ON SOME METHODS OF SUMMABILITY

by

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1. Introduction

The standard definition, due to Cauchy, of convergence of an infinite series of real or complex numbers states that $\sum_{n=0}^{\infty} U_n$ converges to the finite value S if the sequence of partial sums $\{S_n\}$, where $S_n = \sum_{k=0}^{n} U_k$, converges to S. Many series are not convergent in this sense, so it was natural for other definitions of convergence to evolve after Cauchy's time. Series such as $\sum_{k=0}^{\infty}(-1)^k$ and $\sum_{k=0}^{\infty}(-1)^{k+1}k$ are Cauchy divergent, but in Section 3 we note that if one accepts a definition of convergence involving arithmetic means, we would assign the value $\frac{1}{2}$ to the series $\sum_{k=0}^{\infty}(-1)^k$, whereas if we accept a definition involving the limit of a function defined by a power series, we could assign the value $\frac{1}{4}$ to the series $\sum_{k=0}^{\infty}(-1)^{k+1}k$.

Historically all such definitions of convergence or "methods of summing series" are called summability methods. The words convergent and divergent will always be used with reference to the Cauchy definition. In Section 2, the regularity of summability methods will be dicussed, where regularity means that the summability method in question sums every convergent series to the value assigned it by the Cauchy definition. In Section 3, some of the standard methods of summability will be discussed, namely, Cesaro's method and Abel's method. Their applications to Fourier series and multiplication of series will be given in Section 4. Finally, Section 5 gives a short discussion of "reverse regularity" (a Tauberian theorem) in which one asks when summability of a series by method X to the value S implies convergence to the same value S.

The theorems, we shall introduce in this paper, are important in theory of summability. The main purpose of this paper is to give new proofs to these theorems.

2. Regularity

It is obvious that there are some natural reguirements for a useful method of summability. It should be simple in calculation, and it should be reasonably general, in the sense of being applicable to a good variety of important series, and finally it should be consistant or regular. We define this as follows:

Definition. A method of summability is said to be regular, if it sums every convergent series to its sum in Cauchy's sense.

Many methods of summability are special cases of the following procedure. For a given series

$\sum_{k=0}^{\infty}U_k$, let $S_n=\sum_{k=0}^{n}U_k$ and define values T_n by

$$T_n=\sum_{k=0}^{\infty}a_{nk}S_k \qquad (n=0,\ 1,\ 2,\cdots\cdots) \tag{1}$$

where the a_{nk} are real or complex and such that each such series is convergent, so the T_n are well defined. There is an important theorem, due to Silverman, Toeplitz and Schur, which states the necessary and sufficient conditions for regularity of such methods.

Theorem 1. The necessary and sufficient conditions that the sequence $\{T_n\}$ with T_n defined in (1) should be convergent to S as $n\to\infty$ whenever $\{S_k\}$ converges to S as $k\to\infty$ are:

(a) $\sum_{k=0}^{\infty}|a_{nk}|\leq M$ $(M>0)$ independent of n,

(b) $\lim_{n\to\infty} a_{nk}=0$ for every fixed k,

(c) $\sum_{k=0}^{\infty} a_{nk}=A_n\to 1$ as $n\to\infty$.

Proof: (i) Sufficiency

Since $\{S_n\}$ converges, we know that the S_n are bounded, so there exists $K>0$, such that $|S_n|<K$ for all $n=0,\ 1,\ 2,\cdots\cdots$. Combining this with condition (a) we see that all of the infinite series of (1) are absolutely convergent so T_n is well defined.

Let $S_k-S=h_k$, and note that

$$\left|T_n-S\right|=\left|T_n-A_nS+A_nS-S\right|=\left|\sum_{k=0}^{\infty}a_{nk}h_k+S(A_n-1)\right|,$$

so $$\left|T_n-S\right|\leq\left|\sum_{k=0}^{\infty}a_{nk}h_k\right|+\left|S\right|\left|A_n-1\right|.$$

Let arbitrary $h>0$ be given. Now $S_k-S=h_k\to 0$ as $k\to\infty$, so there exists a number $K_1>0$ such that $|h_k|<K_1$ for all $k=0,1,2,\cdots\cdots$, and there exists an integer $P>0$ such that $|h_k|<h/3M$ for all $k>P$. Let $K_2>0$ be such that $|S|<K_2$, then we can use (a) and obtain

$$\left|T_n-S\right|\leq\left|\sum_{k=0}^{p}a_{nk}h_k\right|+\left|\sum_{k=p+1}^{\infty}a_{nk}h_k\right|+\left|S\right|\left|A_n-1\right|<K_1\sum_{k=0}^{p}\left|a_{nk}\right|+M(h/3M)+K_2\left|A_n-1\right|.$$

Condition (b) implies that for each fixed k, there exists an integer $N(k)>0$ such that

$|a_{nk}|<h/[3K_1(P+1)]$ for all $n>N(k)$,

so let $Q=\max_{k=0,1,\cdots\cdots,p} N(k)$, then $\sum_{k=0}^{p}|a_{nk}|<(P+1)h/[3K_1(P+1)]=h/3K_1$ for all $n>Q$. Condition (c) implies that there exists an integer $R>0$, such that

$|A_n-1|<h/3K_2$ for all $n>R$.

Let $N=\max(Q,R)$, then we see that for all $n>N$,

$|T_n-S|<K_1(h/3K_1)+M(h/3M)+K_2(h/3K_2)=h$,

so the conditions are sufficient.

(ii) Necessity

Putting $S_p=1$ and $S_k=0$ for $k\neq p$, we have $\lim_{k\to\infty} S_k=0$,

then $\lim_{n\to\infty} T_n=\lim_{n\to\infty}\sum_{k=0}^{\infty}a_{nk}S_k=\lim_{n\to\infty} a_{np}=0$ for all p.

Hence condition (b) must be satisfied.

Similarly putting $S_k=1$ for every k, we have $\lim_{k\to\infty} S_k=1$, and hence

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} S_k = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \lim_{n\to\infty} A_n = 1.$$

Therefore, condition (c) is a necessary condition.

The last and also the hard part of the proof is to show that condition (a) is necessary.

Assume that one of the series of (a), say $\sum_{k=0}^{\infty} |a_{pk}|$, is divergent. Let a_{pN} be the first term in this series such that $a_{pN} \neq 0$,
and let

$$S_k = \mathrm{sgn}(a_{pk}) / \sum_{m=N}^{k} |a_{pm}| \qquad (k \geq N)$$

where $\mathrm{sgn}(Z) = |Z|/Z$ if $Z \neq 0$,
$= 0$ if $Z = 0$.

