DUAL TOPOLOGICAL SPACES

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1. *Introduction.*

In the family of all topologies on a set X, there is a subfamily F with the property that if $\mathfrak{D} \in F$, then

$$\varphi(\mathfrak{D}) = \{U \mid (X-U) \in \mathfrak{D}\} \in F$$

where, and throughout this paper, $(X-U)$ is the complement of the set U relative to X. The elements of F are called dual topologies on X. For any nonempty set X, each topology on X containing a finite number of elements is a dual topology. So if X is finite, then every topology on X is a dual topology. Moreover, a dual topology can also be determined by the following equivalent condition.

For each $x \in X$, there exists an element $B_x \in \mathfrak{D}$ containing x such that for any $U_x \in \mathfrak{D}$ containing x, it must have $B_x \subset U_x$.

Since a dual topology is itself a cotopology (see Cullen [5], p. 17) the above local characterization of the dual topology can be easily proved with the help of the DeMorgan's laws.

Because of the clearness we shall use this local characterization as the definition of the dual topology. We call $(X, \mathfrak{D})$ or X the dual topological space, and B_x the local base for x. It is not difficult to see that the family $\mathcal{B}$ of all such B_x forms a base for $\mathfrak{D}$. Furthermore, let $\mathcal{E}$ be any base for $\mathfrak{D}$. If $E \in \mathcal{E}$, then for each $x \in E$, the local base B_x for x is a subset of E. Hence we call $\mathcal{B}$ the fundamental base for $\mathfrak{D}$.

The purpose of this paper is to discuss some basic problems about the dual topological spaces. In Section 2, we shall point out a necessary and sufficient condition for a component, the equivalence between the notions of component and quasi-component, and also that the pathwise connectedness is identical with connectedness in a dual topological space. In Section 3, we shall find a number of characterizations of the regularity for a dual topological space. Finally, a method of compactification for a dual T_0-space will be given in Section 4.

2. *Connectedness.*

A space $(X, \mathcal{J})$ is connected iff it contains no proper subsets that are both open and closed. Clearly, connectedness can also be characterized by the concept of separation. Two sets A and B form a separation of X, written $X = A|B$, when (1) $X = A \cup B$ and (2) $A \neq \phi \neq B$, and (3) $\bar{A} \cap B = \bar{B} \cap A = \phi$ where $\bar{A}$ is the closure of A. So, a space $(X, \mathcal{J})$ is connected iff it has no separation. A subset $E \subset X$ is connected iff the subspace $(E, \mathcal{J}_E)$ is connected. From this definition, we find that each element B_x of the fundamental base $\mathcal{B}$ is connected. The maximal connected subsets are called the components of X. The components are always closed (see Bourbaki [3], p.111), but on the other hand, an example shows that the components of X are not open in general.

Example 1. Let $(X,\mathcal{T})$ be the space of all rational numbers with the induced topology from the reals. Evidently, $\mathcal{T}$ is not the discrete topology on X, and hence for each $x\in X$, $\{x\}$ is not an open set. On the other hand, $(X,\mathcal{T})$ is totally disconnected (see Bourbaki [3], p.111), and so every $\{x\}$ is a component of X.

This example also shows that a component and its complement may not form a separation, however, the result will be different in a dual topological space.

Theorem 1. Let E be a connected proper subset of $(X,\mathcal{D})$. Then E is a component of X iff $X=E|(X-E)$.

Proof: (1) Necessity.

If E is a component of X, then as we have mentioned above, E is closed. On the other hand, if $x\in E$, then there exists a local base B_x for x such that $B_x\cup E$ is connected. This implies that $B_x\subset E$, and hence E is open. Since $E\neq\phi\neq X$, it must have $X=E|(X-E)$.

(2) Sufficiency.

