行政院國家科學委員會專題研究計畫　成果報告

以共振穿隧二極體為基礎的細胞神經網路的數學模型之研究

計畫類別：個別型計畫
計畫編號：NSC93-2115-M-029-002-
執行期間：93年08月01日至94年07月31日
執行單位：東海大學數學系

計畫主持人：楊智烜

報告類型：精簡報告

處理方式：本計畫可公開查詢

中　華　民　國　94年9月30日

Stability and Bifurcation of a Two-Neuron Network with Distributed Time Delays*

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June 10, 2005

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Abstract

In this paper we study the stability and bifurcation of the trivial solution of a two-neuron network model with distributed time delays. This model consists of two identical neurons, each possessing nonlinear instantaneous self-feedback and connected to the other neuron with continuously distributed time delays. We first examine the local asymptotic stability of the trivial solution by studying the roots of the corresponding characteristic equation, and then describe the stability and instability regions in the parameter space consisting of the self-feedback strength and the product of the connection strengths between the neurons. It is further shown that the trivial solution may lose its stability via a certain type of bifurcation such as a Hopf bifurcation or a pitchfork bifurcation. In addition, the criticality of Hopf bifurcation is investigated by means of the normal form theory. We also provide numerical evidences to support our theoretical analyses.

Keywords. neural network; distributed time delay; characteristic equation; Hopf bifurcation; pitchfork bifurcation; normal form

AMS subject classifications. 34K18; 34K20; 92B20

1. Introduction

This work is concerned with the stability and bifurcation of the trivial solution of a Hopfield-type neural network model which consists of two identical neurons, each possessing nonlinear instantaneous self-feedback and connected to the other neuron with continuously distributed time delays. The dynamics of this neural netlet is governed by the following equations:

$$\begin{aligned}
\dot{x}(t) &= -x(t) + pf(x(t)) + s\int_0^L k(\theta)f(y(t-\theta))\,d\theta,\\
\dot{y}(t) &= -y(t) + pf(y(t)) + r\int_0^L k(\theta)f(x(t-\theta))\,d\theta,
\end{aligned}$$

where x and y represent the voltages of the neurons; p is the self-feedback strength; r and s are the connection strengths; the nonlinear activation function f is representing the output or firing rate; the continuous density function $k:\ [0,L] \to [0,\infty)$ with $L > 0$ is prescribed which satisfies

$$\int_0^L k(\theta)\,d\theta = 1.$$

As usual, the activation function f we consider is of sigmoidal type. More specifically, we may assume that f is a smooth function possessing the following properties:

$$\text{(H)}\qquad \begin{cases} f:\mathbb{R}\to\mathbb{R} \text{ is an odd function;}\\ f'(0)=1;\\ f(\pm\infty)=\pm M, \text{where } M \text{ is a positive constant;}\\ xf''(x)<0 \text{ for all } x\neq 0. \end{cases}$$

A typical example of f is given by $f(x) = \tanh(x)$.

It has been widely argued and accepted that for various reasons, time delay should be taken into account in the modeling for practical problems. For example, in models of electronic networks, time delays are likely to be present due to the finite switching speed of amplifiers. In biological neurons, there are several types of delays in their performances commonly known as cellular delays, transmission delays, and synaptic delays (see [9] and references cited therein). In practice, although the use of constant discrete time delays in the models serve as a good approximation in simple electronic circuits consisting of a small number of neurons, in biological neural networks, it usually has a spatial extent due to the presence of an amount parallel pathways with a variety of axon sizes and lengths. Thus, there will be a distribution of transmission delays and it cannot be modeled with discrete time delays. In such case, a more appropriate way is to incorporate distributed time delays in the models. Therefore, as pointed out in [15], the studies of models with distributed time delays have more important significance than the ones with discrete delays. For the fundamental theory of delay differential equations, we refer the reader to the book of Hale and Verduyn Lunel [12].

It is well-known that Hopfield-type neural networks with symmetric connection strengths but without time delays exhibit no oscillations and possess global stability (i.e., all trajectories tend to some equilibrium) [13]. Consequently, in the study of Hopfield-type neural networks with or without time delays, sufficient stability conditions of equilibria for asymmetric connection weights and the mechanism for the onset of instability of equilibria are widely considered. See [2]-[9], [14]-[21], [23], and many references therein. Most works mentioned above considered the network models with discrete time delays. However, as we have pointed out above, it is of great importance to discuss the model with distributed time delays. Therefore, in contrast to the two-neuron systems with discrete time delays examined in [9], [19], and [20], in this paper, we will study the two-neuron network system with more realistic distributed time delays.

In the following discussion, for simplicity, we will always assume that $k(\theta) \equiv 1$ with $L = 1$, namely,

$$\begin{aligned} \dot{x}(t) &= -x(t) + pf(x(t)) + s\int_0^1 f(y(t-\theta))\,d\theta, \\ \dot{y}(t) &= -y(t) + pf(y(t)) + r\int_0^1 f(x(t-\theta))\,d\theta. \end{aligned} \tag{1.1}$$

We remark that most of results obtained in this paper might be generalized to more general density functions. By inspection, it is obvious that $(0,0)$ is an equilibrium of (1.1). We will investigate the local asymptotic stability of the trivial solution by studying the corresponding characteristic equation. The trivial solution is locally asymptotically stable if all the roots of the characteristic equation have negative real parts, and whether or not this is true will depend on the values of the parameters involved in the network, namely, r, p and s. In this paper, the stability and instability regions can be completely described in the parameter space. Furthermore, we arrive at the conclusion that similar to the results obtained in [9, 19, 20], the trivial solution may lose its stability through a certain type of bifurcations such as a Hopf bifurcation or a pitchfork bifurcation. We also investigate the criticality of the Hop bifurcation using the normal form theory.

This paper is organized as follows. In section 2, we introduce some notations and the characteristic equation associated with the trivial solution of (1.1). The behaviour of the $i\omega$-curves for the characteristic equation are studied in section 3. In section 4, we describe the local asymptotic stability regions of the trivial solution on the γp-plane. In section 5, the mechanism of Hopf bifurcation as well as the change of criticality are investigated. Finally, in section 6, we examine the steady state bifurcation by studying the existence and local stability of nontrivial equilibria of (1.1). At each stage, we also provide numerical evidences to support our theoretical analyses.

2. The characteristic equation

In this section, we will derive the corresponding characteristic equation of the linearization of (1.1) about the trivial solution $(0,0)$. Some useful notations will also be introduced.

The linearization of (1.1) at the trivial point is given by

$$\begin{aligned} \dot{\bar{x}}(t) &= -\bar{x}(t) + p\bar{x}(t) + s\int_0^1 \bar{y}(t-\theta)\,d\theta, \\ \dot{\bar{y}}(t) &= -\bar{y}(t) + p\bar{y}(t) + r\int_0^1 \bar{x}(t-\theta))\,d\theta. \end{aligned} \tag{2.1}$$

The characteristic equation for this linearized system is obtained by looking for nontrivial solutions of the form $(\bar{x}(t), \bar{y}(t)) = (c_1, c_2)e^{\lambda t}$ where c_1 and c_2 are constants. Such solutions will be nontrivial if and only if

$$P(\lambda) := \det\begin{pmatrix} \lambda - p + 1 & -s\displaystyle\int_0^1 e^{-\lambda\theta}\,d\theta \\ -r\displaystyle\int_0^1 e^{-\lambda\theta}\,d\theta & \lambda - p + 1 \end{pmatrix} = 0, \tag{2.2}$$

which is called the characteristic equation of (1.1) with respect to the trivial solution $(0,0)$. Expanding the determinant in (2.2), we obtain

$$P(\lambda) = (p-1-\lambda)^2 - rs(\int_0^1 e^{-\lambda\theta}\,d\theta)^2 = \Delta_+(\lambda)\Delta_-(\lambda), \tag{2.3}$$

where

$$\Delta_\pm(\lambda) = \begin{cases} (p-1-\lambda) \pm \gamma\displaystyle\int_0^1 e^{-\lambda\theta}\,d\theta & \text{if } rs > 0, \\ (p-1-\lambda) \pm i\gamma\displaystyle\int_0^1 e^{-\lambda\theta}\,d\theta & \text{if } rs < 0, \end{cases} \tag{2.4}$$

with $\gamma = \sqrt{|rs|}$ and $i = \sqrt{-1}$. Now, letting $\lambda = \mu + i\omega$ for some μ, $\omega \in \mathbb{R}$ and separating $\Delta_\pm(\lambda)$ into real and imaginary parts, we further have

$$\Delta_\pm(\lambda) = R_\pm(\mu,\omega) + iI_\pm(\mu,\omega), \tag{2.5}$$

where

$$R_\pm(\mu,\omega) = \begin{cases} (p-1-\mu) \pm \gamma \displaystyle\int_0^1 e^{-\mu\theta}\cos(\omega\theta)\,d\theta & \text{if } rs>0, \\ (p-1-\mu) \pm \gamma \displaystyle\int_0^1 e^{-\mu\theta}\sin(\omega\theta)\,d\theta & \text{if } rs<0, \end{cases} \tag{2.6}$$

$$I_\pm(\mu,\omega) = \begin{cases} -\omega \mp \gamma \displaystyle\int_0^1 e^{-\mu\theta}\sin(\omega\theta)\,d\theta & \text{if } rs>0, \\ -\omega \pm \gamma \displaystyle\int_0^1 e^{-\mu\theta}\cos(\omega\theta)\,d\theta & \text{if } rs<0. \end{cases} \tag{2.7}$$

It has been known that the local stability of the trivial solution $(0,0)$ completely depends on the real parts of the complex roots of (2.2) (cf. [12]). However, the characteristic equation (2.2) has infinitely many complex roots, and it is quite difficult to determine the values of the parameters r, p and s such that the complex roots have positive or negative real parts.

