ON THE PROPERTIES OF LOCALLY COMPACTNESS AND LOCALLY CONNECTEDNESS

BY

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I. Introduction

(1.1) A space X is locally compact if and only if each point of X has a compact neighborhood.

(1.2) A space X is locally compact if and only if each point of X has a closed compact neighborhood.

(1.3) A space X is locally compact if and only if each point of X has a neighborhood basis consisting of compact subsets of X.

The first two definitions, (1.1) and (1.2), of local compactness were discussed in "class-room note"[6]; The third definition (1.3) of local compactness was discussed recently by J. L. Gross [3]. In this paper, we show in section two that if those three definitions are restrict to Hausdorff or regular space are equivalent and any closed subset of those three spaces is hereditary. Since locally connectedness is defined as the same way as (1.3); so that to section three; we can apply this definition to prove that there is a discontinuous function which preserves compactness and connectedness. To the last section; we use this definition to the uniform space and also extend the partitionable in metric space to uniform space, so that we get the conclusion if X is a uniform space which is locally connected and partitionable, then X has property S.

II. Equivalence and Hereditary

(1) Equivalence

Theorem 1. In a Hausdorff space or regular space, the three definition of locally compactness are equivalent. But in an arbitrary spaces need not be equivalent.

Proof:

Since X is (1-1)-locally compact, $P \epsilon X$, has a compact neighborhood K in X. Let N be an arbitrary neighborhood of P in X. we are going to prove the existence of a closed compact neighborhood M of P contained in N. First, assume that X is regular. There exists a closed neighborhood M in X contained in $K \cap N$. As a closed subset of a compact space K, M is also compact. Next, assume X to be a Hausdorff space. then K is a normal Hausdorff space and hence is also regular. Since $K \cap N$ is now a neighborhood of P in a regular space K, there is a closed neighborhood M of P in K contained in $K \cap N$. But K is closed. Hence, M is both closed and compact in X. Since K itself is a neighborhood of P in X, it follows that M is also a neighborhood of P in X. In both cases, M is a closed compact neighborhood of P in X contained in N. This proves (1.1)$\Longrightarrow$(1.2), the following proves (1.2)$\Longrightarrow$(1.3) is seen by [7; p. 90]; it is also easily to get the converse implication (1.3) $\Longrightarrow$ (1.1). Thus, we get the conclusion.

counterexamples:

Example 1 (1.1)$\not\Longleftarrow$(1.2)

Let S denote the set of real numbers and let $\mathcal{J}$ be the topology determined by $\{Up \mid P$ is real and $Up=\{x \mid x \geq p\}\}$ as base. Each point P has a compact neighborhood; namely, Up which, by the way, is not closed; Further, for all neighborhoods of P, the closure of these neighborhoods are also not compact. Thus (1.1) not equivalent to (1.2).

Example 2 (1.2)$\not\Longleftarrow$(1.3)

Let $Y=\{(x,y) \mid 0 \leq x \leq k,\ 0 \leq y \leq k,\ k \epsilon N\}$

Define $B(a,b)=\{(x,y) \epsilon Y \mid y \leq -(\frac{a}{b})x+b,\ a,\ b \epsilon Rp\}$ where $Rp=\{x \mid x \epsilon R,\ x>0\}$

$B(a,b)$ consists of all points of Y which are below or on the straight line through $(a,0)$, $(0,b)$.

then $P=\{B(a,b) \mid a,\ b \epsilon Rp\}$ is a subbase for a topology on Y. Clearly Y is the only open subset of Y which contains the point (k,k). Thus Y itself is a member of every opon cover of Y, and so Y is compact. But then Y also satisfies (1.2). On the other hand, $B(k,k)$ is a neighborhood of the point $(k,0)$. Let U be any neighborhood of $(k,0)$ such that $U \subseteq B(k,k)$. We will show that U is not compact. Suppose that V is a member of the base for the topology such that $V \subseteq U$ and $(k,0) \epsilon V$ say

$$V=B(a_1,\ b_1) \cap \cdots\cdots\cdots\cdots \cap B(a_n,\ b_n)$$

Since $(k,0) \epsilon B(a_j,\ b_j)$ for $j=1,\ 2,\cdots\cdots\cdots\cdots,n$

We have $a_j \geq k$

Hence for each j, $B(k,b_j) \leq B(a_j,\ b_j)$, Take

$b_0=\min\{b_1\cdots\cdots,b_n\}$ Thus $B(k,\ b_0) \subseteq B(a_j,\ b_j)$ for each j,

and so $B(k,\ b_0) \subseteq V$, Now define

$$Q=\{B(k,\ \frac{1}{2}b_0)\} \cup \{B(k-\frac{k}{n},\ n-1) \mid n=k+1,\ k+2,\cdots\cdots\}$$

