行政院國家科學委員會專題研究計畫 成果報告

圖形之標號、著色、覆蓋、分割、與數論、有限幾何之關聯性研究

研究成果報告(精簡版)

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中 文 摘 要 ： 本計畫從事圖形的標號、著色、分割、與覆蓋之相關研究。我們將以有限幾何，數論，群論等工具以及其他標準組合學方法研究以下幾類相關問題:

轩 對一圖形 G 與一交換群(abelian group) A，我們說 A-幻方標號(A -magic labeling)為一組邊標號 f: E(G) -> A - {0}，使得每一頂點之頂點和(vertex sum)皆同，其中 0 為交換群 A 之單位元素。我們記所有可能之 magic vertex sum constants 為 magic sum spectrum，本計畫將針對各類圖形之 magic sum spectrum 研究，特別關於 regular graphs。當 A = Zk （即 modulo k 同餘類所成加法循環群)，注意到刻劃 Zk-magic 圖形仍然為困難之未解問題。此研究方向與 A-coloring (文獻或稱 abelian coloring)相關，特別是 1986 年 Archdeacon 提出之 Conjecture :

Let G be a bridgeless graph and let H be a group of order at least five. Then the edges of G can be colored with the non-identity elements of H such that at each vertex of G the three colors sum to the identity in H.

另外本計畫有關標號著色方向亦有: 圖形的 L(2, 1)-標號(距離二型態標號)，反幻方標號(antimagic labeling)，列表著色(list coloring)問題等等。本計畫將針對相關諸多未解猜想進行研究。

轩 圖的邊集合的團覆蓋(clique covering)問題就是：對於一給定圖形，尋找最少數目的團，使得每一個邊至少(或恰好)包含在某個團之中。Fred Roberts, Walter Wallis 等人考慮那些需要所有的極大團形成邊集合的最小團覆蓋的圖類，並稱之為極大團不可約圖(maximal clique irreducible graphs)。我們推廣定義到一更大範圍之圖類，即弱極大團不可約圖(weakly maximal clique irreducible graphs)。近期與印度 Cochin University 的 Dr. Ambat Vijayakumar 共同研究此方向，獲得 graph product 研究成果。1966 Erdos 指出圖的邊集合的團覆蓋與圖交集表示(intersection representation)有一一對應關係。本計畫亦將研究相關的圖交集表示的唯一性問題。近期與台科大王有禮教授合作發表 diamond-free 圖類之交集表示唯一性。我們將繼續考慮這研究方向的其他問題。

中文關鍵詞： A-幻方標號，可換群著色，L(2,1)-標號，反幻方標號，列表著色，極大團不可約圖，弱極大團不可約圖，團覆蓋，唯一集合表示，有限幾何，數論，群論

英 文 摘 要 ： This project studies the graph labeling, graph coloring, graph partition, and graph covering. We will research on the following related topics using the tools from finite geometry, number theory, group theory, and other standard combinatorial methods:

轩 For a graph G and an abelian group A, we say an edge labeling using non-identity elements is A-magic if all induced vertex sums are constant. The set of all possible such magic vertex sum constants is called magic sum spectrum. We will study the magic sum spectrum of various classes of graphs, in particular regular graphs. Even in the case A = Zk, the finite cyclic additive group consisting of congruence classes modulo k, to characterize Zk-magic graphs is quite hard. This research direction is closely related to A-coloring (a.k.a. abelian coloring in literatures), in particular related to the conjecture raised by Archdeacon in 1986, namely, let G be a bridgeless graph and let H be a group of order at least five, then the edges of G can be colored with the non-identity elements of H such that at each vertex of G the three colors sum to the identity in H. On the other hand, our project will also focus on L(2,1)-labeling, antimagic labeling, and list coloring etc., and we will put special attention on those unsettled conjectures.

轩 The edge clique covering problem is that for a given graph, to find the least number of cliques so that every edge is contained in at least one clique. Fred Roberts and Walter Wallis considered the class of graphs for which the set of all maximal cliques forms an edge clique covering of minimum size, which is called maximal clique irreducible graphs. We extend to a general class of weakly maximal clique irreducible graphs, which is the subject of recent collaboration with Dr. Vijayakumar in Cochin

University in India. In 1966 Paul Erdos pointed out that there is one-to-one correspondence between edge clique covering and intersection representations for graphs. We also study related subjects regarding the uniqueness of set representations of graphs, and I have been collaborating with Professor Wang Yue-Li of National Taiwan University of Science and Technology in Taipei. We just published a paper regarding unique intersectability of diamond-free graphs recently. Therefore we will continue the exploration along this line of research.

英文關鍵詞： A-magic labeling, abelian coloring, L(2,1)-labeling, antimagic labeling, list coloring, maximal clique irreducible, clique covering, unique intersection representation, finite geometry, number theory, group theory

行政院國家科學委員會補助專題研究計畫 ■ 成果報告
□期中進度報告

圖形之標號、著色、覆蓋、分割、與數論、有限幾何之關聯性研究

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成果報告類型(依經費核定清單規定繳交)：■精簡報告 □完整報告

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目錄

A. 報告內容
B. 計畫成果自評
C. 附件(發表之期刊論文以及會議論文)

A. 報告內容

成果發表於下列期刊論文以及會議論文

- Jun-Lin Guo, **Tao-Ming Wang**, and Yue-Li Wang, “Unique Intersectability of Diamond-Free Graphs”, **Discrete Applied Mathematics** 159 (2011) 774–778 **(SCI)**

- **Tao-Ming Wang** and Shi-Wei Hu, “Constant Sum Flows in Regular Graphs”, FAW-AAIM 2011, **Lecture Notes in Computer Science(LNCS)** 6681, pp. 168-175, 2011**(EI)**

- Chih-Hsiuan Liu, Tao-Ming Wang, Ming-I Char, "On Deficiency of Super Edge Magic Labelings for Complete Bipartite Graphs", IEEE Proc. 2nd Intl Conf on Innovative Computing & Communication March 2011(CICC-ITOE 2011), Macao, China pp. 331-334 **(EI)**

B. 計畫成果自評

已達成計畫預定目標

C. 附件

發表之期刊論文以及會議論文

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Unique intersectability of diamond-free graphs☆

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ABSTRACT

For a graph G with vertices $v_1, v_2, \dots, v_n$, a simple set representation of G is a family $\mathcal{F} = \{S_1, S_2, \dots, S_n\}$ of distinct nonempty sets such that $|S_i \cap S_j| = 1$ if v_iv_j is an edge in G, and $|S_i \cap S_j| = 0$ otherwise. Let $\mathbf{S}(\mathcal{F}) = \bigcup_{i=1}^{n} S_i$, and let $\omega_s(G)$ denote the minimum $|\mathbf{S}(\mathcal{F})|$ of a simple set representation $\mathcal{F}$ of G. If, for every two minimum simple set representations $\mathcal{F}$ and $\mathcal{F}'$ of G, $\mathcal{F}$ can be obtained from $\mathcal{F}'$ by a bijective mapping from $\mathbf{S}(\mathcal{F}')$ to $\mathbf{S}(\mathcal{F})$, then G is said to be s-uniquely intersectable. In this paper, we are concerned with the s-unique intersectability of diamond-free graphs, where a diamond is a K_4 with one edge deleted. Moreover, for a diamond-free graph G, we also derive a formula for computing $\omega_s(G)$.

1. Introduction

In this paper, $G = (V, E)$ represents a simple graph of n vertices and m edges; i.e., $|V(G)| = n$ and $|E(G)| = m$, where $V(G) = \{v_1, v_2, \dots, v_n\}$. An edge v_iv_j is in $E(G)$ if vertices v_i and v_j are adjacent. Two adjacent vertices v_i and v_j in $V(G)$ are *twins* if they have the same closed neighborhood. G is *twin free* if it contains no twins. For two graphs G and H, if G has no induced subgraph H, then we say that G is *H-free*. Thus, a graph is *triangle free* (respectively, *diamond free*) if it contains no triangles (respectively, diamonds) as an induced subgraph. Here, a *triangle* means a K_3 and a *diamond* is the graph obtained by deleting an edge from K_4.

The concept of set representation of graphs was first introduced by Szpilrajn-Marczewski [10] and Erdös et al. [4]. A *set representation* of G is a multifamily $\mathcal{F} = \{S_1, S_2, \dots, S_n\}$ of nonempty sets such that, for any $i \neq j$, $S_i \cap S_j \neq \emptyset$ if edge $v_iv_j \in E(G)$, and $S_i \cap S_j = \emptyset$ otherwise, where *multifamily* means that $S_1, S_2, \dots, S_n$ might not be distinct. Note that S_i is a corresponding set of v_i for $i = 1, 2, \dots, n$. A set representation $\mathcal{F}$ is *distinct* if $S_i \neq S_j$ for $i \neq j$, and is *antichain* if $S_i \not\subseteq S_j$ for $i \neq j$. A *simple set representation* is a distinct set representation with $|S_i \cap S_j| = 1$ if $v_iv_j \in E(G)$. For a set representation $\mathcal{F} = \{S_1, S_2, \dots, S_n\}$, let $\mathbf{S}(\mathcal{F}) = \bigcup_{i=1}^{n} S_i$. It is known that any G has a (simple) distinct set representation (see Theorem 2.5 of [5]). Thus we can meaningfully denote by $\omega(G)$ (respectively, $\omega_s(G)$) the minimum size of $|\mathbf{S}(\mathcal{F})|$ among all distinct set representations (respectively, simple set representations) $\mathcal{F}$ of G. A *minimum distinct set representation* (respectively, *minimum simple set representation*) $\mathcal{F}$ of G is a distinct set representation (respectively, simple set representation) with $|\mathbf{S}(\mathcal{F})| = \omega(G)$ (respectively, $\omega_s(G)$). Kou et al. [7] and Poljak et al. [9] proved that finding $\omega(G)$ and $\omega_s(G)$, respectively, for a general graph G is NP-complete. Harary [5] proved that, for a connected graph G of $n(> 3)$ vertices, $\omega(G) = m$ if and only if G is triangle free.

The concept of *unique intersectability* of G was proposed by Alter and Wang [1]. They defined G to be *uniquely intersectable* if, for any two minimum distinct set representations $\mathcal{F}$ and $\mathcal{F}'$ of G, $\mathcal{F}$ can be obtained from $\mathcal{F}'$ by a bijective mapping from

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$\mathbf{S}(\mathcal{F}')$ to $\mathbf{S}(\mathcal{F})$. Based on the above theorem of Harary, they proved that every triangle-free graph G is uniquely intersectable. Later, Tsuchiya [11] studied the unique intersectability with respect to antichains, abbreviated *a-uniquely intersectable*, and showed that being triangle free is also a sufficient condition for a graph to be a-uniquely intersectable. Then, Mahadev and Wang [8] proved that, for every diamond-free graph G, G is uniquely intersectable if and only if G is twin free. This generalizes Alter and Wang's result, since diamond-free graphs are a superset of triangle-free graphs. Kong and Wu [6] defined a superset of diamond-free graphs, called *purple graphs*. They proved that a purple graph is uniquely intersectable if and only if it is twin free and unique intersectable with respect to multifamily representation. This further improves Mahadev and Wang's result, since diamond-free graphs are unique intersectable with respect to multifamily representation [8].