Now in the series

$$\left| T_p \right| = \left| \sum_{k=0}^{\infty} a_{pk} S_k \right| = \sum_{k=N}^{\infty} \left(\left| a_{pk} \right| / \sum_{m=N}^{k} \left| a_{pm} \right| \right),$$

if we put $\sum_{m=0}^{k} \left| a_{pm} \right| = \sum_{m=N}^{k} \left| a_{pm} \right| = d_k$, and $\left| a_{pk} \right| = b_k = d_k - d_{k-1}$,

then $\left| T_p \right| = \sum_{n=N}^{\infty} (b_n/d_n)$, where d_n, b_n are real, $(b_n/d_n) \geq 0$, and d_n is increasing to ∞. Let

$$L_n = \sum_{k=N}^{n} (b_k/d_k).$$

Then for $m > n \geq N$,

$$\begin{aligned} L_m - L_n &= (b_{n+1}/d_{n+1}) + (b_{n+2}/d_{n+2}) + \cdots\cdots + (b_m/d_m) \\ &= [(d_{n+1} - d_n)/d_{n+1}] + [(d_{n+2} - d_{n+1})/d_{n+2}] + \cdots\cdots + [(d_m - d_{m-1})/d_m] \\ &\geq (d_{n+1} - d_n + d_{n+2} - d_{n+1} + \cdots\cdots + d_m - d_{m-1})/d_m \\ &= (d_m - d_n)/d_m \\ &\geq 1 - (d_n/d_m). \end{aligned}$$

Keep n fixed and let $m \to \infty$, then the right side $\to 1$, so the Cauchy criterion for convergence is violated. Hence the series

$$\left| T_p \right| = \sum_{n=N}^{\infty} (b_n/d_n)$$

is divergent. Note that $S_k \to 0$ as $k \to \infty$, and yet $|T_p|$ is divergent, so this method could not be regular.

We have thus shown that each

$$H_n = \sum_{k=0}^{\infty} \left| a_{nk} \right|$$

is finite, for $n = 0, 1, 2, \cdots\cdots$. Now assume that H_n's are unbounded.
Let k_1 be an arbitrary positive integer. Condition (b) says that for each k there is a positive integer N(k) such that

$$|a_{nk}| < (1/k_1) \qquad \text{for all } n > N(k),$$

so if we define $N(k_1) = max\ \{N(0), N(1), N(2), \cdots, N(k_1 - 1)\}$, then

$$\sum_{k=0}^{k_1 - 1} \left| a_{nk} \right| < k_1 (1/k_1) = 1 \qquad \text{for all } n > N(k_1).$$

Now the numbers $H_{N(k_1)+1}$, $H_{N(k_1)+2}, \cdots$ are unbounded, so we can choose an integer $m_1 > N(k_1)$ such that simultaneously

$$\sum_{k=0}^{k_1-1} \left| a_{m_1k} \right| < 1 \qquad \text{and} \qquad H_{m_1} > 1^2+2.$$

Next let k_2 be an integer such that $k_2 > k_1$ and write

$$H_{m_1} = \sum_{k=0}^{k_1-1} \left| a_{m_1k} \right| + \sum_{k=k_1}^{k_2-1} \left| a_{m_1k} \right| + \sum_{k=k_2}^{\infty} \left| a_{m_1k} \right|.$$

Since H_{m_1} is convergent, we can choose $k_2 > k_1$, such that

$$\sum_{k=k_2}^{\infty} \left| a_{m_1k} \right| < 1$$

then

$$1+1+\sum_{k=K_1}^{k_2-1} \left| a_{m_1k} \right| > \sum_{k=0}^{k_1-1} \left| a_{m_1k} \right| + \sum_{k=k_1}^{k_2-1} \left| a_{m_1k} \right| + \sum_{k=k_2}^{\infty} \left| a_{m_1k} \right|$$
$$> 1^2+2,$$

so

$$\sum_{k=k_1}^{k_2-1} \left| a_{m_1k} \right| > 1^2.$$

Again, condition (b) says that there exists

$$N(k_2) = max\ \{N(0),\ N(1),\ N(2),\ \cdots,\ N(k_2-1)\}$$

such that

$$\sum_{k=0}^{k_2-1} \left| a_{nk} \right| < 1 \qquad \text{for all } n > N(k_2),$$

and the numbers $H_{N(k_2)+1}$, $H_{N(k_2)+2}, \cdots$ are unbounded. So, we can choose an integer $m_2 > N(k_2)$, such that simultaneously

$$\sum_{k=0}^{k_2-1} \left| a_{m_2k} \right| < 1 \qquad \text{and} \qquad H_{m_2} > 2^2+2.$$

Again H_{m_2} is convergent, so we choose $k_3 > k_2$, such that

$$\sum_{k=k_3}^{\infty} \left| a_{m_2k} \right| < 1$$

and get

$$1+1+\sum_{k=k_2}^{k_3-1} \left| a_{m_2k} \right| > \sum_{k=0}^{k_2-1} \left| a_{m_2k} \right| + \sum_{k=k_2}^{k_3-1} \left| a_{m_2k} \right| + \sum_{k=k_3}^{\infty} \left| a_{m_2k} \right|$$
$$> 2^2+2,$$

so

$$\sum_{k=k_2}^{k_3-1} \left| a_{m_2k} \right| > 2^2.$$

Continuing this process, we see that there exist positive integers k_r, m_r *and* k_{r+1} such that

$$\sum_{k=0}^{k_r-1} \left| a_{m_rk} \right| < 1, \qquad \sum_{k=k_r}^{k_{r+1}-1} \left| a_{m_rk} \right| > r^2, \qquad \text{and} \sum_{k=k_{r+1}}^{\infty} \left| a_{m_rk} \right| < 1.$$

