行政院國家科學委員會專題研究計畫　期中進度報告

曲線與曲面之時間演化的研究(1/2)

計畫類別：個別型計畫
計畫編號：NSC94-2115-M-029-003-
執行期間：94 年 08 月 01 日至 95 年 07 月 31 日
執行單位：東海大學數學系

計畫主持人：陳文豪

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報告類型：精簡報告

處理方式：本計畫可公開查詢

中　華　民　國 95 年 5 月 17 日

Shortest Paths in Time Evolution of Curves

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Abstract

Consider a smooth curve wriggling and moving in the Euclidean 3-space $\mathbf{R}^3$ as time goes by. How can one say about the geometry in this phenomena? A very natural question is to ask " Does the shortest path from position p at time t_1 to position q at time t_2 exit?" In this paper, we will investigate the existence of the shortest paths, which we call them *feasible paths*. Local existence of feasible paths due to the first variation formula of energy functions. Our main result is to show a long-time existence theorem for feasible paths from p to q. Some interesting example will be mentioned.

This work is partially supported by a Taiwan NSC grant # 94-2115-M-029-003

1 Introduction and local existence

Consider a regular space curve $\gamma : [\alpha, \beta] \to \mathbf{R}^3$. Let $\sigma : [\alpha, \beta] \times [a, b] \to \mathbf{R}^3$ be a smooth map such that $\sigma([\alpha, \beta] \times \{a\}) = \gamma([\alpha, \beta])$ and for each t, $\sigma([\alpha, \beta] \times \{t\})$ is also a regular space curve. Set

$$\Sigma = \{\sigma(u, t) | \alpha \leq u \leq \beta,\ a \leq t \leq b\}$$

and denote $\Sigma_{\bar{t}} = \sigma([\alpha, \beta] \times \{\bar{t}\}$ for some fixed $\bar{t} \in [a, b]$.

A space curve $c : [\bar{a}, \bar{b}] \to \Sigma$ for $0 \leq a \leq \bar{a} < \bar{b} \leq b$ is called a *time-curve* if $c(t) \in \Sigma_t$ for all $t \in [\bar{a}, \bar{b}]$. That is, $c(t)$ is a time-curve provided $c(t) = \sigma(u(t), t)$. A *feasible path* from $p \in \Sigma_{\bar{a}}$ to $q \in \Sigma_{\bar{b}}$ is a time-curve from p to q with the shortest length. By the definition of the energy $\Xi(c)$ of a space curve c from a to b

$$\Xi(c) = \int_a^b \|c'(t)\|^2 dt$$

, a feasible path is a time-curve with minimal energy. Let us first investigate the local existence of feasible paths.

Let $c(t) = \sigma(u(t), t)$ be a time-curve form $p \in \Sigma_{\bar{a}}$ to $q \in \Sigma_{\bar{b}}$. Then $c'(t) = \sigma_u u'(t) + \sigma_u$ and $c''(t) = \sigma_{uu}(u'(t))^2 + 2\sigma_{ut}u'(t) + \sigma_{tt}$. We claim that $c(t)$ is a feasible path if it satisfies the following *feasible path equation*:

$$\frac{d}{dt}(Eu'(t)) - \frac{1}{2}E_u(u'(t))^2 + < \sigma_{tt}, \sigma_u >= 0 \tag{1}$$

, where $E =< \sigma_u, \sigma_u >= \|\sigma_u\|^2$ with the standard Euclidean inner product $<, >$. Note that a curve satisfies equation (1) is in fact a stationary point of the energy function.

Indeed, let

$$c^s(t) = \sigma(u(s,t), t),\ c^s(a) = p\ c^s(b) = q\ \forall s \in (-\epsilon, \epsilon)$$

be a proper variation of $c^0(t) = c(t)$ and denote $\dot{c}^s(t) = \frac{\partial}{\partial t}c^s(t)$. Then the first variation of the energy function is (c.f. [1] or [2])