The following is a useful observation for finding the nonzero roots of $P(\lambda) = 0$. We note that λ is a nonzero root of $\Delta_\pm(\lambda) = 0$ if and only if λ is a nonzero root of

$$0 = \lambda\Delta_\pm(\lambda) = \begin{cases} (p-1-\lambda)\lambda \pm \gamma(1-e^{-\lambda}) & \text{if } rs>0, \\ (p-1-\lambda)\lambda \pm i\gamma(1-e^{-\lambda}) & \text{if } rs<0. \end{cases} \tag{2.8}$$

Again, letting $\lambda = \mu + i\omega$ for some $\mu,\ \omega \in \mathbb{R}$ and then separating $\lambda\Delta_\pm(\lambda)$ into real and imaginary parts, we obtain

$$\lambda\Delta_\pm(\lambda) = R_\pm^\lambda(\mu,\omega) + iI_\pm^\lambda(\mu,\omega), \tag{2.9}$$

where

$$R_\pm^\lambda(\mu,\omega) = \begin{cases} (p-1)\mu - (\mu^2-\omega^2) \pm \gamma(1-e^{-\mu}\cos\omega) & \text{if } rs>0, \\ (p-1)\mu - (\mu^2-\omega^2) \mp \gamma e^{-\mu}\sin\omega & \text{if } rs<0, \end{cases} \tag{2.10}$$

$$I_\pm^\lambda(\mu,\omega) = \begin{cases} (p-1)\omega - 2\mu\omega \pm \gamma e^{-\mu}\sin\omega & \text{if } rs>0, \\ (p-1)\omega - 2\mu\omega \pm \gamma(1-e^{-\mu}\cos\omega) & \text{if } rs<0. \end{cases} \tag{2.11}$$

In the next section, we are going to describe the solution curves in the parameter space such that the nonzero complex roots of the characteristic equation (2.2) have zero real parts.

3. Behaviours of the $i\omega$-curves

Although complex roots of $P(\lambda) = 0$ always come in complex conjugate pairs, we may just consider the case of $\omega > 0$ in this section; see (3.1)-(3.2) and (3.4)-(3.5) below.

We first consider the case $rs > 0$. Let $\lambda = i\omega$ be a nonzero root of $P(\lambda) = 0$. By (2.10) and (2.11), we have

$$\omega^2 = \mp\gamma(1 - \cos\omega), \tag{3.1}$$
$$(p-1)\omega = \mp\gamma \sin\omega. \tag{3.2}$$

Since $\gamma > 0$ and $|\cos\omega| \leq 1$, we only need to consider the plus parts in the above equations with $\omega \neq 2k\pi$, $k \in \mathbb{N}$. Thus, we can represent p and γ in terms of ω and then describe the infinitely many parametric curves in the γp-plane.

Definition 3.1. *Assume $rs > 0$. The jth $i\omega$-curve $(\gamma_j(\omega), p_j(\omega))$ is defined by*

$$\gamma_j(\omega) = \frac{1}{2}\omega^2 \csc^2(\frac{\omega}{2}) \quad \text{and} \quad p_j(\omega) = 1 + \omega\cot(\frac{\omega}{2}), \tag{3.3}$$

for $j = 0, 1, \cdots$ and $\omega \in (2j\pi, (2j+2)\pi)$.

Some properties of the $i\omega$-curves follow immediately from elementary computations.

Lemma 3.2. *Assume $rs > 0$.*

(1). *For each $j \in \{0\} \cup \mathbb{N}$, $p_j((2j+1)\pi) = 1$ and $p_j(\omega)$ is strictly decreasing in variable ω.*

(2). *For each $j \in \mathbb{N}$, $\gamma_j(\omega)$ has a unique minimizer at $\omega = \omega_j$ with $\omega_j \cot\frac{\omega_j}{2} = 2$ (i.e., $p_j(\omega_j) = 3$), and $\gamma_j(\omega)$ is strictly decreasing (resp., strictly increasing) in variable ω when $\omega < \omega_j$ (resp., $\omega > \omega_j$).*

(3). *For $j = 0$, we have*

$$\lim_{\omega\to 0^+} \gamma_j(\omega) = 2 \quad \text{and} \quad \lim_{\omega\to 2\pi^-} \gamma_j(\omega) = \infty,$$
$$\lim_{\omega\to 0^+} p_j(\omega) = 3 \quad \text{and} \quad \lim_{\omega\to 2\pi^-} p_j(\omega) = -\infty.$$

For each $j \in \mathbb{N}$, we have

$$\lim_{\omega\to 2j\pi^+} \gamma_j(\omega) = \infty \quad \text{and} \quad \lim_{\omega\to (2j+2)\pi^-} \gamma_j(\omega) = \infty,$$
$$\lim_{\omega\to 2j\pi^+} p_j(\omega) = \infty \quad \text{and} \quad \lim_{\omega\to (2j+2)\pi^-} p_j(\omega) = -\infty.$$

As a consequence of Lemma 3.2, for $j \geq 1$, the jth $i\omega$-curve has a unique turning point at $p_j = 3$ in the γp-plane. In addition, assume that $p_i(\omega_i) = p_k(\omega_k)$ with $\omega_i \in (2i\pi, (2i+2)\pi)$, $\omega_k \in (2k\pi, (2k+2)\pi)$, and $i < k$. Then

$$\gamma_i(\omega_i) = \frac{1}{2}\big(\omega_i^2 + (p_i - 1)^2\big) < \gamma_k(\omega_k) = \frac{1}{2}\big(\omega_k^2 + (p_k - 1)^2\big).$$

Therefore, all the $i\omega$-curves do not intersect each other. The numerical results of the $i\omega$-curves are depicted in Figure 1 that confirm our analysis.

We now consider the case $rs < 0$. If $\lambda = i\omega$ is a nonzero root of $P(\lambda) = 0$, then we have

$$\omega^2 = \pm\gamma \sin\omega, \tag{3.4}$$
$$(p-1)\omega = \mp\gamma(1-\cos\omega). \tag{3.5}$$

The $i\omega$-curves in the γp-plane are defined as follows.

Definition 3.3. *Assume $rs < 0$. The jth $i\omega$-curves $(\gamma_j(\omega), p_j(\omega))$ is defined by*

$$\gamma_j(\omega) = \omega^2|\csc\omega| \quad \text{and} \quad p_j(\omega) = 1 - \omega\tan(\frac{\omega}{2}), \tag{3.6}$$

for $j = 0, 1, \cdots$ and $\omega \in (j\pi, (j+1)\pi)$.

Note that $p_j(\omega) < 1$ (resp., > 1) when j is even (resp., odd).

Lemma 3.4. *Assume $rs < 0$.*

(1). *For each $j \in \{0\} \cup \mathbb{N}$, $p_j(\omega)$ is strictly decreasing in variable ω.*

(2). *For each $j \in \mathbb{N}$, $\gamma_j(\omega)$ has a unique minimizer at $\omega = \omega_j$ with $\omega_j \cot\omega_j = 2$, and $\gamma_j(\omega)$ is strictly decreasing (resp., strictly increasing) in variable ω when $\omega < \omega_j$ (resp., $\omega > \omega_j$).*

(3). *For $j = 0$, we have*

$$\lim_{\omega\to 0^+} \gamma_j(\omega) = 0 \quad \text{and} \quad \lim_{\omega\to 2\pi^-} \gamma_j(\omega) = \infty,$$
$$\lim_{\omega\to 0^+} p_j(\omega) = 1 \quad \text{and} \quad \lim_{\omega\to 2\pi^-} p_j(\omega) = -\infty.$$

For each $j \in \mathbb{N}$, we have

$$\lim_{\omega\to j\pi^+} \gamma_j(\omega) = \infty \quad \text{and} \quad \lim_{\omega\to (j+1)\pi^-} \gamma_j(\omega) = \infty,$$
$$\lim_{\omega\to 2j\pi^-} p_{2j-1}(\omega) = 1 \quad \text{and} \quad \lim_{\omega\to (2j-1)\pi^+} p_{2j-1}(\omega) = \infty,$$
$$\lim_{\omega\to 2j\pi^+} p_{2j}(\omega) = 1 \quad \text{and} \quad \lim_{\omega\to (2j+1)\pi^-} p_{2j}(\omega) = -\infty.$$

By using a similar argument to the case of $rs > 0$, Lemma 3.4 implies that all the $i\omega$-curves do not intersect each other. The numerical results of the $i\omega$-curves with are illustrated in Figure 2 that confirm our analysis again.