Now we define $\{S_k\}$ by

$$S_k = 0 \qquad k < k_1$$
$$= (1/r)sgn(a_{m_rk}) \quad k_r \leq k < k_{r+1}, \text{ for } r=1,\ 2,\ 3, \cdots,$$

then as $k \to \infty$, $r \to \infty$ *so* $S_k \to 0$. Thus

$$T_{m_r}=\sum_{k=0}^{\infty}a_{m_rk}S_k=\sum_{k=k_1}^{k_2-1}\left(\left|a_{m_1k}\right|/a_{m_1k}\right)a_{m_rk}+\sum_{k=k_2}^{k_3-1}(1/2)\left(\left|a_{m_2k}\right|/a_{m_2k}\right)a_{m_rk}$$
$$+\sum_{k=k_3}^{k_4-1}(1/3)\left(\left|a_{m_3k}\right|/a_{m_3k}\right)a_{m_rk}+\cdots+\sum_{k=k_{r-1}}^{k_r-1}\left(1/(r-1)\right)\left(\left|a_{m_{r-1}k}\right|/a_{m_{r-1}k}\right)a_{m_rk}$$
$$+\sum_{k=k_r}^{k_{r+1}-1}(1/r)\left(\left|a_{m_rk}\right|/a_{m_rk}\right)a_{m_rk}+\sum_{k=k_{r+1}}^{k_{r+2}-1}\left(1/(r+1)\right)\left(\left|a_{m_{r+1}k}\right|/a_{m_{r+1}k}\right)a_{m_rk}$$
$$+\cdots,$$

so

$$\left|T_{m_r}\right|\geqq\sum_{k=k_r}^{k_{r+1}-1}(1/r)\left|a_{m_rk}\right|-\left|\sum_{k=k_1}^{k_2-1}\left(\left|a_{m_1k}\right|/a_{m_1k}\right)a_{m_rk}\right|-\left|\sum_{k=k_2}^{k_3-1}(1/2)\left(\left|a_{m_2k}\right|/a_{m_2k}\right)a_{m_rk}\right|$$
$$-\cdots-\left|\sum_{k=k_{r-1}}^{k_r-1}\left(1/(r-1)\right)\left(\left|a_{m_{r-1}k}\right|/a_{m_{r-1}k}\right)a_{m_rk}\right|$$
$$-\left|\sum_{k=k_{r+1}}^{k_{r+2}-1}\left(1/(r+1)\right)\left(\left|a_{m_{r+1}k}\right|/a_{m_{r+1}k}\right)a_{m_rk}\right|-\cdots$$
$$\geq(1/r)\sum_{k=k_r}^{k_{r+1}-1}\left|a_{m_rk}\right|-\sum_{k=k_1}^{k_2-1}\left|a_{m_rk}\right|-(1/2)\sum_{k=k_2}^{k_3-1}\left|a_{m_rk}\right|-\cdots$$
$$-\left(1/(r-1)\right)\sum_{k=k_{r-1}}^{k_r-1}\left|a_{m_rk}\right|-\left(1/(r+1)\right)\sum_{k=k_{r+1}}^{k_{r+2}-1}\left|a_{m_rk}\right|-\cdots$$
$$>r^2(1/r)-\sum_{k=k_1}^{k_r-1}\left|a_{m_rk}\right|-\sum_{k=k_{r+1}}^{\infty}\left|a_{m_rk}\right|,$$

hence $|T_{m_r}|>r-2$.

Now let $r\to\infty$, $m\to\infty$, then $m_r\to\infty$, and hence $|T_{m_r}|\to 0$ while $S_k\to 0$, so this method cannot be regular. This completes the proof of the necessity of condition (a).

3. Special methods of summability

It this section we shall introduce two methods of summability which are rather simple and useful. Regularity will follow from Theorem 1. Some examples will be given to show their effectiveness.

For convenience, we use the notations of partial sums as follows:

$$S_n=U_0+U_1+U_2+\cdots\cdots+U_n,$$
$$S_n^1=S_0+S_1+S_2+\cdots\cdots+S_n,$$
$$S_n^2=S_0^1+S_1^1+S_2^1+\cdots\cdots+S_n^1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$S_n^p=S_0^{p-1}+S_1^{p-1}+S_2^{p-1}+\cdots\cdots+S_n^{p-1}.$$

Definition (Cesaro summability)

A series $\sum_{K=0}^{\infty}U_k$ is summable (C,p) $(p=1,\ 2,\ 3,\cdots\cdots)$

to the sum Y iff

$$\lim_{n\to\infty}S_n^p/\binom{n+p}{p}=Y\quad\left(\binom{n+p}{p}=(n+p)!/(n!\ p!)\right)$$

and we write $\sum_{k=0}^{\infty}U_k=Y\ (C,p)$.

For example, $(C,1)$ would be

$$\lim_{n\to\infty} S_n^1/\binom{n+1}{1}=\lim_{n\to\infty}(S_0+S_1+\cdots\cdots+S_n)/(n+1).$$

Note that $(C,0)$ is the ordinary Cauchy convergence.

Theorem 2. Summability (C,p) is regular for all positive integer p.

Proof: We will use mathematical induction and Theorem 1 to prove this result.

First we look at $(C,1)$ and put

$$T_n^1=S_n^1/(n+1)=1/(n+1)\sum_{K=0}^{n}S_k.$$

This is a particular case of transformation (1) with $a_{nk}=1/(n+1)$ for $k\leq n$, and $a_{nk}=0$ for all $k>n$.

Thus the matrix (a_{ij}) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdot & \cdot & \cdot \\ 1/3 & 1/3 & 1/3 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{2}$$

It is obvious that

$$\sum_{K=0}^{\infty}\left|a_{nk}\right|=\sum_{K=0}^{\infty}a_{nk}=1 \qquad \text{for all n,}$$

$$\lim_{n\to\infty}a_{nk}=\lim_{n\to\infty}1/(n+1)=0 \qquad \text{for every fixed k.}$$

Hence all conditions of Theorem 1 are satisfied, and so the theorem is true for $(C,1)$.

Suppose that it is true for (C,m), then

$$\lim_{n\to\infty}T_n^m=\lim_{n\to\infty}S_n^m/\binom{n+m}{m}=S$$

whenever $\lim_{k\to\infty}S_k=S$.

Now look at $(C,m+1)$, and

$$T_n^{m+1}=S_n^{m+1}/\binom{n+m+1}{m+1}=\sum_{k=0}^{n}S_k^m/\binom{n+m+1}{m+1}=1/\binom{n+m+1}{m+1}\sum_{k=0}^{n}\binom{m+k}{m}\ S_k^m/\binom{m+k}{m}$$

$$=\sum_{k=0}^{n}\binom{m+k}{m}T_k^m/\binom{n+m+1}{m+1}=\sum_{k=0}^{\infty}b_{nk}\,T_k^m,$$

where $b_{nk}=\binom{m+k}{m}/\binom{n+m+1}{m+1}$ for $k\leq n$, $b_{nk}=0$ for $k>n$.