Inspired by the concept of a-uniquely intersectable, we say that a graph G is *s-uniquely intersectable* if, for any two minimum simple set representations $\mathcal{F}$ and $\mathcal{F}'$ of G, $\mathcal{F}$ can be obtained from $\mathcal{F}'$ by a bijective mapping from $\mathbf{S}(\mathcal{F}')$ to $\mathbf{S}(\mathcal{F})$. Actually, the proof in [1] also reveals that every triangle-free graph G is s-uniquely intersectable. However, Mahadev and Wang's proof [8] cannot be applied directly to the s-unique intersectability of a graph. Therefore, it is interesting to find out a general sufficient condition for the s-unique intersectability of a graph.

2. Preliminaries

First, we introduce some terms which will be used later. A *clique* in G is a set $Q \subseteq V(G)$ whose vertices are pairwise adjacent in G. A *trivial clique* contains only one vertex. A clique in G is *maximal* if it is not properly contained in any other clique in G. A set $\mathbb{Q} = \{Q_1, Q_2, \ldots, Q_p\}$ of cliques in G is called a *clique partition* of G if $E(G) = \bigcup_{i=1}^{p} E(Q_i)$, $E(Q_i) \cap E(Q_j) = \emptyset$ for $i \neq j$, and $\{v\} \in \mathbb{Q}$ for each $v \in V(G)$ of degree 0. Erdös et al. [4] found a bijection between set representations and clique covers of a graph G. Below, we introduce this bijection in detail, and call it the *Erdös bijection*. Let $\mathbb{Q} = \{Q_1, Q_2, \ldots, Q_p\}$ be a clique partition of G. For every v_i, $1 \leqslant i \leqslant n$, construct a set S_i whose elements are those cliques in $\mathbb{Q}$ containing v_i. Clearly, for any $i \neq j$, $|S_i \cap S_j| = 1$ if v_i is adjacent to v_j, and $|S_i \cap S_j| = 0$ otherwise. Hence, $\{S_1, S_2, \ldots, S_n\}$ is a set representation of G where any distinct S_i, S_j have $|S_i \cap S_j| \leqslant 1$. Conversely, given a set representation $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ of G where S_i is the corresponding set of v_i for $1 \leqslant i \leqslant n$ and any distinct S_i, S_j have $|S_i \cap S_j| \leqslant 1$, we can obtain a clique partition of G as follows. For each $s_j \in \mathbf{S}(\mathcal{F}) = \{s_1, s_2, \ldots, s_p\}$, if s_j is in S_i, then let Q_j contain v_i. Clearly, the subgraph of G induced by the vertices in Q_j is a clique, and exactly one Q_j with $1 \leqslant j \leqslant p$ contains $\{v_x, v_y\}$ if $v_x v_y \in E(G)$ and none otherwise. Therefore, the set $\{Q_1, Q_2, \ldots, Q_p\}$ is a clique partition of G. Hereafter, we use *Erdös* $\mathcal{F}$ and *Erdös* $\mathbb{Q}$ to denote the resulting set representation and clique partition, respectively, of the Erdös bijection.

A *finite linear space* (FLS) $\Gamma = (P, \mathcal{L})$ consists of a set P of n *points* and a set $\mathcal{L}$ of *lines*, where a line is a set of points, satisfying the following axioms [2].

(L1) A line contains at least two and at most $n - 1$ points.
(L2) For any two points $x, y \in P$, exactly one line of $\mathcal{L}$ contains $\{x, y\}$.

A *projective plane* (PP) Π is an FLS satisfying further the following two axioms [2].

(P1) Any two distinct lines have exactly one common point.
(P2) There exist four points in which no three points are collinear.

In [3] (see also [2]), de Bruijn and Erdös proved a theorem about FLSs. To employ the theorem in this paper, we note that, with the correspondence between points and vertices, and lines and cliques, there is a bijection between FLSs and clique partitions $\mathbb{Q}$ of complete graphs K_n, $n \geqslant 3$, where the cardinality of every clique in $\mathbb{Q}$ is between 2 and $n - 1$. Therefore we can paraphrase the theorem in terms of clique partition as follows.

Theorem 1 ([3]). *If $\mathbb{Q}$ with $|\mathbb{Q}| > 1$ is a clique partition of K_n with $n \geqslant 3$, and there is no trivial clique in $\mathbb{Q}$, then $|\mathbb{Q}| \geqslant n$, where equality holds if and only if*

(a) *$\mathbb{Q}$ consists of one clique with $n - 1$ vertices and $n - 1$ copies of K_2 or*
(b) *the FLS corresponding to $\mathbb{Q}$ is a PP.*

The FLSs with $n \geqslant 3$, corresponding to clique partitions as in Condition (a) of Theorem 1, are conventionally referred to as *near-pencil* (N-P for short). We will use the two terms N-P and PP to stand for both an FLS and the corresponding clique partition of a complete graph. For example, two clique partitions $\mathbb{Q} = \{Q_1, Q_2, \ldots, Q_7\}$ of K_7 corresponding to N-P and PP are listed in Table 1.

In Table 1, the PP with $n = 7$ is the so-called *Fano plane*, as illustrated in Fig. 1, where the line segments (straight or round) pass through lines $\{v_1, v_2, v_3\}$, $\{v_3, v_4, v_5\}$, $\{v_1, v_5, v_6\}$, $\{v_1, v_4, v_7\}$, $\{v_2, v_5, v_7\}$, $\{v_3, v_6, v_7\}$, $\{v_2, v_4, v_6\}$, respectively.

For the clique partition $\mathbb{Q}$ of K_n with $|\mathbb{Q}| = 1$, i.e., $\mathbb{Q} = \{Q_1\}$ and $Q_1 = \{v_1, v_2, \ldots, v_n\}$, if we add trivial cliques $Q_i = \{v_i\}$ for $i = 2, 3, \ldots, n$ to $\mathbb{Q}$, then the resulting set $\mathbb{Q}' = \{Q_1, Q_2, \ldots, Q_n\}$ is still a clique partition of K_n. Henceforth, we use *Erdös* $\mathcal{F}_{K_n}$ to denote the Erdös $\mathcal{F}$ with respect to $\mathbb{Q}'$, i.e., Erdös $\mathcal{F}_{K_n} = \{S_1, S_2, \ldots, S_n\}$ where $S_1 = \{Q_1\}$ and $S_i = \{Q_1, Q_i\}$ for $i = 2, 3, \ldots, n$. We also use *Erdös* $\mathcal{F}_{\text{N-P}}$ and *Erdös* $\mathcal{F}_{\text{PP}}$ to emphasize set representations of a complete graph which are obtained by Erdös bijection on its N-P and PP, respectively.

Proposition 2. *For a K_n, all of its Erdös $\mathcal{F}_{\text{N-P}}$, Erdös $\mathcal{F}_{\text{PP}}$, and Erdös $\mathcal{F}_{K_n}$ are simple set representations.*

Table 1
Clique partitions $\mathbb{Q}$ of K_7 with $|\mathbb{Q}| = 7$.

$\mathbb{Q}$	N-P	PP
Q_1	$\{v_1, v_2\}$	$\{v_1, v_2, v_3\}$
Q_2	$\{v_1, v_3\}$	$\{v_3, v_4, v_5\}$
Q_3	$\{v_1, v_4\}$	$\{v_1, v_5, v_6\}$
Q_4	$\{v_1, v_5\}$	$\{v_1, v_4, v_7\}$
Q_5	$\{v_1, v_6\}$	$\{v_2, v_5, v_7\}$
Q_6	$\{v_1, v_7\}$	$\{v_3, v_6, v_7\}$
Q_7	$\{v_2, v_3, \ldots, v_7\}$	$\{v_2, v_4, v_6\}$

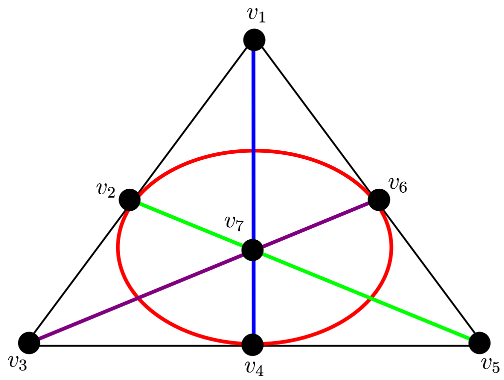

Fig. 1. Fano plane.

3. Diamond-free graphs are *s*-uniquely intersectable

A vertex $v_i \in V(G)$ is called a *monopolized vertex* of Q if v_i is contained in only one maximal clique Q in G; otherwise, v_i is called a *shared vertex*. Similarly, an element in $\mathbf{S}(\mathcal{F})$ is called a *monopolized element* with respect to a set representation $\mathcal{F}$ of G if it appears in only one set of $\mathcal{F}$.

Proposition 3. *In a graph G, the closed neighborhood of a monopolized vertex of a maximal clique Q is contained in Q.*

Theorem 4. *For $n \geqslant 1$, $\omega_s(K_n) = n$. Further, any minimum simple set representation of K_n, for $n \geqslant 3$, is an Erdös $\mathcal{F}_{\text{N-P}}$, Erdös $\mathcal{F}_{\text{PP}}$, or Erdös $\mathcal{F}_{K_n}$.*

Proof. Clearly, $\omega_s(K_1) = 1$ and $\omega_s(K_2) = 2$. We prove that $\omega_s(K_n) = n$ for $n \geqslant 3$. We can easily construct a simple set representation $\mathcal{F}$ of K_n, for $n \geqslant 3$, with $\mathbf{S}(\mathcal{F}) = \{s_1, s_2, \ldots, s_n\}$ by letting $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$, where $S_1 = \{s_1\}$ and $S_i = \{s_1, s_i\}$ for $2 \leqslant i \leqslant n$. Thus $\omega_s(K_n) \leqslant n$ for $n \geqslant 3$. Then we prove that $\omega_s(K_n) \geqslant n$ for $n \geqslant 3$. Suppose to the contrary that K_n, $n \geqslant 3$, has a simple set representation $\mathcal{F}$ with $|\mathbf{S}(\mathcal{F})| \leqslant n-1$. We delete all monopolized elements from all sets in $\mathcal{F}$ and let $\mathcal{F}'$ be the resulting set. Clearly, $\mathcal{F}'$ remains a set representation of K_n, and therefore Erdös $\mathbb{Q}$ with respect to $\mathcal{F}'$ is a clique partition of K_n which contains at most $n-1$ cliques and no trivial ones. By Theorem 1, $|\mathbb{Q}| = 1$. This means that all sets in $\mathcal{F}'$ are the same. Thus, $\mathbf{S}(\mathcal{F})$ consists of an element common to all sets in $\mathcal{F}$ and at most $n-2$ monopolized elements since $|\mathbf{S}(\mathcal{F})| \leqslant n-1$. This implies that at least two sets in $\mathcal{F}$ are the same, a contradiction. Thus we have proved that $\omega_s(K_n) = n$ for $n \geqslant 1$.