Assume that $X=E|(X-E)$. If there exists a connected subset $K\supset E$, then it must have $K=E|(K-E)$. By the connectedness of K, we obtain $(K-E)=\phi$, so $K=E$. This shows that E is a component of X. |

The quasi-components of X are defined by the equivalence classes under the equivalence relation

$$R=\{(x,y)\mid \text{If } X=A|B, \text{ then } (x,y)\in A\times A \text{ or } (x,y)\in B\times B\}.$$

It is easy to see that each component of X is contained in some quasi-component, and every quasi-component is closed in X (see Cullen [5], pp. 225-226). Unfortunately, the quasi-components are not connected in general.

Example 2. Let $(X,\mathcal{T})$ denote the subspace of the plane R^2 consisting of the two points (0,0), (0,1) and all points of the set $\{(\frac{1}{n},y)\mid 0\leq y\leq 1$, n ranges over all natural numbers$\}$. Obviously, no separation $X=A|B$ has the property that $(0,0)\in A$ and $(0,1)\in B$. Thus $Q=\{(0,0),(0,1)\}$ is a quasi-component. However, Q is not a connected subset of X.

In a compact T_2-space $(X,\mathcal{T})$, quasi-components are components (see Cullen [5], pp. 268-269). Further, we shall show in the next theorem that a quasi-component of a dual topological space $(X,\mathcal{D})$ is not only a connected set but also identical with a component of $(X,\mathcal{D})$.

Theorem 2. In a dual topological space $(X,\mathcal{D})$, the two notions of component and quasi-component are equivalent.

Proof: (1) Component⇒quasi-component.

Let E be a component of X. As we have noted above, there is a quasi-component Q of X such that $E\subset Q$. Suppose that $E\neq Q$. Then there must exist two points $x,y\in Q$ such that $x\in E\subset Q$ and $y\in(Q-E)\subset(X-E)$, respectively. From Theorem 1, we obtain $X=E|(X-E)$. This contradicts the definition of a quasi-component. Hence $E=Q$, and therefore E is a quasi-component.

(2) Quasi-component⇒component.

Let Q be a quasi-component of X. If $Q=X$, then X is connected, so Q is a component. Now we assume that $Q\neq X$, it is clear that X is not connected. Let E be a maximal connected subset of Q. Suppose that E is not a component of X. Then there must exist a component K of X such that $E\subset K$ and $K\neq Q$. Since $K\neq X$, it follows from Theorem 1 that $X=K|(X-K)$; therefore, it must have $Q\subset K$. On the other hand, there always exists a quasi-component P of X such that $K\subset P$. Thus we obtain a contradiction that two equivalence classes P and Q have the relation $Q\subsetneqq P$. Hence E is a component of X, so it follows that $X=E|(X-E)$. Thus it must have $E=Q$. |

Let A be a subset of a space $(X,\mathcal{T})$. Then A is pathwise connected iff for every two points $x,y\in A$, there exists a continuous function $f:[0,1]\to A$ such that $f(0)=x$ and $f(1)=y$, respectively. Further, if for each point $x\in A$ and each open set U_x containing x, there exists a pathwise connected open set G_x containing x such that $G_x\subset U_x$, then A is said to be locally pathwise connected. In a topological space $(X,\mathcal{T})$, a pathwise connected set is also connected (see willard [6],p.197), but the converse is not true in general. The following example contains a subset of a space which is connected but not pathwise connected.

Example 3. Let $(X,\mathcal{T})$ be the topological space with the usual topology on the plane $X=R^2$. Let