By induction hypotheses, $T_k^m\to S$ as $k\to\infty$, hence from Theorem 1, it is sufficient to prove that the matrix (b_{nk}) satisfies the conditions (a), (b) and (c).

Obviously,

$$\lim_{n\to\infty}b_{nk}=\lim_{n\to\infty}\binom{m+k}{m}/\binom{n+m+1}{m+1}=0$$

for any fixed k and m. Hence (b) is satisfied. Since $b_{nk}\geq 0$, if we can prove that the condition (c) is satisfied, then (a) follows immediately. Now we show that

$$\sum_{K=0}^{\infty}b_{nk}=\sum_{K=0}^{n}b_{nk}=\sum_{k=0}^{n}\binom{m+k}{m}/\binom{n+m+1}{m+1}=1 \tag{3}$$

for all n, by induction.

Note that

$$b_{00}=\binom{m}{m}/\binom{m+1}{m+1}=1 \quad \text{and} \quad b_{10}+b_{11}=\left[\binom{m}{m}+\binom{m+1}{m}\right]/\binom{m+2}{m+1}=1,$$

so it is true for n=0 and $n=1$. Suppose it is true for $n=p$, i.e.,

$$\sum_{K=0}^{p}\binom{m+k}{m}/\binom{p+m+1}{m+1}=\left[1/\binom{p+m+1}{m+1}\right]\left[\binom{m}{m}+\binom{m+1}{m}+\cdots\cdots\binom{m+p}{m}\right]=1$$

or $$\binom{m}{m}+\binom{m+1}{m}+\cdots\cdots\binom{m+p}{m}=\binom{p+m+1}{m+1}.$$

To complete this induction. there remains to prove that

$$\sum_{k=0}^{p+1}\binom{m+k}{m}/\binom{p+m+2}{m+1}=1.$$

But

$$\binom{m}{m}+\binom{m+1}{m}+\cdots\cdots\binom{m+p}{m}+\binom{m+p+1}{m}=\binom{p+m+1}{m+1}+\binom{m+p+1}{m}=\binom{p+m+2}{m+1}$$

so the result follows. The last equality is known as Pascal's triangle rule, or by the formula of Gamma functions that

$$(p+m+2)/\ (p+1)\ (m+2)+\ (p+m+2)/\ (p+2)\ (m+1)$$
$$=\ (p+m+3)/\ (p+2)\ (m+2).$$

Therefore (3) is true for all n and condition (c) is satisfied.

The induction gives the final conclusion that

$$\lim_{n\to\infty} T^m_n=\lim_{n\to\infty} S^m_n/\binom{n+m}{m}=S \qquad \text{for all integer } m.$$

This completes the proof of our theorem.

We could have noted that

$$a_{nk}=\binom{n-k+p-1}{n-k}/\binom{n+p}{n}$$

for (C,p) summability and conditions (a), (b) and (c) of Theorem 1 are satisfied directly from this form, but the proof given above also shows that if a series $\sum_{k=0}^{\infty}U_k$ is summable to Y (C,p), then $\sum_{k=0}^{\infty}U_k$ is also summable to $Y(C,p+1)$. The converse is not true, and this will become apparent by studying some examples.

Example 1. $\sum_{k=0}^{\infty}(-1)^k=\frac{1}{2}$ (C,p) for all positive integer p.

We need only to show this for $(C,1)$.

$S_n=1,\ 0,\ 1,\ 0\cdots\cdots \qquad n=0,\ 1,\ 2,\cdots\cdots$

$$S^1_n/(n+1)=[1/(n+1)]\sum_{k=1}^{n}S_k=(n+2)/(2n+2) \qquad \text{if n is even,}$$
$$=\tfrac{1}{2} \qquad \text{if n is odd,}$$

so

$$\lim_{n\to\infty} S^1_n/(n+1)=\tfrac{1}{2}\ \ (C,1).$$

Example 2. $\sum_{k=0}^{\infty}(-1)^{k+1}k=\frac{1}{4}$ (C,2), but it is not summable $(C,1)$.

Proof:

$S_0=0,\ S_1=1,\ S_2=-1,\ S_3=2,\ S_4=-2,\cdots\cdots,$

so

$S_n^1/(n+1)=0$ if n is even,

$=\frac{1}{2}$ if n is odd.

Hence $\sum_{k=0}^{\infty}(-1)^{k+1}k$ is not summable $(C,1)$.

Now look at $(C,2)$,

$$S_n^2/\binom{n+2}{2}=\sum_{k=0}^{n} S_k^1/\tfrac{1}{2}(n+1)(n+2)$$

$$=(n+2)(n)/4(n+1)(n+2) \qquad \text{if n is even,}$$

$$=(n+3)(n+1)/4(n+1)(n+2) \qquad \text{if n is odd,}$$

so

$$\lim_{n\to\infty} S_n^2/\binom{n+2}{2}=\tfrac{1}{4} \quad (C,2).$$

Next we want to just mention briefly the Holder summability (H,p) which is closely related to (C,p).

Let

$$H_n=U_0+U_1+U_2+\cdots\cdots+U_n$$

$$H_n^1=(H_0+H_1+H_2+\cdots\cdots+H_n)/(n+1)$$

$$H_n^2=(H_0^1+H_1^1+H_2^1+\cdots\cdots+H_n^1)/(n+1)$$

....

$$H_n^p=(H_0^{p-1}+H_1^{p-1}+H_2^{p-1}+\cdots\cdots+H_n^{p-1})/(n+1).$$

For any positive integer p, the Holder summability (H,p) is defined as follow: if

$$\lim_{n\to\infty} H_n^p=Y$$

we say $\sum_{k=0}^{\infty}U_k$ is Holder summable of order p to Y, and write $\sum_{k=0}^{\infty}U_k=Y\ (H,p)$.

We note that $(C,1)=(H,1)$ but will make no further use of Holder summalbiity in this paper.

Definition. (Abel summability)

Given a series $\sum_{k=0}^{\infty}U_k$, suppose that $\sum_{k=0}^{\infty}U_kx^k$ is convergent for $0\leqq x<1$. If we let

$$f(x)=\sum_{k=0}^{\infty}U_kx^k$$

then the series $\sum_{k=0}^{\infty}U_k$ is called Abel summable to the sum Y iff

$$\lim_{x\to1^-} f(x)=\lim_{x\to1^-}\sum_{k=0}^{\infty}U_kx^k=Y$$

and we write $\sum_{k=0}^{\infty}U_k=Y\ (A)$.