Let $\mathcal{F}$ be a minimum simple set representation of K_n with $n \geqslant 3$, i.e., $|\mathbf{S}(\mathcal{F})| = n$. Delete all monopolized elements from all sets in $\mathcal{F}$ and obtain $\mathcal{F}'$. Then Erdös $\mathbb{Q}$ with respect to $\mathcal{F}'$ is a clique partition of K_n which contains at most n cliques and no trivial ones. By Theorem 1, $\mathbb{Q}$ is an N-P or PP, or has $|\mathbb{Q}| = 1$. In the former two cases, $|\mathbb{Q}| = n$, and therefore $\mathcal{F}' = \mathcal{F}$ is an Erdös $\mathcal{F}_{\text{N-P}}$ or Erdös $\mathcal{F}_{\text{PP}}$. In the last case, $\mathcal{F}$ is an Erdös $\mathcal{F}_{K_n}$. This completes the proof. □

Lemma 5. *If a graph G is diamond free, then any two distinct maximal cliques in G have at most one vertex in common.*

Proof. Suppose to the contrary that two distinct maximal cliques Q and Q' intersect at two vertices v_i and v_j in G. There are at least two nonadjacent vertices v_x and v_y in Q and Q', respectively; otherwise, Q and Q' are contained in one maximal clique. It is clear that the subgraph of G induced by vertices v_i, v_j, v_x, v_y is a diamond in G, a contradiction. □

In the following, unless otherwise stated, we assume that G is a connected diamond-free graph and is not a complete graph, that $\mathbb{Q} = \{Q_1, Q_2, \ldots, Q_p\}$ is the set of all maximal cliques in G, and that $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ is a simple set representation of G, where S_i is a corresponding set of v_i for $i = 1, 2, \ldots, n$. Further, let M_i and H_i be the sets of monopolized and shared vertices, respectively, of Q_i for $1 \leqslant i \leqslant p$, $M = \bigcup_{i=1}^{p} M_i$, $\mathcal{F}_{M_i} = \{S_j : v_j \in M_i\}$ for $1 \leqslant i \leqslant p$, and $\mathcal{F}_M = \{S_j : v_j \in M\}$. Similarly, $H = \bigcup_{i=1}^{p} H_i$, $\mathcal{F}_{H_i} = \{S_j : v_j \in H_i\}$ for $1 \leqslant i \leqslant p$, and $\mathcal{F}_H = \{S_j : v_j \in H\}$. The subgraph of G induced by M is denoted by $G[M]$.

Lemma 6. *If G is a connected graph and is not a complete graph, then $|\mathbf{S}(\mathcal{F}_M)| \geqslant \sum_{i=1}^{p} |M_i|$.*

Proof. Since $G \neq K_n$ and is connected, $|H_i| \geqslant 1$ for $1 \leqslant i \leqslant p$. By Proposition 3, $G[M]$ is the disjoint union of $K_{|M_1|}$, $K_{|M_2|}, \ldots, K_{|M_p|}$. Obviously, $\mathcal{F}_M$ is a simple set representation of $G[M]$. Therefore, by Theorem 4, $|\mathbf{S}(\mathcal{F}_M)| \geqslant \sum_{i=1}^{p} |M_i|$. □

Lemma 7. *If there is a Q_i with $M_i = \emptyset$, then, for any $v_k, v_\ell \in Q_i$, the following statements hold.*

(1) *The element in $S_k \cap S_\ell$ is not in $\mathbf{S}(\mathcal{F}_{M_j})$ for any j with $M_j \neq \emptyset$.*
(2) *If there is a Q_j with $j \neq i$ and $M_j = \emptyset$, then, for any $v_x, v_y \in Q_j$, the element in $S_k \cap S_\ell$ is distinct from the one in $S_x \cap S_y$.*

Proof. Let s be the element in $S_k \cap S_\ell$. To prove statement (1), we suppose to the contrary that s is also in $\mathbf{S}(\mathcal{F}_{M_j})$ for some j with $M_j \neq \emptyset$. This means that both v_k and v_ℓ are adjacent to some vertex, say x, in M_j. Since x is a monopolized vertex of Q_j, by Proposition 3, both v_k and v_ℓ must be also in Q_j. Thus, $|Q_i \cap Q_j| \geqslant 2$ which, by Lemma 5, is a contradiction. This concludes the proof of this statement.

To prove statement (2), suppose to the contrary that there is a Q_j with $j \neq i$ and $M_j = \emptyset$ such that $S_x \cap S_y = \{s\}$ for some $v_x, v_y \in Q_j$. By Lemma 5, $v_x \notin Q_i$ or $v_y \notin Q_i$. For the former, since $s \in S_k \cap S_\ell \cap S_x$, this means that there is a maximal clique, say Q_r, with $r \neq i$ containing vertices v_k, v_ℓ, and v_x. Thus $v_k, v_\ell \in Q_r \cap Q_i$. By Lemma 5, this is a contradiction. The latter case can also be handled similarly. This completes the proof. □

Lemma 8. *Any simple set representation $\mathcal{F}$ of G has $|\mathbf{S}(\mathcal{F})| \geqslant \sum_{i=1}^{p} |M_i| + |\{i : M_i = \emptyset \text{ for } 1 \leqslant i \leqslant p\}|$, where equality holds if and only if $|\mathbf{S}(\mathcal{F}_M)| = \sum_{i=1}^{p} |M_i|$ and $\bigcap \mathcal{F}_{H_i}$, for every $1 \leqslant i \leqslant p$ with $M_i = \emptyset$, contains exactly one element, which is not in $\mathbf{S}(\mathcal{F}_M)$ and $\bigcap \mathcal{F}_{H_i} \neq \mathcal{F}_{H_j}$ for $i \neq j$.*

Proof. Since G is connected, every Q_i, for $1 \leqslant i \leqslant p$, has more than one vertex. Thus, Q_i with $M_i = \emptyset$ has $|H_i| \geqslant 2$. By Lemma 7, every Q_i with $M_i = \emptyset$ has at least one unique element in $\mathbf{S}(\mathcal{F})$ which is not in $\mathbf{S}(\mathcal{F}_M)$. Moreover, $|\mathbf{S}(\mathcal{F}_M)| \geqslant \sum_{i=1}^{p} |M_i|$ by Lemma 6. Therefore,

$$\begin{aligned}|\mathbf{S}(\mathcal{F})| &\geqslant |\mathbf{S}(\mathcal{F}_M)| + |\{i : M_i = \emptyset \text{ for } 1 \leqslant i \leqslant p\}| \\ &\geqslant \sum_{i=1}^{p} |M_i| + |\{i : M_i = \emptyset \text{ for } 1 \leqslant i \leqslant p\}|,\end{aligned}$$

where equality holds if and only if $|\mathbf{S}(\mathcal{F}_M)| = \sum_{i=1}^{p} |M_i|$ and $\bigcap \mathcal{F}_{H_i}$, for every $1 \leqslant i \leqslant p$ with $M_i = \emptyset$, contains exactly one element, which is not in $\mathbf{S}(\mathcal{F}_M)$ and $\bigcap \mathcal{F}_{H_i} \neq \mathcal{F}_{H_j}$ for $i \neq j$. □

Theorem 9. *For a connected diamond-free graph G, $\omega_s(G) = \sum_{i=1}^{p} |M_i| + |\{i : M_i = \emptyset \text{ for } 1 \leqslant i \leqslant p\}|$.*

Proof. Theorem 4 has proved this theorem if G is a complete graph. Thus we assume that G is not a complete graph in the following. By Lemma 8, we can prove this theorem by showing a simple set representation $\mathcal{F}$ of G with $|\mathbf{S}(\mathcal{F})| = \sum_{i=1}^{p} |M_i| + |\{i : M_i = \emptyset \text{ for } 1 \leqslant i \leqslant p\}|$.

For each i with $M_i \neq \emptyset$, let $M_i = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{|M_i|}}\}$, and $S_{i_1} = \{q_{i,1}\}$ and $S_{i_j} = \{q_{i,1}, q_{i,j}\}$ for $2 \leqslant j \leqslant |M_i|$. For each $v_k \in H_i$, where $1 \leqslant i \leqslant p$, assign $S_k = \{q_{x,1} : v_k \in Q_x \text{ and } Q_x \in \mathbb{Q}\}$. Note that, since v_k is a shared vertex, there are at least two cliques in $\mathbb{Q}$ containing v_k. The total number of elements used to construct $\mathcal{F}$ is equal to $\sum_{i=1}^{p} |M_i| + |\{i : |M_i = 0| \text{ for } 1 \leqslant i \leqslant p\}|$.

To complete the proof, we have to show that the constructed $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ is a simple set representation of G. First, we prove that $\mathcal{F}$ is a set representation of G. Clearly, each pair of vertices $v_j, v_k \in V(Q_i)$, for $1 \leqslant i \leqslant p$, has a common element $q_{i,1}$ in their corresponding S_j and S_k. Therefore, $S_j \cap S_k \neq \emptyset$ if edge $v_j v_k \in E(G)$. Now we prove that $S_j \cap S_k = \emptyset$ if there is no edge between v_j and v_k. Since only elements $q_{i,1}$, for $i = 1, 2, \ldots, p$, can appear in the representation sets of two different vertices, it suffices to consider the adjacency of vertices having $q_{i,1}$ in their corresponding sets. By our assignment and Lemma 5, any two distinct maximal cliques, say Q_i and Q_j, have at most one shared vertex v_k, which, if it exists, is the only vertex having $q_{i,1}$ and $q_{j,1}$ in its set representation S_k. All of the other vertices in Q_i cannot have $q_{j,1}$ in their set representations, and vice versa. Therefore, the constructed $\mathcal{F}$ is a set representation of G.

Next, we prove that $\mathcal{F}$ is a distinct set representation of G. Clearly, all vertices v_j in M_i, for $i = 1, 2, \ldots, p$, have different S_j. By Lemma 5 again, if a shared vertex in H_i, for some $1 \leqslant i \leqslant p$, has both $q_{i,1}$ and $q_{j,1}$, for some $1 \leqslant j \leqslant p$ and $j \neq i$, in its set representation, then no other vertex can have both of them in its set representation. Therefore, $\mathcal{F}$ is a distinct set representation of G.

