ON the Structure of Clifford Algebra

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0. Introduction

The purpose of this paper is to determine the structure of the Clifford algebra. We express the results in the table at the end of this article. In order to construct this special kind of algebra, it seems necessary to introduce Graded Algebra, Tensor Algebra and some of their properties. which are mentioned in the first two sections. In the last section, we also mention some properties of Clifford algebra which are considered important.

1. Graded Algebra

1.1 Free Algebra

Let E be an algebra over A generated by a given set of generators $(x_i)_{i\in I}$. Let $\sigma=(i_1\cdots\cdots i_h)$ be a finite sequence of elcments of I and put $y_\sigma=x_{i_1}\cdots\cdots x_{i_h}$. The number h is called the length of σ. Among the "finite sequences" we always admit the empty sequence σ_0, whose length is O, i.e., a sequence with no term, and we put $y_{\sigma_0}=1$. we define the composition of finite sequences $\sigma=(i_1\cdots\cdots\cdots, i_h)$ and $\sigma'=(j_1\cdots\cdots, j_k)$ by $\sigma\cdot\sigma'=(i_1\cdots, i_h, j_1\cdots, j_k)$. For σ_0, we define $\sigma_0\sigma=\sigma\sigma_0=\sigma$. Evidently, $(\sigma\sigma')\sigma''=\sigma(\sigma'\sigma'')$, and we have $y_{\sigma\sigma'}=y_\sigma\cdot y_{\sigma'}$. Under these assumptions, we can prove that every element of E is a linear combination of the y_σ's, σ running over all finite sequences of elements of I.

Definition (1.1.1.): If the y_σ's are linarly independent over A, then E is called a free algebra and the set $(x_i)_{i\in I}$ is called a free system of generators of E.

1.2 Graded Algebra

Definition (1.2.1): Let Γ be an additive group. A Γ—graded algebra is an algebra E which is given together with a direct sum decomposition as a module.

$$E=\sum_{\gamma\in\Gamma} E_\gamma$$

where the E_ν's are submodules of E, in such a way that

$$E_\nu\cdot E_{\nu'}\subset E_{\nu+\nu'}$$

Definition (1.2.2): In a Γ—graded algebra $E=\sum_{\nu\in\Gamma} E_\nu$, an element belonging to E_ν is called *homogeneous of degree* ν. Any element x in E can be expressed uniquely by $x=\sum_{\nu\ni\Gamma} x_\nu$, where the x_ν's are O except for a finite number of ν's.

Each x_ν is called the ν-component of x.

Definition (1.2.3): A submodule M of a Γ—graded algebra $E=\sum E_\nu$ is said to be homogeneous if the homogeneous components of any element of M still belong to M. This is equivalent to the condition $M=\sum_\nu M\cap E_\nu$.

Proposition (1.2.4): If a submodule M or an ideal I of a Γ—graded algebra E is generated by homogeneous elements, then it is homogeneous. Therefore, $M=\sum_\nu M_\nu$ and $I=\sum_\nu I_\nu$, where $M_\nu=M\cap E_\nu$, $I_\nu=I\cap E_\nu$.

PROOF: we just prove the case for I. Let S be a set of homogeneous elements which generates I. As a module, I is spanned by all elements of the form xsy, where $x,y\in E$ and $s\in S$. Putting $x=\sum x_\nu$, $y=\sum y_{\nu'}$, we have $xsy=(\sum x_\nu)s(\sum y_{\nu'})=\sum_{\nu,\nu'} x_\nu s y_{\nu'}$. Since each $x_\nu s\, x_{\nu'}$ is homogeneous, I is also spanned by homogeneous elements $x_\nu s\, x_{\nu'}$ · Let I′ be the set of elements of I whose homogeneous components belong to I. Clearly, $S\subset I'\subset I$. It is not difficult to prove that I′ is a submodule of I. Hence $I'=I$. ||

Proposition (1.2.5): Let I be an ideal of a Γ-graded algebra E generated by homogeneous elements. Then the quotient algebra E/I is also a Γ—graded.

PROOF: Let $\bar{x}\in E/I$ and $x=\sum x_\nu$. Then

$$\bar{x}=\sum x_\nu+I=\sum(x_\nu+I_\nu)\ \in\sum(E_\nu/I_\nu)$$

Hence

$$E/I=\sum(E_\nu/I_\nu).$$ Furthermore, if $x_\nu+I_\nu\in E_\nu/I_\nu$ and

$x_{\nu'}+I_{\nu'}\in E_{\nu'}/I_{\nu'}$,

then $(x_\nu+I)(x_{\nu'}+I)=x_\nu\cdot x_{\nu'}+I\in E_{\nu+\nu'}/I_{\nu+\nu'}$. It follows that $(E_\nu/I_\nu)\cdot(E_{\nu'}/I_{\nu'})\subset E_{\nu+\nu'}/I_{\nu+\nu'}$. By (1.2.1), E/I is a Γ—graded algebra. ||

Remark (1.1): (1) The unit 1 in a Γ—graded algebra is always homogeneous of degree O (O is the zero element of Γ)

(2) Scalars are homogeneous of degree O.

(3) In particular, a free algebra is a Z-graded algebra: Let F be a free algebra. we may write $F=\sum_{i\in Z}F_i$, where $F_h=O$ for all $h<O$ and F_h is the module spanned by the y_σ's, σ being of length h.

(4) The cannonical homomorphism π : E⟶E/I is a homomorphism of Γ—graded algebra, i.e. it is a homomorphism satisfying $\pi(E_\nu)\subset E_\nu/I_\nu$.

1.3 Associated Gradations and the Main Involution.

Let Γ, $\tilde{\Gamma}$ be additive groups and let a homomorphism $\tau:\Gamma\longrightarrow\tilde{\Gamma}$ be given. To any Γ—graded algebra $E=\sum_{\nu\in\Gamma}E_\nu$, we associated the following $\tilde{\Gamma}$—gradation of E ; for each $\tilde{\nu}\in\tilde{\Gamma}$, put

$$E_{\tilde{\nu}}=\sum_{\nu\in\tau^{-1}(\tilde{\nu})}E_\nu\ (E_{\tilde{\nu}}=O \text{ if } \tau^{-1}(\tilde{\nu})=\text{empty})$$

Proposition (1.3.1): Notation as above, we have $E=\sum_{\tilde{\nu}\in\tilde{\Gamma}}E_{\tilde{\nu}}$ and $E_{\tilde{\nu}}\cdot E_{\tilde{\nu}'}\subset E_{\tilde{\nu}+\tilde{\nu}'}$. In this way., $E=\sum_{\tilde{\nu}\in\tilde{\Gamma}}E_{\tilde{\nu}}$ can be considered as a $\tilde{\Gamma}$—graded algebra.