Theorem 3. Abel summability is regular.

Proof: We could give the standard direct proof of this result, but we choose to use our Theorem 1 instead. Assume that $S_k\to S$, *as* $k\to\infty$. Let us note that

$$T(x)=\sum_{k=0}^{\infty}U_kx^k=\sum_{k=0}^{\infty}x^k(1-x)S_k \qquad (0\leq x<1)$$

This follows from

$$\sum_{k=0}^{n} U_k x^k = \sum_{k=0}^{n} (S_k - S_{k-1}) x^k = (1-x) \sum_{k=0}^{n-1} S_k x^k + S_n x^n .$$

Let $\{b_n\}$ be an arbitrary sequence such that $0 \le b_n < 1$ and $b_n \to 1$ as $n \to \infty$. Then $a_{nk} = (b_n)^k (1-b_n)$ in (1).

The conditions of Theorem 1 are verified as follows:

Fix $n = 0, 1, 2, \cdots\cdots$, then

$$\sum_{k=0}^{\infty} \left| a_{nk} \right| = \sum_{k=0}^{\infty} a_{nk} = \sum_{k=0}^{\infty} (b_n)^k (1-b_n) = (1-b_n)/(1-b_n) = 1$$

so (a) and (c) are satisfied. Now fix k ($k = 0, 1, 2, \cdots\cdots$), then $b_n \to 1$ an $n \to \infty$, so $a_{nk} \to 0$ as $n \to \infty$, hence the condition (b) is saisfied.

Finally we know that $\lim_{x \to 1^-} f(x)$ exists iff $\lim_{n \to \infty} f(b_n)$ exists for every sequence $\{b_n\} \to 1$, so Abel summability is regular.

We have pointed out that the "strength" of methods (C,p) increases with p. The following theorem and examples will show that the summability (A) is stronger than any of them.

Theorem 4. If $\sum_{k=0}^{\infty} U_k = Y$ (C,p) for some p, then $\sum_{k=0}^{\infty} U_k = Y$ (A).

Proof: If we repeat the process at the beginning of the proof of Theorem 3, we see that

$$f(x) = \sum_{k=0}^{\infty} U_k x^k = (1-x) \sum_{k=0}^{\infty} S_k x^k = (1-x)^2 \sum_{k=0}^{\infty} S_k^1 x^k$$

$$= \cdots\cdots = (1-x)^p \sum_{k=0}^{\infty} S_k^{p-1} x^k = (1-x)^{p+1} \sum_{k=0}^{\infty} S_k^p x^k, \text{ where } 0 \le x < 1.$$

Our hypothesis says that

$$\lim_{n \to \infty} S_n^p / \binom{n+p}{p} = Y$$

hence for every $h > 0$, there exists an integer $N > 0$, such that $n > N$ implies

$$| S_n^p / \binom{n+p}{p} - Y | < h$$

Let us write

$$\sum_{k=0}^{\infty} S_k^p x^k = \sum_{k=0}^{N} S_k^p x^k + \sum_{k=N+1}^{\infty} S_k^p x^k$$

$$= \sum_{k=0}^{N} S_k^p x^k + \sum_{k=N+1}^{\infty} [S_k^p / \binom{k+p}{p} - Y] \binom{k+p}{p} x^k + Y [\sum_{k=0}^{\infty} \binom{k+p}{p} x^k - \sum_{k=0}^{N} \binom{k+p}{p} x^k].$$

Now we know that

$$\sum_{k=0}^{\infty} \binom{k+p}{p} x^k = 1/(1-x)^{p+1} \qquad (0 \le x < 1)$$

so

$$(1-x)^{p+1} \sum_{k=0}^{\infty} S_k^p x^k - Y = (1-x)^{p+1} \sum_{k=0}^{N} S_k^p x^k + (1-x)^{p+1} \sum_{k=N+1}^{\infty} [S_k^p / \binom{k+p}{p} - Y] \binom{k+p}{p} x^k$$

$$- Y (1-x)^{p+1} \sum_{k=0}^{N} \binom{k+p}{p} x^k,$$

hence

$$| f(x) - Y | = | (1-x)^{p+1} \sum_{k=0}^{\infty} S_k^p x^k - Y |$$

$$\leq (1-x)^{p+1}\left|\sum_{k=0}^{N} S_k^p x^k\right| + h(1-x)^{p+1}\sum_{k=0}^{\infty}\binom{k+p}{p}x^k + |Y|(1-x)^{p+1}\left|\sum_{k=0}^{N}\binom{k+p}{p}x^k\right|$$

$$\leq (1-x)^{p+1}\sum_{k=0}^{N}\left|S_k^p\right| + h + \left|Y\right|(1-x)^{p+1}\sum_{k=0}^{N}\binom{k+p}{p}$$

for all x in $0 \leq x < 1$. This shows that as $x \to 1^-$, $f(x) \to Y$ so the theorem is proved.

Let us look at our previous examples 1 and 2 in terms of Abel summability.

Example 1: $\sum_{k=0}^{\infty}(-1)^k$. We know that

$$1/(1+x) = \sum_{k=0}^{\infty}(-1)^k x^k \qquad (0 \leq x < 1)$$

so as $x \to 1^-$ we see that.

$$\sum_{k=0}^{\infty}(-1)^k = \tfrac{1}{2}\ (A).$$

Example 2: $\sum_{k=0}^{\infty}(-1)^{k+1}k$. we know that

$$\sum_{k=0}^{\infty}(-1)^{k+1}kx^k = x/(1+x)^2 \qquad (0 \leq x < 1)$$

so as $x \to 1^-$ we see that

$$\sum_{k=0}^{\infty}(-1)^{k+1}k = \tfrac{1}{4}\ (A).$$

These examples merely illustrate Theorem 4. Now for an example showing a series which is (A) summable but is not (C,p) summable for any p.

