SOLUTIONS AND APPLICATIONS OF

HIGHER ORDER FIBONACCI SEQUENCES

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1. Fibonacci Sequences

We call a sequence of non-negative integers $a_1, a_2, \cdots\cdots, a_n, \cdots\cdots$ a Fibonacci sequence if it satisfies the recursion relation

$$a_n = a_{n-1} + a_{n-2} \tag{1}$$

The applications of the Fibonacci sequence have been discovered not only in the field of probability theory but also in many other branches.

As in the theory of sequences, we want to have a formula for finding the value of a_n directly in terms of n without having to compute the values of a_1, $a_2, \cdots\cdots a_{n-1}$. One of such formulas for finding a_n of (1) has been found to be (See [1],[5])

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \right] \tag{2}$$

In this paper we consider the modified Fibonacci sequences F_r which satisfy the generalized recursion relation

$$a_n = a_{n-1} + a_{n-r} \qquad r = 1, 2, 3, \cdots\cdots$$

For a positive integer r, we call F_r the r-order Fibonacci sequence (F_2 is the origenal Fibonacci sequence which satisfies (1)). A specific r-order Fibonacci sequence will be determined by a set of r initial conditions. For the purpose of convenience and without loss of generality for applications, we always set up the initial conditions to be

$$a_t = 0, \qquad t = 1, 2, \cdots\cdots, r-1$$

$$a_r = 1.$$

Now we write down the first twenty terms of F_1, F_2, F_3, F_4 and F_5 in the following table:

Order	F_1	F_2	F_3	F_4	F_5
Recursion Relation	$a_n=a_{n-1}+a_{n-1}$	$a_n=a_{n-1}+a_{n-2}$	$a_n=a_{n-1}+a_{n-3}$	$a_n=a_{n-1}+a_{n-4}$	$a_n=a_{n-1}+a_{n-5}$
Initial Conditions	$a_t=0, t<1$ $a_1=1$	$a_t=0, t<2$ $a_2=1$	$a_t=0, t<3$ $a_3=1$	$a_t=0, t<4$ $a_4=1$	$a_t=0, t<5$ $a_5=1$
a_1	1	0	0	0	0
a_2	2	1	0	0	0
a_3	4	1	1	0	0
a_4	8	2	1	1	0
a_5	16	3	1	1	1
a_6	32	5	2	1	1
a_7	64	8	3	1	1
a_8	128	13	4	2	1
a_9	256	21	6	3	1
a_{10}	512	34	9	4	2
a_{11}	1024	55	13	5	3
a_{12}	2048	89	19	7	4
a_{13}	4096	144	28	10	5
a_{14}	8192	233	41	14	6
a_{15}	16384	377	60	19	8
a_{16}	32768	610	88	26	11
a_{17}	65536	987	129	36	15
a_{18}	131072	1597	189	50	20
a_{19}	262144	2584	277	69	26
a_{20}	524288	4181	406	95	34

2. Solution from the Theory of Equations

It is easy to see that

$$a_n=2^{n-1} \tag{3}$$

is the general term of F_1 and that of F_2 has been given in the formula (2). Now let us try to find the general term a_n for F_3 which satisfies the recursion relation

$$a_n=a_{n-1}+a_{n-3} \tag{4}$$

To solve this, let us try a solution of the form

$a_n = cx^n$,

where c is a constant independent of n. This leads to a cubic equation

$$x^3 - x^2 - 1 = 0 \qquad (5)$$

We call (5) the auxiliary equation of (4). The three roots of (5) are

$$x_1 = \frac{1}{3} + P + Q,$$

$$x_2 = \frac{1}{3} + WP + W^2Q$$

$$x_3 = \frac{1}{3} + W^2P + WQ,$$

where $P = \sqrt[3]{\dfrac{29+3\sqrt{93}}{54}}$, $Q = \sqrt[3]{\dfrac{29-3\sqrt{93}}{54}}$, $W = \dfrac{-1+\sqrt{3}\,i}{2}$.