PROOF: Clearly $E\supseteq\sum_{\tilde{\nu}\in\tilde{\Gamma}}E_{\tilde{\nu}}$. Now, let $x=\sum_{\tilde{\nu}\in\Gamma}x_\nu\in E$. Since each $x_\nu\in E_{\tilde{\nu}}$, where $\tilde{\nu}=\tau(\nu)$, it implies that $x\in\sum_{\tilde{\nu}\in\tilde{\Gamma}}E_{\tilde{\nu}}$. Hence $E=\sum_{\tilde{\nu}\in\tilde{\Gamma}}E_{\tilde{\nu}}$. we shall prove $E_{\tilde{\nu}}\cdot E_{\tilde{\nu}'}\subset E_{\tilde{\nu}+\tilde{\nu}'}$. Let $x_{\tilde{\nu}}\in E_{\tilde{\nu}}$ and $x_{\tilde{\nu}'}\in E_{\tilde{\nu}'}$. Then $x_{\tilde{\nu}}=\sum_{\nu\in\tau^{-1}(\tilde{\nu})}x_\nu$ and $x_{\tilde{\nu}'}=\sum_{\nu'\in\tau^{-1}(\tilde{\nu}')}x_{\nu'}$. Since $\tau(\nu+\nu')=\tau(\nu)+\tau(\nu')=\tilde{\nu}+\tilde{\nu}'$, for any $\nu\in\tau^{-1}(\tilde{\nu})$ and $\nu'\in\tau^{-1}(\tilde{\nu}')$, so $\nu+\nu'\in\tau^{-1}(\tilde{\nu}+\tilde{\nu}')$. Therefore $x_{\tilde{\nu}}\cdot x_{\tilde{\nu}'}=\sum_{\nu+\nu'\in\tau^{-1}(\tilde{\nu}+\tilde{\nu}')}x_\nu x_{\nu'}\in E_{\tilde{\nu}+\tilde{\nu}'}$. ||

Definition (1.3.2): The $\tilde{\Gamma}$—gradation $E=\sum_{\tilde{\nu}\in\tilde{\Gamma}}E_{\tilde{\nu}}$ is called the associated $\tilde{\Gamma}$—gradation of E, associated to the Γ—gradation $E=\sum_{\nu\in\Gamma}E_\nu$. For convenience, we denote this associated Γ—gradation of E with E^τ.

Proposition (1.3.3): Every homogeneous element, every homogeneous submodule, and every homogeneous ideal in E are also homogeneous in E, The proof of (1.3. 3) is trivial we omit it.

In particular, if $\tilde{\Gamma}=\{0,1\}$ and τ is onto, E^{τ} is called the associated semi-gradation (a Z_2-gradation), and write $E=E^s=E^s_0 \oplus E^s_1$ instead of E^{τ}. Also, we denote $\tau^{-1}(o)=\Gamma_0$ and $\tau^{-1}(1)=\Gamma_1$. Cleary, Γ_0 is a subgroup of index 2.

Proposition (1.3.4): Let $E=\sum_{\nu\in\Gamma}E_\nu$ be a Γ—graded algebra and let $E^s=E^s_0 \oplus E^s_1$ be the associated semi-gradation of E. Define a map J : $E \longrightarrow E$ by

$$J(x)=x_0-x_1, \text{ where } x=x_0+x_1, \ x_0 \in E^s_0 \text{ and } x_1 \in E^s_1$$

Then J is an automorphism of the Γ—gradation E.

PROOF: Evidently. J is linear, one-to-one, onto, and preserves the degree in the Γ—gradation of E. Now. let $x=x_0+x_1$ and $y=y_0+y_1$, where $x_i \in E^s_i$ *and* $y_i \in E^s_i$, $i=0,1$. Then

$$x\cdot y=(x_0+x_1)(y_0+y_1)$$
$$=(x_0y_0+x_1y_1)+(x_0y_1+x_1y_0), \text{ where } x_0y_0+x_1y_1 \in E^s_0 \ x_0y_1+x_1y_0 \in E_0 \text{ so, we have}$$
$$J(x\cdot y)=(x_0y_0+x_1y_1)-(x_0y_1+x_1y_0)$$
$$=(x_0-x_1)(y_0-y_1)$$
$$=J(x)\cdot J(y)$$

Hence, we proved that J is an automorphism.||

Such J is called the main involution of E ($\because J^2=1$). For convenience, we define symbolic power J^ν of the main involution as follows:

$$(1.3.5):(1^0)\ J^\nu=\begin{cases} J & if\ \nu\in\Gamma_1 \\ 1 & \text{if } \nu\in\Gamma_0 \end{cases} \text{ where 1 is the identity map of E.}$$

$$(2^0)\ (-1)^\nu=\begin{cases} -1 & \text{if } \nu\in\Gamma_1 \\ 1 & if\ \nu\in\Gamma_0 \end{cases}$$

Remark (1.3): we have the following identities:

i) $J^\nu \cdot J^{\nu'}=J^{\nu+\nu'}$

ii) $(-1)^\nu(-1)^{\nu'}=(-1)^{\nu+\nu'}$

iii) $(J^\nu)^{\nu'}=(J^{\nu'})^\nu$

iv) $((-1)^\nu)^{\nu'}=((-1)^{\nu'})^\nu$

v) If $x=\sum_{\nu\in\Gamma}x_\nu$, then $J(x)=\sum_{\nu\in\Gamma}(-1)^\nu x_\nu$

1.4 Derivations.

In this section, we assume that a fixed subgroup Γ_0 of Γ with index 2 has been given. Let E, E′ be two Γ—graded algebras over A and let φ be a homomorphism of E into E′.

Definition (1.4.1): Let λ:E$\longrightarrow$E′ be a linear map and ν any element in Γ. λ is called homogeneous of degree ν if $\lambda(E_\gamma)\subseteq E'_{\gamma+\nu}$ for all $\gamma\in\Gamma$.

Definition (1.4.2): A φ—derivation D of E into E' means a linear mapping D:E$\longrightarrow$E′, homogeneous of some given $\nu\in\Gamma$, such that

$$D(xy)=D(x)\varphi(y)+\varphi(J^\nu x)D(y)$$

where x,y$\in$E and J^ν is the power of the main involution.

Proposition (1.4.3): For every φ—derivation D, we have D(1) = O.

