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An Excursion Into The Topological Sphere Theorems

遊覽拓樸球定理

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所提論文 An excursion into the topological sphere theorems
(遊覽拓樸球定理)

合於碩士班資格水準，業經本委員會評審通過，特此證明。

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- 摘　　要 -

在1926年，Hopf 證明了任意 compact，simply connected 且具有常數曲率的 Riemannian manifold 會跟標準的球保距。根據這個結果，Hopf 提出了一個問題：在曲率的條件爲多少時，一個 compact，simply connected 的 Riemannian manifold 會是一個拓樸上的球？同樣地，在1951年，Rauch 也提出了一個類似的問題。

在本論文中，我們將討論幾何中的拓樸球定理。包括古典球定理，diameter 球定理，Ricci comparsion球定理以及具有正取率的球定理。證明這些定理會運用多種方法，包括 測地線，變分法以及張量分析。最後，我們會使用表格將一些拓樸球定理列出來。

Abstract

In 1926, Hopf proved that any compact, simply connected Riemannian manifold with constant curvature 1 is isometric to the standard sphere. Motivated by this result, Hopf posed the question whether a compact, simply connected manifold with suitably pinched curvature is topologically a sphere. Similarly, in 1951, Rauch also ask some questions about "pinching".

In this paper, we give a survey of various sphere theorems in geometry. These include the classic sphere theorem, the diameter sphere theorem, the Ricci comparison sphere theorem, and the sphere theorem with positive curvature. These theorems employ a variety of methods, including geodesic, variations of energy, and tensor analysis. Finally, we use a table to list all of the sphere theorems.

- 致 謝 辭 -

這篇論文得以完成，需要感謝的人實在太多太多。首先，我要先感謝我的指導教授陳文豪老師在這些日子的教導，在陳文豪老師耐心的教導之下，讓我在幾何這門領域有了基本的認知與了解。 在這碩士的2年半中，我發現老師在學術上的嚴謹是我學習的榜樣，至於做人處事方面，更讓我受益良多。像是在meeting時，老師會教我如何用一些簡單的話去表達很困難的數學式子等等，這些對於我日後當微積分助教有很大的幫助。以及對於我的碩士生涯都是一些非常寶貴的經驗。

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我還要感謝我的微積分老師林雅萍老師，我在當林雅萍老師的助教期間。老師教會了我很多事情，像是如何與學生應對等等，並且要在學業上還是生活上遇到瓶頸，老師總是會耐心的爲我解答，而且也跟我說未來應該要怎麼規劃等等。在當微積分助教期間，對於學業、心境上都有很大的收穫跟成長，讓我在這2年半的碩士期間，更增添精彩的一頁。

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Contents

Chapter 1

Introduction

One of the most beautiful theorem in global differential geometry is the sphere theorem. In 2-dimensional, the simplest case of sphere theorem is $Gauss - Bonnect\ Theorem$, which states is

Theorem 1.0.1. *($Gauss - Bonnet\ Theorem$ [3])*
Let M be a compact 2-dimensional surface equipped with a metric. Then

$$\int_M K dvol = 2\pi\chi(M),$$

where K denotes the Gaussian curvature of M, $dvol$ denotes the induced area measure on M, and $\chi(M)$ denotes the Euler characteristic of M.

In this paper, we investigate the classic topological sphere theorem in Riemannian geometry and its generalizations

The sphere theorem is also known as the quarter-pinched sphere theorem, which determines the topology of manifolds admitting metrics with a particular curvature bounded. The statement of the theorem is

Theorem 1.0.2. *(classic sphere theorem)(M. Berger [4]; W. Klingenberg [34] 1960)*
If M^n is a compact, simply connected Riemannian $n-$manifold with sectional curvature K satisfies

$$\frac{1}{4} < K \leq 1.$$

Then it is homeomorphic to S^n.

The sphere theorem has a long history. In 1951, the sphere theorem was proved for the first time by Rauch [39] for $h < K \leq 1$ and $h \sim \frac{3}{4}$. A fundamental contribution was made by Klingenberg [33] who introduced the problem by the consideration of "cut locus". In the case of a manifold with even dimension, Klingenberg [33] obtained an estimate for the distance from a point to its "cut locus" and he proved the theorem for $h < K \leq 1$ and $h \sim 0.55$. Using Toponogov triangle comparison theorem and the estimation mentioned above, Berger [4] obtained the theorem, still in the case of even dimension, with $h = \frac{1}{4}$. Finally, Klingenberg [34] and Berger [4] extended his estimation from even dimension to odd dimension. the classic sphere theorem as stated above.