$$\begin{aligned}
\frac{\partial}{\partial s}\Xi(c^s(t)) &= \frac{\partial}{\partial s}\int_a^b \|\dot{c}^s(t)\|dt \\
&= \frac{\partial}{\partial s}\int_a^b (E\dot{u}^2 + 2\langle\sigma_u, \sigma_t\rangle\dot{u} + \|\sigma_t\|^2)dt \\
&= \int_a^b \{(E_u\frac{\partial u}{\partial s}\dot{u}^2 + 2E\dot{u}\frac{\partial^2 u}{\partial t\partial s}) + 2(\langle\sigma_u, \sigma_t\rangle_u\dot{u}\frac{\partial u}{\partial s} + \langle\sigma_u, \sigma_t\rangle\frac{\partial^2 u}{\partial t\partial s}) \\
&\quad + \langle\sigma_t, \sigma_t\rangle_u\frac{\partial u}{\partial s}\}dt \\
&= \int_a^b \{(E_u\dot{u}^2 + 2\langle\sigma_u, \sigma_t\rangle_u\dot{u} + \langle\sigma_t, \sigma_t\rangle_u)\frac{\partial u}{\partial s}) + 2\frac{\partial u}{\partial s}\frac{\partial}{\partial t}(E_u\dot{u} + \langle\sigma_u, \sigma_t\rangle)\}dt \\
&= \int_a^b \{E_u\dot{u}^2 + \langle\sigma_t, \sigma_t\rangle_u - 2\frac{\partial}{\partial t}(E\dot{u}) - 2\langle\sigma_{ut}, \sigma_t\rangle - 2\langle\sigma_{tt}, \sigma_u\rangle\}\frac{\partial u}{\partial s}dt \\
&= -2\int_a^b \{\frac{d}{dt}(E\dot{u}) - \frac{1}{2}E_u\dot{u}^2 + < \sigma_u, \sigma_{tt} >\}\frac{\partial u}{\partial s}dt.
\end{aligned}$$

This gives the feasible path equation (1).

Let $p = \sigma(u, t)$ and the affine tangent space $T_p\Sigma_t$ be defined by

$$T_p\Sigma_t \ = \ \{\lambda\sigma_u + \sigma_t | \lambda \in \mathbf{R}\}.$$

If $c(t) = \sigma(u(t), t)$ is a time-curve, then $c'(t) = \sigma_u u'(t) + \sigma_t \in T_p\Sigma_t$ is an affine tangent vector. Now we have the following local existence theorem for feasible paths.

Local Existence Theorem. *Given* $\alpha \leq u_0 \leq \beta$, $0 \leq a \leq t_0 \leq b$ *and*

$p_0 = \sigma(u_0, t_0)$. Then there exits a positive number ϵ_{t_0} depending on the time t_0 such that for $t \in [t_0, t_0 + \epsilon_{t_0})$, $u \in (u_0 - \epsilon_{t_0}, u_0 + \epsilon_{t_0})$ and $\|v\| < \epsilon_{t_0}$, where $v = \sigma_u(u_0, t_0)u'(t_0) + \sigma_t(u_0, t_0) \in T_{p_0}\Sigma_{t_0}$, there exists a unique feasible path $c = c_v(t) : [t_0, t_0+\epsilon_{t_0}) \to \Sigma$ with the initial conditions $c(t_0) = p_0$ and $c'(t_0) = v$. Moreover, the map $c : [t_0, t_0 + \epsilon_{t_0}) \times T_{p_0}\Sigma_{t_0} \to \Sigma$ defined by $c(t, v) = c_v(t)$ is smooth.

2 Some examples

Example 1. Let $\gamma(u) = (\cos u, \sin u)$ and $\sigma(u, t) = (\cos u, \sin u, \phi(t))$. Then $E = 1$, $< \sigma_{tt}, \sigma_u >= 0$ and the feasible path equation is

$$0 = \frac{d}{dt}(Eu'(t)) - \frac{1}{2}E_u(u'(t))^2 + < \sigma_{tt}, \sigma_u >= u''(t).$$

The solution is $u(t) = at + b$ for some constants a and b. In particular, a circular helix $c(t) = (\cos t, \sin t, \phi(t))$ and a time-curve with $u = \textit{constant}$ are both feasible paths.