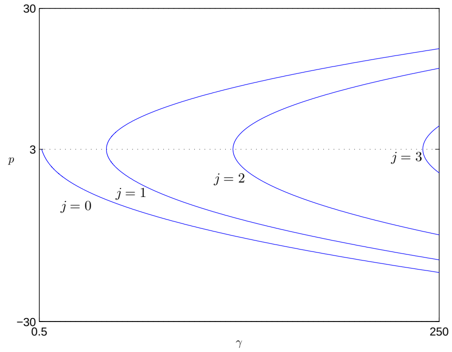

Fig. 1. $i\omega$-curves for $rs > 0$

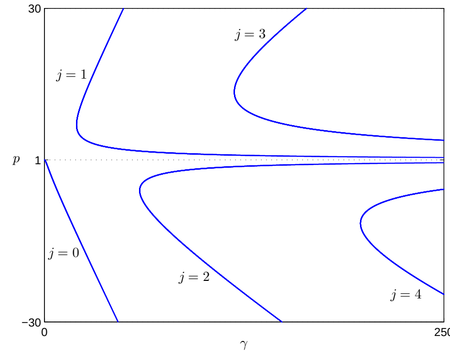

Fig. 2. $i\omega$-curves for $rs < 0$

4. Stability of the trivial solution

In this section, we will investigate the local stability of the trivial solution $(0, 0)$. An equilibrium solution is locally asymptotically stable if all the roots of the corresponding characteristic equation have negative real parts, while it is unstable if at least one root has positive real part. Our goal is to describe two regions of the γp-plane, in which the trivial solution $(0, 0)$ is locally asymptotically stable and unstable, respectively. We shall refer to these subsets as the stability and instability regions of the trivial solution.

First, we consider $rs > 0$. In this case, the stability and instability regions of $(0, 0)$ can be clearly determined, even for a more general density function k.

Theorem 4.1. *Assume $rs > 0$.*

(1). *If $\gamma > 1 - p$, then $P(\lambda) = 0$ has a root with positive real part.*

(2). *If $\gamma < 1 - p$, then all the roots of $P(\lambda) = 0$ have negative real parts.*

Proof. Note that $\gamma = \sqrt{rs} > 0$.

(1). From (2.4), we have

$$\lim_{\mathbb{R}\ni\lambda\to 0} \Delta_+(\lambda) = p - 1 + \gamma > 0 \quad \text{and} \quad \lim_{\mathbb{R}\ni\lambda\to\infty} \Delta_+(\lambda) = -\infty.$$

It follows that there exists a $\lambda^* > 0$ such that $\Delta_+(\lambda^*) = 0$.

(2). According to (2.6), we have

$$\begin{aligned} R_\pm(\mu, \omega) &= p - 1 - \mu \pm \gamma \int_0^1 e^{-\mu\theta} \cos(\omega\theta)\, d\theta \\ &\le p - 1 - \mu + \gamma \\ &< 0, \end{aligned}$$

for all $\mu \geq 0$ and $\omega \in \mathbb{R}$. Therefore all the roots of $\Delta_{\pm}(\lambda) = 0$ have negative real parts.

□

Next, we consider the case $rs < 0$. In this case, the analysis of the stability and instability regions of the trivial solution relies on the following technical lemma with the properties of the $i\omega$-curves studied in section 3.

Lemma 4.2. *Assume $rs \leq 0$. As γ and p vary in parameter space, the number of roots of the characteristic equation $P(\lambda) = 0$ with $\mathrm{Re}(\lambda) > 0$, counting multiplicities, can change only if a characteristic root passes through the imaginary axis in complex plane.*

Proof. Without loss of generality, we may assume p is fixed, and the characteristic equation $P_\gamma(\lambda) = 0$ indicate that γ is the varying parameter. Denote by $\lambda(\gamma)$ a root of $P_\gamma(\lambda) = 0$. Then $\lambda(\gamma)$ is obviously isolated. Let C be a circle surrounding $\lambda(\gamma)$ such that $P_\gamma(\lambda)$ never vanishes in the interior of C except at $\lambda(\gamma)$. Furthermore, there exists a $\varepsilon > 0$ such that for $\lambda \in C$, we have

$$|P_\gamma(\lambda)| > |P_{\gamma'}(\lambda) - P_\gamma(\lambda)| \quad \text{whenever } |\gamma - \gamma'| < \varepsilon.$$

Now, Rouché's theorem [1] implies that $P_\gamma(\lambda)$ and $P_{\gamma'}(\lambda)$ has the same number of zeros inside C whenever $|\gamma - \gamma'| < \varepsilon$. In other words, the multiplicity of $\lambda(\gamma)$ does not change for a small perturbation of γ. Thus as γ varies, the number of roots of $P_\gamma(\lambda) = 0$ in the right-half plane, counting multiplicity, can change only in the case that either $\lambda(\gamma)$ pass through the imaginary axis or there exists a sequence $\lambda(\gamma_j)$ with $\mathrm{Re}(\lambda(\gamma_j)) > 0$ satisfying

$$\lim_{j\to\infty} |\lambda(\gamma_j)| = \infty \quad \text{and} \quad \lim_{j\to\infty} \gamma_j < \infty.$$

However, we can verify that the latter case is impossible. Indeed, the above implies that

$$\begin{aligned} |P_{\gamma_j}(\lambda(\gamma_j))| &= \left| \big(p - 1 - \lambda(\gamma_j)\big)^2 + \gamma_j^2 \Big(\int_0^1 e^{-\lambda(\gamma_j)\theta}\, d\theta \Big)^2 \right| \\ &\geq \big|p - 1 - \lambda(\gamma_j)\big|^2 - \gamma_j^2 \to \infty \text{ as } j \to \infty \end{aligned}$$

which contradicts to $P_{\gamma_j}(\lambda(\gamma_j)) = 0$. This completes the proof. □

We remark that in Lemma 4.2, we allow the degenerate case $rs = 0$, i.e., $\gamma = 0$. Now, the stability analysis of the trivial solution for $rs < 0$ can be stated as follows:

Theorem 4.3. *Assume $rs < 0$.*

(1). *If $p \neq 1$ and (γ, p) lies in the right-hand side of the 0th $i\omega$-curve, then $P(\lambda) = 0$ has a root with positive real part.*

(2). *If $p < 1$ and (γ, p) lies in the left-hand side of the 0th $i\omega$-curve, then all the roots of $P(\lambda) = 0$ have negative real parts.*

Proof. Keep it in mind that $\gamma = \sqrt{|rs|}$.

(1). First, from (2.3), we note that $P(\lambda) = 0$ have positive real root $\lambda = p - 1$ of multiplicity 2, whenever $\gamma = 0$ and $p > 1$. Next, let (γ^*, p^*) be a given point on some $i\omega$-curve and let $\lambda = 0 + i\omega^*$ with $\omega^* > 0$ be the corresponding pure imaginary root of $P(\lambda) = 0$. According to (2.10) and (2.11), if $R^\lambda_\pm(\mu, \omega) = 0$ and $I^\lambda_\pm(\mu, \omega) = 0$ then we have

$$(p-1)\mu - (\mu^2 - \omega^2) \mp \gamma e^{-\mu} \sin\omega = 0, \tag{4.1}$$
$$(p-1)\omega - 2\mu\omega \pm \gamma(1 - e^{-\mu}\cos\omega) = 0. \tag{4.2}$$

An implicit differentiation based on (4.1) and (4.2) leads to

$$\frac{d\mu}{d\gamma}\Big|_{\mu=0,\omega=\omega^*} = \frac{\gamma^*(1-\cos\omega^*)(1+\frac{\sin\omega^*}{\omega^*})}{(p^*-1\pm\gamma^*\sin\omega^*)^2 + (2\omega^* \mp \gamma^*\cos\omega^*)^2} > 0. \tag{4.3}$$

Therefore, the real parts of characteristic roots of $P(\lambda) = 0$ are always positive, provided $\gamma > \gamma^*$ and $\gamma \simeq \gamma^*$. Now, combining the above facts with Lemma 4.2, we can conclude that if $p \neq 1$ and (γ, p) lies in the right-hand side of the 0th $i\omega$-curve, then $P(\lambda) = 0$ has at least one root with positive real part.

(2). According to (2.6), if $\mu + i\omega$ is a root of $P(\lambda) = 0$ with non-negative real part, we have

$$\begin{aligned} 0 \leq \mu &= p - 1 \pm \gamma \int_0^1 e^{-\mu\theta} \sin(\omega\theta)\, d\theta \\ &\leq p - 1 + \gamma, \end{aligned}$$

which implies that if $\gamma < 1-p$ then all the roots of $P(\lambda) = 0$ have negative real parts. Next, it is easy to check that if $\gamma < 1 - p$ then (γ, p) lies in the left-hand side of the 0th $i\omega$-curve (cf. Figure 2). Furthermore, based on the principle of Lemma 4.2, one can verify that, for (γ, p) lying in the left-hand side of the 0th $i\omega$-curve, the number of characteristic roots with positive real parts is zero. This completes the proof.