Example 3: If $f(x) = e^{1/(1+x)}$ and $U_k = f^{(k)}(0)/k!$

then

$$\sum_{k=0}^{\infty} U_k = e^{\frac{1}{2}}(A).$$

Note that the $f(x)$ is well defined in $|x|<1$. Assume that

$$\sum_{k=0}^{\infty} U_k = Y\ (C,p) \text{ for some p.}$$

Now

$$U_k = S_k^p - S_{k-1}^p - S_{k-1}^{p-1} - \cdots\cdots - S_{k-1}^1 - S_{k-1},$$

$\lim_{n\to\infty} S_n^p/\binom{n+p}{p} = Y$, and $\lim_{n\to\infty} S_n^m/\binom{n+p}{p} = 0$ if $m<p$, so there exists a constant $M>0$ such that

$$|U_k| \leq M\binom{k+p}{p} \text{ for all } k=0,\ 1,\ 2, \cdots\cdots\cdots\cdots.$$

Hence

$$f(x) = \sum_{k=0}^{\infty} U_k x^k \leq M\sum_{k=0}^{\infty}\binom{k+p}{p}|x|^k = M/(1-|x|)^{p+1}$$

for $|x|<1$. But we know that

$$(1-|x|)^{p+1}f(x) = (1-|x|)^{p+1}e^{1/(1+x)} \to \infty \text{ as } x \to 1^-.$$

so this contradiction completes our proof.

There are other methods of summability which are closely related to Abel summability. We shall mention, but not use, the general Abel summability method denoted by (A, b).

Let $\{b_n\}$ be a sequence such that

$0 \leq b_0 \leq b_1 \leq b_2 \leq \cdots\cdots \leq b_k \leq \cdots\cdots$

with $\lim_{k\to\infty} b_k = \infty$, and let

$$f(x) = \sum_{k=0}^{\infty} U_k e^{-b_k x} \quad \text{be convergent for all } x>0$$

If $\lim_{x\to 0^+} f(x) = Y$, then we define

$$\sum_{k=0}^{\infty} U_k = Y \ (A, b)$$

As a special case, when $b_k = k$ and $e^{-x} = y$, we have

$$\lim_{x\to 0^+} \sum_{k=0}^{\infty} U_k e^{-kx} = \lim_{y\to 1^-} \sum_{k=0}^{\infty} U_k y^k = Y$$

which is just the summability (A).

4. Applications

In this section we give some of the standard applications of summability to Fourier series, and multiplication of series.

(i) Fourier series.

In the theory of Fourier series, it is well-known that the Fourier series of a continuous function need not converge to the value of the function at every point in the Cauchy sense. However this situation can be remedied by using $(C, 1)$ summability. This famous result is due to Fejer.

Theorem 5. If $f(x)$ is real valued, continuous, and periodic of period 2π for all real x, then the Fourier series

$$\sum_{n=-\infty}^{+\infty} C_n e^{inx}, \text{ with } C_n = {}^1/(2\pi) \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

is uniformly $(C,1)$ summable to $f(x)$ for all x.

Proof: Let

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx} \qquad \text{and} \qquad K_n(x) = [{}^1/(n+1)] \sum_{m=0}^{n} D_m(x)$$

for $n = 0, 1, 2, \cdots\cdots$ If we multiply $D_n(x)$ *by* e^{ix}, then subtract $D_n(x)$ we obtain

$$(e^{ix}-1)D_n(x) = e^{i(n+1)x} - e^{-inx},$$

so

$$D_n(x) = (e^{i(n+1)x} - e^{-inx})/(e^{ix}-1).$$

Also

$$\begin{aligned} K_n(x) &= [{}^1/(n+1)] \sum_{m=0}^{n} D_m(x) = [{}^1/(n+1)(e^{ix}-1)](e^{ix} \sum_{m=0}^{n} e^{imx} - \sum_{m=0}^{n} e^{-imx}) \\ &= [{}^1/(n+1)(e^{ix}-1)][e^{ix}(1-e^{i(n+1)x})/(1-e^{ix}) - (1-e^{-i(n+1)x})/(1-e^{-ix})] \\ &= [{}^1/(n+1)(e^{ix}-1)(e^{-ix}-1)](2-e^{i(n+1)x}-e^{-i(n+1)x}) \\ &= (2-e^{i(n+1)x}-e^{-i(n+1)x})/[(n+1)(2-e^{ix}-e^{-ix})] \end{aligned}$$

$$=(2-2\cos(n+1)x)/[(n+1)(2-2\cos x)]$$
$$=(1-\cos(n+1)x)/[(n+1)(1-\cos x)]$$

Thus for $0<b\leq|x|\leq\pi$,

$$0\leq K_n(x)\leq 2/[(n+1)(1-\cos b)]. \qquad (4)$$

Note that

$$\int_{-\pi}^{\pi}e^{ikx}dx=0 \text{ if } k\neq 0,$$
$$=2\pi \text{ if } k=0,$$

so

$$(1/2\pi)\int_{-\pi}^{\pi}K_n(x)dx=(1/2\pi)\int_{-\pi}^{\pi}[(1/(n+1))\sum_{m=0}^{n}D_m(x)]dx$$
$$=(1/(n+1))\sum_{m=0}^{n}(1/2\pi)\int_{-\pi}^{\pi}D_m(x)dx$$
$$=(n+1)/(n+1)=1. \qquad (5)$$

Next, the nth partial sum $S_n(x)$ of the Fourier series of $f(x)$ becomes

$$S_n(x)=\sum_{m=-n}^{n}C_me^{imx}=\sum_{m=-n}^{n}[(1/2\pi)\int_{-\pi}^{\pi}f(t)e^{-imt}dt]e^{imx}$$
$$=(1/2\pi)\int_{-\pi}^{\pi}f(t)\sum_{m=-n}^{n}e^{im(x-t)}dt$$
$$=(1/2\pi)\int_{-\pi}^{\pi}f(t)D_n(x-t)dt.$$

If we substitute $u=x-t$ and use the 2π periodicity of the integrand, we have

$$S_n(x)=(1/2\pi)\int_{x-\pi}^{x+\pi}f(x-u)D_n(u)du=(1/2\pi)\int_{-\pi}^{\pi}f(x-u)D_n(u)du,$$

so we will write

$$S_n(x)=(1/2\pi)\int_{-\pi}^{\pi}f(x-t)D_n(t)dt$$

Note that

$$S_n^1(x)=(1/(n+1))\sum_{m=0}^{n}S_m(x)=(1/(n+1))\sum_{m=0}^{n}(1/2\pi)\int_{-\pi}^{\pi}f(x-t)D_m(t)dt$$
$$=(1/2\pi)\int_{-\pi}^{\pi}f(x-t)[(1/(n+1))\sum_{m=0}^{n}D_m(t)]dt$$
$$=(1/2\pi)\int_{-\pi}^{\pi}f(x-t)K_n(t)dt \qquad (6)$$