The general solution of (4), by the method for solving a homogeneous difference equation (See [8], p.40), is

$$a_n = c_1x_1^n + c_2x_2^n + c_3x_3^n,$$

and the initial conditions $a_1 = a_2 = 0$, $a_3 = 1$ give us a system of equations

$$c_1x_1 + c_2x_2 + c_3x_3 = 0,$$

$$c_1x_1^2 + c_2x_2^2 + c_3x_3^2 = 0,$$

$$c_1x_1^3 + c_2x_2^3 + c_3x_3^3 = 1.$$

Let

$$\triangle = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_3^3 \end{vmatrix}.$$

Then by the elementary symmetric functions in x_1, x_2, x_3, we have

$$\triangle = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_3^3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & \sum x_i \\ x_1^2 & x_2^2 & \sum x_i^2 \\ x_1^3 & x_2^3 & \sum x_i^3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & 1 \\ x_1^2 & x_2^2 & 1 \\ x_1^3 & x_2^3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & x_2 & 1 \\ x_1^2 & x_2^2 & 1 \\ x_1^3 - x_1^2 & x_2^3 - x_2^2 & (4-1) \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & 1 \\ x_1^2 & x_2^2 & 1 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= (x_2 - x_1)[3x_1x_2 + 1 - (x_2 + x_1)] = (x_2 - x_1)(3x_1x_2 + x_3)$$

$$= \frac{(x_2 - x_1)(3 + x_3^2)}{x_3}$$

Similarly, we can derive alternatively

$$\triangle = \frac{(x_3 - x_2)(3 + x_1^2)}{x_1} = \frac{(x_1 - x_3)(3 + x_2^2)}{x_2}$$

Hence

$$c_1 = \frac{\begin{vmatrix} x_2 & x_3 \\ x_2^2 & x_3^2 \end{vmatrix}}{\triangle} = \frac{1}{3 + x_1^2}, \qquad c_2 = \frac{\begin{vmatrix} x_3 & x_1 \\ x_3^2 & x_1^2 \end{vmatrix}}{\triangle} = \frac{1}{3 + x_2^2},$$

$$c_3 = \frac{\begin{vmatrix} x_1 & x_2 \\ x_1^2 & x_2^2 \end{vmatrix}}{\triangle} = \frac{1}{3 + x_3^2}$$

Thus we obtain a solution for (4),

$$a_n=\frac{x_1^n}{3+x_1^2}+\frac{x_2^n}{3+x_2^2}+\frac{x_3^n}{3+x_3^2} \quad\cdots\cdots (6)$$

We note that the solution (2) of F_2 can be written as in the form analogous to (6),

$$a_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$$

$$=\frac{x_1^n}{2+x_1}+\frac{x_2^n}{2+x_2} \quad\cdots\cdots (7)$$

where x_1 and x_2 are the roots of the auxiliary equation

$$x^2-x-1=0.$$

Similarly, the solution (3) of F_1 can be written as

$$a_2=2^{n-1}=\frac{x_1^n}{1+x_1^0}, \quad\cdots\cdots (8)$$

where x_1 is the root of the auxiliary equation $x-1-1=0$.

The solutions (6), (7) and (8) give us a suggestion to establish the following theorem about the general solution of the higher order Fibonacci sequences F_r.

Theorem 1. Let F_r $(r=1,2,3,\cdots\cdots,)$ be the Fibonacci sequences of order r specified by

$$\left.\begin{array}{ll} a_n=a_{n-1}+a_{n-r} & n>r, \\ a_n=1 & n=r, \\ a_n=0 & n<r. \end{array}\right\} \quad\cdots\cdots (9)$$

Then a solution of a_n is of the form

$$a_n=\sum_{i=1}^{r}\frac{x_i^n}{r+x_i^{r-1}}, \quad\cdots\cdots (10)$$

where x_i's are roots of the auxiliary equation $x^r-x^{r-1}-1=0$.

Proof: As a matter of fact, for a known positive integer r (not very large), the candiate-form (10) for the solutin of (9) can be found not difficultly by either the method given above or the generating function of (9) (See [8], pp.68-73), and also we can check that it is a solution of (9). However, the point is how to know that it is the required solution of (9) for every positive integer r.