□

5. Hopf bifurcations

In this section, we will investigate the mechanism for Hopf bifurcation of the trivial equilibrium $(0,0)$ of (1.1). We first consider the case $rs > 0$. Let $\lambda = \mu + i\omega \neq 0$. In virtue of (2.10) and (2.11), if $R^{\lambda}_{\pm}(\mu,\omega) = 0$ and $I^{\lambda}_{\pm}(\mu,\omega) = 0$ then we have

$$\begin{aligned} (p-1)\mu - (\mu^2-\omega^2) \pm \gamma(1-e^{-\mu}\cos\omega) &= 0, && (5.1)\\ (p-1)\omega - 2\mu\omega \pm \gamma e^{-\mu}\sin\omega &= 0. && (5.2) \end{aligned}$$

Theorem 5.1. *Assume $rs > 0$. Let $(\gamma_c, p_c) = (\gamma_j(\omega_c), p(\omega_c))$ for some j and $\omega_c > 0$. Then for fixed $p = p_c \neq 3$, (1.1) undergoes a Hopf bifurcation at $\gamma = \gamma_c$. Furthermore, If $f'''(0)(\gamma_c\cos\omega_c - 2 < 0)$ (> 0, resp.), then the Hopf bifurcation is supercritical (subcritical, resp.).*

Proof. According to the definition of $i\omega$-curves, (2.2) has the purely imaginary root $i\omega_c$ when $(\gamma,p) = (\gamma_c,p_c)$. To prove the assertion, it suffices to show that the $i\omega_c$ is simple, the roots of (2.2) are transversal to the imaginary axis as γ varies near γ_c, and no other characteristic root is an integral multiple of $i\omega_c$ except $-i\omega_c$ (cf. [12]).

First, from (2.8), we can verify that $i\omega \neq 0$ is a root of $P(\lambda) = 0$ if and only if $\Delta_-(i\omega) = 0$. A simple computation shows

$$\begin{aligned} \Delta'_-(i\omega_c) &= -1 - \gamma\Big(\frac{(\lambda+1)e^{-\lambda}-1}{\lambda^2}\Big)_{\lambda=i\omega_c}\\ &= -1 + \frac{\gamma_c(\cos\omega_c + \omega_c\sin\omega_c - 1)}{\omega_c^2} + i\frac{\gamma_c(\omega_c\cos\omega_c - \sin\omega_c)}{\omega_c^2}. \qquad (5.3) \end{aligned}$$

The imaginary part of (5.3) is equal to zero if and only if $\sin\omega_c = \omega_c\cos\omega_c$, which together with (3.1) and (3.2) will force the real part to be

$$\mathrm{Re}\Delta'_-(i\omega_c) = p_c - 3.$$

Hence if $p_c \neq 3$ then $i\omega_c$ is simple.

Secondly, at $\lambda = i\omega_c$, the derivative of real part of the characteristic root with respect to γ may be obtained by implicit differentiation based on (2.10) and (2.11). Indeed, we can derive

$$\frac{d\mu}{d\gamma}\Big|_{\lambda=i\omega_c} = \frac{1}{\gamma_c}\frac{\omega_c^2(\gamma_c - p_c + 1)}{(p_c - 1 - \gamma_c\cos\omega_c)^2 + (2\omega_c - \gamma_c\sin\omega_c)^2} > 0. \qquad (5.4)$$

Hence the complex roots of (2.2) are transversal to the imaginary axis as γ varies near γ_c.

Thirdly, Lemma 3.2 (1) shows that p_j is strictly decreasing in variable ω. Therefore, it is not even possible to have another pure imaginary characteristic root except $-i\omega_c$ for $p = p_c$ and $\gamma = \gamma_c$. Thus, Hopf Bifurcation occurs at (γ_c, p_c).

According to [6, 7] and the computations in the Appendix, it can be shown that for $p = p_c$ and $\gamma \approx \gamma_c$ the dynamics of (1.1) for (x, y) near $(0, 0)$ is determined by the following differential equations in polar coordinates

$$\begin{aligned} \dot{r} &= \mu'(\gamma_c)r + ar^3, \\ \dot{\theta} &= \omega_c. \end{aligned}$$

If $a < 0$ ($a > 0$ resp.), the Hopf bifurcation is supercritical (subcritical resp.). The sign of a is determined as follows:

$$\operatorname{sign}(a) = \operatorname{sign}\big(f'''(0)(\gamma_c \cos\omega_c - 2)\big).$$

This completes the proof. □

Next, we are going to investigate the Hopf bifurcation for the case of $rs < 0$ that can be treated by a similar argument.

Theorem 5.2. *Assume $rs < 0$. Let $(\gamma_c, p_c) = (\gamma_j(\omega_c), p(\omega_c))$ for some j and $\omega_c > 0$. Then for fixed $p = p_c$, (1.1) undergoes a Hopf bifurcation at $\gamma = \gamma_c$. Furthermore, If $f'''(0)(\frac{\omega_c}{\sin\omega_c} - 2 - \omega_c^2) < 0$ (> 0, resp.), then the Hopf Bifurcation is supercritical (subcritical, resp.). In particular, if (γ_c, p_c) lies in the 0th $i\omega$-curve and $f'''(0)(\frac{\omega_c}{\sin\omega_c} - 2 - \omega_c^2 < 0)$, then the periodic solution induced by the Hopf bifurcation is attracting.*

Proof. The proof is similar to Theorem 5.1.

First, a direct computation shows that

$$\begin{aligned} \Delta'_\pm(i\omega_c) &= -1 \pm i\gamma\Big(\frac{(\lambda+1)e^{-\lambda} - 1}{\lambda^2}\Big)_{\lambda = i\omega_c} \\ &= -1 \pm \gamma_c \frac{(\omega_c\cos\omega_c - \sin\omega_c)}{\omega_c^2} \mp i\gamma_c \frac{(\cos\omega_c + \omega_c\sin\omega_c - 1)}{\omega_c^2}. \end{aligned}$$

Suppose $\Delta'_\pm(i\omega_c) = 0$. Applying (3.4) and (3.5), we have

$$\sin\omega_c = \frac{\omega_c}{2 + \omega_c^2} \quad \text{and} \quad \cos\omega_c = \frac{2}{2 + \omega_c^2}, \tag{5.5}$$

which implies $\omega_c = 0$. This leads to a contradiction and $i\omega_c$ is therefore simple.

Secondly, for $\lambda = i\omega_c$, from (4.3) we obtain

$$\frac{d\mu}{d\gamma}\Big|_{\mu=0,\omega=\omega_c} = \frac{\gamma_c(1-\cos\omega_c)(1+\frac{\sin\omega_c}{\omega_c})}{(p_c - 1 \pm \gamma_c \sin\omega_c)^2 + (2\omega_c \mp \gamma_c\cos\omega_c)^2}, \tag{5.6}$$

and since the numerator of (5.6) is always positive, the complex roots of (2.2) are transversal to the imaginary axis as γ varies near γ_c.

According to Lemma 3.4 (1), we have $\frac{dp_j}{d\omega} < 0$. Therefore, no other characteristic root is an integral multiple of $i\omega_c$ except $-i\omega_c$. Thus, Hopf Bifurcation occurs at (γ_c, p_c).

Similar to the proof of Theorem 5.1, the sign of a is determined as follows:

$$\text{sign}(a) = \text{sign}\big(f'''(0)(\frac{\omega_c}{\sin\omega_c} - 2 - \omega_c^2)\big).$$

This completes the proof. □

Finally, we provide an example with $rs < 0$ to demonstrate the occurrence of the Hopf bifurcation.

Example 5.3. We consider the system (1.1) with $f(x) = \tanh(x)$ and $rs < 0$. Choosing $\omega = \frac{\pi}{6}$ and $j = 0$ in (3.6), then we have a point $(\gamma_0(\frac{\pi}{6}), p_0(\frac{\pi}{6}))$ located on the 0th $i\omega$-curve. According to Theorem 5.2, for such fixed $p_0(\frac{\pi}{6})$, a Hopf bifurcation should be occurred for $\gamma = \gamma_0(\frac{\pi}{6})$ and the induced periodic solution is attracting. We choose $r_L = -s_L = \gamma_0(\frac{\pi}{6}) - 0.1$ and $r_R = -s_R = \gamma_0(\frac{\pi}{6}) + 0.1$. Then we have $\gamma_L := \sqrt{|r_L s_L|} \lesssim \gamma_0(\frac{\pi}{6}) \lesssim \gamma_R := \sqrt{|r_R s_R|}$. For the case $(\gamma, p) = (\gamma_L, p_0(\frac{\pi}{6}))$, the trivial solution $(0,0)$ is locally asymptotically stable which can be confirmed numerically (cf. Figure 3), while for $(\gamma, p) = (\gamma_R, p_0(\frac{\pi}{6}))$, a periodic solution occurs (cf. Figure 4). All numerical simulations in this example are mainly based on the MATLAB codes provided by [22].