PROOF: $D(1)=D(1)\varphi(1)+\varphi(J^\nu 1)D(1)$

$$=D(1)+D(1)$$

This implies immediately that D(1) = O.||

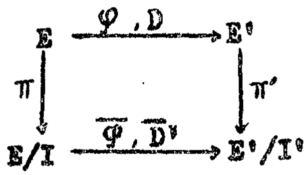
Proposition (*1.4.4*) :Let D be *a* φ—derivation of E into E′, F (or I) a homogeneous subalgebra (or ideal) of E, S a set of homogeneous generators of F (or I), and let F′ (or I′) be a homogeneous subalgebra (or ideal) of E′. Then if $D(S) \subset F'$ and $\varphi(S) \subset F'$, we have $D(F) \subset F'$ (or $D(I') \subset I'$) and $\varphi(F) \subset F'$ (or $\varphi(I) \subset I'$).

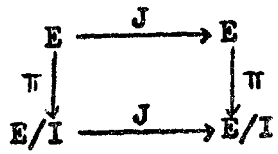
PROOF: we just prove the case for subalgebras. Clearly $\varphi(F) \subset F'$, for φ is a homomorphism. To prove $D(F) \subset F'$, let $F_1 = \{x \epsilon F \mid D(x) \epsilon F'\}$ · Then F_1 is closed under addition and scalar multiplication. Also if $x = \sum x_\gamma \epsilon F_1$, then $D(x) = \sum D(x_\gamma) \epsilon F'$. By hypothesis, each γ-component of x must be in F_1 · Therefore F_1 is a homogeneous submodule of F. And so $x \epsilon F_1$ implies $J^\nu(x) \epsilon F_1$, where ν is the degree of D. Now, for $x, y \epsilon F_1$, we have

$$D(xy) = D(x)\varphi(y) + \varphi(J^\nu x) D(y)$$

and since $D(x)$, $\varphi(y)$, $\varphi(J^\nu x)$, $D(y)$ all belong to F′, we have $xy \epsilon F_1$. Thus F_1 is a subalgebra of F containning S. Hence $F_1 = F$.||

Proposition (*1.4.5*) *:* Let E, E′ be Γ—graded algebras, φ a homomorphism $E \longrightarrow E'$, and D a φ-derivation of $E \longrightarrow E'$. Also let I and I′ be homogeneous ideals in E and E′ respectively such that $D(I) \subset I'$ and $\varphi(I) \subset I'$. Under these assumptions, the induced mapping $\bar{D}$: $E/I \longrightarrow E'/I'$ obtained from D is a $\bar{\varphi}$—derivation. where $\bar{\varphi}$ means the induced homomorphism $E/I \longrightarrow E'/I'$ obtained from φ.





PROOF: Obviously, $\bar{\varphi}$ aand $\bar{D}$ are well defined, linear and make the right diagram commute. First, we shall prove $\bar{\varphi}$ is a homomorphism. Let $\bar{x}, \bar{y} \epsilon E/I$. Then

$$\bar{\varphi}(\bar{x} . \bar{y}) = \bar{\varphi}(\overline{xy}) = \overline{\varphi(xy)} = \overline{\varphi(x)\varphi(y)}$$
$$= \overline{\varphi(x)} \cdot \overline{\varphi(y)} = \bar{\varphi}(\bar{x})\bar{\varphi}(\bar{y})$$

Also, if $\bar{x} \ \epsilon E_\gamma / I_\gamma$, then $x \epsilon E_\gamma$. Since $\varphi(x) \epsilon E'_\gamma$, $\bar{\varphi}(\bar{x}) = \overline{\varphi(x)} \epsilon \ E'_\gamma / I'_\gamma$. Hence $\bar{\varphi}$ is a homomorphism. Next, let us consider the induced map $\bar{J}$: $E/I \longrightarrow E/I$, obtained from J· Since I is homogeneous, $J(I) = I$. Therefore $\bar{J}$ is also an automorphism of E/I. we shall prove that $\bar{J}$ is also a main involution of E/I. For any $\bar{x} \epsilon E/I$, it can be expressed by

$$\bar{x} = \overline{x_0 + x_1} = \bar{x}_0 + \bar{x}_1 \text{ , where } x = x_0 + x_1,\ x \epsilon E,\ x_i \epsilon E^s_i \text{ , } i = 0, 1.$$

According the statements in (1.3), we have

$$\bar{x}_0 \ \epsilon (E/I)^s_0 \text{ and } \bar{x}_1 \ \epsilon \ (E/I)^s_1.$$

Since $\bar{J}(\bar{x}) = \overline{J(x)} = \overline{x_0 - x_1} = \bar{x}_0 - \bar{x}_1$

we proved that $\bar{J}$ is also a main involution. From (1.3.5), it is easily seen that $(\bar{J})^\nu = \bar{J}^\nu$.

Finally, we shall prove that $\bar{D}$ is a $\bar{\varphi}$-derivation Let D be homogeneous of degree ν. Then evidently $\bar{D}$ is also homogeneous of degree ν . For any $\bar{x}, \bar{y} \epsilon E/I$, we have

$$\bar{D}(\bar{x} . \bar{y}) = \overline{D(xy)}$$
$$= \overline{D(x)\varphi(y) + \varphi(J^\nu x) D(y)}$$
$$= \overline{D(x)}\,\overline{\varphi(y)} + \overline{\varphi(J^\nu x)} . \overline{D(y)}$$
$$= \bar{D}(\bar{x})\bar{\varphi}(\bar{y}) \ + \bar{\varphi}[(\bar{J})^\nu \bar{x}]\bar{D}(\bar{y})$$

This completes the proof. ||

Remark (*1.4*) *:* (1^0) In case $E = E'$ and $\varphi = identity$, D is called a derivation.

(2^0) If $\Gamma = Z$ and D is a derivation, then

$D(xy)=D(x)y+(-1)^{h\nu}x\cdot D(y)$; $x\epsilon E_h$, $y\epsilon E$.

1.5 Existence of derivations in free Algebras.

Let $F=\sum_{h\epsilon Z}F_h$ be a free algebra with a free system of generators $(x_i)_{i\epsilon I}$ over a ring A, as mentioned in Remark (1.1). Let E be a Z-graded algebra over A and φ a homomorphism $F\longrightarrow E$.

Proposition (1.5.1): Assume that for each $i\epsilon I$, a homogeneous element $y_i\epsilon E$ of degree $\nu+1$ is preassigned arbitrarily, where ν is a fixed integer. There exists one and only one φ—derivation D of F into E, which is of degree ν and satisfies $D(x_i)=y_i$.

PROOF: Uniqueness; Assume D′ is another φ—derivation of E into E′, having the same degree as D′s and satisfying $D'(x_i)=y_i$ · Then $(D-D')(x_i)=O$ for each generator x_i. Let $F_1=\{x\epsilon F|(D-D')(x)=O\}$. Using the same method as we did in the proof (1.4.4), we can prove that F_1 is a subalgebra of F containing the set of generators, $(x_i)_{i\epsilon I}$. Since F is generated by (x_i), F_1 should be equal to F. Hence $D=D'$.

