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黎曼流形上之高階固有值的研究(II)

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計畫主持人：陳文豪

計畫參與人員：大專學生研究助理--胡雅婷

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一、中文摘要

本計畫以研究黎曼流形上的 Dirichlet 固有值問題，特別是關於高階固有值的估計問題，以及相關的應用為主要研究對象。我們藉由探討 P.Li 教授與丘成桐教授對於歐式空間中有界區域的 Dirichlet 問題與高階固有值的估計所採用的方法，研究其推廣至一般黎曼流形的可能性。同時，我們也討論有關黎曼流行上的隨機測地方程式的相關問題，利用 S.Helgason 教授對於對稱空間上之傅立葉轉換的相關探討，我們給出這類流形上之隨機測地方程式的初步結果。

關鍵詞：高階固有值，傅立葉轉換，測地方程式、黎曼流形。

Abstract

This project focuses on the investigations of the Dirichlet eigenvalue problem in Riemannian manifolds, especially for the estimate of higher eigenvalues of Laplace operator and its applivations. By the methods in P.Li and Yau’s work about studying the Dirichlet problem and the estimate of higher eigenvalues in a bounded domain of Euclidean space, we investigate the possibility of extending their work to general Riemannian manifolds. Meanwhile, we also discuss the related problems about the stochastic geodesic on Riemannian manifolds. By S.Helgason’s results about Fourier transforms on symmetric spaces, we give partial results about the stochastic geodesic equations on these manifolds.

Key words: higher eigenvalues, Fourier transform, geodesic equation Riemannian manifolds.

二、緣由與目的

Consider the Dirichlet problem for bounded domains Ω in the standard Euclidean space $\Re^n$. Then the Dirichlet eigenvalue problem is as follows:

$$\begin{cases} \Delta\phi &= -\lambda\phi \\ \phi|_{\partial\Omega} &= 0 \end{cases}$$

, where Δ denote the Laplacian operator.

By use of heat kernel and Tauberian theorem, H. Weyl proved in 1912 the asymptotic formula

$$\lambda_k \sim c_n\left(\frac{k}{V}\right)^{2/n}$$

$$(k \to \infty)$$

, where $c_n = (2\pi)^2 \Big/ \left(\frac{\omega_{n-1}}{n}\right)^{n/2}$,

$\omega_{n-1} = Area(S^{n-1})$ and $V = Vol(\Omega)$.

Based on this formula, Polya conjectured in 1960 that for each k

$$\lambda_k \geq c_n\left(\frac{k}{V}\right)^{2/n}.$$

In the case $n = 2$, Polya proved that the conjecture holds for some special planar domains. In 1980, E. Leib proved that there exists a constant $\tilde{c}_n < c_n$ such that

$$\lambda_k \geq \tilde{c}_n \left(\frac{k}{V}\right)^{2/n}.$$

The most recent result on this conjecture is the following theorem shown by Li and Yau (c.f. **[LY]**).

Theorem (Li-Yau): *For any* $k > 0$, *we have*

$$\sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+2} c_n \cdot k \left(\frac{k}{V}\right)^{2/n}.$$

Since $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$, as a corollary of the above theorem, one has

$$\lambda_k \geq \frac{n}{n+2} c_n \left(\frac{k}{V}\right)^{2/n}.$$

The main tool in the proof of Li and Yau's theorem is the theory of Fourier transforms. Especially, the proof depends heavily on the Planchel theorem. Here we mention them as follows: Let $\{\phi_i\}_{i=1}^{k}$ be an orthonormal family of eigenfunctions corresponding to $\{x_i\}_{i=1}^{k}$, and let

$$\phi(x,y) = \sum_{i=1}^{k} \phi_i(x)\phi_i(y).$$

Denote by $\hat{\phi}(z,y)$ the Fourier transform of $\phi(x,y)$, i.e.

$$\hat{\phi}(z,y) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\Omega} \phi(x,y) e^{ix\cdot z} dx.$$

Then the Planchel theorem shows that

$$\int_{\Re^n} |\phi(x,y)|^2 \, dx = \int_{\Re^n} |\hat{\phi}(x,z)|^2 \, dz.$$

That is, the Fourier transform $\hat{\phi}$ of ϕ keeps the L^2-norm.

Our goal is to extend these results from the Euclidean spaces to Riemannian manifolds and then to estimate the higher eigenvalues of the Laplacian operator Δ. Here

$$\Delta = \frac{1}{\sqrt{G}} \sum \frac{\partial}{\partial x^i}\left(\sqrt{G} \cdot g^{ij} \frac{\partial}{\partial x^i}\right)$$

, where $(x^1, \cdots, x^n)$ is a local coordinate and then the metric can be expressed as $ds^2 = \sum g_{ij} dx^i dx^j$ with $(g^{ij}) = (g_{ij})^{-1}$ and $G = \det(g_{ij})$. To our end, we will follow a similar method as in Li-Yau's theorem. Therefore, it is necessary to develop the related tools and then investigate their application.

In fact, Chung, Grigor'yan and Yau gave in [CGY] the estimation of eigenvalues on Riemannian manifolds. However, they used the method of studying the isoperimetric inequalities instead of considering the Fourier transforms on Riemannian manifolds. Their result is

Theorem: ([CGY])

Let M be a complete Riemannian manifolds of dimension >1. Let $\rho_{\xi(x)}$ be a distance function on M such that, for some $R_o \in (0,\infty]$ and all $\xi \in M$ the inequalities $|\nabla \rho_\xi|$ and $\Delta \rho_\xi^2 \geq 2n$ hold inside the ball $B_\xi(R_o)$. If $\Omega \subset M$ is a precompact and with smooth boundary and there exists a positive integer such that $Vol(\Omega) \leq \varepsilon R_o^n K$, where $\varepsilon = \varepsilon(n) > 0$ is a constant, then $\lambda_k(\Omega) \geq a\left(\frac{k}{vol(\Omega)}\right)^{\frac{2}{n}}$ with some $a = a(n) > 0$.