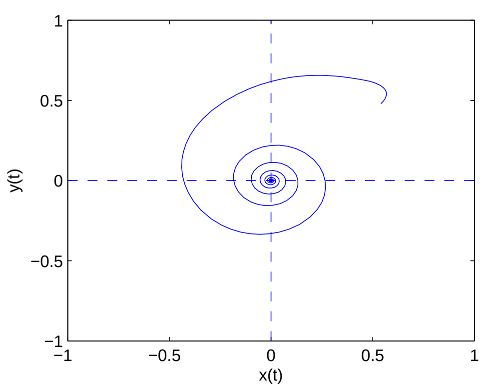

Fig. 3. $(0,0)$ is loc. asympt. stable

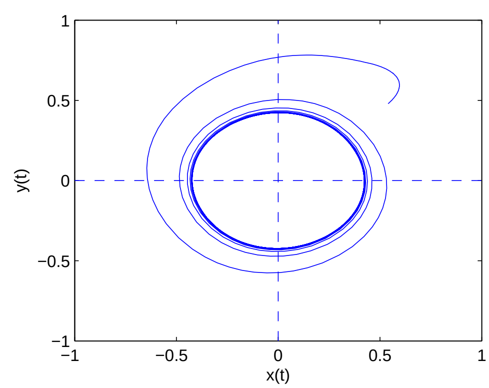

Fig. 4. a periodic solution occurs

6. Steady state bifurcations

Following the results obtained in sections 4 and 5, we conclude that the local stability of the trivial solution $(0, 0)$ might be changed when the parameter values varied. In this section, we attempt to explore the steady state bifurcation of the trivial solution by studying the existence and local stability of nontrivial equilibria of (1.1).

6.1. Existence and multiplicity of nontrivial equilibria

Notice that an equilibrium (x^*, y^*) of (1.1) must satisfy the following system of equations:

$$\begin{array}{rcl} x^* - pf(x^*) & = & sf(y^*), \\ y^* - pf(y^*) & = & rf(x^*). \end{array} \tag{6.1}$$

First, we investigate the existence with multiplicity of the nontrivial equilibria. To this aim, the following two observations are useful in our analyses and their proofs are straightforward.

Lemma 6.1. *(x^*, y^*) is an equilibrium of (1.1) if and only if $(-x^*, -y^*)$ is an equilibrium of (1.1).*

Lemma 6.2. *Let $h(x) := x - pf(x)$. Then $h(x)$ is a C^2 odd function with $h(0) = 0$. Moreover, we have*

(1). *If $p < 1$, then $h(x)$ is a strictly increasing function.*

(2). *If $p > 1$, then there exist constants $B > A > 0$ such that the minimum value of $h(x)$ for $x \in (0, \infty)$ occurs at $x = A$, $h(B) = 0$, and $h(x)$ is negative on $(0, B)$ while positive on (B, ∞).*

For the case $rs > 0$, we have the following results:

Theorem 6.3. *Assume $rs > 0$.*

(1). *Let $p < 1$.*

(1-1). *If $\gamma < 1 - p$ and $p > 0$, then there is no nontrivial equilibrium of (1.1).*

(1-2). *If $\gamma < 1 - p$ and $\max\{|s|, |r|\} < 1 - p$, then there is no nontrivial equilibrium of (1.1).*

(1-3). *If $\gamma > 1-p$, then there are nontrivial equilibria of (1.1). In addition, if $p > 0$ then there are exactly two nontrivial equilibria of (1.1).*

(2). *Let* $p > 1$. *Then there are at least two nontrivial equilibria of* (1.1). *Furthermore, we have*

(2-1). *If* $\min\{|s|,|r|\} > p-1$, *then there are exactly two nontrivial equilibria of* (1.1).

(2-2). *If* $\gamma < p-1$, *then there are at least eight nontrivial equilibria of* (1.1).

Proof. We will focus on the case of $r > 0$ and $s > 0$. The case of $r < 0$ and $s < 0$ can be treated in a similar way. By Lemma 6.1, we only need to consider the equilibrium (x^*, y^*) with $x^* \geq 0$. For convenience, we rewrite the isoclines of (1.1) as $y = Y(x)$ and $x = X(y)$, i.e., $Y(x)$ and $X(y)$ satisfies

$$x - pf(x) = sf(Y(x)), \qquad (6.2)$$
$$y - pf(y) = rf(X(y)). \qquad (6.3)$$

Differentiating equations (6.2) and (6.3) with respect to x and y respectively, we have

$$1 - pf'(x) = sf'(Y(x))Y'(x), \qquad (6.4)$$
$$1 - pf'(y) = rf'(X(y))X'(y), \qquad (6.5)$$
$$-pf''(x) = sf''(Y(x))Y'^2(x) + sf'(Y(x))Y''(x), \qquad (6.6)$$
$$-pf''(y) = rf''(X(y))X'^2(y) + rf'(X(y))X''(y), \qquad (6.7)$$

and thus

$$m_1 := Y'(0) = \frac{1-p}{s}, \quad m_2 := X'(0) = \frac{1-p}{r}.$$

On the other hand, due to the boundedness of f, there exist $x_- < 0 < x_+$ and $y_- < 0 < y_+$ such that

$$x_\pm - pf(x_\pm) = \pm sM, \quad \lim_{x \to x_\pm} Y(x) = \pm\infty,$$
$$y_\pm - pf(y_\pm) = \pm rM, \quad \lim_{y \to y_\pm} X(y) = \pm\infty.$$

(1). Under the assumption $p < 1$, it follows from Lemma 6.2 (1) that the two isoclines $y = Y(x)$ and $x = X(y)$ lie in the first and third quadrants. Thus, we are allowed to consider the nontrivial equilibrium (x^*, y^*) in first quadrant only.

(1-1). (cf. Figure 5) If $p > 0$, then (6.6) and (6.7) imply that

$$Y''(x) > 0 \text{ for } x > 0 \quad \text{and} \quad X''(y) > 0 \text{ for } y > 0.$$

It immediately follows that

$$Y(x) > m_1 x \text{ for } x > 0 \quad \text{and} \quad X(y) > m_2 y \text{ for } y > 0.$$

The assumption $\gamma < 1 - p$ implies $m_1 m_2 > 1$. Therefore, the two isoclines $y = Y(x)$ and $x = X(y)$ do not intersect in the first quadrant. This proves the assertion (1-1).

(1-2). (cf. Figure 5) If $(s+p) < 1$ and $(r+p) < 1$, then $Y'(0) > 1$ and $X'(0) > 1$. We claim that

$$Y(x) > x \text{ for } x > 0 \quad \text{and} \quad X(y) < y \text{ for } y > 0.$$

Suppose it is not true, say $Y(x) > x$ for $x > 0$ is not true. Then there must exist $x_0 > 0$ such that $Y(x_0) = x_0$ and $Y'(x_0) \leq 1$. From (6.4), we obtain

$$1 - pf'(x_0) = sf'(Y(x_0))Y'(x_0) \leq sf'(x_0),$$

i.e., $(p+s)f'(x_0) \geq 1$. However, this contradicts to our assumptions (H), since $0 < f'(x_0) <$. Hence, the two isoclines are separated by $y = x$, i.e., there is no nontrivial equilibrium of (1.1).

(1-3). (cf. Figure 6) It suffices to show that the isoclines intersect exactly once in the first quadrant. If $\gamma > 1-p$, we have $m_1 m_2 < 1$. This together with $Y(x_+) = +\infty$ and $X(y_+) = +\infty$ implies that the two isoclines intersect in the first quadrant. In addition, if $p > 0$ then both $Y(x)$ and $X(y)$ are increasing and convex for $x > 0$ and $y > 0$, respectively. Hence, the intersection is unique.

(2). (cf. Figure 7) By Lemma 6.2 (2), if $p > 1$ then

$$Y(x) > 0 \text{ for } x > B \quad \text{and} \quad X(y) > 0 \text{ for } y > B.$$

This together with $Y(x_+) = +\infty$ and $X(y_+) = +\infty$ implies that the two isoclines intersect in the first quadrant. Moreover, form (6.4) - (6.7), $Y(x)$ and $X(y)$ are increasing and convex for $x > 0$ and $y > 0$, respectively. Hence, the intersection is unique in the first quadrant.

(2-1). (cf. Figure 7) It suffices to show that the isoclines never intersect in the four quadrant. By the assumptions, it is obvious that $-1 < m_1 < 0$ and $-1 < m_2 < 0$. By the same arguments as the proof of (1-2), we have

$$Y(x) > -x \text{ for } 0 < x < A \quad \text{and} \quad X(y) < -y \text{ for } -A < y < 0.$$

Since $Y(A)$ is the minimum for $0 < x < B$ and $X(A)$ is the maximum for $0 < x < B$, the isoclines are separated by $y = -x$ in the fourth quadrant. Hence, there are no nontrivial equilibria in the second and fourth quadrants.

(2-2). (cf. Figure 8 and Figure 9) By the assumptions, it is obvious that $m_1 m_2 > 1$. Our goal is to prove that the isoclines $Y(x)$ and $X(y)$ intersect at least once in the second and fourth quadrants. According to Lemma 6.2 (2), we have

$$Y(x) < 0 \text{ for } 0 < x < B \quad \text{and} \quad X(y) > 0 \text{ for } -B < x < 0,$$

$Y(A)$ is the minimum for $0 < x < B$ and $X(-A)$ is the maximum for $-B < x < 0$. It may happen that $Y(x_+) = -\infty$ or $X(y_+) = +\infty$ (cf. Figure 11). However, no matter they are finite or not, the two isoclines intersect at least at three points in the fourth quadrant.