$$A_0=\{(x,y)\mid |y|\leqq 1,\ x=0\},$$

and for each natural number n, let

$$A_n=\{(x,y)\mid |y|\leqq 1,\ x=\frac{1}{n}\},$$

$$B_n=\{(x,y)\mid y=-1,\ \frac{1}{2n+1}\leqq x\leqq\frac{1}{2n}\},$$

$$C_n=\{(x,y)\mid y=1,\ \frac{1}{2n}\leqq x\leqq\frac{1}{2n-1}\}.$$

It is clear that $H_1=A_0$ and $H_2=\bigcup_{n=1}^{\infty}(A_n\cup B_n\cup C_n)$ are both connected subsets of X. Further, every point of H_1 is an accumulation point of H_2, and $\overline{H}_2=H_1\cup H_2$. Hence $\overline{H}_2$ is a connected subset of X (see Pervin [2],p.52). Now suppose that $\overline{H}_2$ is pathwise connected. Let us consider the two points $(0,1)\in H_1$ and $(1,1)\in H_2$. Then there exists a continuous function f from $[0,1]$ into $\overline{H}_2$ such that $f(0)=(0,1)$ and $f(1)=(1,1)$. Let $f([0,1])=K$. Since H_1 is closed in X, $K\cap H_1\neq\phi$ is closed in K. It follows from the continuity of f that $f^{-1}(K\cap H_1)$ is closed in $[0,1]$. Further, let t be an arbitrary point of $f^{-1}(K\cap H_1)\subset[0,1]$ and $f(t)=(0,a)\in(K\cap H_1)$. Let $V=\{(x,y)\in K\mid |x|<0.1,\ |y-a|<0.1\}$ be an open set in K which contains $(0,a)$. Then there exists an open interval U of t in $[0,1]$ such that $f(U)\subset V$. Since U is connected in $[0,1]$ (see Bourbaki [3],pp. 336-337), $f(U)$ is also a connected subset of V under the continuous function f. Let G be a component of V which contains $(0,a)$. It must have $G\subset H_1$. Thus $f(U)\subset G\subset K\cap H_1$ or $U\subset f^{-1}(K\cap H_1)$. This means that $f^{-1}(K\cap H_1)$ is open in $[0,1]$

as well.

We have shown that $f^{-1}(K \cap H_1) \neq \phi$ is both open and closed in $[0,1]$. Because of the connectedness of $[0,1]$, it must have $f^{-1}(K \cap H_1)=[0,1]$ or $f([0,1])=K \subset H_1$. But $f(1)=(1,1) \notin H_1$. This contradiction shows that $\overline{H}_2$ is not pathwise connected.

This counterexample shows the fact that connectedness does not imply pathwise connectedness in general. However, these two concepts are identical in a dual topological space.

Theorem 3. A subset in a dual topological space is pathwise connected iff it is connected.

Proof: As we have pointed out above, a pathwise connected subset is also connected. On the other hand, a connected subset which is also locally pathwise connected must be pathwise connected (see willard [6], p.199). Thus to show the converse part of our theorem is sufficiently to show that a dual topological space is always locally pathwise connected.

For an arbitrary $x \in X$, by the nature of the fnudamental base, the proof will be complete if we can show that B_x is pathwise connected. We shall prove this by the following three possible cases.

(1) If $p \neq x$ and $q \neq x$ are two distinct points of B_x, then we define a function f from $[0,1]$ into B_x by

$f([0,1/2))=\{p\}$,

$f(1/2)=x$,

$f((1/2,1])=\{q\}$.

Now we shall show that f is a continuous function on $[0,1]$. Let U_x be an open set in B_x which contains x. Then it must have $B_x=U_x$. The entire set $[0,1]$ is the open set containing $1/2$ such that $f([0,1])=\{p,q,x\} \subset B_x$. Hence f is continuous at $1/2$. For $y \in [0,1/2)$, the constant function $f([0,1/2))=\{p\} \subset B_p$ ensures that f is continuous on $[0,1/2)$. Similarly, f is also continuous on $(1/2,1]$. Hence f is a continuous function from $[0,1]$ into B_x such that $f(0)=p$ and $f(1)=q$.

(2) If $p \neq x$ and $q=x$ (or $p=x$ and $q \neq x$) are two distinct points of B_x, then we define

$f([0,1))=\{p\}$,

$f(1)=x=q$.

The proof for the continuity of f in this case is similar to that in the case (1), and $f(0)=p$ and $f(1)=q$.