It remains to show that $|S_i \cap S_j| = 1$ if $v_i v_j \in E(G)$. Clearly, for any two $v_i, v_j \in Q_k$, $q_{k,1}$ is the only common element between S_i and S_j. This concludes the proof of this theorem. □

Lemma 10. *If there exists a simple set representation $\mathcal{F}$ of G with $|\mathbf{S}(\mathcal{F}_M)| = \sum_{i=1}^{p} |M_i|$, then, for every nonempty M_i, $1 \leqslant i \leqslant p$, $\mathcal{F}_{M_i}$ is an Erdös $\mathcal{F}_{K_{|M_i|}}$ of $G[M_i]$.*

Proof. Note that $|\mathbf{S}(\mathcal{F}_M)| = \sum_{i=1}^{p} |M_i|$ implies that $|\mathbf{S}(\mathcal{F}_{M_i})| = |M_i|$. If there exists $|M_i| = 1$ or 2 for $1 \leqslant i \leqslant p$, then $\mathcal{F}_{M_i}$ can only be an Erdös $\mathcal{F}_{K_{M_i}}$. Thus, this lemma holds for $|M_i| = 1$ and 2. In the following, we consider the case where $|M_i| \geqslant 3$ if it exists.

Suppose to the contrary that there is an $\mathcal{F}_{M_i} = \{S_1, S_2, \ldots, S_{|M_i|}\}$ which is not an Erdös $\mathcal{F}_{K_{|M_i|}}$ of $G[M_i]$. By Theorem 4, $\mathcal{F}_{M_i}$ is an Erdös $\mathcal{F}_{\text{N-P}}$ or an Erdös $\mathcal{F}_{\text{PP}}$ of $G[M_i]$. We only consider the former case since the latter can be handled similarly. Let $\mathbb{Q}_{\text{N-P}} = \{Q'_1, Q'_2, \ldots, Q'_{|M_i|}\}$ be an N-P of $G[M_i]$ so that the Erdös $\mathcal{F}$ with respect to it is $\mathcal{F}_{M_i}$. By the definition of N-P, there is no clique in $\mathbb{Q}_{\text{N-P}}$ containing all vertices of M_i. Therefore, there is also no common element among all S_i for $i = 1, 2, \ldots, |M_i|$.

Since $|H_i| \neq 0$ and every $v_k \in H_i$ is adjacent to every vertex in M_i, the intersection between S_k and $\mathbf{S}(\mathcal{F}_{M_i})$ has at least two elements, say e_1 and e_2. We claim that there exists a vertex $v_\ell \in M_i$ whose corresponding S_ℓ also contains both e_1 and e_2. Note that v_ℓ is the vertex in $Q_{e_1} \cap Q_{e_2}$, where Q_{e_j}, for $j = 1$ or 2, is the clique containing all vertices v_x with $e_j \in S_x$. Consequently, $|S_k \cap S_\ell| \geqslant 2$, which contradicts that $\mathcal{F}_{M_i}$ is a simple set representation of $G[M_i]$. This establishes the lemma. □

Theorem 11. *Every connected diamond-free graph G is s-uniquely intersectable except K_n for $n \geqslant 3$.*

Proof. By Theorem 9, for a connected diamond-free graph G except K_n with $n \geqslant 3$, $\omega_s(G) = \sum_{i=1}^{p} |M_i| + |\{i : M_i = \emptyset \text{ for } 1 \leqslant i \leqslant p\}|$. By Lemma 8, $|\mathbf{S}(\mathcal{F}_M)| = \sum_{i=1}^{p} |M_i|$ for any minimum simple set representation $\mathcal{F}$ of G. By Lemma 10, every nonempty $\mathcal{F}_{M_i}$ is an Erdös $\mathcal{F}_{K_{|M_i|}}$ of $G[M_i]$.

Thus, for every i with $M_i \neq \emptyset$, the common element in all sets of $\mathcal{F}_{M_i}$ is also in every $S_j \in \mathcal{F}_{H_i}$ as v_j is adjacent to any vertex in M_i. Moreover, since $|\mathbf{S}(\mathcal{F})| = \sum_{i=1}^{p} |M_i| + |\{i : M_i = \emptyset \text{ for } 1 \leqslant i \leqslant p\}|$ and $|\mathbf{S}(\mathcal{F}_M)| = \sum_{i=1}^{p} |M_i|$, by Lemma 7 and the pigeonhole principle, for every i with $M_i = \emptyset$, all S_j for $v_j \in Q_i$ contain a common element, say e_i, which is not in $\mathbf{S}(\mathcal{F}_M)$, and e_i and e_j are distinct for $1 \leqslant i, j \leqslant p$ and $i \neq j$. From above, for any $v_j \in H$, $|S_j|$ is equal to the number of cliques containing v_j, and $S_j = \{e_i : e_i \text{ is the common element in } \mathcal{F}_{M_i} \cup \mathcal{F}_{H_i} \text{ and } v_j \text{ is in } Q_i \text{ for } i = 1, 2, \ldots, p\}$. Therefore, for any two minimum simple set representations $\mathcal{F}$ and $\mathcal{F}'$ of G, $\mathcal{F}$ can be obtained from $\mathcal{F}'$ by a bijective mapping from $\mathbf{S}(\mathcal{F}')$ to $\mathbf{S}(\mathcal{F})$. That is, the common element in $\mathcal{F}_{M_i} \cup \mathcal{F}_{H_i}$ has a unique corresponding common element in $\mathcal{F}'_{M_i} \cup \mathcal{F}'_{H_i}$, and every monopolized element in $\mathcal{F}_{M_i}$ has a unique corresponding monopolized element in $\mathcal{F}'_{M_i}$ for each $1 \leqslant i \leqslant p$. Thus G is s-uniquely intersectable. □

A similar proof to Theorem 7.6 in [6] (except replacing Lemma 7.1 in [6] by Theorem 4) establishes the following theorem.

Theorem 12. *For any graph G, $\omega_s(G) \geqslant c + \sum_{i=1}^{p} |M_i|$, where c is the number of maximal cliques Q_i in G not only with $M_i = \emptyset$ but having an edge not in any other Q_j.*

As a further study, it is interesting to find a sufficient and necessary condition for graphs having $\omega_s(G) = c + \sum_{i=1}^{p} |M_i|$ and study their s-uniquely intersectability.

References

[1] R. Alter, C.C. Wang, Uniquely intersectable graphs, Discrete Mathematics 18 (1977) 217–226.
[2] L.M. Batten, A. Beutelspacher, The Theory of Finite Linear Spaces, Cambridge University Press, 1993.
[3] N.G. de Bruijn, P. Erdös, On a combinatorial problem, Proceedings Koninklijke Nederlandse Akademie van Wetenschappen 51 (1948) 1277–1279.
[4] P. Erdös, A. Goodman, L. Pósa, The representation of a graph by set intersections, Canadian Journal Mathematics 18 (1966) 106–112.
[5] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1972, pp. 19–21.
[6] J. Kong, Y. Wu, On economical set representations of graphs, Discrete Mathematics and Theoretical Computer Science 11 (2009) 71–96.
[7] L.T. Kou, L.J. Stockmeyer, C.K. Wong, Covering edges by clique with regard to keyword conflicts and intersection graphs, Communications of the ACM 21 (1978) 135–139.
[8] N.V.R. Mahadev, T.M. Wang, On uniquely intersectable graphs, Discrete Mathematics 207 (1999) 149–159.
[9] S. Poljak, V Rödl, D. Turzík, Complexity of representation of graphs by set systems, Discrete Applied Mathematics 3 (1981) 301–312.
[10] E. Szpilrajn-Marczewski, Sur deux proprietes des classes d'ensembles, Fundamenta Mathematicae 33 (1945) 303–307.
[11] M. Tsuchiya, On intersection graphs with respect to antichains (II), Utilitas Mathematica 37 (1990) 29–44.

Constant Sum Flows in Regular Graphs

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Abstract. For an undirected graph G, a zero-sum flow is an assignment of non-zero integers to the edges such that the sum of the values of all edges incident with each vertex is zero. We extend this notion to a more general one in this paper, namely a **constant-sum flow**. The constant under a constant-sum flow is called an **index** of G, and $I(G)$ is denoted as the **index set** of all possible indices of G. Among others we obtain that the index set of a regular graph admitting a perfect matching is the set of all integers. We also completely determine the index sets of all r-regular graphs except that of $4k$-regular graphs of even order, $k \geq 1$.

1 Introduction and Preliminaries

Throughout this paper, all terminologies and notations on graph theory can be referred to the textbook by West[8]. We use $\mathbf{Z}$ to stand for the set of all integers, and $\mathbf{Z}^*$ the set of all non-zero integers.

Let G be a directed graph. A *k-flow* on G is an assignment of integers with maximum absolute value $k-1$ to each edge such that for every vertex, the sum of the values of incoming edges is equal to the sum of the values of outgoing edges. A *nowhere-zero k-flow* is a k-flow with no zero edge labels. A celebrated conjecture of Tutte says that:

(Tutte's 5-flow Conjecture[7]) Every bridgeless graph has a nowhere-zero 5-flow.

Jaeger showed that every bridgeless graph has a nowhere-zero 8-flow[4]. Next Seymour proved that every bridgeless graph has a nowhere-zero-6-flow[6].

One may study the elements of null space of the incidence matrix of an undirected graph. For an undirected graph G, the incidence matrix of G, $W(G)$, is defined as follows:

$$W(G)_{i,j} = \begin{cases} 1 & \text{if } e_j \text{ and } v_i \text{ are incident,} \\ 0 & \text{otherwise.} \end{cases}$$

An element of the null space of $W(G)$ is a function $f : E(G) \longrightarrow \mathbf{Z}$ such that for all vertices $v \in V(G)$ we have

$$\sum_{u \in N(v)} f(uv) = 0,$$

where $N(v)$ denotes the set of adjacent vertices to vertex v. If f never takes the value zero, then it is called a **zero-sum flow** on G. A **zero-sum k-flow** is a zero-sum flow whose values are integers with absolute value less than k. There is a conjecture for zero-sum flows similar to the Tutte's 5-flow Conjecture for nowhere-zero flows as follows. Let G be an undirected graph with incidence matrix W. If there exists a vector in the null space of W whose entries are non-zero real numbers, then there also exists a vector in that space, whose entries are non-zero integers with absolute value less than 6, or equivalently,

(Zero-Sum Conjecture[1]) If G is a graph with a zero-sum flow, then G admits a zero-sum 6-flow.

It was proved by Akbari et al. [1] that the above Zero-Sum Conjecture is equivalent to the Bouchet's Conjecture for bidirected graphs[3]. For regular graphs they obtained the following theorem:

Theorem 1 *Let G be an r-regular graph ($r \geq 3$). Then G has a zero-sum 7-flow. If $3|r$, then G has a zero-sum 5-flow.*

We extend the notion zero-sum flows to a more general one, namely **constant-sum flows** as follows:

Definition 2 *For an undirected graph G, if there exits $f : E(G) \to \mathbf{Z}^*$ such that $\sum_{u \in N(v)} f(uv) = C$ for each $v \in V(G)$, where C is an integer constant (called an* **index***), then we call f a* **constant-sum flow** *of G, or simply a C-sum flow of G.*

Denote by $I(G)$ the set of all possible indices for G, and call it the **index set** *of G.*

Remark. Note that $0 \in I(G)$ if and only if G admits a zero-sum flow.