The proof of Theorem 6.3 is complete. □

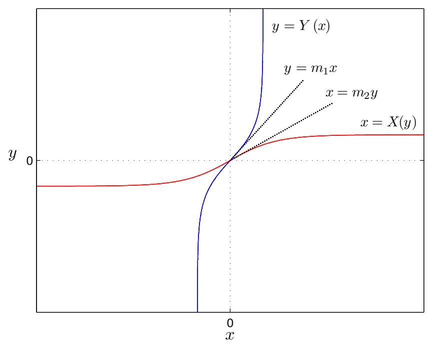

Fig. 5. Theorem 6.3 (1-1) and (1-2)

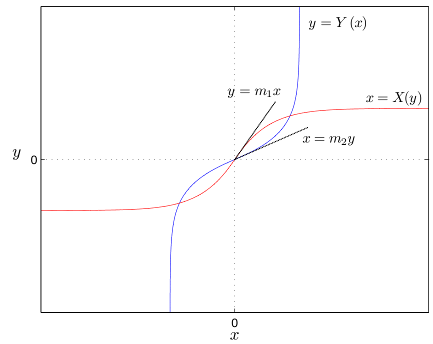

Fig. 6. Theorem 6.3 (1-3)

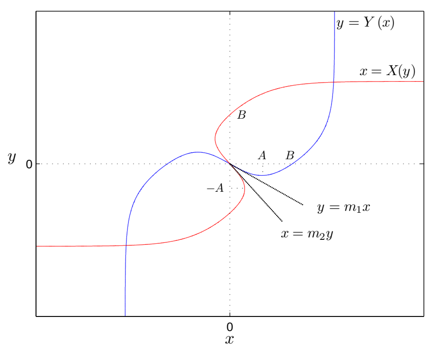

Fig. 7. Theorem 6.3 (2) and (2-1)

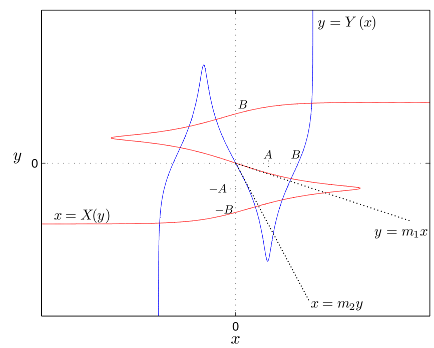

Fig. 8. Theorem 6.3 (2-2)

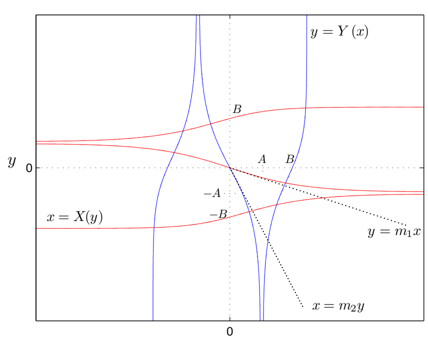

Fig. 9. Theorem 6.3 (2-2)

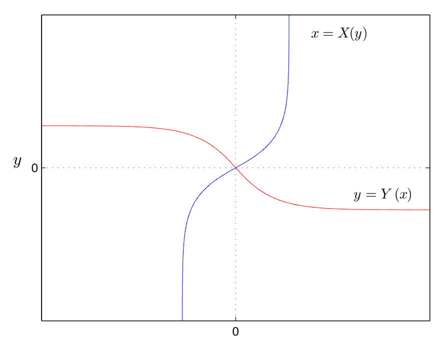

Fig. 10. Theorem 6.4 (1)

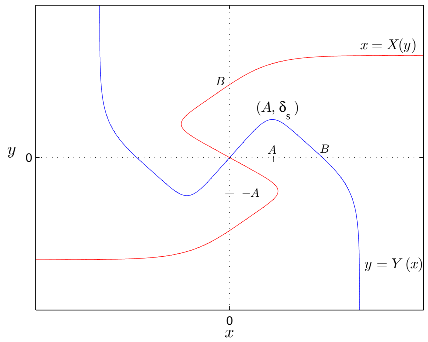

Fig. 11. Theorem 6.4 (2-1)

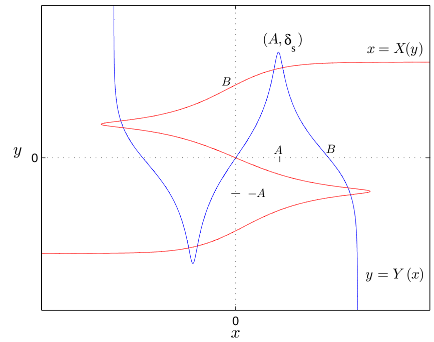

Fig. 12. Theorem 6.4 (2-2)

Next, we study the existence and multiplicity of nontrivial equilibria of (1.1) when $rs < 0$.

Theorem 6.4. *Assume* $rs < 0$.

(1). *Let* $p < 1$. *Then there is no nontrivial equilibrium of* (1.1).

(2). *Let* $p > 1$.

(2-1). *If* $\min\{|r|, |s|\} \geq \frac{p}{B}(pf(A) - A)$, *then there is no nontrivial equilibrium of* (1.1).

(2-2). *Define* $\delta_s := f^{-1}(\frac{1}{s}(A - pf(A)))$ *and* $\delta_r := f^{-1}(\frac{1}{r}(A - pf(A)))$. *If*

$$\begin{aligned} &|\delta_s| > |r|f(A) + \frac{p}{|s|}|A - pf(A)| \\ \text{or} \quad &|\delta_r| > |s|f(A) + \frac{p}{|r|}|A - pf(A)|, \end{aligned} \tag{6.8}$$

then there are at least four nontrivial equilibria of (1.1).

Proof. Without loss of generality, we may assume that $r > 0 > s$.

(1). (cf. Figure 10) If $p < 1$, then it follows from Lemma 6.2 that the isocline $y = Y(x)$ lies in the second and fourth quadrants, while another isocline $x = X(y)$ lies in the first and third quadrants. Thus there is no nontrivial equilibrium of (1.1).

(2-1). (cf. Figure 11) The proof of this part can be found in [14]. We omit the details.

(2-2). (cf. Figure 12) As $r > 0 > s$, (6.8) becomes

$$\delta_s > rf(A) + \frac{p}{s}(A - pf(A)).$$

By the definition of δ_s and A, it is obvious that (A, δ_s) lies on the first isocline. Furthermore, the isocline has maximum value at this point. Hence, if condition (6.8) holds, then

$$\delta_s - pf(\delta_s) > rf(A), \tag{6.9}$$

and the two isoclines defined by (6.2) and (6.3) will intersect exactly at two points in the first quadrant.

□

We conclude this subsection with the following example.

Example 6.5. Consider the example that $f(x) = \tanh(x)$. Then we have

$$\begin{aligned}
A &= \ln(\sqrt{p} + \sqrt{p-1}), \quad f(A) = \sqrt{(p-1)/p}, \\
\delta_s &= \frac{1}{2} \ln \frac{s + \ln(\sqrt{p} + \sqrt{p-1}) - \sqrt{p(p-1)}}{s - \ln(\sqrt{p} + \sqrt{p-1}) + \sqrt{p(p-1)}}, \\
\delta_r &= \frac{1}{2} \ln \frac{r + \ln(\sqrt{p} + \sqrt{p-1}) - \sqrt{p(p-1)}}{r - \ln(\sqrt{p} + \sqrt{p-1}) + \sqrt{p(p-1)}}.
\end{aligned}$$

6.2. Local stability of the nontrivial equilibria

In this subsection, we will provide some sufficient conditions for ensuring the stability of the nontrivial equilibria.

Lemma 6.6. *Let (x^*, y^*) be a nontrivial equilibrium of* (1.1)*.*

(1). *If $(1 - pf'(x^*))(1 - pf'(y^*)) < rsf'(x^*)f'(y^*)$, then (x^*, y^*) is unstable.*

(2). *Assume $|(1-pf'(x^*))(1-pf'(y^*))| > |rs|f'(x^*)f'(y^*)$. If $pf'(x^*) < 1$ and $pf'(y^*) < 1$, then (x^*, y^*) is locally asymptotically stable; If $pf'(x^*) > 1$ or $pf'(y^*) > 1$, then (x^*, y^*) is unstable.*

Proof.

(1). Our goal is to show that the corresponding characteristic equation of (1.1) at (x^*, y^*) has roots with positive real parts. Similar to (2.2), the characteristic equation at (x^*, y^*)

$$\begin{aligned}
P^*(\lambda) &= (\lambda + 1 - pf'(x^*))(\lambda + 1 - pf'(y^*)) \\
&\quad - rsf'(x^*)f'(y^*)(\int_0^1 e^{-\lambda\theta}\, d\theta)^2 = 0.
\end{aligned} \tag{6.10}$$

Since

$$\lim_{\lambda\to 0} P^*(\lambda) = (1 - pf'(x^*))(1 - pf'(y^*)) - rsf'(x^*)f'(y^*) < 0,$$
$$\lim_{\lambda\to \infty} P^*(\lambda) = \infty,$$

there must exist a real $\lambda^* > 0$ such that $P^*(\lambda^*) = 0$.