In Chapter 2, we review some basic definitions and theorems in Riemannian Geometry. The content of this chapter contains Riemannian manifold, geodesic, curvature, Rauch comparison theorem, Toponogov's triangle comparison theorem, Hopf-Rinow theorem etc.

In Chapter 3, we present the classic sphere theorem in Riemannian geometry, diameter sphere theorem, Ricci comparison sphere theorem, and sphere theorem of positive curved with nontrivial Killing field. Moreover, we introduce some other topological sphere theorems in the last section of chapter 3. Finally, we use a table to list all of the sphere theorems.

Table 1.1: The Chronology of Topological Sphere Theorem

Year	Theorem	Mathematician
1951	M^n: compact, simply connected condition: $h < K \leq 1$ and $h \sim \frac{3}{4}$ conclusion: it is homeomorphic to S^n	H.E. Rauch
1960	M^n: compact, simply connected condition: $\frac{1}{4} < K \leq 1$ conclusion: it is homeomorphic to S^n	M. Berger & W.Klingenberg
	M^n: compact, $n \geq 5,\ n \in odd$ condition: positive $\frac{n-3}{4n-3}$ − pinched conclusion: $b_2 = 0$	M. Berger
1975	M^n: complete condition: $Ric \geq (n-1)k > 0$ and $diam(M) = \frac{\pi}{\sqrt{k}}$ conclusion: it is isometric to S^n	S.Y. Cheng
1977	M^n: compact, $n \geq 4$ condition: $K \geq 1$ and $diam(M) > \frac{\pi}{2}$ conclusion: it is homeomorphic to S^n	Grove & Shiohama
1988	M^n: compact, simply connected condition: positive isotropic curvature conclusion: it is homotopic to S^n	M. Micallef & J.D. Moore
1989	M^4: 4-dimensional orientable complete positive curvature condition: nontrivial Killing field conclusion: it is homeomorphic to S^4or $\mathbb{CP}^2$	Wu-Yi Hsiang & Bruce Kleniner
	M^n: compact, orientable, $n = 4$ condition: δ-pinched in the global sense and $\delta \sim 0.188$ conclusion: it is homeomorphic to S^4or $\mathbb{CP}^2$	W. Seaman & M.Ville
1993	M^n: compact, $n \geq 4,\ n \in even$ condition: positive isotropic curvature conclusion: $b_2 = 0$	M. Micallef & M. Wang
1997	M^n: compact, $n \geq 2,\ \psi(n) \in (0,1)$ condition: $Ric \geq (n-1)g$ and $vol(M,g) \geq (1-\psi(n))vol(S^n(1))$ conclusion: it is diffeomorphic to S^n	J. Cheeger & T. Colding

Chapter 2

Preliminaries

In this chapter, we review some consequences in Riemannian geometry.

2.1 Introduction to Riemannian geometry

Let M be a *differentiable* $manifold$. A Riemannian manifold is a differentiable manifold M equipped with a *Riemannian metric* $g(\cdot,\cdot)$ denoted by the pair (M,g).

Given a function $f : M \to \mathbb{R}$. Define $\Im(M)$={all vector fields of class C^∞ in M} and $D(M)$={all real-valued functions of class C^∞ in M}.

Definition 2.1.1. [22] An *affine connection* ∇ on a differentiable manifold M is a bilinear mapping

$$\nabla : \Im(M) \times \Im(M) \to \Im(M)$$

defined by $(X,Y) \to \nabla_X Y$ and satisfing the following properties:

1. ∇ is linear in the first variable and second variable.

2. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$,

in which $X, Y \in \Im(M)$ and $f \in D(M)$.

If an affine connection ∇ on M satisfies symmetric and compatible with the Riemannian metric, then ∇ is called the $Levi-Civita$(or *Riemannian*) *connection* on M.

Let M be a differentiable manifold with an affine connection ∇. A vector field V along a curve $c : I \to M$ is called *parallel* when $\frac{DV}{dt} = 0$, $t \in I$.

We will introduce the *geodesic* on a Riemannian manifold. A geodesic is a generalization of the notion of a "straight line" to "curve space".

Definition 2.1.2. [22] A parametrized curve $\gamma : I \to M$ is a *geodesic* for $t \in I$ if $\frac{D}{dt}(\frac{d\gamma}{dt}) = 0$ for $t \in I$.

We shall define the exponential map. Let TM be the *tangent bundle*, the tangent bundle of M is the disjoint union of the tangent space T_pM of $p \in M$.