Now for an arbitrary integer r, it is easy to see that the solution (10) satisfies (9) when $n>r$. Before we prove that a_n satisfies the initial conditions, it is necessary to note several facts from the theory of equations (See [7], pp.69-71).

If $x_1, x_2, \cdots\cdots, x_r$ are the distinct roots of the polynomial

$$f(x)=x^r+p_1x^{r-1}+p_2x^{r-2}+\cdots\cdots+p_r,$$

then, by Taylor's formula, there is an important identity established as

$$\frac{f'(x)}{f(x)}=\sum_{i=1}^{r}\frac{1}{x-x_i}. \quad\cdots\cdots (11)$$

Furthermore, if we write

$$S_k=\sum_{i=1}^{r}x_i^k \qquad k=1,2,\cdots\cdots r,$$

then the Newton's formulas

$$S_k+p_1S_{k-1}+p_2S_{k-2}+\cdots\cdots+p_{k-1}S_1+Kp_k=0 \quad\cdots\cdots (12)$$

hold for $k=1,2,\cdots\cdots\cdots,r$.

Now in the case of Theorem 1, let r be an arbitrary positive integer, then the polynomial function

$$f(x)=x^r-x^{r-1}-1 \qquad (13)$$

has all its roots distinct. First, we shall show that $a_r=1$ and $a_{r-1}=0$. By using the equation $x^r-x^{r-1}-1=0$ and the formula (11),

$$a_r=\sum_{i=1}^{r}\frac{x_i^r}{r+x_i^{r-1}}=\sum_{i=1}^{r}\frac{x_i^{r-1}+1}{r+x_i^{r-1}}=\sum_{i=1}^{r}\frac{x_i^{r-1}+r+1-r}{r+x_i^{r-1}}$$

$$=r+(1-r)\sum_{i=1}^{r}\frac{1}{r+x_i^{r-1}}=r+(1-r)\sum_{i=1}^{r}\frac{1}{r-\frac{1}{1-x_i}}$$

$$=r+(1-r)\sum_{i=1}^{r}\frac{1-x_i}{r(1-x_i-\frac{1}{r})}=r+(1-r)\left[1+\frac{1}{r^2}\sum_{i=1}^{r}\frac{1}{(1-\frac{1}{r}-x_i)}\right]$$

$$=r+(1-r)\left[1+\frac{1}{r^2}\frac{f'(1-\frac{1}{r})}{f(1-\frac{1}{r})}\right]$$

$$=r+(1-r)\left[1+\frac{1}{r^2}\frac{r(\frac{r-1}{r})^{r-1}-(r-1)(\frac{r-1}{r})^{r-2}}{(\frac{r-1}{r})^r-(\frac{r-1}{r})^{r-1}-1}\right]$$

$$=r+(1-r)\left[1+\frac{1}{r^2}\frac{\frac{(r-1)^{r-1}}{r^{r-2}}-\frac{(r-1)^{r-1}}{r^{r-2}}}{(\frac{r-1}{r})^r-(\frac{r-1}{r})^{r-1}-1}\right]$$

$$=r+(1-r)=1$$

In the proof above, we note that $\sum_{i=1}^{r}\frac{1}{r+x_i^{r-1}}=1$,

thus, by the equatiou $x^r-x^{r-1}-1=0$, we also have

$$a_{r-1}=\sum_{i=1}^{r}\frac{x_i^{r-1}}{r+x_i^{r-1}}=0.$$

In order to show that $a_t=0$, for $t=1,2,\cdots\cdots\cdots,(r-2)$, we observe that

$$p_1=-1,\qquad p_2=p_3=\cdots\cdots p_{r-1}=0,\qquad p_r=-1.$$

From (12), we have $S_k=1$ for $k=1,2,\cdots\cdots,(r-2)$.