Example 2. Consider the unit-speed curve $\gamma(u) = (f(u), 0, g(u))$ with $(f'(u))^2 + (g'(u))^2 = 1$ rotating about the $z-$axis. Suppose γ rotates an angle $\phi(t)$ at time t. So

$$\sigma(u, t) = (f(u)\cos\phi(t), f(u)\sin\phi(t), g(u))$$

and then $E = 1$ and, $\sigma_{tt} =$
$(-\phi''(t)f(u)\sin\phi(t) - (\phi'(t))^2 f(u)\cos\phi(t), \phi''(t)f(u)\cos\phi(t) - (\phi'(t))^2 f(u)\sin\phi(t), 0)$.
Therefore, the feasible path equation is

$$u''(t) - f'(u(t))f(u(t))(\phi'(t))^2 = 0.$$

In particular, let $\gamma(u) = (1, 0, u)$ and compare with Example 1. Then the feasible equation is $u''(t) = 0$. Hence the parallel curve with $u = \textit{constant}$ and helix are feasible paths. Moreover, let $f(u) = 2 + \cos u$ and $g(u) = \sin u$ (compare with a torus). Then the feasible path equation is

$$u''(t) + \sin u(t)(2 + \cos u(t))(\phi'(t))^2 = 0.$$

Example 3. Consider the circle $\gamma(u) = (a\cos u, a + a\sin u)$ of radius $a > 0$ rolling without slipping along the x-axis as time goes by. Suppose the circle rolls a distance $a\phi(t)$ at time (t). Then

$$\sigma(u(t), t) = (a\cos(u(t) - \phi(t)), a + a\sin(u(t) - \phi(t)), 0).$$

Then $E = a^2$, $< \sigma_{tt}, \sigma_u >= -a^2\phi''(t)$ and the feasible path equation is

$$0 = \frac{d}{dt}(Eu'(t)) - \frac{1}{2}E_u(u'(t))^2 + < \sigma_{tt}, \sigma_u >= a^2(u''(t) - \phi''(t)).$$

In particular, if $\phi(t) = t$ then a *cycloid*, which is a curve with $u = constant$, is a feasible path. That is, a time-curve with $u = constant$ is a feasible path if the rolling velocity is constant.

Example 4. Consider time evolution of the curve $\gamma(u) = (a + \cos u, \sin u)$, $a \geq 1$ with

$$\sigma(u, t) = e^{-t}(\cos u + a\cos t, \sin u + a\sin t).$$

That is, the center of γ is moving along the logarithmic spiral $(e^{-t}\cos t, e^{-t}\sin t)$ and the radius is e^{-t} at time t. Then $E = e^{-2t}$, $< \sigma_{tt} = e^{-t}(2a\sin t + \cos u, -2a\cos t + \sin u)$ and then

$$< \sigma_{tt}, \sigma_u >= -2ae^{-2t}(\sin t\sin u + \cos t\cos u) = -2ae^{-2t}\cos(u - t).$$

So the feasible path equation is

$$u''(t) - 2u'(t) - 2a\cos(u - t) = 0.$$

A very special case is when $a = 1$ and $u = \pi + t$ then $c(t) = 0$. Significantly, a time-curve with $u = constant$ is never a feasible path.

3 Long-time existence of feasible paths

Now we sketch a proof of the long-time existence of feasible paths. Our main tool is the Morse theory and the technique presented in Jost's book [3]. Let

$$\Lambda_{pq} = \{c(t) = \sigma(u(t), t) | c(a) = p, \ c(b) = q\}$$

be the set of all time-curves with end point p, q and $c(\bar{t}) \in \Sigma_{\bar{t}}$ for $a \leq \bar{t} \leq b$. Then the energy functional $\Xi(c) = \frac{1}{2}\int_a^b |c'(t)|^2 dt$ is continuous on Λ_{pq}. For $\eta > 0$ define

$$\Lambda_{pq}^{\eta} = \{c \in \Lambda_{pq} | \Xi(c) \leq \eta.\}$$

Let $a = t_0 < t_1 < t_2 < \ldots < t_k = b$ be a partition of $[a, b]$. Define

$$\Lambda_{pq}(t_1, \ldots, t_{k-1}) = \{c \in \Lambda_{pq} | c|_{[t_{i-1}, t_i]} \ is \ feasible.\}$$

Now we denote

$$\Lambda_{pq}^{\eta}(t_1, \ldots, t_{k-1}) = \Lambda_{pq}(t_1, \ldots, t_{k-1}) \cap \Lambda_{pq}^{\eta}.$$