Definition 2.1.3. [22] Let $\mathcal{U} = \{(q, w) \in TM | q \in M, v \in T_qM, |v| < \epsilon\} \subset TM$ be an open set. Then the map $\exp : \mathcal{U} \to M$ given by

$$\exp(q, v) = \gamma(1, q, v) = \gamma(|v|, q, \frac{v}{|v|}), \quad (q, v) \in \mu,$$

is called the *exponential map* on $\mathcal{U}$

The *exponential map* is a generalize the ordinary exponential function of mathematical analysis to a differentiable manifold with an affine connection.

Definition 2.1.4. [22] The *Riemannian curvature operator* R of a Riemannian manifold M is a correspondence that associate to every pair $X, Y \in \Im(M)$ a mapping $R(X, Y) : \Im(M) \to \Im(M)$ defined by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

where $Z \in \Im(M)$ and ∇ is the Riemannian connection of M.

From now on, we shall write $g(R(X, Y)Z, T) = R(X, Y, Z, T)$.

Definition 2.1.5. [22] Given a point $p \in M$ and a two-dimensional subspace $\sigma \subset T_pM$. Then

$$K_p(X, Y) = K_p(\sigma) = \frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}$$

is called the *sectional curvature* of σ at p, where $\{X, Y\}$ is any basis of σ.

We are going to introduce the *Ricci curvature* and *scalar curvature* on a Riemannian manifold M. Ricci curvature is the average of the Riemannian curvature, and Scalar curvature is the average of the Ricci curvature. The concise definitions are as following:

Definition 2.1.6. [22] Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis. Then

(1) $Ric_p(x) = \frac{1}{n-1}\sum\limits_i g\big(R(x, e_i)x, e_i\big), \quad i = 1, 2, \ldots, n-1,$

(2) $Sca(p) = \frac{1}{n}\sum\limits_j Ric_p(e_j) = \frac{1}{n(n-1)}\sum\limits_{ij} g\big(R(e_i, e_j)e_i, e_j\big), \quad j = 1, 2, \ldots, n.$

Then the expression (1) is called the *Ricci curvature* in the direction x at p and (2) is called the *scalar curvature* at p, respectively.

We shall introduced the *Rauch comparison theorem*. The Rauch comparison theorem is the fundamental result which relates the sectional curvature of a Riemannian manifold to the rate at which geodesics spread apart. Here a *Jacobi field* means a vector field J along a geodesic γ and satisfies

$$\frac{D^2 J}{dt^2} + R(\gamma^{'}(t), J(t))\gamma^{'}(t) = 0.$$

Two points p and q are *conjugate* along a geodesic γ if there exists a non-zero Jacobi field along γ that vanishes at p and q.

Theorem 2.1.7. *(Rauch comparison theorem [3])*
Let $\gamma : [0, a] \to M^n$ and $\tilde{\gamma} : [0, a] \to \tilde{M}^{n+k}, k \geq 0$, be geodesics with the same velocity (i.e.,$|\gamma^{'}(t)| = |\tilde{\gamma}^{'}(t)|$), and let J and $\tilde{J}$ be Jacobi fields along γ and $\tilde{\gamma}$, respectively, such that

$$J(0) = \tilde{J}(0) = 0, \quad < J^{'}(0), \gamma^{'}(0) >=< \tilde{J}^{'}(0), \tilde{\gamma}^{'}(0) >, \quad |J^{'}(0)| = |\tilde{J}^{'}(0)|.$$

Assume that $\tilde{\gamma}$ does not have conjugate points on $(0, a]$ and that, for all t and all $x \in T_{\gamma(t)}(M), \tilde{x} \in T_{\tilde{\gamma}(t)}(M)$, we have

$$\tilde{K}(\tilde{x}, \tilde{\gamma}^{'}(t)) \geq K(x, \gamma^{'}(t)),$$

where $K(x,y)$ denotes the sectional curvature with respect to the plane generated by x and y. Then

$$|\tilde{J}| \leq |J|.$$

In addition, if for some $t_0 \in (0,a]$, we have $|\tilde{J}(t_0)| = |J(t_0)|$, then $\tilde{K}(\tilde{J}(t), \tilde{\gamma}'(t)) = K(J(t), \gamma'(t))$, for all $t \in [0, t_0]$.