(2). Let Γ be the half-circle in the right-half complex plane with sufficient large radius and center at origin. If $\lambda \in \Gamma$ or λ is a pure imaginary number, we have

$$|(\lambda + 1 - pf'(x^*))(\lambda + 1 - pf'(y^*))| \geq |1 - pf'(x^*)||1 - pf'(y^*)|$$

and

$$|rsf'(x^*)f'(y^*)(\int_0^1 e^{-\lambda\theta}\, d\theta)^2| \leq |rs|f'(x^*)f'(y^*).$$

Therefore, for $\lambda \in \Gamma$ or λ is a pure imaginary number, we have

$$|(\lambda + 1 - pf'(x^*))(\lambda + 1 - pf'(y^*))| > |rsf'(x^*)f'(y^*)(\int_0^1 e^{-\lambda\theta}\, d\theta)^2|.$$

According to Rouche's theorem [1], $P^*(\lambda)$ and $(\lambda + 1 - pf'(x^*))(\lambda + 1 - pf'(y^*))$ must have the same number of zeros in the right-half complex plane. Thus the assertions follow immediately.

□

According to Lemma 6.6, a classification of local stability for the nontrivial equilibrium (x^*, y^*) is given in Table 1.

sufficient conditions	signs of $(1 - p_1, 1 - p_2)$	rs	stability
$(1 - p_1)(1 - p_2) < rsa_1a_2$	$(+, +)$	$+$	unstable
$(1 - p_1)(1 - p_2) < rsa_1a_2$	$(+, -)$	$+$	unstable
$(1 - p_1)(1 - p_2) < rsa_1a_2$	$(-, +)$	$+$	unstable
$(1 - p_1)(1 - p_2) < rsa_1a_2$	$(-, -)$	$+$	unstable
$\|(1 - p_1)(1 - p_2)\| > \|rs\|a_1a_2$	$(+, +)$	$\pm$	stable
$\|(1 - p_1)(1 - p_2)\| > \|rs\|a_1a_2$	$(+, -)$	$\pm$	unstable
$\|(1 - p_1)(1 - p_2)\| > \|rs\|a_1a_2$	$(-, +)$	$\pm$	unstable
$\|(1 - p_1)(1 - p_2)\| > \|rs\|a_1a_2$	$(-, -)$	$\pm$	unstable

Table 1. a classification of local stability for the nontrivial equilibrium (x^*, y^*) with $(a_1, a_2) := (f'(x^*), f'(y^*))$

Finally, if $r = s$ then the stability of the nontrivial equilibria can be examined in detail.

Theorem 6.7. *Assume $r = s$. Let (x^*, y^*) be a nontrivial equilibrium of* (1.1)*.*

(1). *If $|x^*| \neq |y^*|$ then (x^*, y^*) is unstable.*

(2). *Assume $|x^*| = |y^*|$. If $sf'(x^*) > 1 - pf'(x^*)$, then (x^*, y^*) is unstable; If $sf'(x^*) < 1 - pf'(x^*)$, then (x^*, y^*) is locally asymptotically stable.*

Proof. Since $r = s$, by (6.2) and (6.3), the isoclines $Y(\bar{x})$ and $X(\bar{y})$ are symmetric with respect to the curves: $\bar{y} = \pm\bar{x}$. Therefore, if (x^*, y^*) belongs to the first or the third quadrant, then we have $|x^*| = |y^*|$.

(1). If $|x^*| \neq |y^*|$ then (x^*, y^*) belongs to the second or the fourth quadrants and the proof of part (2-2) of Theorem 5.3 implies that either $|x^*| < A < |y^*|$ or $|y^*| < A < |x^*|$. Since $f'(A) = 1/p$, in either case, we have $f'(x^*) > 1/p > f'(y^*)$ or $f'(y^*) > 1/p > f'(x^*)$. Therefore, the inequality in Lemma 5.6 (1) holds. Hence, (x^*, y^*) is an unstable equilibrium.

(2). The proof is similar to that for Theorem 4.1 by replacing $P(\lambda)$ with

$$P^*(\lambda) = (pf'(x^*) - 1 - \lambda)^2 - rsf'(x^*)^2 (\int_0^1 e^{-\lambda\theta}\, d\theta)^2.$$

$\square$

Figure 13 is the bifurcation diagram of the equilibria of (1.1) with $f(x) = \tanh(x)$ and $r = s = 0.5$. We note that pitchfork bifurcations appear.

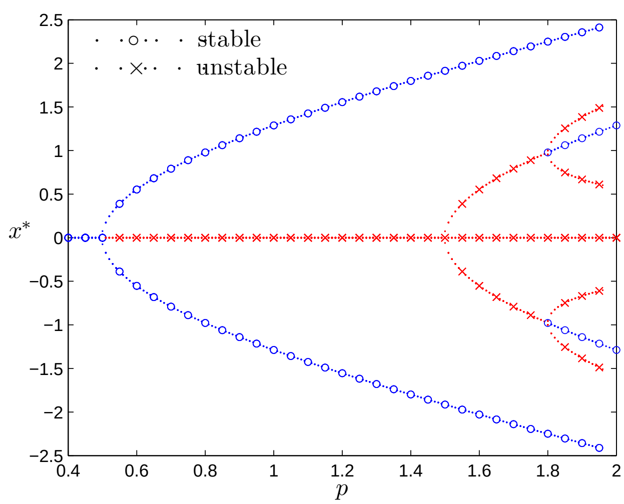

Fig. 13. bifurcation diagram of equilibria of (1.1)

Appendix

Rewrite (1.1) as

$$\dot{u}(t) \quad = \quad -u(t) + pf(u(t)) + \mathcal{M}\int_{-1}^{0} f(u(t+\theta))d\theta \qquad (6.11)$$

where $u = (x, y)^T$, $\mathcal{M}$ is the coupling matrix and $f(u)$ stands for $(f(x), f(y))^T$. Let

$$f(u) = u + g(u) \quad \text{with} \quad g(u) := \frac{f'''(0)u^3}{3!} + O(u^4).$$

Denote by C the Banach space of all continuous functions from $[-1, 0]$ to the 2-dimensional space of column vectors and $u_t \in C$ with $u_t(\theta) = u(t + \theta)$ for $\theta \in [-1, 0]$. Splitting the right-hand side of (6.11) into linear and nonlinear parts, we have

$$\dot{u}(t) = Lu_t + F(u_t), \qquad (6.12)$$

where L is the linear functional on C defined by

$$Lu := (p-1)u(0) + \mathcal{M}\int_{-1}^{0} u(\theta)d\theta,$$

and

$$F(u) := pg(u(0)) + \mathcal{M}\int_{-1}^{0} g(u(\theta))d(\theta).$$

Following the notations of [12], we may write

$$Lu = \int_{-1}^{0} d\eta(\theta)u(\theta),$$

where $\eta(\theta)$ is a function of bounded variation from $[-1, 0]$ to the space of 2×2 matrices. The eigenvalue λ of the infinitesmall generator of the C_0-semigroup A, which is defined by the solution of (6.11), satisfies

$$\det \Delta(\lambda) = 0$$

with

$$\Delta(\lambda) := \lambda I - (p-1)I - \mathcal{M}\int_{-1}^{0} e^{\lambda\theta} d\theta.$$

Let (γ_c, p_c) be the parameter where Hopf bifurcation occurs (cf. Theorem 5.1 or Theorem 5.2) and $\mathcal{M}_c$ be the coupling matrix corresponding to γ_c, then the eigenvector $\boldsymbol{v}$ of $\Delta(i\omega_c)$ satisfies

$$\mathcal{M}_c \int_{-1}^{0} e^{\lambda\theta} d\theta \boldsymbol{v} = -(p_c - 1 - i\omega_c)\boldsymbol{v}. \qquad (6.13)$$

Let

$$\phi_1(\theta) := e^{i\theta\omega_c}\boldsymbol{v}, \quad \phi_2(\theta) := e^{-i\theta\omega_c}\overline{\boldsymbol{v}}. \tag{6.14}$$

Then $P := \text{span}(\phi_1, \phi_2)$ is the eigenspace corresponding to $\pm i\omega_c$. According to the adjoint system theory developed by Hale (cf. [13]), C admits a direct sum decomposition associated with P. Indeed, denote by C' the Banach space of all continuous functions from $[0, 1]$ to the 2-dimensional space of row vectors. Let $\boldsymbol{w}^T$ be a left eigenvector of $\Delta(i\omega_c)$, i.e.

$$\boldsymbol{w}^T \mathcal{M}_c \int_{-1}^{0} e^{i\theta\omega_c} d\theta = -(p_c - 1 - i\omega_c)\boldsymbol{w}^T \tag{6.15}$$

and

$$\psi_1(\xi) := \boldsymbol{w}^T e^{-i\xi\omega_c} \text{ and } \psi_2(\xi) := \overline{\boldsymbol{w}}^T e^{i\xi\omega_c}, \quad \xi \in [0, 1],$$

then $\psi_i \in C'$. Consider the adjoint bilinear form on $C' \times C$ given by

$$(\psi, \phi) := \psi(0)\phi(0) - \int_{-1}^{0} \int_{0}^{\theta} \psi(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi,$$

and define the subspace $Q := \{\phi \in C | (\psi_i, \phi) = 0, \quad i = 1, 2\}$, then we have

$$C = P \oplus Q.$$

For convenience, let

$$\Psi := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \text{ and } \Phi := (\phi_1, \phi_2).$$