Existence: Obviously, the elements $p_\sigma=x_{i_1}\cdots\cdots,.x_{i_h}$ form a base of F, where $\sigma=(i_1,\cdots\cdots, i_h)$ runs over the set Σ consisting of all finite sequences taken from I we define $\delta(p_\sigma)\epsilon$ E such that

(1) $\delta(p_{\sigma_o})=\delta(1)=0$ if σ_o=empty sequence

(2) $\delta(p_\sigma)=\delta(x_{i_1},\ \cdots\cdots,\ x_{i_h})=\delta(x_{i_1},\cdots\cdots x_{i_{h-1}})\varphi(x_{i_h})+\varphi[J^\nu(x_{i_1}\cdots\cdots,\ x_{i_{h-1}})]y_{i_h}$ if $\delta(p_{\sigma'})$ has already been defined for every σ' with length less than h.

In the case where $h=1$, we have $\delta(x_i)=y_i$. From the definition, $\delta(p_\sigma)$ is homogeneous of degree $h+\nu$ if σ has length h. For, if $h=1$, $\delta(x_i)=y_i$ is of degree $\nu+1$ by hypothesis, and if this property has already been proved up to h-1, the degrees of the two terms on the right side of (2) are $(h-1+\nu)+1$ and $(h-1)+\nu+1$ which are both equal. Hence $\delta(p_\sigma)$ is of degree $h+\nu$.

Now, we define a mapping D: F⟶E such that $D(p_\sigma)=\delta(p_\sigma)$ for all $\sigma\epsilon\Sigma$. Since (p_σ) forms a base of F, such D always exists and is uniquely determined. From the definition, D is linear and homogeneous of degree ν . Next, we shall prove D holds the following condition.

(3) $D(uv)=D(u)\varphi(v)+\varphi(J^\nu u)D(v)$ for any $u,v\epsilon F$.

Let us first consider the image of ux_i under D, where $u=\sum_{\sigma\epsilon\Sigma}a_\sigma p_\sigma$ is in F and x_i is an arbitrarily generator of F. Then, we have

$$\begin{aligned}D(ux_i)&=D(\Sigma a_\sigma p_\sigma x_i)\\&=\Sigma a_\sigma D(p_\sigma x_i)\\&=\Sigma a_\sigma[D(p_\sigma)\varphi(x_i)+\varphi(J^\nu p_\sigma)D(x_i)] \qquad (by (2))\\&=D(\Sigma a_\sigma p_\sigma)\varphi(x_i)+\varphi[J^\nu(\Sigma a_\sigma p_\sigma)]\cdot D(x_i)\\&=D(u)\cdot\varphi(x_i)+\varphi(J^\nu u)\cdot D(x_i)\end{aligned}$$

i.e.

(4) $D(ux_i)=D(u)\cdot\varphi(x_i)+\varphi(J^\nu u)\cdot D(x_i)$

Now, we denote by F_1 the set of all elements of F which satisfy the condition (3) for all u. And we shall prove F_1 =F. From (4), we see that each generator x_i is in F_1 and also 1 is in F_1 . We claim that $v\ \epsilon F_1$ implies $vx_i\ \epsilon F_1$. For if $v\epsilon F_1$, then

$$D[u(vx_i)]=D[(uv)x_i]$$

$$=D(uv)\varphi(x_i)+\varphi(J^\nu(uv))D(x_i)$$
$$=[D(u)\varphi(v)+\varphi(J^\nu u)Dv]\varphi(x_i)+\varphi(J^\nu u)\varphi(J^\nu v)D(x_i)$$
$$=D(u)\varphi(vx_i)+\varphi(J^\nu u)[Dv\varphi(x_i)+\varphi(J^\nu v)D(x_i)]$$
$$=D(u)\varphi(vx_i)+\varphi(J^\nu u)\cdot D(vx_i)$$

It follows that $vx_i \epsilon F_1$. Applying this fact and that each $x_i \epsilon F_1$, finally we can prove F_1 contains all the bases $p_\sigma = x_{i_1} \cdots\cdots x_{i_h}$. Hence $F_1 = F$. And hence D is a φ—derivation of F into E.||

2. Tensor Algebra.

2.1 Graded Structure of Tensor Algebras.

Let M be a module over A. Let $F=\sum_{h\epsilon Z} F_h$ be a free algebra over A freely generated by M. In this case, we may identify A with F_0 and M with F_1. To distinguish the addition, substruction and scalar multiplication in F from those in M, we denote the formers by $\dot{+}$, $\dot{-}$, and $\alpha \cdot x (\alpha \epsilon A)$. Also, let I be the ideal of F generated by the set of all the elements of the forms;

(1) $(x \dot{+} y) \dot{-} (x+y)$ for any x,y ϵ M

(2) $\alpha \cdot x \dot{-} (\alpha x)$

From (1.2.4), we see that I is homogeneous. Hence we may define a tensor algebra as follows.

Definition (*2.1.1*): The quotient algebra $T = F/I = \sum_{h\epsilon Z} T_h$, where $T_h = F_h / I \cap F_h$, is called a tensor algebra over M.

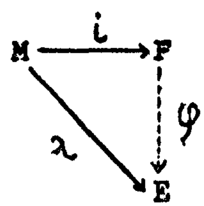
According (1.2.5). T is a Z-graded algebra. Furthermore, it preserves the following universal property.

Proposition (*2.1.2*): If λ is a linear mapping of M into an algebra E over A, then λ can be extented to a homomorphism of T into E which makes the diagram (II) commute.

PROOF: Define a mapping φ: F⟶E such that

$$\varphi(p_\sigma) = \varphi(x_{i_1} \cdots\cdots, x_{i_h})$$
$$= \lambda(x_{i_1}) \cdots\cdots \lambda(x_{i_h})$$

where $p_\sigma = x_{i_1} \cdots, x_{i_h}$ is a basis in F. By the consruction of a free algebra in 1.1., it is easily seen that φ is a homomorphism of F⟶E. which extends λ.



(I)

Next, we shall prove φ (I) $=0$, where I is an ideal of F defined as above. Since

$\varphi(x \dot{+} y \dot{-} (x+y))$ $(x,y \epsilon M)$

$= \varphi(x) + \varphi(y) - \varphi(x+y)$

$= \lambda(x) + \lambda(y) - \lambda(x+y)$

$= 0$ $(\because \lambda$ *ix linear*$)$

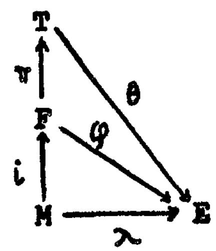
and

$\varphi(\alpha \cdot x \dot{-} \alpha x)$

$= \varphi(\alpha \cdot x) - \varphi(\alpha x)$

$= \alpha \cdot \varphi(x) - \varphi(\alpha x)$

$= \alpha \lambda(x) - \lambda(\alpha x) = 0$



(II)

we have φ $(I) = 0$. Hence φ defines a homomorphism θ : T⟶E which extends φ, i.e.