And then, we shall introduce an application of the Rauch comparison Theorem, the *Toponogov triangle comparison theorem.* The concise statement is as following:

Theorem 2.1.8. *(Toponogov triangle comparison theorem [3])*
Assume that the sectional curvature of a Riemannian manifold M satisfies

$$K \geq \delta > 0.$$

1. *Let $\{p,q,r\}$ be any triangle in M and in $S^n(\delta)$ take some triangle $\{p', q', r'\}$ such that the corresponding side from p and p' have equal lengths and equal angles at p and p'. Then*

   $$d_M(q,r) \leq d_{S^n(\delta)}(q', r').$$

   Where $S^n(\delta)$ is the space form of dimension n and sectional curvature k. (see Figure 2.1.1)

2. *Given any triangle with vertices $p, q, r \in M$, it follows that the interior angles are larger than the corresponding interior angles for a comparison triangle in $S^n(\delta)$. (see Figure 2.1.2)*

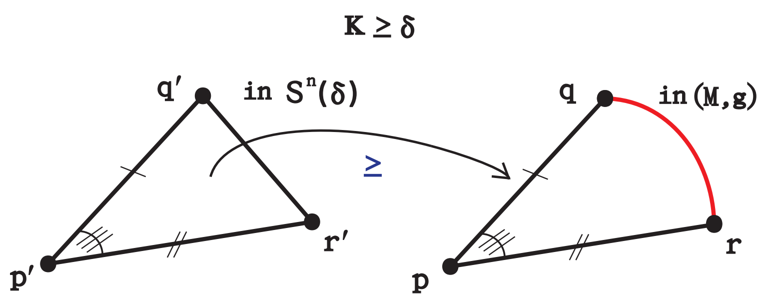

Figure 2.1.1: The Toponogov triangle comparison theorem.

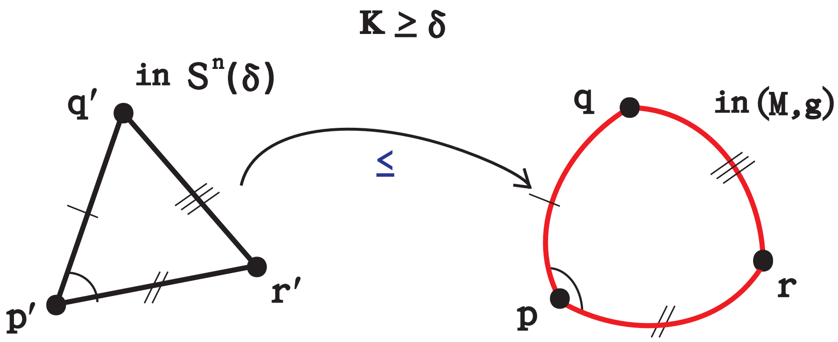

Figure 2.1.2: The Toponogov triangle comparison theorem.

Hopf-Rinow theorem is one of the most important theorem for the complete manifolds, which state any two points in a complete manifold admits a minimizing geodesic joining these two points.

Theorem 2.1.9. *(Hopf-Rinow theorem [22])*

Let M be a Riemannian manifold and let $p \in M$. The following assertions are equivalent:

(a) exp_p is defined on all of T_pM.

(b) The closed and bounded sets of M are compact.

(c) M is complete as a metric space.

(d) M is geodesically complete. In addition, any of the statements above implies that

(e) For any $q \in M$ there exists a geodesic γ joining p to q with $l(\gamma) = d(p, q)$.

We shall define the formula for the first and second variation. A *prop variation* is a variation with the same initial point and endpoint. Let $c : [0, a] \to M$ be a piecewise differentiable curve in M and $f : (-\epsilon, \epsilon) \times [0, a] \to M$ be a variation of c with $f(0, t) = c(t)$. We define $E(s)$ be *energy function* by

$$E(s) = \int_0^a |\frac{\partial f}{\partial t}(s,t)|^2 dt, \qquad s \in (-\epsilon, \epsilon).$$

Proposition 2.1.10. *(variation formulas [22])*

Let $c : [0, a] \to M$ be a piecewise differentiable curve. Let $f : (-\epsilon, \epsilon) \times [0, a] \to M$ be a proper variation of c, and let $E : (-\epsilon, \epsilon) \to \mathbb{R}$ be the energy of f. Then

