABLE 4.5. The value of BIC of two models for two companies

BIC	HTC	Acer
BSM	-10185	-11174
JDM	-10189	-11209

CHAPTER 5

Conclusion

The aim of this paper is to estimate parameters of stochastic model, Vasicek model, Black-scholes model and Jump- diffusion model by using maximum likelihood estimation.

In chapter 2 we present three stochastic models and thier exact solusion by using Itô lemma for JDM. Finally we introduce our estimation method, maximum likelihood estimation.

In Chapter 3 we use the one year U.S. treasury data to estimate the parameter and simulate in Vasicek model. We find the trend of simulation result is bad when we use the whole data, so we divide the data and re-estimate the parameter. In this paper we divide the data into 4 intervals and the trend is more fit the data then the local trend.

In Chapter 4 we estimate the parameters in BSM and JDM. In BSM model there are expilit solution, we can easily estimate the parameters, standard error, but in JDM it became more difficult, so we use the matlab code then give some initial value to estimate the parameters, standard error, and the mean, variance, skewness and kurtosis between data and simulation are. very close.

This thesis use the maximum likelihood function to estimete the parameters in interest rate model, BSM and JDM which are often used to price the European option. The furture study are that in Vasicek model we find it is not good if we use the entire data, so we use the annual inflation rate when the rate rise or down rapidly, so we can add the jump diffusion in the model (Baz and Das 1996). In JDM model, we can change the jump size to the double-exponential distribution (Kou 2002).

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