(3) If $p=q=x$, then there is nothing to prove.

We have shown that B_x is really a pathwise connected subset. Thus the proof of our theorem is complete. |

As we have pointed out in the beginning of this section and in the proof of the last theorem that the local base B_x of a point x in a dual topological space $(X,\mathcal{D})$ is connected, so $(X,\mathcal{D})$ is also locally connected. Thus the necessity part of the Theorem 1 is an immediate consequence of the fact that the components of a locally connected space are both open and closed (see Willard [6],p.200).

3. Regularity.

Separation axioms give many interesting and important results which make points and sets of a space topologically distinguishable. Unfortunately, a dual topological space is T_1 iff it is a discrete space. This follows from the fact that each subset which contains one point in such space is both open and closed (see Pervin [2], p.70). Thus we shall discuss, in this section, the axioms which do not connect with the T_1-axiom. A space $(X,\mathcal{T})$ is regular iff for any closed subset H of X and any point $x\in X$ not in H, there exist two disjoint open sets U and V such that $x\in U$ and $H\subset V$. One of the characterizations of the regularity for a dual topological space will be given in the following theorem, and another will be used as a lemma in the proving of this theorem.

Theorem 4. A dual topological space is regular iff every open set is closed.

Proof: Let us first establish a local characterization of the regularity for a dual topological space which will be useful in the proving of our theorem. A dual topological space is regular iff each local base B_x is both open and closed.

To show this characterization we assume that $(X,\mathcal{D})$ is regular. Then for each B_x, $H=(X-B_x)$ is a closed subset and $x\notin H$. Thus there exist two disjoint open sets U and V such that $x\in U$ and $H\subset V$. Since B_x is the smallest open set containing x it must have $B_x\subset U$. Suppose that $H\neq V$. Then we have $(X-V)\subsetneq(X-H)=B_x\subset U$. This means that there exists a point $y\in U$ such that $y\notin(X-V)$, and so $y\in U\cap V$. This contradiction follows that $H=(X-B_x)=V$ is open. Hence B_x is both open and closed.

Conversely, let us assume that each local base B_x is both open and closed, and let x be a point not in a closed set H. Then x belongs to the open set $(X-H)$. By the fact that B_x is the smallest open set containing x, it must have $B_x\subset(X-H)$, and hence $H\subset(X-B_x)$. It is clear that B_x and $(X-B_x)$ are the desired disjoint open sets containing x and H, respectively. Hence $(X,\mathcal{D})$ is regular.

Now we shall give a general proof about the necessary condition of our theorem. Let G be an arbitrary open set of the regular space $(X,\mathcal{D})$. Suppose that there exists a point $x\in(\bar{G}-G)$. Then it must have $x\notin G$ and $B_x\cap G\neq\phi$ with the local base B_x of x. Thus there exists a point $z\neq x$ and its local base B_z such that $z\in B_z\subset B_x\cap G$. From the local characterization we have just proved above, B_z is a closed subset and $x\notin B_z$. Thus, by regularity, there exist two disjoint open sets U and V such that $B_z\subset U$ and $x\in B_x\subset V$. But $z\in B_z\subset U$ and $z\in B_z\subset V$. This contradiction shows that $\bar{G}=G$, and hence G is both open and closed.

The sufficient condition of our theorem can be easily proved by the local characterization. Hence the proof of our theorem is complete. |

Remarks:

(1) The last theorem shows that a regular dual topological space which is not indiscrete will never be connected.

(2) An important characterization of the regularity for a general topological

space $(X,\mathcal{J})$ is that every point $x\in X$ and open set G containing x, there exists an open set E such that $x\in E\subset\bar{E}\subset G$ (see pervin [2],pp.87-88). The local characterization of regularity for a dual topological space can be proved as an immediate consequence of this fact.