$(\theta\circ\pi)(x)=\varphi(x)=\lambda(x) \qquad \forall x\epsilon M$

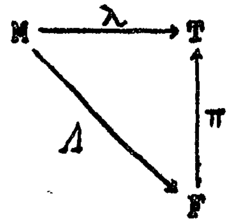
Remark (2.1): (1°) The definition of a tensor algebra in (2.1.1) coincides with that in [2] . For convenience, we prefer to use the later one in the next section. (3)

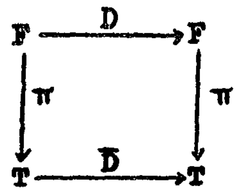
2.2 Derivations in a tensor Algebra.

Let M be a module over A and $T=\sum_{h\epsilon Z} T_h$ the tensor algebra over M. we shall prove the following.

Proposition (2.2.1): If λ is a linear mapping : $M\longrightarrow T_{\nu+1}$ (ν : any integer ≥ -1), then λ can be extended uniquely to a derivation in T(of degree ν).

PROOF: Evidently, M generates T. Using the same method in the proof of (1.5.1), we can prove the uniqueness. we shall prove the existence. Denote by π the canonical map : $F_{\nu+1}\longrightarrow T_{\nu+1}$, where $T_{\nu+1}=F_{\nu+1}/I\cap F_{\nu+1}$. For each $x\epsilon M$, we select an element $\Lambda(x)\epsilon F_{\nu+1}$ such that $\lambda(x)=\pi(\Lambda(x))$. This defines a map Λ: $M\longrightarrow F_{\nu+1}$ such that the diagram in (I) is commutative. Since M is a free system of generators of F, by (1.5.1), there exists uniquely a derivation D of F such that $D(x)=\Lambda(x)$ for every $x\epsilon M$. Note that D is of degree ν. Now, we shall prove that





(1) $D(I)\subset I$

In fact, we have

$$\begin{aligned}&\pi(D(x\dot{+}y\dot{-}(x+y))),\quad x,y\epsilon M\\ &=\pi[D(x)+D(y)-D(x+y)]\\ &=\pi[\Lambda(x)+\Lambda(y)-\Lambda(x+y)]\\ &=(\pi\circ\Lambda)(x)+(\pi\circ\Lambda)(y)-(\pi\circ\Lambda)(x+y)\\ &=\lambda(x)+\lambda(y)-\lambda(x+y)\\ &=0 \qquad (\because \lambda \text{ is linear})\end{aligned}$$

This implies that $D(x\dot{+}y\dot{-}(x+y))\epsilon\ ker\pi=I$. Similarly $D(\alpha\cdot x-\alpha x)\epsilon$ I. By (1.4.4), we have $D(I)\subset I$. From (1.4.5), we see that the induced map $\overline{D}$: $T\longrightarrow T$, obtained from D, is a derivation of T. To prove that $\overline{D}$ extends λ, let $x\epsilon M$. Then $x=\pi(x)$ and

$$\overline{D}(x)=\overline{D}(\pi(x))=\pi(D(x))=\pi(\Lambda(x))=\lambda(x). ||$$

3. Clifford Algebra

3.1 Quadratic Forms

Definition (3.1.1): Let M be a module over the besic ring A. A quadratic form over M is a mapping Q : M⟶A such that

(1) $Q(\alpha x)=\alpha^2 Q(x)$

(2) The mapping B : MxM⟶A defined by

$B(x,y)=Q(x+y)-Q(x)-Q(y)$

is bilinear. B is called bilinear form associated to Q.

Definition (3.1.2): Let x,y ϵ M. They are called orthogonal each other if $B(x,y)=0$, i.e. $Q(x+y)=Q(x)+Q(y)$.

Definition (3.1.3): Let R^k be a k-space over real numbers R. The mapping $Q_k:R^k\longrightarrow R$, defined by

$$Q_k(x_1\cdots\cdots,x_k)=-\sum_{i=1}^{k}x_i^2,\ \ x_i\epsilon R$$

is a quadratic form over R. It is called *definite negative form.*

Another quadratic form $Q_k'=-Q_k$ is called *definite positive form.*

Definition (3.1.4): Let $E=E_1\oplus E_2$ be a module, where E_i are submodules of E. Also let Q be a quadratic form over E. Then $E=E_1\oplus E_2$ is called an orthogonal decomposition of E relative to Q if for every $x\epsilon E_1$ and every $y\epsilon E_2$, x and y are orthogonal each other.

3.2. Clifford Algebra.

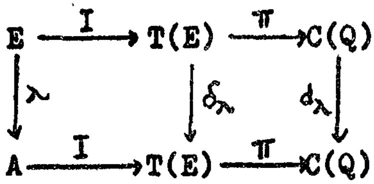
Let k be a commutative field and Q a quadratic form over a k-module E. Let $T(E)=\sum_{i\epsilon Z}T^iE$ be the tensor algebra over E, where $T^\circ E\approx k$, $T^1E\approx E$, and $T^iE=\overbrace{E\otimes\cdots\otimes E}^{i\ factors}$. Also, let I(Q) be the ideal of T(E) generated by the set of all elements of the form $x\otimes x-Q(x)\cdot 1$, $x\epsilon E$. Then I(Q) is homogeneous, by (1.2.4). Hence we may define the following.

Definition (3.3.1): The quotient algebra $C(Q)=T(E)/I(Q)$ is called the Clifford algebra of Q.

Proposition (3.2.2): Define $i_Q: E\longrightarrow C(Q)$ to be the canonical map given by the composition $E\xrightarrow{I}T(E)\xrightarrow{\pi}C(Q)$, where I is inclusion and π is the canonical homomorphism. Then i_Q is injective.