We will show that Q covers $B(k,k)$ and hence U.

clearly, $(k,0) \epsilon B(k,\frac{1}{2}b_0)$

if (x,y) is any other point of $B(k,k)$ then $x<k$ choose an integer

$S>\frac{k}{k-x}+k$, Then

$$k-x=\frac{s}{k}(k-x)-(\frac{s}{k}-1)(k-x)<\frac{s}{k}(k-x)-1$$

$$= s - \frac{s}{k}x - 1$$

$$= -\frac{s}{k}x + s - 1$$

$$= -\frac{s-1}{k-\frac{k}{s}}x + (s-1)$$

But $(x,y)\epsilon B(k,k)$ implies $Y \leqslant k-x$. Hence

$$Y \leqslant -\frac{s-1}{k-\frac{k}{s}}x + (s-1)$$

and so $(x,y)\epsilon B(k-\frac{k}{s}, s-1)$. This shows Q covers $B(k,k)$ and therefore Q covers U.

If $m_1 \leqslant m_2$, then $B(k-\frac{k}{m_1}, m_1-1) \subseteq B(k-\frac{1}{m_2}, m_2-1)$,

hence,if Q contains a finite subcover of U,

then there is an integer $M \geq k+1$ such that

$U \subseteq B(k, \frac{1}{2}b_0) \cup B(k-\frac{k}{M}, M-1)$.

But the point $(k-\frac{k}{2M}, \frac{b_0}{2M})$ is in $B(k, b_0)$, and therefore in U.

Yet $(k-\frac{k}{2M}, \frac{b_0}{2M})$ is neither in $B(k, \frac{b_0}{2})$ nor in $B(k-\frac{k}{M}, M-1)$

Hence Q does not contain a finite subcover of U, so U is not compact,

Thus Y is not locally compact: (1.3)

Example 3 (1.3) $\not\Rightarrow$ (1.2)

Let X be the interval $[-1, 1]$in R. Consider the following equivalence relation S on R; if $x \neq \pm 1$, the equivalence class of x contains of x and $-x$; the equivalence class of 1 (resp. -1) consists of 1 (resp. -1) alone. Thus S is open and that the quotient space X/S is accessible [7;p134] but not Hausdorff, then every point of X/S has a compact neighborhood; and that the images α,β of 1 and -1 in Y have compact neighborhoods which are not closed. Those three above counterexamples tell us that in arbitrary spaces, they are not equivalent.

(2) Hereditary

Let $(R,\mathfrak{J})$ be the space of rationals with the relative topology and let $(x, \mathfrak{J}^*)$ be the one point compactification of $(R,\mathfrak{J})$. We know that $(X, \mathfrak{J}^*)$ is (1-1)-locally compact space, but the subspace $(R,\mathfrak{J})$ is not (1.1) -locally compact space. Since $(X, \mathfrak{J}^*)$ is a C-C space [example 1, 12], so that $(X, \mathfrak{J}^*)$ is also (1-2)- locally compact space, and that $(R, \mathfrak{J})$ is also not (1-2)- locally compact space. From [Exercise 2, p. 89, 11], we also know that an arbitrary subspace of (1-3)- locally compact space need not be (1-3)- locally compact space. Combine the above statement, we can get the following conclusion.

Theorem 2. Any closed subspace of each (1-i)-locally compact space, where (i=1,2,3), is hereditary. But that an arbitrary subspace of a (1-i)-locally compact space, where (i=1,2,3), need not be (1-i)-locally compact space. Proof:

(i) Let E be a closed subset of (1.1)-space X and let P be an arbitrary point in E. Since X is (1.1)-space, there is a compact neighborhood K of P in X. Let $H=K\cap E$. As a closed subset of K, H is compact. Since K is a neighborhood of P in X, it follows that H is a neighborhood of P in E. This implies that E is locally compact at P.

(ii) Let R^* be a closed set of (1.2)-space R and $a\epsilon R^*$. select neighborhood U of a in the space R such that $\bar{U}$ is compact, we shall show that the neighborhood $U^*=U\cap R^*$ of the point a in the space R^* has compact closure $\tilde{U}^*$ in R^*. Indeed $\tilde{U}^*=\bar{U}\cap R^*$ is the intersection of two closed subsets of the space R and is itself closed. But also $\tilde{U}^*\subset\bar{U}$, so that $\tilde{U}^*$ is a closed set in the space $\bar{U}$. Since the latter is compact, it follows that $\tilde{U}^*$ is compact too.