(first variation formula)

$$\begin{aligned} \tfrac{1}{2}E'(0) = \; & -\textstyle\int_0^a g\big(V(t), \frac{D}{dt}\frac{dc}{dt}\big)dt \\ & -\textstyle\sum_{i=1}^k g\big(V(t_i), \frac{dc}{dt}(t_i^+) - \frac{dc}{dt}(t_i^-)\big) \\ & -g\big(V(0), \tfrac{dc}{dt}(0)\big) + g\big(V(a), \tfrac{dc}{dt}(a)\big) \end{aligned}$$

where $V(t) = \frac{\partial f}{\partial s}(0, t)$ is the variational field of f, and

$$\frac{dc}{dt}(t_i^+) = \lim_{t \to t_i^+} \frac{dc}{dt}, \qquad \frac{dc}{dt}(t_i^-) = \lim_{t \to t_i^-} \frac{dc}{dt}.$$

(second variation formula)

$$\frac{1}{2}E''(0) = -\int_0^a g\Big(V(t), \frac{D^2V}{dt^2} + R(\frac{d\gamma}{dt}, V)\frac{d\gamma}{dt}\Big)dt - \sum_{i=1}^k g\Big(V(t_i), \frac{DV}{dt}(t_i^+) - \frac{DV}{dt}(t_i^-)\Big),$$

where $V(t) = \frac{\partial f}{\partial s}(0, t)$ is the variation field of f, R is the Riemannian curvature operator of M and

$$\frac{DV}{dt}(t_i^+) = \lim_{t \to t_i^+} \frac{DV}{dt}, \qquad \frac{DV}{dt}(t_i^-) = \lim_{t \to t_i^-} \frac{DV}{dt}.$$

Corollary 2.1.11. *[22] If the variation is not proper, then we obtain following expression:*

$$\begin{aligned}\tfrac{1}{2}E''(0) = \quad & -\int_0^a g\big(V(t), \tfrac{D^2V}{dt^2} + R(\tfrac{d\gamma}{dt}, V)\tfrac{d\gamma}{dt}\big)dt \\ & -\sum_{i=1}^k g\big(V(t_i), \tfrac{DV}{dt}(t_i^+) - \tfrac{DV}{dt}(t_i^-)\big) - g\big(\tfrac{D}{ds}\tfrac{\partial f}{\partial s}, \tfrac{d\gamma}{dt}\big)(0,0) \\ & +g\big(\tfrac{D}{ds}\tfrac{\partial f}{\partial s}, \tfrac{d\gamma}{dt}\big)(0,a) - g\big(V(0), \tfrac{DV}{dt}(0)\big) + g\big(V(a), \tfrac{DV}{dt}(a)\big).\end{aligned}$$

We shall introduce some concepts of classical Morse theory. Let f be a smooth real-value function on a manifold M. A point $p \in M$ is called a *critical point* of f if the induced map $f_* : T_pM \to T_{f(p)}\mathbb{R}$ is zero. (i.e. $\frac{\partial f}{\partial x_1}(p) = \frac{\partial f}{\partial x_2}(p) = \cdots = \frac{\partial f}{\partial x_n}(p) = 0$, where $(x_1, \ldots, x_n)$ is a local coordinate system in a neighborhood U of p.) The real number $f(p)$ is called a *critical value* of f. A critical point p is called $non-degenerate$ if and only if the matrix

$$A = \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\Big)$$

is non-singular, (i.e. $det(A) \neq 0$.)

Lemma 2.1.12. *(Morse Lemma [37])*
Let p be a non-degenerate critical point for f. Then there is a local coordinate system $(y_1, y_2, \cdots, y_n)$ in a neighborhood U of p with $y_i(p) = 0$ for all i and such that the identity

$$f = f(p) - (y_1)^2 - \ldots - (y_\lambda)^2 + (y_{\lambda+1})^2 + \ldots + (y_n)^2$$

holds throughout U, where λ is the index of f at p.

We shall introduce the *cut locus* $C_m(p)$ with $p \in M$. It is roughly the set of all other points for which there are multiple minimizing geodesics connecting them from p. We know that exp_p is an injective mapping to an open ball $B_r(p)$ if and only if the radius r is less than or equal to the minimizing distance from p to $C_m(p)$. In the following, we shall define *injectivity radius* of M.