We have the following observation for the index set of an r-regular graph:

Theorem 3 *Let G be r-regular($r \geq 2$) with a perfect matching, then $I(G) = \mathbf{Z}$.*

Proof.

Let M be the perfect matching. Note that we have the factorization $G = M \oplus (G \backslash M)$, where $G \backslash M$ is an $(r-1)$-regular graph. Since for the perfect matching $I(M) = \mathbf{Z}^*$, and $G \backslash M$ has indices $r-1$ and $1-r$ by labeling 1 and -1

respectively on edges, we have that $I(G) = I(M \oplus (G \backslash M)) \supseteq (r-1) + \mathbf{Z}^*$ and $I(G) = I(M \oplus (G \backslash M)) \supseteq (1-r) + \mathbf{Z}^*$. If $(r-1) + \mathbf{Z}^* \neq (1-r) + \mathbf{Z}^*$, then we are done with $I(G) = \mathbf{Z}$, since $(r-1) + \mathbf{Z}^* = \mathbf{Z}^* \backslash \{r-1\}$ and $(1-r) + \mathbf{Z}^* = \mathbf{Z}^* \backslash \{1-r\}$. In case $(r-1) + \mathbf{Z}^* = (1-r) + \mathbf{Z}^*$, which implies $r-1 = 1-r$ that is $r = 1$, a contradiction. $\square$

Moreover, we see that $I(G) = \mathbf{Z}^*$ for 1-regular graphs G, and $I(G) = \mathbf{Z}$ or $\mathbf{2Z}^*$ for 2-regular graphs G based upon the following observation:

Lemma 4 *Let C_n be an n-cycle, where $n \geq 3$. We have the following:*
(1) $I(C_n) = \mathbf{2Z}^*$, for n odd.
(2) $I(C_n) = \mathbf{Z}$, for n even.

Proof.
(1) Note that in any constant-sum flow of a cycle, the edges should alternatively be labeled the same. Therefore, for n odd, the labels on all edges are all the same. Therefore $I(C_n) = \mathbf{2Z}^*$.
(2) For n even, we label the edges $1, x-1, 1, x-1, \cdots, 1, x-1$ for $x \in \mathbf{Z} \backslash \{1\}$ to obtain the index x, and $2, -1, 2, -1, \cdots, 2, -1$ to obtain the index 1. Therefore $I(C_n) = \mathbf{Z}$.

$\square$

Corollary 5 *Let G be a 2-regular graph. Then $I(G) = \mathbf{2Z}^*$ if G contains an odd component (a connected component consisting of an odd cycle), and $I(G) = \mathbf{Z}$ otherwise.*

We determine completely the index sets of r-regular graphs in later sections for $r \geq 3$, except the index sets of $4k$-regular graphs, $k \geq 1$.

2 Constant-Sum Flows for Regular Graphs

Lemma 6 *Suppose G is a graph and $\{0, 1\} \subseteq I(G)$, then $I(G) = \mathbf{Z}$.*

Proof. Let $V(G) = \{1, 2, \cdots, n\}$, and $a_{ij} \neq 0$ be the edge labeling from vertex i to vertex j. Since $1 \in I(G)$,

$$\sum_{j \in N(i)} a_{ij} = 1, \quad \forall i \in V(G).$$

Pick some $x \in \mathbf{Z}^*$, then

$$x \cdot \sum_{j \in N(i)} a_{ij} = \sum_{j \in N(i)} x a_{ij} = x, \quad \forall i \in V(G).$$

Therefore $x \in I(G)$, and $I(G) = \mathbf{Z}$. $\square$

Remark. In [1] it was proved that all r-regular graphs G admit zero-sum flows if $r \geq 3$, that is, $0 \in I(G)$. Therefore it suffices to show $I(G) = \mathbf{Z}$ by verifying $1 \in I(G)$. Also it is not hard to see that if $m \in I(G)$ then $\mathbf{mZ} \subseteq I(G)$ for each positive integer $m \geq 2$.

Pull Back Labeling Construction:

In the following we propose a pull back labeling construction mentioned in [1]. First for an undirected loopless graph G, we define a new graph G' as follows. Suppose that $V(G) = \{1, 2, \cdots, n\}$, then G' is a bipartite graph with two parts $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_n\}$. Join u_i to v_j in G' if and only if the two vertices i and j are adjacent in G. Assume that G' admits a constant x-sum flow f'. If $f'(u_iv_j) + f'(u_jv_i) \neq 0$ for any pair of edges u_iv_j and u_jv_i in G', then we construct a constant $2x$-sum flow f for G, in the following way. For two adjacent vertices i and j in G, let ij be the edge connecting them in G. Then we may define f via f' by $f(ij) = f'(u_iv_j) + f'(u_jv_i)$. By our assumption, $f(ij) \in \mathbf{Z}^*$. Pick some x in $\mathbf{Z}^*$, we have

$$\sum_{v_j \in N(u_i)} f'(u_iv_j) = x, \sum_{u_j \in N(v_i)} f'(u_jv_i) = x,$$

thus we find

$$\sum_{j \in N(i)} f(ij) = \sum_{v_j \in N(u_i)} f'(u_iv_j) + \sum_{u_j \in N(v_i)} f'(u_jv_i) = 2x, \quad \forall i \in V(G).$$

This defines a $2x$-sum flow for G. If G is r-regular, then G' is an r-regular bipartite graph. Thus by Hall's Marriage Theorem, all edges of G' can be partitioned into r perfect matchings. Let $E_1, \cdots, E_r$ be the set of edges of these matchings. We will use this construction and notations throughout the remaining of this article.

2.1 Odd Regular Graphs

We deal with general odd regular graphs here:

Lemma 7 *If G is a $(2k+1)$-regular graph, then $I(G) = \mathbf{Z}$ for all $k \geq 2$.*

Proof.
Construct G' as before, we change the definition of $f'(e)$ as follows:

Let
$$f'_0(e) = \begin{cases} k+1\ , & e \in E_1 \cup \cdots \cup E_k \\ -k \quad , & e \in E_{k+1} \cup \cdots \cup E_{2k+1} \end{cases}$$

This gives that f'_0 is a zero-sum flow for G' and satisfies $f'_0(u_iv_j) + f'_0(u_jv_i) \neq 0$. So $f_0(e) \in \{2k+2, -2k, 1\}$ is a zero-sum flow for G.

Let

$$f_1'(e) = \begin{cases} k & , \quad e \in E_1 \cup \cdots \cup E_k \\ 1-k \ , & e \in E_{k+1} \cup \cdots \cup E_{2k+1} \end{cases}$$

This gives that f_1' is a 1-sum flow for G' and satisfies $f_1'(u_iv_j) + f_1'(u_jv_i) \neq 0$. So $f_2(e) \in \{2k, 2-2k, 1\}$ is a 2-sum flow labeling for G.

Now we set $f(ij) = \dfrac{f_0(ij) + f_2(ij)}{2}$ for all $i, j \in V(G)$. Then $f(e) \in \{2k+1, 1-2k, 1\}$ and

$$\sum_{j \in N(i)} f(ij) = \frac{1}{2}(\sum_{j \in N(i)} f_0(ij) + \sum_{j \in N(i)} f_2(ij)) = \frac{1}{2}(0+2) = 1.$$

That is, f is a 1-sum flow for G and $1 \in I(G)$. □

Lemma 8 *If G is a 3-regular graph, then $I(G) = \mathbf{Z}$.*

Proof.
As mentioned in previous remark, we see that it suffices to show $I(G) = \mathbf{Z}$ by verifying $1 \in I(G)$ for regular graphs G.

Construct G' as before, we define the $f_0'(e)$ as follows:

Let

$$f_0'(e) = \begin{cases} -2 \ , & e \in E_1 \\ 1 \ , & e \in E_2 \\ 1 \ , & e \in E_3 \end{cases}$$

This gives $f_0(e) \in \{-4, 2, -1\}$ which is a zero-sum flow for G. Define the $f_1'(e)$ as follows:

Let

$$f_1'(e) = \begin{cases} -3 \ , & e \in E_1 \\ 2 \ , & e \in E_2 \\ 2 \ , & e \in E_3 \end{cases}$$

This gives $f_2(e) \in \{-6, 4, -1\}$ which is a 2-sum flow for G.

Now, we set $f(e) = \dfrac{1}{2}(f_0(e) + f_2(e))$. Then $f(e) \in \{-5, 3, -1\}$ and

$$\sum_{j \in N(i)} f(ij) = \frac{1}{2}(\sum_{j \in N(i)} f_0(ij) + \sum_{j \in N(i)} f_2(ij)) = \frac{1}{2}(0+2) = 1.$$

That is, f is a 1-sum flow for G, same as saying $1 \in I(G)$, and hence $I(G) = \mathbf{Z}$. □

2.2 Even Regular Graphs

Note that for any regular graph of odd degree, the number of vertices is always even. Also Petersen[5] proved the following two well known Theorems in 1891:

Theorem 9 (Petersen, 1891) *Every regular graph of even degree is 2-factorable.*

Theorem 10 (Petersen, 1891) *Let k be a positive integer. If a connected $2k$-regular graph G has an even number of vertices, then it may be k-factored. That is, G can be factored into the sum of two k-regular spanning subgraphs.*

Therefore we observe the following for even regular graphs with odd orders:

Lemma 11 *If G is r-regular graph with odd vertices (therefore r is even), then $I(G) = \mathbf{2Z}$, for all $r \geq 3$.*

Proof. We show that $I(G) \subseteq \mathbf{2Z}$ first. Suppose $c \in I(G)$, then we have

$$2 \sum_{e \in E(G)} f(e) = c|V(G)|.$$

Since $|V(G)|$ is odd, therefore r must be even, thus $I(G) \subseteq \mathbf{2Z}$.

Conversely, we show $I(G) \supseteq \mathbf{2Z}$. Let $r = 2k$. G will have a 2-factor, namely E_1 by Theorem 9. We now define a 2-flow by $f_2(e) = k$, if $e \in E_1$, and $f_2(e) = -1$ for others edges. Set $f_0(e) = k - 1$, if $e \in E_1$, and $f_0(e) = -1$ for others edges would gives a 0-flow. □

To complete the picture, we need one more Lemma:

Lemma 12 *If G is a $2k$-regular graph with even vertices, where k is odd, then $I(G) = \mathbf{Z}$ for all $k \geq 3$.*

Proof.

Without loss of generality we may assume G is connected. By Petersen's Theorem 10, $G = K_1 \oplus K_2$, where K_1 and K_2 are two k-factors. Since k is odd, $I(K_1) = I(K_2) = \mathbf{Z}$ by Lemma 7 and Lemma 8. Therefore, $I(G) = I(K_1 \oplus K_2) \supseteq I(K_1) + I(K_2) = \mathbf{Z}$. □

Remark. We complete the picture for all even regular graphs except $4k$-regular graphs of even order.

In below we present examples of index sets of 4-regular graphs. However the index set of a general 4-regular graph is not known yet.

Example 1. $I(C_m \Box C_n) = \mathbf{Z}$ for even m and even n, where $C_m \Box C_n$ is Cartesian product.