(iii) Prove that a closed subset of (1.3)-locally compact space is also (1.3)-locally compact space is similar as (i).

III. Locally Connectedness in Discontinuous Function

In ordinary general topological books, we know that a function f from a topological space X to a topological space Y preserves compactness if for each compact set $K\subset X$ the set $f(K)$ is compact. Similarly we define functions that preserve connectedness. Howeves in[5] it is shown that if X is locally-connected-first countable space and $Y$is regular, then every function f: $X\to Y$ that preserves compactness and connectedness is necessarily continuous. It is also show in [5] that this result fails if any one of these hypothesis on X, Y is omitted. But, it is well known that a function which preserves compactness and connectedness may not be continuous [7; p. 61, Exercise 23]. This section works on that if on some suitable conditions, we can find a discontinuous function which preserve compactness and connectedness.

Theorem 3. If X is any completely regular space that is not locally connected, then there is a discontinuous function f: $X\to R$ which preserves compactness and connectedness.

Proof:

Since X is not locally connected. We can choose a point $x_0\in X$ and an open neighborhood U of x_0 that does not contain a connected neighborhood of x_0. Since X is completely regular, we can choose a continuous function ϕ: $X\to R$ such that $\phi(x_0)=0$ $\phi(X\backslash U)=\{5\}$ $\phi(X)\subseteq[0,5]$

Let F be the component of $\phi^{-1}([0,4])$ that contains x_0. Then F is closed, and F is not a neighborhood of x_0 since it is a connected subset of U. Put

$$V=\phi^{-1}([0,1))\backslash F$$

and define α: $R\to R$ by $\alpha(t)=\cos(\pi t/2)$.

Now define f:$X\to R$ by

$$f(x)=\begin{cases}\alpha\phi(x), & \text{if } x\in X\backslash V\\ 0, & \text{if } x\in V\end{cases}$$

Clearly, f is discontinuous at x_0, since $f(x_0)=1$ and every neighborhood of x_0 contains a point of V. Also note that f is continuous when restricted to $X\backslash F$, since f is continuous on

$\phi^{-1}([1,5])$ and $f=0$ on $\phi^{-1}([0,1))\setminus F$

Now suppose that K is a compact set in X.

then $f(K)=f(K\cap V)\cup f(K\setminus V)$

Since K is compact and V is open, $K\setminus V$ is compact and therefore $f(K\setminus V)$ is compact because f is continuous on $K\setminus V$. Also $f(K\cap V)=\phi$ or $\{0\}$ so it now follows that f is compact. Hence f preserves compactness.

Let C be a connected set in X. We show that $f(C)$ is connected by considering three cases:

(1) $C\cap V=\phi$ in this case $f(C)$ is connected because f is continuous on C.

(2) $C\cap F=\phi$ we have already observed that f is continuous on $X\setminus F$, which in this case consists C, so again $f(C)$ is connected.

(3) $C\cap V\neq\phi$ and $C\cap F\neq\phi$

In this case $F\cup C$ is connected and properly larger than F and must therefore contain a point $u\in X\setminus\phi^{-1}([0,4])=\phi^{-1}((4,5])$, because F is the component of x_0 in $\phi^{-1}([0,4])$. Choose any point $v\in C\cap V$. Since ϕ is continuous we have $\phi(C)\supset[\phi(v),\phi(u)]\supset[1,4]$. Finally, since $f=\alpha\phi$ in $\phi^{-1}([1,4])$ we deduce that $f(C)\supset[-1,1]$ and therefore $f(C)=[-1,1]$ Hence f preserves connectedness.

VI. Locally Connectedness In Uniform Space

Let X be a non-empty set. For each finite partition $\overline{W}=(A_j)$ $1\leq j\leq k$ of X, let $V_{\overline{w}}$ denote $\underset{j}{U}\ A_j\times A_j$ The sets $V_{\overline{w}}$ form a fundamental system of entourages of a uniformity $\mathscr{U}$ on X. For if $\overline{W}$ is any finite partition of X we have $\Delta\subset V_{\overline{w}}$ and $V_{\overline{w}}\,o V_{\overline{w}}=\overset{-1}{V_{\overline{w}}}=V_{\overline{w}}$ [7, *p.* 17 1]; and if $\overline{W}'=(B_j)$ and $\overline{W}''=(C_k)$ are two finite partitions of X, then those of the sets $B_j\cap C_k$ which are not empty form a finite partitions $\overline{W}$ of X, and we have $V_{\overline{w}}\subset V_{\overline{w}'}\cap V_{\overline{w}''}$

Definition (VI. 1) A uniform space X is said to have property S if for each $U\in\mathscr{U}$, the set X is covered by a finite family of connected U-small sets.