For $j=0,1,2,\cdots\cdots,(r-2)$, let P(j) be the statement:

$$\sum_{i=1}^{r}\frac{x_i^{r+j}}{r+x_i^{r-1}}=1.$$

For $j>(r-2)$, let P(j) be any true statement.

Now, for $j=0,1,2,\cdots\cdots,(r-2)$,

$$\sum_{i=1}^{r}\frac{x_i^{r+j}}{r+x_i^{r-1}}=\sum_{i=1}^{r}x_i^{j+1}-r\sum_{i=1}^{r}\frac{x_i^{j+1}}{r+x_i^{r-1}}$$
$$=S_{j+1}-ra_{j+1}$$

We have shown that P(0) is true, But

$$\sum_{i=1}^{r}\frac{x_i^{r}}{r+x_i^{r-1}}=S_1-ra_1=1$$

This implies that $a_1=\sum_{i=1}^{r}\frac{x_i}{r+x_i^{r-1}}=0$. For j=1, by the equation $x^{r+1}-x^r-x=0$, we have

$$\sum_{i=1}^{r}\frac{x_i^{r+1}}{r+x_i^{r-1}}=\sum_{i=1}^{r}\frac{x_i^{r}}{r+x_i^{r-1}}+\sum_{i=1}^{r}\frac{x_i}{r+x_i^{r-1}}=1.$$

Hence P(1) is true. Also

$$\sum_{i=1}^{r}\frac{x_i^{r+1}}{r+x_i^{r-1}}=S_2-ra_2=1$$

This implies $a_2=\sum_{i=1}^{r}\frac{x_i^2}{r+x_i^{r-1}}=0$

Further, suppose that P(j) is true for some $j<(r-2)$, i.e.,

$$\sum_{i=1}^{r}\frac{x_i^{r+j}}{r+x_i^{r-1}}=S_{j+1}-ra_{j+1}=1$$

This implies that $a_{j+1}=\sum_{i=1}^{r}\frac{x_i^{j+1}}{r+x_i^{r-1}}=0$. Look at

$$\sum_{i=1}^{r}\frac{x_i^{r+j+1}}{r+x_i^{r-1}}=\sum_{i=1}^{r}\frac{x_i^{r+j}}{r+x_i^{r-1}}+\sum_{i=1}^{r}\frac{x_i^{j+1}}{r+x_i^{r-1}}=1,$$

we see that P(j+1) is true. By induction, P(j) is true for all non-negative integer j. Furthermore, in the process we have carried above, we also see that

$$a_1=a_2=\cdots\cdots=a_{r-2}=0$$

Therefore the proof of our theorem is complete.

3. *Solutions from the Theory of Probability*

In this section, we shall give F_r an alternative solution with the form

$$a_n=\sum_{m=1}^{[\frac{n}{r}]}\binom{n-m(r-1)-1}{m-1},$$

where $[\frac{n}{r}]$ is the greatest integer in $\frac{n}{r}$, and

$$\binom{n}{j}=\frac{n!}{(n-j)!\,j!}$$

Also, by convention, $a_n=0$ if $\frac{n}{r}<1$, and $0!=1$. The motivational reasons for considering such a solution are from the theory of probability. But the discussion of these reasons will be given in the next section.

Theorem 2. Let F_r $(r=1,2,3,\cdots\cdots\cdots)$ be the Fibonacci sequence of order r specified by

$$a_n=a_{n-1}+a_{n-r}, \qquad n=r+1,r+2,\cdots\cdots\cdots;$$
$$a_n=1, \qquad n=r;$$
$$a_n=0, \qquad n=1,2,\cdots\cdots\cdots,r-1.$$

Then

$$a_n=\sum_{m=1}^{[\frac{n}{r}]}\binom{n-m(r-1)-1}{m-1} \quad \cdots\cdots (14)$$

is a solution for F_r.

Proof: Let r be an arbitrary positive integer. By our convention, when $n<r$,

$$a_n=\sum_{m=1}^{[\frac{n}{r}]}\binom{n-m(r-1)-1}{m-1}=0;$$

when $n=r$,

$$a_n=\binom{r-(r-1)-1}{0}=\frac{0!}{0!}=1;$$

when $n>r$, we shall break the proof into two parts.