Example 2. $I(G) = \mathbf{Z}$ for the following 4-regular graph G without perfect matching. Note that we give the 0-sum and 1-sum flows.

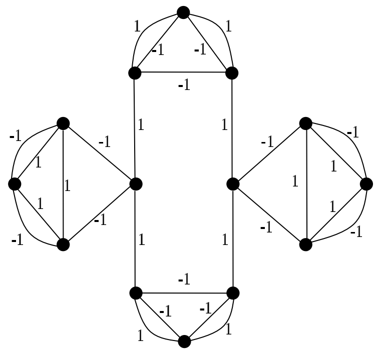

Fig. 1. A 4-regular graph with 0-sum flow

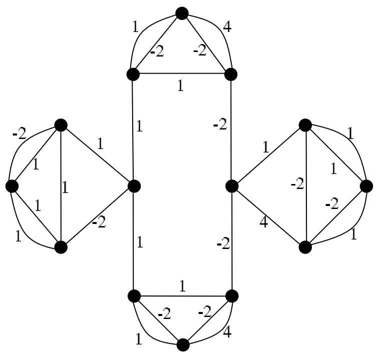

Fig. 2. A 4-regular graph with 1-sum flow

3 Concluding Remarks

To summarize up, we have obtained all index sets of r-regular graphs except $4k$-regular graphs, $k \geq 1$, with even number of vertices as follows:

Theorem 13 *The index sets of r-regular graphs G of order n, are as follows:*

$$I(G) = \begin{cases} \mathbf{Z}^*, & r = 1. \\ \mathbf{Z}, & r = 2 \textit{ and } G \textit{ contains even cycles only.} \\ \mathbf{2Z}^*, & r = 2 \textit{ and } G \textit{ contains an odd cycle.} \\ \mathbf{2Z}, & r \geq 3,\ r \textit{ even and } n \textit{ odd }. \\ \mathbf{Z}, & r \geq 3,\ r \neq 4k,\ k \geq 1,\ \textit{and } n \textit{ even }. \end{cases}$$

Even further one may consider the concept constant sum k-flow similar to that of zero-sum k-flow. It would be interesting to study the relationship among these related notions. Calculating the index sets of other graph classes are obviously next sets of research problems to be explored.

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References

1. S. Akbari, A. Daemi, O. Hatami, A. Javanmard, A. Mehrabian, Zero-Sum Flows in Regular Graphs, *Graphs and Combinatorics* **26** (2010), 603-615.
2. S. Akbari, N. Ghareghani, G.B. Khosrovshahi, A. Mahmoody, On zero-sum 6-flows of graphs, *Linear Algebra Appl.* 430, 3047-3052 (2009)
3. A. Bouchet, Nowhere-zero integral flows on a bidirected graph. *J. Combin. Theory Ser. B* 34, 279-292 (1983)
4. F. Jaeger, Flows and generalized coloring theorems in graphs. *J. Combin. Theory Ser. B* 26(2), 205-216 (1979)
5. J. Petersen, Die Theorie der regularen graphs, *Acta Mathematica* (15) 193-220 (1891)
6. P.D. Seymour, Nowhere-zero 6-flows. *J. Combin. Theory Ser. B* 30(2), 130-135 (1981)
7. W.T. Tutte, A contribution to the theory of chromatic polynomials. *Can. J. Math.* 6, 80-91 (1954)
8. D.B. West, Introduction to Graph Theory, 2nd edn. Prentice Hall, Englewood Cliffs (2001)

On Deficiency of Super Edge Magic Labelings for Complete Bipartite Graphs

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***Abstract*—It is known that edge magic labeling can be applied to the arrangement of devices of a wireless network. The concept of edge magic labeling was introduced by A. Kotzig and A. Rosa in 1970. A (p,q)-graph G with p vertices and q edges is called edge magic if there exists a bijective function $f: V(G)\cup E(G)\to\{1,2,\ldots,p+q\}$ such that $f(u)+f(v)+f(uv)$ is constant for any edge $uv\in E(G)$. Moreover, G is called super edge magic if $f(V(G))=\{1,2,\ldots,p\}$. In 1970 A. Kotzig and A. Rosa also defined the edge-magic deficiency, $\mu(G)$, of a graph G as the minimum number of isolated vertices added to the graph so that the resulting disconnected graph is edge magic. In 1999 Figueroa-Centeno et al. introduced and studied similar notion of super edge magic deficiency $\mu_s(G)$ of a graph G for super edge magic labeling. Calculating the deficiency provides with more detailed information regarding related graph labeling. In this paper we completely determine the super edge magic deficiency of complete bipartite graphs $\mu_s(K_{m,n})=(m-1)(n-1)$, which justifies a conjecture raised by R. Figueroa-Centeno et al. and Hegde et al. independently.**

I. Introduction

Consider a wireless network in which every device must be able to connect to a subset of the other devices in the network using a unique channel to prevent collisions. One way to create such a channel assignment is to give numeric labels to the devices and channels in such a way that the labels of two devices and the communication line between them sum to a consistent value across every pair of devices in the network. In this case, knowing the labels of the two communicating devices gives the identification number of the communication line between them[9]. This solution is an example of an edge-magic labeling, which we introduce in the following.

In this article we consider finite undirected graphs without loops or multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G respectively. We say that G is a (p,q)-graph if $|V(G)|=p$ and $|E(G)|=q$.

Edge magic labeling were first introduced by A. Kotzig and A. Rosa[8] in 1970. Super edge magic labeling were first introduced by Enomoto, Llad'o, Nakamigawa and Ringel[2] in 1998. We define these labelings below:

Definition 1.1: An **edge magic** labeling of a (p,q) graph G is a bijective function $f: V(G)\cup E(G)\to\{1,2,\ldots,p+q\}$ such that $f(u)+f(v)+f(uv)$ is constant for every edge $uv\in E(G)$. In such case, G is said edge magic. If moreover $f(V(G))=\{1,2,\ldots p\}$, then f is called a **super edge magic** labeling, and G is said to be super edge magic.

In 2001, it was observed by R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle [4] the following:

Lemma 1.1: A (p,q) graph G is super edge magic if only only if there exists a bijective function $f: V(G)\to\{1,2,\ldots,p\}$ such that the set

$$S=\{f(u)+f(v): uv\in E(G)\}$$

consists of q consecutive integers.

Therefore, one may consider the super edge magic labeling problems using the vertex labeling in the above proposition. The following are necessary conditions for being super edge magic:

Lemma 1.2: If a (p,q)-graph G with p vertices and q edges is super edge magic, then $q\le 2p-3$.

Proof: Let G be super edge magic with vertex labels $\{1,2,\ldots,p\}$. Then the set of induced edge labels is $\{k,k+1,\ldots,k+q-1\}$ for some integer k. Therefore $k+(q-1)\le(p-1)+p$ and $1+2\le k$, thus $q\le 2p-3$.

Q.E.D.

More generally, for a (p,q)-graph G, by an **edge-antimagic** vertex labeling we mean a one-to-one mapping f from $V(G)$ into $\{1,2,\cdots,p\}$ such that the edge-weights $f^+(uv)=f(u)+f(v)$ of edges $uv\in E(G)$ are distinct. The vertex labeling f is called (a,d)**-edge-antimagic** if moreover the set of distinct edge-weights forms an arithmetic progression $a,a+d,\cdots,a+(q-1)d$ with initial term a and common difference d, where a and d are two fixed positive integers. A graph G is called edge-antimagic ((a,d)-edge-antimagic, respectively) if it admits an edge-antimagic ((a,d)-edge-antimagic, respectively) vertex labeling.

Lemma 1.3: If a (p,q)-graph G with p vertices and q edges is edge-antimagic, then $q\le 2p-3$. In particular this is true for (a,d)-edge-antimagic graphs.

Proof: Let G be (a,d)-edge-antimagic with vertex labels $\{1,2,\ldots,p\}$. Then the set of induced edge labels is $\{a,a+d,\ldots,a+(q-1)d\}$ for some integer a and d. Therefore if $q>2p-3$, and note that the edge weights are numbers among the $2p-3$ numbers ranging from $3,4,\ldots,2p-1$, thus by pigeonhole principle, there must be two edge weights are the same, a contradiction. Hence $q \leq 2p-3$. **Q.E.D.**

On the other hand, for a (p, q)-graph $G = (V(G), E(G))$, a bijection g from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p+q\}$ is called (a, d)**-edge-antimagic total** labeling if the edge-weights $w(xy) = g(x) + g(y) + g(xy)$, for $xy \in E(G)$, form an arithmetic progression starting from a and having common difference d, where a and d are two fixed integers. Note that d is allowed to be 0. An (a, d)-edge-antimagic total labeling is called *super* (a, d)*-edge-antimagic total* if $g(V(G)) = \{1, 2, \ldots, p\}$. A graph G is called (a, d)-edge-antimagic total (super (a, d)-edge-antimagic total, respectively) if it admits an (a, d)-edge-antimagic total (super (a, d)-edge-antimagic total, respectively) labeling.

The following lemma shows that every (a, d)-edge-antimagic vertex labeling can be extended to a super (a, d)-edge-antimagic total labeling.

Lemma 1.4: A (p, q) graph G is super (a, d)-edge-antimagic total if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set

$$S = \{f(u) + f(v) : uv \in E(G)\}$$

consists of an arithmetic progression $\{b, b+d', \cdots, b+(q-1)d'\}$, where $d' = d+1$.

Proof: Assume there exists a bijective function $f : V(G) \to \{1, \ldots, p\}$ such that the set $S = \{f(u) + f(v) : uv \in E(G)\}$ consists of an arithmetic progression $\{b, b+d', \cdots, b+(q-1)d'\}$. Then one may extend the vertex labeling to $V(G) \cup E(G)$ by assigning values $p+1, \ldots, \ldots, p+(q-1), p+q$ to the edges with the sum of endpoint labels in the reversing order $b+(q-1)d', \ldots, b+d', b$ respectively, thus we get the arithmetic progression $\{a, a+d, \cdots, a+(q-1)d\}$ over the edges with their two endpoints, where $a = b+(p+q), a+d = b+d'+(p+q-1), \cdots, a+(q-1)d = b+(q-1)d'+(p+1)$. Therefore $d' = d+1$ and $a = b+(p+q)$. **Q.E.D.**

Corollary 1.1: Every super edge-magic and edge-antimagic (p, q)-graph contains at least two vertices of degree less than 4.

Proof: Assume on the contrary that $p-1$ vertices of G are of degrees at least 4. Then, by Lemma 1.2 and Lemma 1.3

$$4p - 4 = \sum_{i=1}^{p-1} 4 \leq \sum_{v \in V(G)} degv = 2q \leq 2(2p-3) = 4p - 6$$

which is a contradiction. **Q.E.D.**

Remark. Note that the relationships among above mentioned labelings are as follows: an (a, d)-edge-antimagic vertex labeling is a super $(a', d-1)$-edge-antimagic total labeling, and when $d = 1$, it is a super edge magic labeling.