By the local existence of feasible paths, there exists $\rho_0 > 0$ such that for $a \le t_j,\ t_m \le b$, $|t_j - t_m| < \rho_0$ and $x \in \Sigma_{t_j}$, $y \in \Sigma_{t_m}$ there is a unique feasible path from x to y. Now we choose a partition $a = t_0 < t_1 < \ldots t_k = b$ of $[a, b]$ with

$$t_i - t_{i-1} < \frac{\rho_0^2}{2\eta}$$

for $i = 1, 2, \ldots, k$. Then, for each $c \in \Lambda^{\eta}_{pq}(t_1, \ldots, t_{k-1})$,

$$\begin{aligned} d(c(t_{i-1}), t_i)^2 &\le Length(c|_{[t_{i-1},t_i]})^2 \\ &= 2(t_i - t_{i-1})\Xi(c|_{[t_{i-1},t_i]}) \\ &\le 2(t_i - t_{i-1})\Xi(c) \\ &\le 2(t_i - t_{i-1})\eta < \rho_0^2. \end{aligned}$$

Therefore, the feasible path from $c(t_{i-1})$ to $c(t_i)$ is unique and hence coincides with $c|_{[t_{i-1},t_i]}$.

Moreover, the piecewise feasible path c is uniquely determined by

$$(c(t_1), \ldots, c(t_k)) \in \Sigma \times \ldots \times \Sigma = \Sigma^{k-1}.$$

Thus,

$$c \to (c(t_1), \ldots, c(t_k))$$

defines a homeomorphism of the interior of $\Lambda^{\eta}_{pq}(t_1, \ldots, t_{k-1})$ onto an open subset of Σ^{k-1} and hence the interior of $\Lambda^{\eta}_{pq}(t_1, \ldots, t_{k-1})$ may be equipped with the structure of a differentiable manifold. Then for $c \in \Lambda^{\eta}_{pq}(t_1, \ldots, t_{k-1})$, we have the formula

$$\Xi(c) = \sum_{i=1}^{k} \Xi(c|_{[t_{i-1},t_i]}) = \sum_{i=1}^{k} \frac{d(c(t_{i-1}), c(t_i))^2}{2(t_i - t_{i-1})}.$$

In particular, The restriction of Ξ to $\Lambda^{\eta}_{pq}(t_1, \ldots, t_{k-1})$ is differentiable. Moreover, it can be shown that (c.f. [3]) the energy function Ξ is differentiable on Λ_{pq}

Lemma 3.1. *All critical points of* Ξ *is on* Λ^{η}_{pq} *are contained in* $\Lambda^{\eta}_{pq}(t_1, \ldots, t_{k-1})$.

Proof. Let $c \in \Lambda^{\eta}_{pq}$. Then

$$d(c(t_{i-1}), c(t_i))^2 \le 2(t_i - t_{i-1})\Xi(c) < \rho_0^2.$$

This implies that the map

$$r : \Lambda^{\eta}_{pq} \to \Lambda^{\eta}_{pq}(t_1, \ldots, t_{k-1})$$

is well-defined. Moreover, r is continuous and

$$\Xi(r(c)) \le \Xi(c)$$

Now we define a family $(r_t)_{0\le t\le 1}$ of maps $r_t : \Lambda^\eta_{pq} \to \Lambda^\eta_{pq}$ by the following. For $i = 1, \ldots, k-1$, let

$$r_t(c)|_{[t_{i-1}, t_{i-1}+t(t_i - t_{i-1})]}$$

be the feasible path from $c(t_{i-1})$ to $c(t_{i-1}+t(t_i-t_{i-1}))$ and let

$$r_t(c)|_{[t_{i-1}+t(t_i-t_{i-1})]} = c|_{[t_{i-1}+t(t_i-t_{i-1})]}.$$

Then we have $r_0(c) = c$, $r_1(c) = r(c)$ and $r_t(c)$ is continuous in t and c. This proves that $\Lambda^\eta_{pq}(t_1, \ldots, t_{k-1})$ is a deformation retract of Λ^η_{pq}. Since the critical points of Ξ are feasible paths and so are piecewise feasible paths, they lie in $\Lambda^\eta_{pq}(t_1, \ldots, t_{k-1})$ if their energy is $\le \eta$. □

Let another partition $(\tau_1, \ldots, \tau_k)$ be given by

$$t_0 < \tau_1 < t_1 < \tau_2 < \ldots < \tau_k < t_k$$

and we also assume that

$$\tau_i - \tau_{i-1} < \frac{\rho_0^2}{2\eta}$$

for $i = 1, \ldots, k$ with $\tau_0 = \tau_k - b$.