(a) If $n=rp$, p is an arbitrary positive integer, then

$$a_n+a_{n+r-1}=\sum_{m=1}^{p}\binom{rp-m(r-1)-1}{m-1}+\sum_{m=1}^{p}\binom{rp+r-1-m(r-1)-1}{m-1}$$

$$=\sum_{m=1}^{p-1}\binom{rp-m(r-1)-1}{m-1}+\binom{rp-p(r-1)-1}{p-1}+$$

$$\sum_{m=2}^{p}\binom{rp+r-m(r-1)-2}{m-1}+\binom{rp-1}{0}$$

$$=\sum_{m=1}^{p-1}\left[\binom{rp-m(r-1)-1}{m-1}+\binom{rp+r-(m+1)(r-1)-2}{m+1-1}\right]+$$

$$\binom{rp}{0}+\binom{p-1}{p-1}$$

$$=\sum_{m=1}^{p-1}\left[\binom{rp-m(r-1)-1}{m-1}+\binom{rp-m(r-1)-1}{m+1-1}\right]+\binom{rp}{0}+\binom{p}{p}$$

$$=\sum_{m=1}^{p-1}\binom{rp-m(r-1)}{m}+\binom{rp}{o}+\binom{p}{p}$$

$$=\sum_{m=2}^{p}\binom{rp-(m-1)(r-1)}{m-1}+\binom{rp}{0}+\binom{p}{p}$$

$$=\sum_{m=1}^{p+1}\binom{rp+r-m(r-1)-1}{m-1}=\sum_{m=1}^{[\frac{n+r}{r}]}\binom{n+r-m(r-1)-1}{m-1}=a_{n+r}$$

(b) If $n=rp+j$, where $j=1,2,\cdots\cdots,(r-1)$, then

$$a_n+a_{n+r-1}=\sum_{m=1}^{p}\binom{rp+j-m(r-1)-1}{m-1}+\sum_{m=1}^{p+1}\binom{rp+j+r-1-m(r-1)-1}{m-1}$$

$$=\sum_{m=1}^{p}\binom{rp+j-m(r-1)-1}{m-1}+\sum_{m=2}^{p+1}\binom{rp+j+r-1-m(r-1)-1}{m-1}+\binom{rp+j-1}{0}$$

$$=\sum_{m=1}^{p}\left[\binom{rp+j-m(r-1)-1}{m-1}+\binom{rp+j+r-1-(m+1)(r-1)-1}{m+1-1}\right]+\binom{rp+j-1}{0}$$

$$=\sum_{m=1}^{p}\binom{rp+j-m(r-1)}{m}+\binom{rp+j-1}{0}$$

$$=\sum_{m=2}^{p+1}\binom{rp+j-(m-1)(r-1)}{m-1}+\binom{rp+j-1}{0}$$

$$=\sum_{m=2}^{p+1}\binom{rp+j+r-m(r-1)-1}{m-1}+\binom{rp+j}{0}$$

$$=\sum_{m=1}^{p+1}\binom{n+r-m(r-1)-1}{m-1}$$

$$=a_{n+r}.$$

From (a) and (b), we see that

$$a_n+a_{n+r-1}=a_{n+r}$$

is true for all $n>r$. Hence we have concluded the proof of the theorem.

A consequence of Theorem 1 and Theorem 2 is a set of identities between those elements of probability theory and theory of equations which seem to be entirely unrelated, however, they are connected by the Fibonacci sequences. This is another point of this paper. We shall state it as the following corollary.

Corollary For any positive integer r, the relation

$$\sum_{m=1}^{[\frac{n}{r}]}\binom{n-m(r-1)-1}{m-1}=\sum_{i=1}^{r}\frac{x_i^{\,n}}{r+x_i^{\,r-1}}$$

holds for all positive integer n, where $[\frac{n}{r}]$ is the greatest integer in $\frac{n}{r}$, and x_i's are roots of $x^r-x^{r-1}-1=0$.