Although the problem about the estimation of higher eigenvalues of Laplace operator on Riemannian manifolds has been done, the investigation of Fourier transforms on Riemannian manifolds is still important and interest. In the next paragraph, we will state our observation about stochastic geodesic equations on symmetric spaces.

三. 結果與討論

We consider the problem about how to define the "stochastic" geodesic equations on Riemannian manifolds. As we know, for a given Riemannian manifold (M,g) with induced Riemannian connection ∇, a curve $\gamma(s)$ with $\gamma(0) = x_0$ and $\gamma'(0) = v$ is called a geodesic in (M,g) if it satisfies the geodesic equation $\nabla_{\gamma'}\gamma' = 0$.

Now we add stochastic effects on (M,g). In [P], the author considered the stochastic Riemannian geodesic $\gamma(t,x)$ on (M,g) with initial data $(\gamma(0,x), v(0,x))$, where $\gamma(0,x)$ is a random variable with variance 0 and $v(0,x)$ is a random variable with transition density $\mu(y)$. Then it can be shown that the stochastic geodesic equation is

$$\begin{cases} a^i : drift\ \ vector\ \ of\ \ v(t,x) \\ B^{ij} : diffusion\ \ matrix\ \ of\ \ v(t,x) \\ \dfrac{d\mu}{dt} = -\dfrac{1}{\sqrt{G}}\dfrac{\partial}{\partial x_i}[\tilde{a}^i\sqrt{G}\mu] + \dfrac{1}{2}\dfrac{1}{\sqrt{G}}\dfrac{\partial}{\partial x_i}(\sqrt{G}B^{ij}\dfrac{\partial \mu}{\partial x_j}) \end{cases}$$

, where $\tilde{a} = a^i - \frac{1}{2\sqrt{G}}\frac{\partial}{\partial x_i}(\sqrt{G}B^{ij})$ and $G = \det g_{ij}$ as above. However, this result only holds locally since the main tool in [Pa] is to use Fourier transform to solve the corresponding Fokker-Plank equation on Euclidean spaces. Therefore, a necessary work to establish a theory of stochastic Riemannian geodesic is to develop a suitable Fourier transform theory on Riemannian manifolds.

S.Helgason showed in [S] a theory of a Fourier transform $f \longrightarrow \tilde{f}$ on a symmetric space of the noncompact type and investigated its operational properties. Let G be a connected semisimple Lie group with finite center and $X = G/K$ be the associated symmetric space. The Fourier transform on $\mathbb{R}^n$ in polar coordinate is

$$\widetilde{F}(\lambda\omega)=\int_{\mathbb{R}^n}F(x)e^{-i\lambda(x,w)}dx, |w|=1, \lambda\in\mathbb{R}.$$

Geometrically, the scalar product (x,w) represents the signed distance from the origin to the hyperplane passing through x, having unit normal w. In the symmetric space, the scalar product is the vector $A(x,b)\in a$ which is the composite distance from the origin $0=\{k\}$ in x to the horocycle $\xi(x,b)$ through $x\in X$ with normal $b\in B={K}/{M}$. Then he defined the Fourier transform on X by

Definition: If f is a function on X, then the Fourier transform $\widetilde{f}$ is defined by

$$\widetilde{f}(\lambda,b)=\int_x f(x)e^{(-i\lambda+e)(A(x.b))}dx$$ for all $\lambda\in a_c^*$, $b\in B$ for which this integral exists.

Moreover, he gave the Plancherel formula in this case. Let $a_+^*=\{\lambda\in a^* \,|\, A_\lambda\in a^+\}$ be the preimage of the positive Weyl chamber a^+. Here $B(A_\lambda,H)=\lambda H$ for $H\in a$. Then,

Theorem: The Fourier transform $f(x)\longrightarrow\widetilde{f}(\lambda,b)$ extends to an isometry of $L^2(x)$ onto $L^2(a_+^*\times B)$ (With the measure $|c(\lambda)^{-2}|d\lambda db$ on $a_+^*\times B$). Moreover,

$$\int_x f_1(x)\overline{f_2(x)}dx=\frac{1}{\omega}\int_B\widetilde{f_1}(\lambda,b)\overline{\widetilde{f_2}(\lambda,b)}|c(\lambda)|^{-2}d\lambda db$$

By use of the above definition and formula, we may extend the local stochastic geodesic equation to a global one on symmetric spaces. It result from that Helgason's theory provides us an idea to slove the corresponding Fokker-Plan equation on symmetric spaces. This result is in preparation.

四. 計畫結果自評：

Initially, we expect to find the estimation of higher eigenvalues of Laplace operator on Riemannian manifold. We want to extend Professors Li and You's method to obtained some results. Although this estimation is done in [CGY] by another method, it is still important to study the Fourier transform on Riemannian manifolds. We discuss the stochastic Riemannian geometry and give an idea about the stochastic geodesic equations on symmetric spaces by Helgason's theory. We look forward to finding a good way to definded the Fourier transform on Riemannian manifolds and then extending our results to general Riemannian manifolds.

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