PROOF: we shall prove that $o\neq x\epsilon E$ implies $i_Q(x)\neq o$. Let $(x_i)_{i\epsilon I}$ be a base of E. Also, let $o\neq x=\sum a_ix_i\epsilon E$, $a_i\epsilon K$. Then $a_i\neq o$ for some $a_i\epsilon K$. Now, we define a mapping λ : E$\longrightarrow$K such that the image of an element in E under λ is the coefficient of x_i in that element; i.e. if $y=\sum_{i\epsilon I}b_ix_i$, then $\lambda(y)=b_i$ • Evidently λ is linear and $\lambda(x)=a_i\neq$o. By (2.2.1), there exists a derivation δ_λ in T(E) (of degree-1), which extends λ. Since

$$\delta_\lambda(x\otimes x-Q(x).1),\ x\epsilon E=T^1(E)$$
$$=\delta_\lambda(x\otimes x)-Q(x)\delta_\lambda(1)$$
$$=\delta_\lambda(x\otimes x)\qquad(\because\delta_\lambda(1)=0,\text{by }(1.4.3))$$
$$=\delta_\lambda(x)\otimes x-x\otimes\delta_\lambda(x)\qquad(by\ (1.3.5),$$
$$J^{-1}=J,\therefore J^{-1}(x)=J(x)=-x)$$
$$=\delta_\lambda(x)\ (1\otimes x-x\otimes 1)$$
$$=\delta_\lambda(x)\ (x-x)=0$$



so, $\delta_\lambda(I(Q))=0$. Therefore, by (1.4.5), δ_λ induces a derivatron d_λ of $C(Q)$, which makes the diagram commute. Hence, for this given x, we have,

$$(d_\lambda\circ\pi)\ (I(x))=(\pi\circ\delta_\lambda)\ (I(x))$$
$$=\pi[(\delta_\lambda\circ I)\ (x)]$$
$$=\pi[(I\circ\lambda)(x)]$$
$$=\pi(a_i\cdot 1)\neq 0\qquad(\because T^\circ E\cap I(Q)=0\text{ and }a_i\neq 0)$$

i.e. $d_\lambda(i_Q(x))\neq 0$

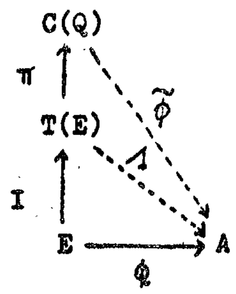
It follows that $i_Q(x)\neq 0$. Hence i_Q is injective.

From (3.2.2), *we may identify E with* $i_Q(E)\subset C(Q)$.

Proposition (3.2.3): Let ϕ : E$\longrightarrow$A be a linear map of E into a k-algebra with unit, say A, such that for all $x\epsilon E$, the identity $\phi(x)^2=Q(x)\cdot 1$ is valid. Then there exists a unique homomorphism $\tilde{\phi}$: $C(Q)\longrightarrow A$, such that $\tilde{\phi}\circ i_Q=\phi$. Such $\tilde{\phi}$ is called the exension of ϕ •

PROOF: By (2.1.2), ϕ can be extended to a homomorphism Λ: T(E)$\longrightarrow$A. If $x\epsilon E$, we have,

$$\Lambda(x \otimes x - Q(x)\cdot 1)$$
$$= [\Lambda(x)]^2 - Q(x)\cdot\Lambda(1)$$
$$= [\phi(x)]^2 - Q(x)\cdot 1 = 0$$



This implies that $\Lambda(I(Q))=0$. Thus Λ defines a homomorphism

$$\tilde{\phi}: C(Q) \longrightarrow \Lambda, \text{ such that } \tilde{\phi}\cdot\pi = \Lambda$$

It follows that $\tilde{\phi}\circ i_0 = \tilde{\phi}\circ\pi\circ I = \Lambda\circ I = \phi$. ||

Applying the argucment in (1.3) The following becomes trivial.

Proposition (*3.2.4*): C(Q) canbe decomposed as a Z_2-graded algebra. That is

(1) $C(Q) = C^0(Q) \oplus C^1(Q)$ where $C^0(Q)$ = *image of* $\sum_{i=0}^{\infty} T^{2i}E$

$$C^1(Q) = \textit{image of } \sum_{i=0}^{\infty} T^{2i+1}E$$

(2) $C(Q) = \sum_{i \epsilon z} C^i(Q)$, $C^i(Q) = T^iE / I(Q) \cap T^iE$

If $x_i \epsilon C^i(Q)$, $y_j \epsilon C^j(Q)$, then $x_i y_j \epsilon C^k(Q)$, $k = i+j$ (*mod 2*)

Definition (*3.2.5*): Let $E=E_0 \oplus E_1$ and $E'=E'_0 \oplus E'_1$ be two Z_2-graded algebras. Then $E\otimes E' = [E_0 \otimes E'_0 \oplus E_1 \otimes E'_1] \oplus [E_0 \otimes E'_1 \oplus E_1 \otimes E'_0]$ is also a Z_2-graded algebra. A skew tensor product of E and E′ is defined as follows:

(1) As a module, it has the same structure with $E \otimes E'$.

(2) For any $u\otimes u'$, $v\otimes v'$ in it, we define its multiplication by

$$(u\otimes u').(v\otimes v') = (-1)^{ij} uv \otimes u'v', \quad u' \epsilon E'_i \text{ and } v \epsilon E_j$$

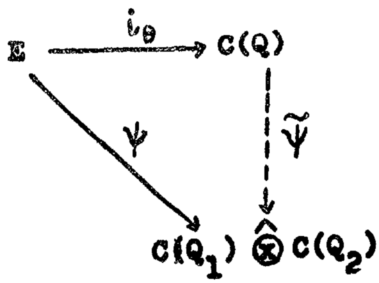
we denote it by $E \hat{\otimes} E'$. Such an algebra is also a Z_2-graded algebra.

Proposition (*3.2.6*): Suppose that $E=E_1 \oplus E_2$ is an orthogonal decomposition of E relative to Q and let Q_i denote the restriction of Q to E_i. Then there is an isomorphism.

$$\tilde{\psi}: C(Q) \longrightarrow C(Q_1) \hat{\otimes}_K C(Q_2)$$

PROOF: Define a mapping $\psi: E \longrightarrow C(Q_1) \otimes C(Q_2)$ by

$$\psi(x) = x_1 \otimes 1 + 1 \otimes x_2$$



where $x = x_1 + x_2$ and $x_i \epsilon E_i$. Then, we have

$$\psi(x)^2 = (x_1 \otimes 1 + 1 \otimes x_2).(x_1 \otimes 1 + 1 \otimes x_2)$$
$$= (x_1.x_1) \otimes 1 + x_1 \otimes x_2 + 1 \otimes (x_1.x_2)$$
$$- x_1 \otimes x_2 \quad (by\ (3.2.5))$$
$$= Q_1(x_1)(1\otimes 1) + Q_2(x_2).(1\otimes 1)$$
$$= [Q_1(x_1) + Q_2(x_2)](1\otimes 1) \quad \cdots\cdots\cdots\cdots (1)$$

where $1\otimes 1$ is the identity in $C(Q_1) \hat{\otimes} C(Q_2)$. By hypothesis,

$$B(x_1, x_2) = Q(x) - Q_1(x_1) - Q_2(x_2) = 0$$

Therefore, (1) becomes $\psi(x)^2 = Q(x)\cdot 1$. From (3.2.3), ψ can be extended to a homomorphism $\tilde{\psi}: C(Q) \longrightarrow C(Q_1) \hat{\otimes} C(Q_2)$ such that $\tilde{\psi}\circ i_0 = \psi$. Next, we define a mapping