(3) In a general topological space, there is no implication between regularity and normality (see Willard [6],p.100). However, in a dual topological space, Theorem 4 shows that regularity implies normality. But the converse implication is not true. For example, let $X=\{a,b,c\}$ and $\mathcal{D}=\{\phi,\{a\},\{b\},\{a,b\},X\}$. Then $(X,\mathcal{D})$ is a normal dual topological space because no two proper closed subsets are disjoint. On the other hand, we consider the point $b\in X$ and the closed subset $H=\{a,c\}$. It is clear that the only one open set containing the point c is X. Hence $(X,\mathcal{D})$ is not regular.

4. *Compactification.*

Compact spaces play an important role in all branches of mathematics. Though not every topological space is compact, yet it is possible to bring noncompact spaces within reach of some advantages of compactness by the device called compactification. A compactification of a space $(X,\mathcal{J})$ is a compact space $(X^*,\mathcal{J}^*)$ such that $(X,\mathcal{J})$ is homeomorphic to a dense subset of $(X^*,\mathcal{J}^*)$. The Wallman compactification and the Stone-Cech compactification are established on the basis of T_1-spaces and Tychonoff spaces, respectively (see Thron [4], pp.139-142 ; Steiner [10], pp. 295-304). As we have pointed out that a dual topological space which is T_1 is the discrete space, so that it is not of much interest for us to study these compactifications here. Another important compactification is the one-point compactification which is meaningful for all topological spaces (see Bushaw [1], p.90). However,this compactification is somewhat unsatisfactory since, for instance, the one-point compactification of a T_2-space needs not be a T_2-space. Furthermore, the following example shows that the one-point compactification of a dual topological space is not a dual topological space.

Example 4. Let $X=N$, the set of all natural numbers, and $\mathcal{D}$ be the discrete topology on X. It is obvious that $(X,\mathcal{D})$ is a dual noncompact space.

Now $X^*=X\cup\{\infty\}$ and $\mathcal{D}^*=\{G^*|G^*=G\in\mathcal{D}$ or $G^*=G\cup\{\infty\}$ where $(X-G)$ is a closed and compact set in $X\}$. We note that $\{\infty\}\notin\mathcal{D}^*$; and for each $n\in N$, $G^*=(X-\{n\})\cup\{\infty\}$ is an open set in X^*, It is clear that the smallest open set containing the point ∞ must be $\{\infty\}$. Hence $(X^*,\mathcal{D}^*)$ is not a dual topological space by definition. We shall give a compactification of a dual topological space, in this section, under which the properties of a dual topological space are preserved.

Theorem 5. Let $(X,\mathcal{D})$ be a dual T_0-space. Then there exists a compactification which is also a dual T_0-space.

Proof: We shall break our proof into a number of parts.

(1) Let us first consider the subcollection $\Theta=\{B_x\ x\in X,\ B_x\in\mathcal{B}\}\subset\mathcal{D}$. Then we define the function $\varphi:X\to\Theta$ by $\varphi(x)=B_x$. It is clear that φ is onto. We now show that φ is 1-1. Assume that $B_x=B_z$ with $x\neq z$, then by the fundamental base, all open sets containing one of x and z must contain the other. This contradicts

to the T_0-axiom. Thus we have shown that φ is a 1-1 and onto function.

(2) Let us next consider the family $\mathcal{U}$ of all sets $P[G]=\{H \mid H=\bigcup_{x\in I} B_x, \phi\neq I\subset G\}$, where $G\neq\phi$ ranges over all of $\mathfrak{D}$. We note that $P[\phi]$ is empty, and each $P[G]$ is a subset of $\Theta^*=(\mathfrak{D}-\{\phi\})$. Furthermore, the family $\mathcal{U}$ forms a base for a topology $\mathcal{T}^*$ on Θ^* because it satisfies the following conditions:

(a) $P[X]=\Theta^*$,

(b) $P[G_1]\cap P[G_2]=P[G_1\cap G_2]$

Since $\mathcal{B}=\{B_x \mid x\in X\}$ is a base for $\mathfrak{D}$, $P[X]=\{H \mid H=\bigcap_{x\in I} B_x, \phi\neq I\subset X\}=(\mathfrak{D}-\{\phi\})=\Theta^*$. Hence the condition (a) is true.