Definition (VI.2) A uniform space X is said to be partitionable if and only if there exists a finite partition $\overline{W}=(A_j)$ such that each A_j with induced uniformity has property S.

Using the above definition (VI. 1) and (VI. 2), we can prove directly the following.

Theorem 4: *If* $(X,\mathscr{U})$ is locally connected and partitionable then X has property S.

Proof:

Suppose that $U\in\mathscr{U}$, We want to show that X is covered by a finite family of connected U-small sets. By hypothesis, there exists a finite partition $\overline{W}=(A_j)_{j=1}^{k}$ such that each A_j with induced uniformity. has property S. It is easy to verify that there exists a symmetric entourage $V\in\mathscr{U}$ such that $VoV\subset U$ Let $V_j=V\cap A_j\times A_j$ where $1\leq j\leq k$, then $\bigcup_{j=1}^{k}V_j\subset V$ Now consider each V_j. since V_j is an entourage of induced uniformity $\mathscr{U}_j$ of A_j and A_j has property S, we have A_j is covered by a finite family of connected V_j -small sets. i. e. there exists a finite number of points $x_{1j}, x_{2j},\cdots\cdots x_{n_j j}\in A_j$ such that $A_j=\bigcup_{i=1}^{n_j}Cx_{ij}$ where Cx_{ij} is a connected neighborhood of x_{ij} in A_j and $Cx_{ij}\times Cx_{ij}\subset V_j$ for $1\leq i\leq n_j$. Therefore $X=\bigcup_{j=1}^{k}\bigcup_{i=1}^{n_j}Cx_{ij}$ and $Cx_{ij}\subset V_j\ (x_{ij})\subset V\ (x_{ij})$ for all $1\leq j\leq k$, $1\leq i\leq n_j$. Moreover, for each pair (i,j), there exists an open set O_{ij} in X such that $Cx_{ij}\subset O_{ij}\subset V(x_{ij})$. Let Dx_{ij} be the component of O_{ij} which x_{ij} belongs to, then Dx_{ij} is open for each component of an open set in a locally connected space is also open. Hence Dx_{ij} is an open connected neighborhood of x_{ij} in X. Since Cx_{ij} is a connected set containing x_{ij} in A_j which implies, it is a connected set containing x_{ij} in X and $Cx_{ij} \subset O_{ij}$, by the definition of component of O_{ij} to which x_{ij} belongs, we obtain that $Cx_{ij} \subset Dx_{ij}$. It follows immediately that $X = \bigcup_{i=1}^{k} \bigcup_{j=1}^{n_j} Cx_{ij} \subset \bigcup_{j=1}^{k} \bigcup_{i=1}^{n_j} Dx_{ij}$ and Dx_{ij} is a connected neighborhood of x_{ij} and $Dx_{ij} \times Dx_{ij} \subset V(x_{ij}) \times V(x_{ij}) \subset V \circ V \subset U$ for all $1 \leq j \leq k$, $1 \leq i \leq n_j$ Q. E. D.

Remarks:

(1) Notice that the space Y of (II.1.2) is normal but not regular. It is normal because all nonvoid closed subsets of Y contain the point (k,k). It is not regular because (0,0) is in every open subset of Y.

(2) Using the condition (VI.2) we can simplify Lemma 1 and Lemma 2. [4]

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Locally compactness was discussed in three different ways. They can be proved equivalent in Hausdorff space and regular space.

Ordinarily, continuous function preserve compactness and connectedness. But in this paper, one will find a discontinuous function which preserve these properties. Since partitionable was defined in uniform space and it joins the condition of locally connectedness, the important result was derived that uniform space has property S.

局部緊緻和局部連通之性質

黎 廣 福

本文就局部緊緻 (Locally compactness) 中三種不同的定義討論在赫斯道夫空間 (Hausdorff space) 和正則空間 (regular space) 裡此三種定義全等。通常，祇有連續函數可以保持局部緊緻和局部連通之性質，本文中提出一非連續性函數亦可保持此種性質，由於在均勻空間 (uniform space) 裡定義了劃分 (partitionable)，吾人可連合劃分和局部連通得到一重要結論，即均勻空間具有性質 S。