Definition 2.1.13. [22] We define the *injectivity radius* of M by

$$i(M) = \inf_{p \in M} d(p, C_m(p)).$$

In the following, we shall introduce the fundamental *tensor analysis*. A *tensor* S of order r on a Riemannian manifold is a functional linear mapping. Then we can define a *covariant derivative tenser* ∇S by

$$\nabla S(X, Y) = (\nabla_X S)(Y) = \nabla_X(S(Y)) - S(\nabla_X Y).$$

More generally, define

$$\begin{aligned}\nabla S(X, Y_1, \ldots, Y_r) &= (\nabla_X S)(Y_1, \ldots, Y_r)\\ &= \nabla_X(S(Y_1, \ldots, Y_r)) - \sum_{i=1}^{r} S(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_r).\end{aligned}$$

If $f : M \to \mathbb{R}$ is a smooth function, then we already have ∇f defined as the vector field satisfying

$$g(\nabla f, v) = D_v f = df(v).$$

We will define the Hessian of f and Laplacian of f :

Definition 2.1.14. [38] The *Hessian* $Hessf$ is defined by

$$Hessf(X, Y) = g(S(X), Y).$$

Where $S(X) = \nabla_X \nabla f$ be self-adjoint $(1, 1)-$tensor.

Let M be a Riemannian manifold, $X, Y \in \Im(M)$ and $f \in D(M)$. Define the divergence of X as a function $divX : M \to \mathbb{R}$ given by $divX(p) = tr(\nabla_Y X(p))$, $p \in M$.

Definition 2.1.15. [38] Let M be Riemannian manifold. Define the *Laplacian operator* $\triangle : D(M) \to D(M)$ by

$$\triangle f = div \nabla f,$$

where $f \in D(M)$.

We will define the *killing field* on a Riemannian manifold. A vector field X on a Riemannian manifold (M, g) is called a *Killing field* if the local flows generated by X

act by isometries. The concise definition is as following:

Definition 2.1.16. [38] A vector field X on a Riemannian manifold (M, g) is a *Killing field* if and only if $L_X g = 0$. (i.e. $g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0$, for all $Y, Z \in \Im(M)$.)

The vector field on a circle that points clockwise and has the same length at each point is a Killing vector field, since moving each point on the circle along this vector field simply rotates the circle.

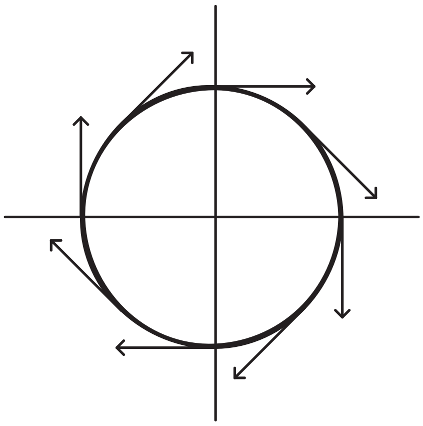

Figure 2.1.3: The killing vector field on a circle.

2.2 Introduction to algebraic topology

In this section, we shall introduce the fundamental property of algebraic topology. Include *homotopy group*, $n-$*connection*, *universal covering*, *homology group*, and *Betti number* etc.

We are going to introduce the homotopy group, which is two continuous functions from one topological space to another. If one can be "continuous deformed" into the other, then such a deformation being called a homotopy between the two functions.

We next introduce the definition of the homotopy group and fundamental group as following:

By a *path* in a topological space X we mean a continuous map $f : I \to X$ where I is the unit interval $[0, 1]$. A *homotopy* of paths in X is a family $f_t : I \to X$, $0 \le t \le 1$, such that

1. The endpoint $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t.

2. The associated map $F : I \times I \to X$ defined by $F(s, t) = f_t(s)$ is continuous.

When two paths f_0 and f_1 are connected in this way by a homotopy f_t, they are said to be *homotopic*. The notation for this is $f_0 \simeq f_1$.

The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation. An equivalence class will be denoted $[f]$ and called the homotopy class of f.

Definition 2.2.1. [30] Given two path $f, g : I \to X$ such that $f(1) = g(0)$, there is a composition or product path $f \cdot g$ that traverses first f and then g, defined by the formula:

$$(f \cdot g)(s) = \begin{cases} f(2s), & s_1 \in [0, \frac{1}{2}] \\ g(2s-1) & s_1 \in [\frac{1}{2}, 1] \end{cases}$$

This product operation respects homotopy classes. Above homotopy classes form a group, these groups are called fundamental groups, denoted by $\pi_1(X, x_0)$.

Definition 2.2.2. [30] Let I^n be the n-dimensional unit cube, the product of n copies of the interval $[0, 1]$. The boundary ∂I^n of I^n is the subspace consisting of points with at least one coordinate equal to 0 or 1. For a space X with basepoint $x_0 \in X$, define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$, where homotopies f_t are required to satisfy $f_t(\partial I^n) = x_0$ for all t. The definition extend to the following case:

1. When $n = 0$, by taking I^0 to be a point and ∂I^0 to be empty, so $\pi_0(X, x_0)$ is just the set of path-components of X.