If $H\in P[G_1]\cap P[G_2]$, then H is a nonempty subset of X. We note that for any $K\in P[G]$, if a chain $C=\{I_j \mid K=\bigcup_{x\in I_j} B_x, I_j\subset G\}$ is ordered by ascending inclusion, then $\bigcup I_j$ is again an element of C. Hence, by Zorn's lemma, there is a maximal nonempty subset $I_k\subset G_k (k=1,2)$ respectively such that $H=\bigcup_{x\in I_1} B_x=\bigcup_{z\in I_2} B_z$. Now for any $x\in I_1$, suppose that $x\notin I_2$, by the fact that $x\in B_x\subset H=\bigcup_{z\in I_2} B_z$, there exists $z\in I_2$ such that $x\in B_z$. It follows from the nature of the fundamental base that $x\in B_x\subset B_z\subset G_2$ and $H=\bigcup_{z\in I_2} B_z \cup B_x$. This contradicts to the maximality of I_2. Thus it must have $x\in I_2$. This means that $I_1\subset I_2$. Obviousely, we can conclude tha $I_1=I_2$. This fact ensures that $P[G_1]\cap P[G_2]\subset P[G_1\cap G_2]$. The proof of the converse inclusion is simple. Thus the condition (b) holds.

(3) To show that φ is a homeomorphism from X onto Θ, we first establish that, for any open set $G\subset X$,

$$\varphi(G)=\Theta\cap P[G].$$

Let $x\in G$. Then $B_x\in\Theta$ and also $B_x\in P[G]$. Hence $\varphi(G)\subset\Theta\cap P[G]$. On the other hand, if $B_x\in\Theta\cap P[G]$, then $B_x\in P[G]$. Thus there exists a subset $I\subset G$ such that $B_x=\bigcup_{z\in I} B_z$, and therefore, $x\in B_z\subset B_x$ for some $z\in I$. But $B_x\subset B_z$, so $B_x=B_z$. As we have noted before, it must have $x=z\in I\subset G$. This means that $B_x\in\varphi(G)$, and thus $\varphi(G)\supset\Theta\cap P[G]$.

Now since the 1-1 condition of φ has been proved in the part (1), the equality $\varphi(G)=\Theta\cap P[G]$ ensures that φ^{-1} is a continuous function from Θ onto X. Applying φ^{-1} to both sides of $\varphi(G)=\Theta\cap P[G]$, we obtain $\varphi^{-1}(\Theta\cap P[G])=G$. Since $\Theta\cap P[G]$ forms a base for the subspace $(\Theta,\mathcal{T})$, we conclude that φ is continuous (see Cullen [3], p.31).

(4) In this part we shall show that Θ is a dense subset of $(\Theta^*,\mathcal{T}^*)$. If $\mathcal{L}$ is a nonempty open set of $(\Theta^*,\mathcal{T}^*)$, then $\mathcal{L}$ contains at least one base element P[G]. By the definition of P[G], every P[G] contains all $B_x\in\Theta$ with $x\in G$. Since $G\neq\phi$, it must have $P[G]\cap\Theta\subset\mathcal{L}\cap\Theta\neq\phi$. Thus Θ is a dense subset of $(\Theta^*,\mathcal{T}^*)$. Up to this part, we have shown that $(\varphi, (\Theta^*,\mathcal{T}^*))$ is an extension of $(X,\mathfrak{D})$ (see Thron [3],p.132).