4. Applications

In this section, we shall give two problems in the theory of probability to explain the applications of the Fibonacci sequences.

Problem 1. We toss an unbiased coin n times so that the two elementary outcomes, heads and tails, are equally likely at each throw. Find the number of sequences of heads and tails:

(a) Of all possible cases.

(b) Which have no isolated outcomes.

(c) Which have their consecutive outcomes unchanged at least r times.

The question (a) can be easily seen in the theory of probability that $a_n=2^n$. However, we shall give an instructive answer so that it will act as a clue for solving (b) and (c)

Let H=head and T=tail. Suppose we have known a_n, then the (n+1)-th outcome is either H or T. Hence we have

$$a_{n+1}=2a_n=a_n+a_n$$

From Theorem 1, we have $a_n=2^{n-1}$. But the initial condition of (a) is $a_1=2$, and therefore $a_n=2^n$.

To solve the question (b), let us consider a sequence of length n+2. The tail portion with no isolated outcomes must be:

(i)………THH, ………HTT ;

or (ii)………HHH, ………TTT.

In (i) we allow sequences of length n followed by HH if the n-th outcome is T, and TT otherwise. In (ii) we allow sequences of length n+1 followed by H if the n-th and (n+1)-th outcomes are HH, and followed by T if preceded by TT. Since the sequences in (i) and (ii) behave as sequences with no isolated outcomoes of length n and n+1, respectively, and these are the only possible ways to get a_{n+2}. Thus we have

$$a_{n+2}=a_n+a_{n+1}.$$

Here the initial conditions are $a_1=0$, $a_1=2$, by Theorem 1, we have

$$a_n=\frac{2}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right].$$

As to the question (c), we note that it is a general case of (a) and (b). By the similar discussion, we have

$$a_{n+r}=a_n+a_{n+r-1}, \qquad n=1,2,\cdots\cdots;$$

$$a_1=a_2=\cdots\cdots=a_{r-1}=0;$$

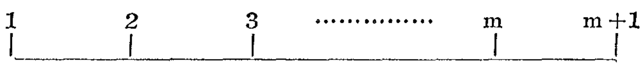
$$a_r=2.$$

Hence by Theorem 1 again,

$$a_n=2\sum_{i=1}^{r}\frac{x_i^{\,n}}{r+x_i^{\,r-1}},$$

where x_i's are roots of $x^r-x^{r-1}-1=0$.

Problem 2. On a long rectangular tray with partitioning dividers and cross-section



we place n balls in the m troughs between m+1 dividers. Find the ways for all possible m if the restriction is (See [6]):

(a) At least 1 ball in each trough.

(b) At least 2 balls in each trough.

(c) At least r balls in each trough.

To solve these questions, it is convenient to look at the trivial case that if we make no restriction on the number of balls in each trough, we are merely arranging n balls and m-1 dividers arbitarily. This can be done in

$$\frac{(n+m-1)!}{n!(m-1)!}=\binom{m+m-1}{m-1}$$

ways for a fixed m.

Now, in question (a), we have n−m balls and m-1 dividers to arrange, for which there are

$$\frac{(n-m+m-1)}{(n-m)!(m-1)!}=\binom{n-1}{m-1}$$

ways for each possible m. As m=1,2,……n, the total number of ways is

$$a_n=\sum_{m=1}^{n}\binom{n-1}{m-1}$$

We note that such a_n is the general term of the Fibonacci sequence F_1, thus by Theorem 2,

$$a_n=a_{n-1}+a_{n-1}.$$

In question (b), we have n-2m balls and m-1 dividers to arrange, thus for each possible m, there are

$$\binom{n-2m+m-1}{m-1}=\binom{n-m(2-1)-1}{m-1}=\binom{n-m-1}{m-1}$$

ways and therefore

$$a_n=\sum_{m=1}^{[\frac{n}{2}]}\binom{n-m-1}{m-1}.$$

By Theorem 2 again,

$$a_n=a_{n-1}+a_{n-2}.$$

The question (c) is the general case of (a) and (b). By a similar discussion, we have

$$a_n=\sum_{m=1}^{[\frac{n}{r}]}\binom{n-m(r-1)-1}{m-1},$$

and in this case the connection between a_n's is

$$a_n=a_{n-1}+a_{n-r}.$$

We note that both solutions of Problem 1 and Problem 2 satisfy the Fibonacci sequences. Thus there must be a similarity between them.