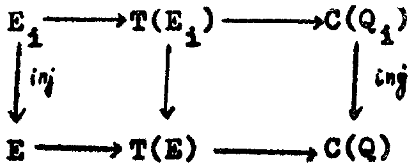
$$\phi: C(Q_1) \hat{\otimes} C(Q_2) \longrightarrow C(Q)$$

by $\phi(u\otimes v) = u\cdot v$, where $u \epsilon C(Q_1)$ and $v \epsilon C(Q_2)$. From the diagram next, we see that $u\cdot v \epsilon C(Q)$. Hence ϕ is well-defined. We shall prove that ϕ is a homomorphism of k-algcbra.

Let $u \otimes v,\ u' \otimes v' \epsilon\ C(Q_1) \hat{\otimes} C(Q_2)$, where $v \epsilon\ C^i(Q_2)$, $u' \epsilon\ C^j(Q_1)$, $i,j=0,1$.

Then, we have

$$\phi[(u \otimes v)\cdot(u' \otimes v')] = (-1)^{ij} uu'.vv' \cdots (2)$$



We need the following lemma;

Lemma: For any $x \epsilon E_1$ and $y \epsilon E_2$, we have $xy = (-1)yx$.

Pf.: $(x+y)^2 = x^2 + y^2 + xy + yx$

$= Q_1(x).1 + Q_2(y).1 + yx + xy$

$= Q(x+y) \cdot 1 + yx + xy$

$= Q(x+y) \cdot 1$

It follows that $xy = (-1)yx$.

Now, let us consider u′ and v in (2).

(1^0) If $v \epsilon\ C^0(Q_2)$, we may write $v = v_1 \cdots\cdots v_{2h}$, where $v_i \epsilon E_2$ and $v_i \neq v_j$. Applying Lemma, we got

$u'v = u'(v_1 \cdots\cdots v_{2h})$

$= (-1)^{2hj}(v_1 \cdots\cdots v_{2h})u' \qquad (\because\ u' \epsilon C^j(Q_1))$

$= vu'$

Therefore, (2) becomes

$\phi[(u \otimes v).(u' \otimes v')] = uu'.vv' = (uv).(u'v')$

$= \phi(u \otimes v) \cdot \phi(u' \otimes v')$.

Similarly for $u' \epsilon C^0(Q_1)$.

(2^0) If $u' \epsilon C^1(Q_1)$, and $v\ \epsilon\ C^1(Q_2)$ again using the same method in (1°). We got $u'v = -vu'$. Therefore

(2) becomes

$\phi[(u \otimes v).(u' \otimes v')]$

$= (-1)\ uu'.vv' = uv.\ u'v' = \phi(u \otimes v).\ \phi(u' \otimes v')$.

Thus, we proved that ϕ is a homomorphism of k-algebra. It remains to prove $\tilde{\psi} \circ \phi = 1_{C(Q_1) \hat{\otimes} C(Q_2)}$ and $\phi \circ \tilde{\psi} = 1_{C(Q)}$. Let $u = u_1 \cdots\cdots u_m \epsilon C(Q)$, $u_i \epsilon E$. Also, let $u_i = u_{i1} + u_{i2}$, $u_{ji} \epsilon E_j$, Then

$\phi\tilde{\psi}(u) = \phi\tilde{\psi}(u_1, \cdots\cdots u_m)$

$= \phi[\psi(u_1) \cdots\cdots \psi(u_m)]$

$= [(\phi\psi)(u_1)] \cdots\cdots [(\phi\psi)(u_m)]$

$= [\phi(u_{11} \otimes 1 + 1 \otimes u_{12})] \cdots\cdots [\phi(u_{m1} \otimes 1 + 1 \otimes u_{m2})]$

$= (u_{11} + u_{12}) \cdots\cdots (u_{m1} + u_{m2})$

$= u_1 \cdots\cdots u_m$

$= u$

Therefore, $\phi \circ \tilde{\psi} = 1_{C(Q)}$. Let $v = v_1 \cdots\cdots v_n$. Then

$\tilde{\psi} \circ \phi(u \otimes v) = \tilde{\psi}(u.v)$

$= \tilde{\psi}(u_1 \cdots\cdots u_m.v_1 \cdots\cdots v_n) \qquad (v_i \epsilon E_2)$

$= \psi(u_1) \cdots\cdots \psi(u_m)\psi(v_1) \cdots\cdots \psi(v_n)$

$= (u_1 \otimes 1) \cdots\cdots (u_m \otimes 1)(1 \otimes v_1) \cdots\cdots (1 \otimes v_n)$

$= [(u_1 \cdots\cdots u_m) \otimes 1][1 \otimes (v_1 \cdots\cdots v_n)]$

$= (u \otimes 1).(1 \otimes v)$

$= u \otimes v \quad (by\ (3.2.5))$

Therefore, $\tilde{\psi}\circ\phi = 1_{C(Q_1)} \hat{\otimes}_{C(Q_2)}$. Hence $\tilde{\psi}$ is an isomorphism.

The following property is trivial.

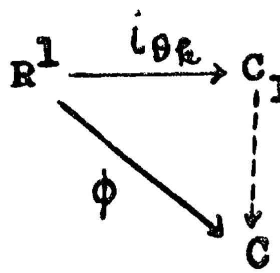
Proposition (3.2.7): Define a map $\alpha : E \longrightarrow C(Q)$ by $\alpha(x) = -i_Q(x)$ for all $x \epsilon E$. Then α extends to an automorphism of C(Q), denoted also by α. It is called the canonical automorphism of C(Q).

3.3 The Algebra C_k.

We are more interested in the algebras $C(Q_k)$, where Q_k is the definite negative form as we defined in (3.1.3). From now on, we denote by C_k the Clifford algebra of Q_k, and identify R_k with $i_{Q_k}(R^k) \subset C_k$ and R with $R\cdot 1 \subset C_k$. For $k=0$, $C_k=R$. Also, denote the k-tuple $(0,\cdots\cdots, 1,\cdots\cdots, 0)$ with 1 in the ith position by e_i.