(5) To show that $(\Theta^*,\mathcal{T}^*)$ is a compactification of $(X,\mathfrak{D})$, it remains to show that $(\Theta^*,\mathcal{T}^*)$ is a compact space. We note that $X\notin P[G]$ for any element $G\in(\mathfrak{D}-$

$\{X\}$). This means that X belongs to one and only one base element $P[X]=\Theta^*$ which is the ground set. Thus every open set containing $P[X]$ is $P[X]$ itself. It is clear that any open covering of Θ^* must contain $P[X]$. Hence $(\Theta^*,\mathcal{T}^*)$ is compact.

(6) In this part we shall show that $(\Theta^*,\mathcal{T}^*)$ is a T_0-space. Let $G_1\neq G_2$ be any two distinct points in Θ^*. Without loss of generality, we assume that there exists $x\in G_1$ such that $x\notin G_2$. Then $P[G_2]$ is an open set which contains G_2 but not G_1. To prove this fact, we suppose that $G_1\in P[G_2]$. Then there exists a subset $I\subset G_2$ such that $G_1=\bigcup_{z\in I} B_z$. From the nature of the fundamental base, every B_z with $z\in I$ is a subset of G_2. Thus we have $G_1=\bigcup_{z\in I} B_z\subset G_2$. This is a contradiction. Hence $(\Theta^*,\mathcal{T}^*)$ is a T_0-space.

(7) Finally, the proof of our theorem will be complete if we can show that $(\Theta^*,\mathcal{T}^*)$ is a dual topologicol space. To do this is to find the fundamental base for the space $(\Theta^*,\mathcal{T}^*)$. Now we consider the base $\mathcal{U}=\{P[G]\,|\,G\in\mathcal{D},G\neq\phi\}$ which is given in the part (2). Let $G\in\Theta^*$ arbitrarily. Then $G\neq\phi$ is an open set in X. As we have pointed out in the part (6), for any open set $H\subset X$ which does not contain G as a subset, it must have $G\notin P[H]$. On the other hand, if $G\subset H$, then $G\in P[G]\subset P[H]$. Thus $P[G]$ is the smallest open set containing G. We have proved that $\mathcal{U}$ is not only a base but also the fundamental base for $(\Theta^*,\mathcal{T}^*)$. Hence $(\Theta^*,\mathcal{T}^*)$ is a dual topologicol space. ∎

Remarks:

(1) In establishing the base $\mathcal{U}$ for $\mathcal{T}^*$ in the part (2) of the last theorem, one may naturally hope that the family $\mathcal{V}=\{P[B_x]\,|\,x\in X,\ B_x\in\mathcal{B}\}$ could form a base for $\mathcal{T}^*$ so that there would be more closer relations between the two spaces $(X,\mathcal{D})$ and $(\Theta^*,\mathcal{T}^*)$. Unfortunately, this is not true. For example, we consider the set $X=\{a, b, c, d\}$ and the fundamental base $\mathcal{B}=\{\{b\},\{c\},\{a,b,c\},\{b,c,d\}\}$ for a dual topology $\mathcal{D}$ on X. It is clear that $(X,\mathcal{D})$ is a T_0-space. Now $P[\{a,b,c\}]\cap P[\{b,c,d\}]=P[\{b,c\}]=\{\{b\},\{c\},\{b,c\}\}$. It is obvious that the only possible way to make $P[\{b,c\}]$ as a union of the elements of $\mathcal{V}$ is to consider the elements $P[\{b\}]$ and $P[\{c\}]$. However, $P[\{b\}]\cup P[\{c\}]=\{\{b\},\{c\}\}\neq P[\{b,c\}]$, so $\mathcal{V}$ is not a base for $\mathcal{T}^*$ (see Bushaw [1],p.47).

(2) Once we have established that the family $\mathcal{U}=\{P[G]\,|\,G\in\mathcal{D},G\neq\phi\}$ is a base for $\mathcal{T}^*$, it seems reasonable to hope that $\mathcal{U}$ might be not only a base but a topology on Θ^* because $\mathcal{D}$ is a topology already. However, this is not true. For the same example given in the remark (1), $X=\{a,b,c,d\}$ and $\mathcal{D}=\{\phi,\{b\},\{c\},\{b,c\},\{a,b,c\},\{a,b,d\},X\}$. Now we consider the union $P[\{b\}]\cup P[\{c\}]=\{\{b\},\{c\}\}$. Then it is clear that $\{\{b\},\{c\}\}\notin\mathcal{U}$, and so $\mathcal{U}$ is not a topology.