Now let arbitrary $h>0$ be given, and by the continuity of $f(x)$, we can choose $M>0$ such that

$$|f(x)|<M \quad \text{for all } x. \qquad (7)$$

Also, since f(x) is uniformly continuous, there exists $d_1>0$ such that $|x-y|<d_1$ implies

$$|f(x)-f(y)|<h/2. \qquad (8)$$

Note that these statements are possible because f(x) is of period 2π, so we could just use the boundedness and uniform continuity on $[-\pi,\pi]$ for all x. Furthermore, by (4), we can choose N and $d=\min(b, d_1)$ such that $n\geq N$ and $d\leq|t|\leq\pi$ imply

$$K_n(t)\leq h/(4M) \qquad (9)$$

Thus from (4) through (9), we have

$$\begin{aligned}
\left| S_n^1(x)-f(x)\right| &= \left|(1/2\pi)\int_{-\pi}^{\pi} f(x-t)K_n(t)dt - f(x)\right| \\
&= \left|(1/2\pi)\int_{-\pi}^{\pi} f(x-t)K_n(t)dt - f(x)(1/2\pi)\int_{-\pi}^{\pi}K_n(t)dt\right| \\
&\leq (1/2\pi)\int_{-\pi}^{\pi}\left|f(x-t)-f(x)\right|K_n(t)dt \\
&\leq (1/2\pi)\int_{-\pi}^{-d}\left|f(x-t)-f(x)\right|K_n(t)dt \\
&\qquad +(1/2\pi)\int_{-d}^{d}\left|f(x-t)-f(x)\right|K_n(t)dt \\
&\qquad +(1/2\pi)\int_{d}^{\pi}\left|f(x-t)-f(x)\right|K_n(t)dt \\
&\leq (h/(8M\pi))\int_{-\pi}^{-d}\left|f(x-t)-f(x)\right|dt \\
&\qquad +(h/(4\pi))\int_{-d}^{d}K_n(t)dt \\
&\qquad +(h/(8M\pi))\int_{d}^{\pi}\left|f(x-t)-f(x)\right|dt \\
&\leq (h/(8M\pi))\int_{-\pi}^{0}2Mdt + (h/(4\pi))\int_{-\pi}^{\pi}K_n(t)dt + (h/(8M\pi))\int_{0}^{\pi}2Mdt \\
&\leq (h/4)+((2h)/4)+(h/4)=h
\end{aligned}$$

for all x, and all $n\geq N$. Hence

$$\lim_{n\to\infty} S_n^1(x)=f(x)$$

uniformly, and by the definition of Cesaro smmability,

$$\sum_{n=-\infty}^{\infty}C_n e^{inx}=f(x) \quad (C,1)$$

uniformly for all x; This completes the proof.

A very important consequence of this theorem is that any two continuous functions f(x) and g(x) having the same Fourier series must be f(x)=g(x) for every x.

(ii) Multiplication of series

The Cauchy product of two series $\sum_{r=0}^{\infty}U_r$, $\sum_{r=0}^{\infty}Vr$; is defined as the series

$$\sum_{r=0}^{\infty}Wr \quad \text{where} \quad W_r=\sum_{K=0}^{r}U_kV_{r-k}.$$

A natural question is to assume that

$$\sum_{r=0}^{\infty}U_r=E \quad \text{and} \quad \sum_{r=0}^{\infty}V_r=F$$

in the ordinary Cauchy sense and inquire as to whether $\sum_{r=0}^{\infty}W_r$ converges; and if so does it converge to EF. Abel, in his classical theorem, showed shat

$$\sum_{r=0}^{\infty}W_r=EF$$

under the assumption that $\sum_{r=0}^{\infty}W_r$ is convergent but the fact that $\sum_{r=0}^{\infty}W_r$ might not converge is well known. Theorem 6 will show how (C,p) summability can be brought to bear on such problems,

Theorem 6. If $\sum_{r=0}^{\infty}U_r=E(C,p)$ and $\sum_{r=0}^{\infty}V_r=F(C,q)$, then $\sum_{r=0}^{\infty}W_r=EF(C,\ p+q+1)$.

Proof: Let us consider the power series relation

$$(\sum_{r=0}^{\infty}U_r x^r)(\sum_{r=0}^{\infty}V_r x^r)=\sum_{r=0}^{\infty}W_r x^r \text{ for } |x|<1.$$

This is valid because $\sum_{r=0}^{\infty}U_r$ and $\sum_{r=0}^{\infty}V_r$ are (A) summable if they are (C,p) and (C,q) summable, so the series are absolutely convergent.

Multiplying by appropriate power of $(1-x)$ we have

$$(1-x)^{-p-1}(\sum_{r=0}^{\infty}U_r x^r)(1-x)^{-q-1}(\sum_{r=0}^{\infty}V_r x^r)=(1-x)^{-p-q-2}(\sum_{r=0}^{\infty}W_r x^r)$$

and we recall that this was equivalent to

$$(\sum_{r=0}^{\infty}S_r^p(U)x^r)(\sum_{r=0}^{\infty}S_r^q(V)x^r)=\sum_{r=0}^{\infty}S_r^{p+q+1}(W)x^r,$$

thus we equate coefficients and have

$$S_n^{p+q+1}(W)=\sum_{r=0}^{n}S_r^p(U)S_{n-r}^q(V) \tag{10}$$

Since $\lim_{n\to\infty} S_n^p(U)/\binom{n+p}{p}=E$ and $\lim_{n\to\infty} S_n^q(V)/\binom{n+q}{q}=F$,

we may write

$$S_n^p(U)\sim\binom{n+p}{p}\mathrm{E} \quad \text{and } S_n^q(V)\sim\binom{n+q}{q}\mathrm{F},$$

where $f\sim g$ means $(f/g)\to 1$ as $n\to\infty$.

The equation (10) implies that

$$S_n^{p+q+1}(W)\sim EF\sum_{r=0}^{n}\binom{r+p}{p}\binom{n-r+q}{q}.$$

We shall show that

$$\sum_{r=0}^{n}\binom{r+p}{p}\binom{n-r+q}{q}=\binom{n+p+q+1}{p+q+1} \tag{11}$$

for all non-negative integer n by use of induction.