2. When $n \geq 2$, a product operation in $\pi_n(X, x_0)$, generalizing the composition operation in π_1, is defined by

$$(f \cdot g)(s_1, s_2, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots, s_n), & s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, \cdots, s_n), & s_1 \in [\frac{1}{2}, 1] \end{cases}$$

   It is evident that this sum is well-defined on homotopy classes. Above equivalence classes form a group, these groups are called homotopy groups.

We mention the application of homotopy group, namely, $n-$*connected*, N-connectedness is a way to say that a space vanishes.

Definition 2.2.3. [30] A space X is a topological space with basepoint x_0 is said to be n-connected if $\pi_i(X, x_0) = 0$ for $i \leq n$.

Thus, 0-connected means path-connected and 1-connected means simply-connected.

We are going to introduce the universal cover space.

Definition 2.2.4. [30] Let X be the topological space. A covering space of a space X is a space C together with a continuous surjective map

$$p : C \to X$$

satisfies the following condition: There exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$(the inverse image of U under p) is a disjoint union of open sets in C, each of which is mapped homeomorphically onto U by p.

A connected covering space is a universal covering if it is simply connected.

We shall define the two important topological properties, namely, homology group and Betti number. The concise definition is as following:

Definition 2.2.5. [40] Let X be the topological space. The $k-$th homology group of X is defined to be the quotient group

$$H_k(M) = Z_k(M)/B_k(M)$$

where $Z_k(M)$ is a closed $k-$chain group defined by $Z_k(M) = \{c_k | \partial c_k = 0\}$ and $B_k(M)$ is a boundary $k-$chain group defined by $B_k(M) = \{b_k = \partial c_{k+1} | c_{k+1} \in C_{k+1}(M)\}$. Here the $\partial c_k = 0$ is called a closed $k-$chain, the $b_k = \partial c_{k+1}$ is called the boundary chain and c_k is the continuous $k-$chain.

Definition 2.2.6. [40] Let X be the topological space. The rank of the the $k-$dimensional homology group $H_k(M)$ of X is called the Betti number of order k, denoted by

$$b_k = dim(H_k(M)).$$

Chapter 3

The Topological Sphere Theorem and Its Generalizations

The topological sphere theorem is a beautiful theorem in Riemannian geometry, which has a long history, dating back to s paper by H.E. Rauch in 1951. In that paper [39], Rauch posed the question of whether a compact, simply connected Riemannian manifold M with sectional curvature lie in the interval $(\frac{1}{4}, 1]$ is necessarily homeomorphic to the sphere. Around 1960, M.Berger and W.Klingenberg gave an certain answer to this question:

Theorem 3.0.7. *(M. Berger [4]; W. Klingenberg [34] 1960)*
Let M be a compact, simply connected Riemannian manifold whose sectional curvatures lie in the interval $(\frac{1}{4}, 1]$. Then M is homeomorphic to S^n.

In this chapter, we shall introduce the topological sphere theorems in Riemannian geometry, and shall discuss the relations under the different curvature conditions. In section 1, we are going to introduce the classic sphere theorem in Riemannian geometry. In section 2, we shall replace the curvature condition to $K \geq 1$ and $diam(M) \geq \frac{\pi}{2}$. Then M is still homeomorphic to S^n. In section 3, similarly, we are going to replace the curvature condition to $Ric \geq (n-1)k$, where k is any positive constant, and $diam(M) = \frac{\pi}{\sqrt{k}}$. Then M is isometric to S^n. In section 4, we consider positive curved manifold M which has nontrivial killing field. Then M is homeomorphic to S^4 or $\mathbb{CP}^2$. In section 5, we consider a Riemannian manifold with positive isotropic curvature, and shall discuss other sphere theorems in Riemannian geometry.

3.1 The sphere theorem in Riemannian geometry

In this section, we shall discuss the classic sphere theorem in Riemannian geometry. We are going to introduce the Bonnet-Myers theorem.

Theorem 3.1.1. *(Bonnet – Myers Theorem [22])*

Let M^n be a complete Riemannian manifold. Suppose that the Ricci curvature of M satisfies

$$Ric_p(v) \geq \frac{1}{r^2} > 0,$$

for all $p \in M$ and for all $v \in T_pM$. Then M is compact and the diameter $diam(M) \leq \pi r$.

Proof.