(3) It should be noted that if the property "$\phi\neq I\subset G$" for defining the set $P[G]$ in the part (2) is replaced by "$I\subset G$", the family $\mathcal{U}$ also forms a topology $\mathcal{T}^{**}$ on the set $\Theta^{**}=\mathcal{D}$ but not on Θ^*. However in this case, Θ is not a dense subset of $(\Theta^{**},\mathcal{T}^{**})$ because $P[\phi]=\{\phi\}$ is an open subset of Θ^{**} and $P[\phi]\cap\Theta$ is empty.

REFERENCES

〔1〕 D. Bushaw, Elements of General Topology, John Wiley and Suns, New York (1963).

〔2〕 W. J. Pervin, Foundations of General Topology, Academic Press, New York (1964).

〔3〕 N. Bourbaki, General Topology, Part I (transl.), Addison-Wesley, Reading (1966).

〔4〕 W. J. Thron, Topological Structures, Holt, Rinehart and Winston, New York (1966).

〔5〕 H. F. Cullen, Introduction to General Topology, D.C. Heath, Boston (1968).

〔6〕 S. Willard, General Topology, Addison-Wesley (1971).

〔7〕 R. M. Brooks, On Wallman Compactification, Fund. Math. 60, pp. 157-173 (1957).

〔8〕 H. Tamano, On Compactification, J. Math. Kyoto Univ. I, pp. 162-193 (1962).

〔9〕 W. C. Hsieh, On the Family of All Topologies on A Set, Tunghai J. VII-2, pp. 61-64 (1966).

〔10〕 E. F. Steiner, Wallman Spaces and Compactifications, Fund. Math. 61, pp. 295-304 (1968).

對偶拓樸空間

解 萬 臣

在集合 X 上之拓樸族中有一種拓樸 $\mathfrak{D}$ ，其諸元素之餘集合所形成之集合仍為一拓樸，稱 $\mathfrak{D}$ 為對偶拓樸，且稱 (X,$\mathfrak{D}$) 為對偶拓樸空間。本文就對偶拓樸空間某些基本問題提出討論之。

第二節中指出對偶拓樸空間上 (1) 分支 (component) 之充要條件，(2) 分支與擬分支 (quasi-component) 互為等價，及 (3) 連通 (connectedness) 與路線連通 (pathwise connectedness) 二者同義。第三節中將舉出數種有關正則對偶拓樸空間 (regular dual topological space) 之定性 (characterization) 。最後在第四節中就 T_0—對偶拓樸空間提出一種緊緻化的方法。

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DUAL TOPOLOGICAL SPACES

Wan-chen Hsieh

In the family of all topologies on a set X, there is a subfamily $\mathfrak{F}$ with the property that if $\mathfrak{D}\in\mathfrak{F}$ then the collection $\varphi(\mathfrak{D})$ of all complements of the elements of $\mathfrak{D}$ is also a topology on X, i.e., $\varphi(\mathfrak{D})=\{U\mid(X-U)\in\mathfrak{D}\}\in\mathfrak{F}$. We call $\mathfrak{D}$ a dual topology on X, and (X,$\mathfrak{D}$) a dual topological space. This paper is to discuss some basic problems about the dual topological spaces.

In the section 2, we shall point out on a dual toqoiogical space (1) a necessary and sufficient condition for a component, (2) the equivalence between the notions of component and quasi-component, and (3) the equivalence between the notions of connectedness and pathwise connectedness. In the section 3, we shall find a number of characterizations of the regularity of a dual topological space. Finally, a method of compactification for a dual T_0-space will be given in the section 4.