Reset the coefficient of ψ_i by

$$\psi_1 := \left((\boldsymbol{w}^T (I - \mathcal{M}_c \alpha) \boldsymbol{v} \right)^{-1} \boldsymbol{w}^T e^{-i\xi\omega_c} \text{ and } \psi_2 := \overline{\psi}_1 \text{ with } \alpha := \int_{-1}^{0} \theta e^{i\theta\omega_c} d\theta.$$

Then $((\psi_i, \phi_j))$ is an identity matrix and the projection operator of C on P is given by

$$\pi : C \to P \text{ with } \pi(u) = \Phi \boldsymbol{z},$$

where $\boldsymbol{z} := (\Psi, u)$ is the coordinates of u with respect to the basis $\{\phi_1, \phi_2\}$ of P. Furthermore, by considering (6.12) as an abstract ordinary differential equation in C (cf. [7]), we have the following differential equations for variables $\Phi\boldsymbol{z} \in P$ and $y := u - \Phi\boldsymbol{z} \in Q$ respectively,

$$\dot{\boldsymbol{z}} = B\boldsymbol{z} + \Psi(0)F(\Phi\boldsymbol{z} + y), \tag{6.16}$$

$$\frac{dy}{dt} = Ay + (I - \pi)X_0 F(\Phi\boldsymbol{z} + y), \tag{6.17}$$

where $B = \text{diag}(i\omega_c, -i\omega_c)$ and

$$X_0(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ 1, & \theta = 0. \end{cases}$$

Thus, (6.16) and (6.17) are equal to

$$\dot{\boldsymbol{z}} = B\boldsymbol{z} + \Psi(0)\frac{f'''(0)}{3!}\Big(p_c(\Phi(0)\boldsymbol{z} + y)^3 + \mathcal{M}_c\int_{-1}^{0}(\Phi\boldsymbol{z} + y)^3 d\theta + \cdots\Big), \tag{6.18}$$

$$\frac{dy}{dt} = Ay + (I - \pi)X_0\frac{f'''(0)}{3!}\Big(p_c(\Phi(0)\boldsymbol{z} + y)^3 + \mathcal{M}_c\int_{-1}^{0}(\Phi\boldsymbol{z} + y)^3 d\theta + \cdots\Big). \tag{6.19}$$

The nonlinear terms in (6.18) and (6.19) are of order ≥ 3, therefore the center manifold near $(0, 0)$ can be identified approximately.

Lemma A.1. Let (γ_c, p_c) be described in Theorem 5.1 or 5.2, then there is a 2-dimensional center manifold near $(0, 0)$ given by $y = h(\Phi\boldsymbol{z}) := 0 + O((\boldsymbol{z}, y)^3)$.

According Lemma A.1, the projection of the vector field (6.18) and (6.19) on the center manifold into the center space is

$$\dot{\boldsymbol{z}} \;=\; B\boldsymbol{z} + \Psi(0)\frac{f'''(0)}{3!}\left(p_c(\Phi(0)\boldsymbol{z})^3 + \mathcal{M}_c\int_{-1}^{0}(\Phi\boldsymbol{z})^3 d\theta\right),$$

where the terms of order > 3 are ignored. Notice that the power of a vector in the above equations stands for the power of each component in this vector. The vector $\boldsymbol{z}$ can be replaced by $(z, \bar{z})^T$ with $z \in \mathbb{C}$ and it suffices to consider the dynamics of z,

$$\dot{z} = i\omega z + \psi_1(0)\frac{f'''(0)}{3!}\left(p_c(\phi_1(0)z + \phi_2(0)\bar{z})^3 + \int_{-1}^{0}(\phi_1 z + \phi_2\bar{z})^3 d\theta\right). \tag{6.20}$$

The coefficient of $z^2\bar{z}$ in (6.20) are concerned. Indeed, it can be shown that for $\gamma \approx \gamma_c$ and $p = p_c$, the dynamics of (6.11) for (x, y) near $(0, 0)$ is determined by the differential equations in polar coordinates,

$$\dot{r} \;=\; \mu'(\gamma_c)r + ar^3, \tag{6.21}$$

$$\dot{\theta} \;=\; \omega_c, \tag{6.22}$$

where a is the real part of the coefficient of $z^2\bar{z}$ in (6.20) and $\mu'(\gamma_c)$ is the derivative of the real part of the root of $\Delta(\lambda) = 0$ with respect to γ at γ_c. Since

$$\begin{aligned}(\phi_1 z + \phi_2\bar{z})^3 \;&=\; \left(\begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix} e^{i\theta\omega_c} z + \begin{pmatrix} \bar{\upsilon}_1 \\ \bar{\upsilon}_2 \end{pmatrix} e^{-i\theta\omega_c}\bar{z}\right)^3 \\ &=\; \begin{pmatrix} \upsilon_1|\upsilon_1|^2 \\ \upsilon_2|\upsilon_n|^2 \end{pmatrix} 3e^{i\theta\omega_c} z^2\bar{z} + \cdots,\end{aligned}$$

the coefficient of $z^2\bar{z}$ in (6.20) is

$$
\begin{aligned}
&\frac{f'''(0)}{2}\left((\boldsymbol{w}^T(I-\mathcal{M}_c\alpha)\boldsymbol{v}\right)^{-1}\boldsymbol{w}^T\left(p_cI+\mathcal{M}_c\int_{-1}^{0}e^{i\theta\omega_c}d\theta\right)\begin{pmatrix}v_1|v_1|^2\\ v_2|v_n|^2\end{pmatrix}z^2\bar{z}\\
&=\frac{f'''(0)}{2}(1+i\omega_c)\left((\boldsymbol{w}^T(I-\mathcal{M}_c\alpha)\boldsymbol{v}\right)^{-1}\boldsymbol{w}^T\begin{pmatrix}v_1|v_1|^2\\ v_2|v_n|^2\end{pmatrix}z^2\bar{z}\\
&=\frac{f'''(0)}{2}(1+i\omega_c)\left((\boldsymbol{w}^T(I-\mathcal{M}_c\alpha)\boldsymbol{v}\right)^{-1}\left(\sum_{j=1}^{2}w_jv_j|v_j|^2\right)z^2\bar{z}.
\end{aligned}
$$

Thus, we have

$$
a=\mathrm{Re}\left(\frac{f'''(0)}{2}(1+i\omega_c)\left((\boldsymbol{w}^T(I-\mathcal{M}_c\alpha)\boldsymbol{v}\right)^{-1}\sum_{j=1}^{2}w_jv_j|v_j|^2\right). \tag{6.23}
$$

From (6.13) and (6.15), $\boldsymbol{v}$ is an eigenvector of $\mathcal{M}_c$ and $\boldsymbol{w}^T$ is a left eigenvector of $\mathcal{M}_c$, we obtain

$$
\boldsymbol{v}_\pm=(\pm\frac{\sqrt{rs}}{r},1)^T \quad \text{with } \mathcal{M}_c\boldsymbol{v}_\pm=\pm\sqrt{rs}\boldsymbol{v}_\pm, \tag{6.24}
$$

and

$$
\boldsymbol{w}_\pm^T=(\pm\frac{\sqrt{rs}}{s},1) \quad \text{with } \boldsymbol{w}_\pm^T\mathcal{M}_c=\pm\sqrt{rs}\boldsymbol{w}_\pm^T. \tag{6.25}
$$

Therefore,

$$
\boldsymbol{w}_\pm^T(I-\mathcal{M}_c)\alpha)\boldsymbol{v}_\pm=\boldsymbol{w}_\pm^T\boldsymbol{v}_\pm(1\mp\alpha\sqrt{rs})=\begin{cases}2(1\mp\gamma_c\alpha), & rs>0,\\ 2(1\mp i\gamma_c\alpha), & rs<0.\end{cases} \tag{6.26}
$$

Substituting (6.24)∼(6.26) into (6.23) and simplifying for the case $rs>0$ and $rs<0$ respectively, we obtain

$$
a=\begin{cases}f'''(0)(\gamma_c\cos\omega_c-2)(1+\frac{s}{r})(\frac{1}{4|1+\gamma_c\alpha|^2}|), & rs>0,\\ f'''(0)(\frac{\omega_c}{\sin\omega_c}-2-\omega_c^2)(1-\frac{s}{r})(\frac{1}{4|1\mp i\gamma_c\alpha|^2}|), & rs<0.\end{cases}
$$

Notice that for the case $rs>0$, only the $\boldsymbol{v}_-$ and $\boldsymbol{w}_-^T$ in (6.24)∼(6.26) need to be substituted into (6.23) since $\boldsymbol{v}_+$ corresponds to the equation $\Delta_+(\lambda)=0$ (cf. (2.4) and (6.13)), which have no purely imaginary roots (cf. Section 3).

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