Now we are in a position to consider that, how far a graph is away from being super edge magic. Therefore a more general notion of deficiency is introduced as follows.

Definition 1.2: The **super edge magic deficiency** $\mu_s(G)$ of a graph G is defined as $\mu_s(G) = \min\{n \geq 0 : G \cup nK_1$ is super edge magic$\}$. If G is not super edge magic by adding any number of isolated vertices, then $\mu_s(G) = \infty$, and $\mu_s(G) = 0$ if G is super edge magic.

In 1970 Kotzig and Rosa[8] defined the **edge-magic deficiency**, $\mu(G)$, of a graph G as the minimum n such that $G \cup nK_1$ is edge-magic total. If no such n exists, they define $\mu(G) = \infty$. In 1999 Figueroa-Centeno, Ichishima, and Muntaner-Batle[3] extended this notion to super edge-magic deficiency, $\mu_s(G)$, in the analogous way. They conjectured that $\mu_s(K_{m,n}) = (m-1)(n-1)$. This conjecture was also studied independently by Hegde and Shetty[6], who used the notions of **strongly** k**-indexable** labelings and **vertex characteristics** in 2009. They both proved $K_{m,n}$ is super edge magic if and only if $m = 1$ or $n = 1$. They observed that $\mu_s(K_{m,n}) \leq (m-1)(n-1)$ and conjectured that $\mu_s(K_{m,n}) = (m-1)(n-1)$ by giving several supporting examples such as $K_{2,n}$, $K_{3,n}$ and $K_{4,n}$ etc. We calculate the super edge magic deficiency of $K_{m,n}$ for any m and n in later sections, thus confirm the conjecture completely:

Theorem 1.1: The super edge magic deficiency of complete bipartite graphs is $\mu_s(K_{m,n}) = (m-1)(n-1)$ for positive integers m, n.

We define the following more general concept:

Definition 1.3: The **(a, d)-edge-antimagic deficiency** $\mu_d(G)$ of a graph G is defined as $\mu_d(G) = \min\{n \geq 0 : G \cup nK_1$ is (a, d)-edge-antimagic $\}$, where a and d are two fixed positive integers. If G is not (a, d)-edge-antimagic by adding any number of isolated vertices, then $\mu_d(G) = \infty$, and $\mu_d(G) = 0$ if G is (a, d)-edge-antimagic.

More examples and discussions about (a, d)-edge-antimagic vertex labeling, (a, d)-edge-antimagic total labeling and their deficiency problems can be referred to [1].

However in this article, we focus on the deficiency problem of super edge magic labeling for complete bipartite graphs. We will show Theorem 1.1 in later sections.

II. Basic Labeling Matrix Construction

Let f be a super edge magic labeling of $K_{m,n} \cup zK_1$ and z be a non-negative integer. We let $A = \{a_1, a_2, \cdots, a_n\}$ and $B = \{b_1, b_2, \cdots, b_m\}$ be the two partite sets of $K_{m,n}$. Without loss of generality we assume $f(a_1) < f(a_2) < \cdots\cdots < f(a_n)$ and $f(b_1) < f(b_2) < \cdots\cdots < f(b_m)$. For simplicity, we abuse the language and express these labellings $f(a_i)$ and $f(b_j)$ by a_i and b_j, respectively. That is to say that we assume that $a_1 < a_2 < \cdots\cdots < a_n$ and $b_1 < b_2 < \cdots\cdots < b_m$, respectively.

Note that we may assume $m, n \geq 2$, since in case either $m = 1$ or $n = 1$, then it is not hard to see that $\mu_s = 0$, which means the star graphs $K_{1,n}$ are super edge magic. In the following we put the induced edge labels in a matrix form, where the entry along the a_i column and b_j row is the sum $a_i + b_j$, which is the induced label over the edge $a_i b_j$.

We suppose $a_1 + b_1 = k$, the smallest possible induced labeling over the edge $a_1 b_1$. Then without loss of generality may assume $a_1+b_2=k+1$. Since f is super edge magic, we may assume further that there is a longest consecutive integer sequence of induced edge labelings $k,k+1,\cdots,k+(t_0-1)$ with length $t_0\geq 2$, while we label consecutively $b_1,b_2,\cdots,b_{t_0}$ along the partite set B.

$B\backslash A$	a_1	a_2	$\cdots\cdots\cdots$	a_{r_0}
b_1	k	$k+t_0$	$\cdots\cdots\cdots$	$k+(r_0-1)t_0$
b_2	$k+1$	$k+t_0+1$	$\cdots\cdots\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots\cdots\cdots$	$\cdots\cdots\cdots$	$\cdots$
b_{t_0}	$k+(t_0-1)$	$k+2t_0-1$	$\cdots\cdots\cdots$	$k+t_0r_0-1$

Then $a_2+b_1=k+t_0$ should be the next smallest edge label. Since the sequence of vertex labels $b_1,b_2,\cdots,b_{t_0}$ is consecutive and given, it can be seen that the edge labels over $a_2b_1,a_2b_2,\cdots$ until $a_2b_{t_0}$ are $k+t_0,k+t_0+1,\cdots,k+2t_0-1$ respectively.

Now we may assume further that along the partite set A, there is an arithmetic progression $a_1,a_2,\cdots,a_{r_0}$ with the common difference t_0 and the longest length $r_0\geq 2$. Therefore we obtain all the consecutive edge labels from $a_1+b_1=k$ to $a_{r_0}+b_{t_0}=k+t_0r_0-1$, and in this fashion we say they form a **basic matrix**.

We continue growing the basic matrix along partite sets A and B, and the next smallest edge label is $b_{t_0+1}+a_1=k+r_0t_0$. Note that the vertex labels $a_1,a_2,\cdots,a_{r_0}$ are fixed, therefore the edge labels along the row $b_{t_0+1}+a_1,b_{t_0+1}+a_2,\cdots,b_{t_0+1}+a_{r_0}$ are also fixed as $k+r_0t_0,k+r_0t_0+t_0,\cdots,k+r_0t_0+(r_0-1)t_0$.

Now there are two possibilities for the induced edge label $k+r_0t_0+1$, which are $b_{t_0+2}+a_1$ and $a_{r_0+1}+b_1$. If $a_{r_0+1}+b_1=k+r_0t_0+1$, then $b_{t_0+1}+a_2=a_{r_0+1}+b_{t_0}=k+r_0t_0+t_0$, a contradiction. Therefore it should be the case $b_{t_0+2}+a_1=k+r_0t_0+1$.

Similarly, we have $b_{t_0+3}+a_1=k+r_0t_0+2,\cdots$, until $b_{2t_0}+a_1=k+r_0t_0+t_0-1$, again the edge labels along the rows are fixed. In this fashion, another basic matrix of consecutive edge labels, of size $t_0\times r_0$, is obtained. Keep growing similar matrices, we may assume that there is a largest $t_1\geq 2$ of the same size $t_0\times r_0$ basic matrices.

$B\backslash A$	a_1	$\cdots\cdots\cdots$	a_{r_0}
b_1	k	$\cdots\cdots\cdots$	$k+(r_0-1)t_0$
$\cdots$	$\cdots$	$\cdots\cdots\cdots$	$\cdots$
b_{t_0}	$k+(t_0-1)$	$\cdots\cdots\cdots$	$k+t_0r_0-1$
b_{t_0+1}	$k+t_0r_0$	$\cdots\cdots\cdots$	$k+(2r_0-1)t_0$
$\cdots$	$\cdots$	$\cdots\cdots\cdots$	$\cdots$
b_{2t_0}	$k+t_0r_0+(t_0-1)$	$\cdots\cdots\cdots$	$k+2t_0r_0-1$
$\cdots$	$\cdots$	$\cdots\cdots\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots\cdots\cdots$	$\cdots$
$b_{t_1t_0}$	$k+(t_1-1)t_0r_0+(t_0-1)$	$\cdots\cdots\cdots$	$k+t_1t_0r_0-1$

Keep growing the edge labeling matrix of $K_{m,n}$ in similar fashions, we may see it grows alternatively from the basic matrix of size $t_0\times r_0$, then t_1 basic matrices of size $t_0\times r_0$, and then r_1 matrices of size $t_1t_0\times r_0$, then t_2 matrices of size $t_1t_0\times r_1r_0$, $\cdots\cdots$, recursively until either t_{k+1} matrices of size $t_k\cdots t_0\times r_k\cdots r_0$ (with $r_{k+1}=1$), or r_{k+1} matrices of size $t_{k+1}t_k\cdots t_0\times r_k\cdots r_0$ (with $t_{k+2}=1$), for some $k\geq 0$.

We conclude that if $K_{m,n}\cup zK_1$ is super edge magic, then the induced edge labeling matrix must grow in the above fashion, and

$$m=\prod_{i\geq 0}t_i,\ n=\prod_{j\geq 0}r_j.$$

Note that eventually t_i or r_j will reach 1, which means the matrices stop growing, since m and n are finite.

III. Proof of Main Result

Proof of the Theorem 1.1:

In 2006 [3], Figueroa-Centeno et al. showed that $\mu_s(K_{m,n})\leq(m-1)(n-1)$ by giving a super edge magic labeling with extra $(m-1)(n-1)$ isolated vertices. Therefore it suffices to show that $\mu_s(K_{m,n})\geq(m-1)(n-1)$.

Let $[m,n]$ be the set of integers $\{r\mid m\leq r\leq n\}$. We claim that $[a_1,a_n]\cap[b_1,b_m]=\emptyset$. If the claim is true, then either $a_n-b_1\geq mn$ or $b_m-a_1\geq mn$. Therefore if $K_{m,n}\cup zK_1$ is super edge magic, then $z\geq(m-1)(n-1)$, and thus $\mu_s(K_{m,n})\geq(m-1)(n-1)$, then the Theorem 1.1 is proved.

To prove the claim, we see from the conclusion of previous section, either $mn=t_0r_0\cdots t_kr_kt_{k+1}$ or $mn=t_0r_0\cdots t_{k+1}r_{k+1}$, that is, either $m=t_0\cdots t_kt_{k+1}$ and $n=r_0\cdots r_k$, or $m=t_0\cdots t_{k+1}$ and $n=r_0\cdots r_{k+1}$. We proceed by mathematical induction. When $k=0$, it is not hard to see the claim is true. We assume $k\geq 0$.

Case 1: $m=t_0\cdots t_kt_{k+1}$ and $n=r_0\cdots r_k$.

Assume to the contrary that $[a_1,a_n]\cap[b_1,b_m]\neq\emptyset$, we have 4 different cases. Since the arguments are similar, we use the case $b_1<a_1<b_m<a_n$ as an example.