It is due to this similarity, in Theorem 2, we had the motivation to try the solution (14) for F_r.

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高階范氏數列之通解及其應用提要

解 萬 臣

數論中之范氏數列 $\{a_n\}$ 是由下列之遞歸式所定義之

$$F_2 \quad a_n=\begin{cases} 0, & n=1; \\ 1, & n=2; \\ a_{n-1}+a_{n-2}, & n=3,4,5 \end{cases}$$

F_2 之通項解

$$a_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$$

及其應用早被發現。因而廣義之 r－ 階范氏數列

$$F_r: \quad a_n=\begin{cases} 0, & n=1,2,\cdots,(r-1); \\ 1, & n=r; \\ a_{n-1}+a_{n-r}, & n=(r+1),(r+2),\cdots\cdots \end{cases}$$

之討論亦頗具價值。當$r>2$時，則稱F_r為高階范氏數列。

本文分別在第二、三節中提出並證明F_r之二種不同型類之通解：

$$a_n=\sum_{i=1}^{r}\frac{x_i{}^n}{r+x_i{}^{r-1}}, \qquad a_n=\sum_{m=1}^{\left[\frac{n}{r}\right]}\binom{n-m(r-1)-1}{m-1}。$$

上式中諸 x_i 為方程式 $x^r-x^{r-1}-1=0$ 之根，$\left[\frac{n}{r}\right]$ 為最大整數函數，及 $\binom{n}{m}=\frac{n!}{(n-m)!\,m!}$ 。
並且由此而得恒等式：

$$\sum_{i=1}^{r}\frac{x_i{}^n}{r+x_i{}^{r-1}}=\sum_{m=1}^{\left[\frac{n}{r}\right]}\binom{n-m(r-1)-1}{m-1}。$$

有關 F_r 之應用亦在第四節討論之。

※ 本著作之完成得國家科學委員會之補助。

SOLUTIONS AND APPLICATIONS OF HIGHER ORDER FIBONACCI SEQUENCES

by Wan-chen Hsieh

The Fibonacci seguence $\{a_n\}$ is defined by the recursion relation

$$a_n=\begin{cases}0, & n=1;\\ 1, & n=2;\\ a_{n-1}+a_{n-2}, & n=3,4,\cdots\end{cases}$$

The solution of a_n in terms of n has been found to be

$$a_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right],$$

and its applications have been discovered not only in the field of probability but also in many other branches.

In this paper, we consider the r-order Fibonacci sequences F_r which satisfy the generalized recursion relations

$$a_n=\begin{cases}0, & n=1,2,\cdots(r-1);\\ 1, & n=r;\\ a_{n-1}+a_{n-r}, & n=(r+1),(r+2),\cdots\end{cases}$$

We give and prove two entirely different types of solutions for F_r,

$$a_n=\sum_{i=1}^{r}\frac{x_i^n}{r+x_i^{r-1}}$$

and

$$a_n=\sum_{m=1}^{[\frac{n}{r}]}\binom{n-m(r-1)-1}{m-1},$$

in Section 2 and Section 3 respectively, where x_i's are roots of the equation $x^r-x^{r-1}-1=0$, $[\frac{n}{r}]$ is the greatest integer function, and

$$\binom{n}{m}=\frac{n!}{(n-m)!\,m!}.$$

As a consequence we get a set of identities

$$\sum_{i=1}^{r}\frac{x_i^n}{r+x_i^{r-1}}=\sum_{m=1}^{[\frac{n}{r}]}\binom{n-m(r-1)-1}{m-1}$$

for a given positive integer r.

The applications of F_r are discussed in the Section 4.