We know that it is true for $n=0$, $n=1$. Suppose it is true for $n=m$ that

$$\sum_{r=0}^{m}\binom{r+p}{p}\binom{m-r+q}{q}=\binom{m+p+q+1}{p+q+1}$$

Look at

$$\sum_{r=0}^{m+1}\binom{r+p}{p}\binom{m-r+q+1}{q}-\sum_{r=0}^{m}\binom{r+p}{p}\binom{m-r+q}{q}$$

$$=\sum_{r=0}^{m}\binom{r+p}{q}\left[\binom{m-r+q+1}{q}-\binom{m-r+q}{q}\right]+\binom{m+1+p}{p}\binom{q}{q}$$

$$=\sum_{r=0}^{m}\binom{r+p}{p}\binom{m-r+q}{q-1}+\binom{m+1+p}{q}\binom{q-1}{q-1}$$

$$=\sum_{r=0}^{m+1}\binom{r+p}{p}\binom{m+1-r+q-1}{q-1}=\binom{m+p+q+1}{p+q}. \tag{12}$$

The last step (12) can be verified by induction on q and by using (3). Then we have

$$\sum_{r=0}^{m+1}\binom{r+p}{p}\binom{m-r+q+1}{q}=\binom{m+p+q+1}{p+q+1}+\binom{m+p+q+1}{p+q}$$

$$=\binom{m+p+q+2}{p+q+1}.$$

therefore, (11) is true for all n. Consequently

$$\lim_{n\to\infty} S_n^{p+q+1}(W)/\binom{n+p+q+1}{p+q+1}=FF$$

and

$$\sum_{r=0}^{\infty} W_r=EF\ (C,p+q+1).$$

Note that $p=q=0$ means Cauchy convergence so we have

Corollary 1. If $\sum_{r=0}^{\infty} U_r=E$ and $\sum_{r=0}^{\infty} V_r=F$ (ordinary convergence), then

$$\sum_{r=0}^{\infty} W_r=EF\ (C,1).$$

This was first proved by Cesaro in 1890. Next we get Abel's result as

Corollary 2. If $\sum_{r=0}^{\infty} U_r=E$, $\sum_{r=0}^{\infty} V_r=F$, and $\sum_{r=0}^{\infty} W_r$ is convergent, then

$$\sum_{r=0}^{\infty} W_r=EF.$$

This follows from Corollary 1 since $(C,1)$ summability is regular.

5. A Tauberian Theorem

There are many theorems which deal with the problem of finding conditions under which the summability of a series to Y by a new method implies ordinary Cauchy convergence to the same number Y. Alfred Tauber proved the first result of the type in 1897 so all such theorems have come to be known as Tauberian theorem. We will prove just one such result, namely the original theorem of Tauber.

Theorem 7. If $\sum_{n=0}^{\infty} U_n = Y(A)$ and $nU_n\to 0$ as $n\to\infty$, then $\sum_{n=0}^{\infty} U_n=Y$ in the Cauchy sense.

Proof: Without loss of generality we can assume that $U_0=0$. We write

$$\sum_{k=0}^{n} U_k-Y=\sum_{k=0}^{n} U_k-\sum_{k=0}^{\infty} U_k x^k+f(x)-Y$$

$$=\sum_{k=0}^{n} U_k(1-x^k)-\sum_{k=n+1}^{\infty} U_k x^k+f(x)-Y$$

Now

$$(1-x^k)=(1-x)(1+x+x^2+\cdots\cdots+x^{k-1})\le k(1-x)$$

for $0\le x<1$, so

$$\left|\sum_{k=0}^{n} U_k(1-x^k)\right|\le(1-x)\sum_{k=0}^{n} k|U_k|.$$

Since we are assuming that $kU_k\to 0$ as $k\to\infty$, (and hence $k|U_k|\to 0$), we know that the sequence $\{k|U_k|\}$ is also $(C,1)$ summable to 0 so

$$\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} k|U_k|=0$$

Let arbitrary $h>0$ be given, then there exists an integer $N_1>0$ such that

$$(1/n)\sum_{k=1}^{n} k|U_k| < h/3 \qquad \text{for all } n > N_1,$$

so if we set $x=(n-1)/n$ we have

$$\left|\sum_{k=1}^{n} U_k[1-((n-1)/n)^k]\right| \leq (1/n)\sum_{k=1}^{n} k|U_k| < h/3$$

for all $n>N_1$. Next we estimate

$$\sum_{k=n+1}^{\infty} U_k x^k = \sum_{k=n+1}^{\infty} U_k((n-1)n/)^k$$

Since $k|U_k| \to 0$ as $k \to \infty$, there exists an integer $N_2 > 0$ such that $k|N_k| < h/3$ for all $k > N_2$, so we can write

$$\left|\sum_{k=n+1}^{\infty}((kU_k)/k)((n-1)/n)^k\right| \leq \sum_{k=n+1}^{\infty}((k|U_k|)/k)((n-1)/n)^k$$

$$\leq (h/(3n))\sum_{k=n+1}^{\infty}((n-1)/n)^k$$

$$< (h/(3n))\sum_{k=0}^{\infty}((n-1)/n)^k = h/3$$

for all $n > N_2$.

Also $f((n-1)/n) \to Y$ as $n \to \infty$ so there exists integer $N_3 > 0$ such that

$$|f((n-1)/n) - Y| < h/3 \text{ for all } n > N_3,$$

so if we set $N = \max(N_1, N_2, N_3)$, then

$$\left|\sum_{k=0}^{n} U_k - Y\right| < h/3 + h/3 + h/3 = h$$

for all $n > N$. Hence

$$\sum_{k=0}^{\infty} U_k = Y \text{ in Cauchy sense.}$$

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有關級數求和法

解 萬 臣

級數 $\sum_{k=0}^{\infty} U_k$ 之一般求和法，即 Cauchy 之收歛定義，爲當其"部份和"（$S_n=\sum_{k=0}^{n} U_k$）之數列 $\{S_n\}$ 趨近某一定値 S 時，則此級數謂之收歛，而其和爲 S。甚多級數在此定義下爲發散級數，因而無法探究其性質。故此自 Cauchy 以來，其他不同之"級數求和法"繼而產生。任一"級數求和法"其"一致性"尤爲重要，卽某一級數若在 Cauchy 定義下其和爲 S 時，而以此不同之求和法求其和時亦應爲 S。

本文藉介紹二種簡單而常用之"級數求和法"，卽 Cesaro 及 Abel 所提出之求和法，討論其一致性及彼此關係，並提出其在 Fourier 級數及級數乘積上之應用。文中所提出有關"級數求和論"中之重要定理多予以新的證明，亦爲本文另一目的。

ON SOME METHODS OF SUMMABILITY

Wan-chen Hsieh

In the theory of summability there are two more useful and simple methods of summability, namely, Cesaro's method and Abel's method. The close relation between them, the regularity of these two methods, and their applicaions to Fourier series and multiplication of series are discussed here.