By induction hypothesis, we know for example a_1 could lie between $b_{t_0\cdots t_k}$ and $b_{t_0\cdots t_k+1}$, that is the gap between the first matrix and the second matrix of sizes $t_k\cdots t_0\times r_k\cdots r_0$. Look at $|a_{r_0\cdots r_k}-a_1|=(r_k-1)t_0\cdots t_k+(r_{k-1}-1)t_0\cdots t_{k-1}+\cdots+(r_0-1)t_0$ and $|b_{t_0\cdots t_k+1}-b_{t_0\cdots t_k}|=(r_k-1)t_0\cdots t_k+(r_{k-1}-1)t_0\cdots t_{k-1}+\cdots+(r_0-1)t_0-1$. Note that since $|a_{r_0\cdots r_k}-a_1|$ and $|b_{t_0\cdots t_k+1}-b_{t_0\cdots t_k}|$ differ by 1, there must be repeated labellings $a_i=b_j$, a contradiction.

Similar situations for other following possible gap positions for a_1 such as, a_1 could lie between $b_{2t_0\cdots t_k+1}$ and $b_{2t_0\cdots t_k+2}$, or between $b_{3t_0\cdots t_k+2}$ and $b_{3t_0\cdots t_k+3}$,$\cdots\cdots\cdots\cdots$, or between $b_{(t_{k+1}-1)t_0\cdots t_k}$ and $b_{(t_{k+1}-1)t_0\cdots t_k+1}$, we got a contradiction in each case.

As for other possibilities $b_1<a_1<a_n<b_m$, $a_1<b_1<b_m<a_n$, or $a_1<b_1<a_n<b_m$, one may reach contradictions by similar arguments.

Case 2: $m=t_0\cdots t_{k+1}$ and $n=r_0\cdots r_{k+1}$.

Assume to the contrary that $[a_1,a_n]\cap[b_1,b_m]\neq\emptyset$, we have 4 cases similar to that in Case 1. We use $a_1<b_1<a_n<b_m$ as an example.

By induction hypothesis, we know for example b_1 could lie between $a_{r_0\cdots r_k}$ and $a_{r_0\cdots r_k+1}$ as before, and we look at $|a_{r_0\cdots r_k}-a_{r_0\cdots r_k+1}|=(t_{k+1}-1)t_0r_0\cdots t_kr_k+(t_{k-1}-1)t_0r_0\cdots t_{k-1}r_{k-1}+\cdots+(t_1-1)t_0r_0+t_0$ and $|b_m-b_1|=(t_{k+1}-1)t_0r_0\cdots t_kr_k+(t_k-1)t_0r_0\cdots t_{k-1}r_{k-1}+\cdots+(t_1-$

$1)t_0r_0 + t_0 - 1$. Note that $|a_{r_0\cdots r_k} - a_{r_0\cdots r_k+1}|$ and $|b_m - b_1|$ differ by 1, therefore again there must be repeated labellings $a_i = b_j$, a contradiction.

Other possibilities can be done similarly. Hence we are done with the proof of the claim. **Q.E.D.**

IV. Examples

We give an algorithm below on super edge magic labeling of $K_{m,n} \cup (m-1)(n-1)K_1$, since $K_{m,n}$ has the super edge magic deficiency $(m-1)(n-1)$. Starting with the basic matrix of size $t_0 \times r_0$, then growing the matrix of induced edge labels by giving the vertex labeling along one partite set to be continuous, while along the other partite set to be an arithmetic progression. Let also the smallest vertex labeling along one partite set be 1, and the smallest vertex labeling along the other partite set is a.

Algorithm 4.1: Super Edge Magic Labeling of $K_{m,n}$

Input: Given a complete bipartite graph $K_{m,n}$.

1) For m, n, fix factorizations of $m = t_0t_1\cdots t_kt_{k+1}$ and $n = r_0r_1\cdots r_{k-1}r_k$.
2) if $t_{k+1} = 1$, then the matrix is fixed for $mn = t_0r_0\cdots t_kr_k$.
3) if $r_k = 1$, then the matrix is fixed for $mn = t_0r_0\cdots t_{k-1}r_{k-1}t_k$.
4) solve a by the formula $a + (r_k-1)t_0r_0\cdots r_{k-1}t_k + (r_{k-1}-1)t_0r_0\cdots r_{k-2}t_{k-1} + \cdots + (r_0-1)t_0 = mn+1$.

Output:

The super edge magic labeling of $K_{m,n} \cup (m-1)(n-1)K_1$ in matrix form.

Example 4.1: Note that $\mu_s(K_{6,8}) = 35$. We give below 3 possible cases, out of 14 possible cases totally, while calculating the super edge magic deficiency for $K_{6,8}$. We assume without loss of generality that $m = 6, n = 8$.

Case 1: $mn = t_0r_0$, $t_0 = 6$, $r_0 = 8$, $t_1 = 1$, $a = 7$.

$m\backslash n$	7	13	19	25	31	37	43	49
1	8	14	20	26	32	38	44	50
2	9	15	21	27	33	39	45	51
3	10	16	22	28	34	40	46	52
4	11	17	23	29	35	41	47	53
5	12	18	24	30	36	42	48	54
6	13	19	25	31	37	43	49	55

Case 2: $mn = t_0r_0t_1$, $t_0 = 3$, $r_0 = 8$, $t_1 = 2$, $r_1 = 1$, $a = 28$.

$m\backslash n$	28	31	34	37	40	43	46	49
1	29	32	35	38	41	44	47	50
2	30	33	36	39	42	45	48	51
3	31	34	37	40	43	46	49	52
25	53	56	59	62	65	68	71	74
26	54	57	60	63	66	69	72	75
27	55	58	61	64	67	70	73	76

Case 3: $mn = t_0r_0t_1r_1$, $t_0 = 3$, $r_0 = 4$, $t_1 = 2$, $r_1 = 2$, $t_2 = 1$, $a = 16$.

$m\backslash n$	16	19	22	25	40	43	46	49
1	17	20	23	26	41	44	47	50
2	18	21	24	27	42	45	48	51
3	19	22	25	28	43	46	49	52
13	29	32	35	38	53	56	59	62
14	30	33	36	39	54	57	60	63
15	31	34	37	40	55	58	61	64

V. Concluding Remarks

For a graph determining the deficiency is the same as determining the super edge magic labeling by assigning labels to vertices in a relaxed way. Therefore it is applicable in the arrangement of devices of a wireless network as mentioned in the beginning section.

For future research directions we propose that the work in this paper can be explored further for special types of bipartite graphs, including those of trees. Also we have found certain classes of complete multi-partite graphs which can not be made super edge magic adding any number of vertices.

For more generalization of super edge magic labeling and their deficiency problems, one may consider (a, d)-edge-antimagic vertex/total labeling as mentioned in the introduction section and look up the recent book [1].

Acknowledgment

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References

[1] M. Bača and M. Miller, Super Edge-Antimagic Graphs: A Wealth of Problems and Some Solutions. Brown Walker Press, Boca Raton (2008)

[2] H. Enomoto, A. S. Llado, T. Nakamigawa, and G. Ringel, Super edge-magic graphs, *SUT J. Math.* 34 (1998) 105-109.

[3] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, "On the super edge magic deficiency of graphs", *Ars Combin.,* 78 (2006) 33-45.

[4] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, "The place of super edgemagic labelings among other classes of labelings", *Discrete Mathematics,* 231(2001),153-168.

[5] Gallian, J.A., A Dynamic survey of graph labeling, 2010.

[6] S. M. Hegde, S. Shetty, Strongly indexable graphs and applications, *Discrete Mathematics* 309(21): 6160-6168 (2009)

[7] A. Kotzig and A. Rosa, Magic valuations of complete graphs, Centre de Recherches Mathematiques, Universite de Montreal, (1972) CRM-175.

[8] A. Kotzig and A. Rosa, Magic valuations of finite graphs, Canad. Math. Bull., 13 (1970) 451-461.

[9] W.D. Wallis, Magic Graphs. Birkhauser, New York, NY, USA (2001)

國科會補助專題研究計畫項下出席國際學術會議心得報告

日期:100 年 3 月 7 日

計畫編號	NSC 99-2115-M-029-001		
計畫名稱	圖形之標號、著色、覆蓋、分割、與數論、有限幾何之關聯性研究		
出國人員姓名	王道明	服務機構及職稱	東海大學數學系
會議時間	100 年 3 月 5 日至 100 年 3 月 6 日	會議地點	澳門
會議名稱	2011 2ND International Conference on Innovative Computing and Communication and 2011 Asia-Pacific Conference on Information Technology and Ocean Engineering		
發表論文題目	On Deficiency of Super Edge Magic Labelings for Complete Bipartite Graphs		

一、參加會議經過

I attended the ITOE 2011 conference held in Macau, China, in March 5th and 6th 2011. I arrived in the Regency Hotel around noon in March 5th, and registered with the conference.

I met many other scholars in the field of communications and computing. We exchange ideas about research and talked about advances most recently. There are 4 main keynote speakers invited in the conference, and one of them is the Professor Chang Chin Cheng from Feng Chia University and National Chung Cheng University, who is well known in the field of cryptography and number theory applications. There were scholars who came from other counties and regions such as Hong Kong, Macau, China, India, Japan etc.

The talks I have listened to were mostly around communications and computing fields, although there is another joint conference about Ocean Engineering. I presented in the 2nd day of the conference, which attracted attentions of many other scholars and we talked about co-work opportunities afterwards.

二、與會心得

This is a very successful conference, at which I met old friends and also made new friends in the research field, and listened to many nice talks.

三、考察參觀活動(無是項活動者略)

四、建議

In Taiwan, the research groups regarding theory and application parts in the fields of communications and computing usually have no as much interaction as foreign scholars. Hopefully NSC can promote such joint activities here, and it will benefit the research in Taiwan.

五、攜回資料名稱及內容

Conference Proceeding in CD format, which includes all papers presented.

六、其他

國科會補助計畫衍生研發成果推廣資料表

日期:2011/11/18

國科會補助計畫	計畫名稱: 圖形之標號、著色、覆蓋、分割、與數論、有限幾何之關聯性研究	
	計畫主持人: 王道明	
	計畫編號: 99-2115-M-029-001-	學門領域: 離散數學

無研發成果推廣資料

99 年度專題研究計畫研究成果彙整表

計畫主持人：王道明	計畫編號：99-2115-M-029-001-
計畫名稱：圖形之標號、著色、覆蓋、分割、與數論、有限幾何之關聯性研究	

成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	2	2	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	1	1	50%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	2	2	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	重要國際合作: 第二屆印度台灣離散數學會議(The 2nd India-Taiwan Conf. on Discrete Math.), 受邀至印度 Invited 演講’’Nowhere-Zero Constant Sum Flows’’ 並擔任分組討論主席 September 8-11, 2011, Coimbatore, India

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

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□未達成目標（請說明，以 100 字為限）

□實驗失敗

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說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：■已發表 □未發表之文稿 □撰寫中 □無

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3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

我們研究圖形零和邊標號問題(Zero-Sum Flow Problem), 決定所有正則圖(regular graph)的指標集(index set), 並且計算各類圖形之零和流指標(Zero-Sum Flow Index). 圖形零和邊標號問題(Zero-Sum Flow Problem)是一類網路流問題(Network Flow Problem), 對各類網路應用有理論及應用價值.