Let γ be a time-curve from $p \in \Sigma_a$ to $q \in \Sigma_b$. Note that this curve always exists due to the local existence theorem. Assume

$$\Xi(\gamma) \le \eta.$$

Now we use the curve shortening process to prove the long time existence of the feasible paths. Let $r_1(c)$ be the piecewise feasible path for which $r_1(c)|_[t_{i-1}, t_i]$ is the feasible path from $c(t_{i-1})$ to $c(t_i)$. By the local existence theorem and the choice of rho_0, this determines $r_1(c)$ uniquely. Then define $r_2(c)$ as the piecewise feasible path for which $r_2(c)|_[\tau_{i-1}, \tau_i]$ is the feasible path from $c(\tau_{i-1})$ to $c(\tau_i)$. Note that $r_2(c)$ is likewise uniquely determined. Define

$$P(c) \equiv r_2 \circ r_1(c).$$

Lemma 3.2. *We have the inequality*

$$\Xi(P(c)) \le \Xi(c),$$

with equality holds if and only if c is a feasible path.

Proof. Note that

$$\Xi(r_1(c)) \le \Xi(c),$$

which the equality holds if and only if c is a piecewise feasible path from p to q with nodes $c(t_1), \ldots, c(t_{k-1})$. Likewise, for each time-curve $\tilde{c}$

$$\Xi(r_2(\tilde{c})) \le \Xi(\tilde{c}),$$

with equality holds if and only if $\tilde{c}$ is a piecewise feasible path from p to q with nodes $\widetilde{c(\tau_1)}, \ldots, \tilde{c}(\tau_{k-1})$. Thus , if $\Xi(P(\gamma)) = \Xi(\gamma)$, then all segments $c_[t_i, t_{i-1}]$ as well as all segments $c_[\tau_i, \tau_{i-1}]$ are feasible paths. Hence c is a feasible path from p to q. □

Lemma. 3.3 *Let c be a time-curve with energy $\Xi(c) \leq \eta$. Then a subsequence of $P^n(c) \equiv P \circ \ldots \circ P(c)$ converges uniformly to a feasible path.*

Proof. First note that each curve $P^n(c)$ for $n = 1, 2, \ldots$ is a piecewise feasible path with nodes $P^n(c(\tau_1)), \ldots, P^n(c(\tau_k))$ where the individual segments are feasible paths between those nodes. Hence, as in, each such curve may be identified with a $k-$tuple

$$P^n(c(\tau_1)), \ldots, P^n(c(\tau_k)) \in \Sigma^k.$$

Since Σ^k is compact, $P^n(c(\tau_1)), \ldots, P^n(c(\tau_k))$ converges to some $(p_1, \ldots, p_k) \in \Sigma^k$ and hence $P^n(c)$ converges uniformly to the piecewise feasible path c_0 with nodes $c_0(\tau_i) = p_i$ for $i = 1, \ldots, k$, whose individual segments $c_0|_{[\tau_{i-1}, \tau_i]}$ again are the feasible path from $c_0(\tau_{i-1})$ to $c_0(\tau_i)$ since the limit of feasible paths is a feasible path. Denote the convergent subsequence of $(P^n(c))_{n \in N}$ by $c_m \equiv (P^{n_m}(c))_{m \in N}$. Then

$$\Xi(c_0) = \lim_{m \to \infty} \Xi(c_m),$$

as follows from. Moreover,

$$\Xi(c_0) = \lim \Xi(c_{m+1}) = \lim \Xi(P^{n_m} c_m) \leq \lim \Xi(P(c_m)) \leq \lim \Xi(c_m) = \Xi(c_0).$$

Therefore, equality must hold throughout. Hence $P(c_m)$ converges to $P(c_0)$ and we have

$$\Xi(P(c_0)) = \lim_{m \to \infty} \Xi(P(c_m)) = \Xi(c_0).$$

Finally, by Lemma, c_0 is a feasible path from p to q. □

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