Proposition (3.3.1): The algebra C_1 is isomorphic to C (the complex numbers) considered as an algebra over R. Further

$$C_k \cong C_1 \hat{\otimes} C_1 \hat{\otimes} \cdots\cdots \hat{\otimes} C_1 \qquad \text{(k factors)}$$



PROOF: Define a linear map $\phi: R^1 \longrightarrow C$ such that

$$\phi(e_1) = i, \text{ where } i^2 = -1,\ e_1 = (1)$$

Then, we have

$$\phi(x)^2 = a^2\phi(e_1)^2,\quad x = ae_1 \epsilon R^1,\ a \epsilon R$$
$$= -a^2 \cdot 1$$
$$= Q_k(x) \cdot 1$$

By (3.2.3), ∃ a homomorphism $\tilde{\phi}: C_1 \longrightarrow C$ which extends ϕ. Trivially, $\tilde{\phi}$ is an isomorphism.

$C_k \cong C_1 \hat{\otimes} \cdots \hat{\otimes} C_1$ follows from repeated application of (3.2.6).

Proposition (3.3.2): The basis $\{e_i\}_{i=1,\cdots,k}$ of R^k satisfies the relations

$$e_j^2 = -1,\quad e_ie_j + e_je_i = 0 \text{ if } i \neq j$$

PROOF:

$$e_j^2 = Q_k(e_j).1 = -1$$

and

$$(e_i + e_j)^2 = e_i^2 + e_j^2 + e_ie_j + e_je_i$$
$$= -1 + (-1) + e_ie_j + e_je_i$$
$$= Q_k(e_i + e_j).1$$
$$= -2$$

Hence $e_ie_j + e_je_i = 0.$ ||

Remark (3.3):

(1°) $\{e_i\}$ generates C_k.

(2°) Dim $(C_k) = 2^k$.

3.4. Determination of the Algebras C_k.

Let R, C and H respectively be real, complex and quarternion number fields. If F is any one of these fields, F(n) will be the full nxn matrix algebra over F. The following are well-known identities among these:

$$(3.4.1)\begin{cases} F(n)\cong R(n) \underset{R}{\otimes} F,\ R(n) \underset{R}{\otimes} R(m)\cong R(nm) \\ C\otimes C\cong C\oplus C \\ H\otimes C\cong C(2) \\ H\otimes H\cong R(4) \end{cases}$$

Let $C'_k=C(Q'_k)$, where Q'_k is the positive definite form. Now, we will determine C_k with respect to R, C and H.

Proposition *(3.4.2)*: There exist isomorphisms:

$$C_k\otimes C'_2\cong C'_{k+2}$$
$$C'_k\otimes C_2\cong C_{k+2}$$

PROOF: Consider the linear map : ψ : $R^{k+2}\longrightarrow C_k\underset{R}{\otimes} C'_2$ defined by

$$\psi(e_i)=\begin{cases} e_{i-2}\otimes e_1e_2 & if\ 2<i\leq k+2 \\ 1\otimes e_i & if\ 1\leq i\leq 2 \end{cases}$$

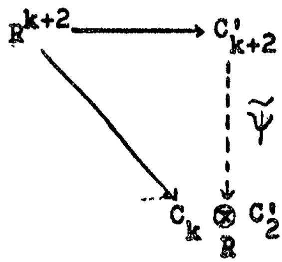
Since

$$\psi(e_i)^2=(e_{i-2}\otimes e_1e_2)^2,\ 2<i\leq k+2$$
$$=e^2_{i-2}\otimes (e_1e_2)^2$$
$$=e^2_{i-2}\otimes (-e^2_1 e^2_2) \qquad \text{(by (3.3.2))}$$
$$=[Q_k(e_{i-2})\cdot 1]\otimes[-(Q'_2(e_1)\cdot Q'_2(e_2))\cdot 1]$$
$$=1\otimes 1$$
$$=Q'_{k+2}(e_i)\cdot(1\otimes 1)$$

and

$$\psi(e_i)^2=1\otimes e^2_i \qquad ,\ 1\leq i\leq 2$$
$$=1\otimes[Q'_2(e_i)\cdot 1]$$
$$=1\otimes 1=Q'_{k+2}(e_i)\cdot(1\otimes 1)$$

by (3.2.3), ψ can be extended to a homomorphism $\tilde{\psi}$: $C'_{k+2}\longrightarrow C_k\otimes C'_2$. Because Dim $(C'_{k+2})=$ Dim $(C_k\otimes C'_2)=2^{k+2}$, $\tilde{\psi}$ must be an isomorphism.



Similarly, we can prove $C'_k\otimes C_2\cong C_{k+2}$

The following are trivial.

$$(3.4.3)\begin{cases} C_1\cong C \\ C_2\cong H \\ C'_1\cong R\oplus R \\ C'_2\cong R(2) \end{cases}$$

Now, applying (3.4.1), (3.4.2) and (3.4.3), we got the followings:

$$C_3\cong C'_1\otimes C_2\cong (R\oplus R)\otimes H\cong H\oplus H$$
$$C_4\cong C'_2\otimes C_2\cong R(2)\otimes H\cong H(2)$$
$$C'_4\cong C_2\otimes C'_2\cong C'_2\otimes C_2\cong C_4$$
$$C_8\cong C'_6\otimes C_2\cong C_4\otimes C'_2\otimes C_2$$
$$\cong C_4\otimes C_4$$
$$\cong H(2)\otimes H(2)$$

$$\cong R(2) \otimes H \otimes H \otimes R(2)$$
$$\cong R(2) \otimes R(4) \otimes R(2)$$
$$\cong R(16)$$

$$C_{k+4} \cong C'_{k+2} \otimes C_2 \cong C_k \otimes C'_2 \otimes C_2 \cong C_k \otimes C_4$$
$$C_{k+8} \cong C'_{k+6} \otimes C_2 \cong C_{k+4} \otimes C'_2 \otimes C_2 \cong C_k \otimes C_4 \otimes C'_2 \otimes C_2$$
$$\cong C_k \otimes C'_6 \otimes C_2$$
$$\cong C_k \otimes C_8$$

Propositon (3.4.4): If $C_k \cong F(m)$, *then* $C_{k+3} \cong F(16m)$

PROOF: $C_{k+8} \cong C_k \otimes C_8 \cong F(m) \otimes R(16) \cong F(16m)$.

Now, we summarize the results in the following table:

k	C_k	C'_k
1	C	$R \oplus R$
2	$H(1)$	$R(2)$
3	$H(I) \oplus H(1)$	$C(2)$
4	$H(2)$	$H(2)$
5	$C(4)$	$H(2) \oplus H(2)$
6	$R(8)$	$H(4)$
7	$R(8) \oplus R(8)$	$C(8)$
8	$R(16)$	$R(16)$

From (3.4.4), we see that both columns above are in a quite definite sense of period 8. If we move up eight steps, the field is left unaltered, while the dimension is multiplied by 16.

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