Let p, q be any two point in M. Since M is complete, by Hopf-Rinow Theorem 2.1.9, there exists a minimizing geodesic $\gamma : [0,1] \to M$ joining p to q. (i.e. $l(\gamma) = d(p,q)$) Then we claim that $l(\gamma) \leq \pi r$, because M is bounded and complete, therefore M is compact; besides, if $l(\gamma) \leq \pi r$, then $d(p,q) \leq \pi r$, for all $p, q \in M$ which implies $diam(M) \leq \pi r$ and we complete this proof.

If not, suppose that $l(\gamma) > \pi r$. Consider parallel fields $e_1(t), \dots e_{n-1}(t)$ along γ be orthonormal, for each $t \in [0,1]$ and $\{e_1(t), \dots e_{n-1}(t)\} \in \{\gamma(t)\}^{\perp}$. Let $e_n(t) = \frac{\gamma'(t)}{l}$ and let V_j be vector field along γ given by

$$V_j(t) = (\sin \pi t) e_j(t), \qquad j = 1, \dots, n-1.$$

Clearly, $V_j(0) = V_j(1) = 0$, therefore V_j can construct a proper variation of γ, whose energy function denote by E_j.

By second variation formula 2.1.10 and $e_j(t)$ is parallel, (i.e. $e_j^{'}(t) = 0$) we have

$$
\begin{aligned}
\frac{1}{2}E_j^{''}(0) &= -\int_0^1 g\Big(V_j(t), V_j^{'}(t) + R\big(\gamma^{'}(t), V_j(t)\big)\gamma^{'}(t)\Big)dt - \sum_{i=1}^{k} g\Big(V_j(t), \frac{DV}{dt}(t_i^+) - \frac{DV}{dt}(t_i^-)\Big) \\
&= -\int_0^1 g\Big(V_j(t), V_j^{'}(t) + R\big(\gamma^{'}(t), V_j(t)\big)\gamma^{'}(t)\Big)dt \\
&= -\int_0^1 g\Big((\sin\pi t)e_j(t), \pi^2(-\sin\pi t)e_j(t) + R\Big(\big(l\cdot e_n(t)\big), (sin\pi t)e_j(t)\Big)\big(l\cdot e_n(t)\big)\Big)dt \\
&= \int_0^1 (\sin\pi t)^2\pi^2 g\Big(e_j(t), e_j(t)\Big)dt + \int_0^1 l^2(\sin\pi t)^2 g\Big(e_j(t), R\big(e_n(t), e_j(t)\big)e_n(t)\Big)dt \\
&= \int_0^1 (\sin\pi t)^2\pi^2 dt - \int_0^1 l^2(\sin\pi t)^2 K\big(e_n(t), e_j(t)\big)dt \\
&= \int_0^1 (sin^2\pi t)\Big(\pi^2 - l^2 K\big(e_n(t), e_j(t)\big)\Big)dt
\end{aligned}
$$

where $K(e_n(t), e_j(t))$ is sectional curvature at $\gamma(t)$ with respect to the plane generated by $e_n(t)$, $e_j(t)$. Summing on j and by definition of Ricci curvature, we obtain

$$
\begin{aligned}
\frac{1}{2}\sum_{j=1}^{n-1} E_j^{''}(0) &= \sum_{j=1}^{n-1}\int_0^1 (\sin^2\pi t)\Big(\pi^2 - l^2 K\big(e_n(t), e_j(t)\big)\Big)dt \\
&= \int_0^1 \{\sin^2\pi t\Big((n-1)\pi^2 - l^2\sum_{j=1}^{n-1} K\big(e_n(t), e_j(t)\big)\Big)\}dt \\
&= \int_0^1 \{\sin^2\pi t\Big((n-1)\pi^2 - (n-1)l^2 Ric_{\gamma(t)}\big(e_n(t)\big)\Big)\}dt.
\end{aligned}
$$

Since $Ric_{\gamma(t)}(e_n(t)) > \frac{1}{r^2} > 0$ and $l > \pi r$, which implies

$$(n-1)l^2 Ric_{\gamma(t)}(e_n(t)) > (n-1)\pi^2(r^2\cdot\frac{1}{r^2}) = (n-1)\pi^2,$$

hence,

$$\frac{1}{2}\sum_{j=1}^{n-1} E_j^{''}(0) < \int_0^1 sin^2\pi t((n-1)\pi^2 - (n-1)\pi^2)dt = 0.$$

As a result, there exists an index j such that $E_j^{''}(0) < 0$, which implies $E_j^{'}(s)$ is strictly decreasing at zero. Since $E_j(s)$ has local minimum at zero, there exists a curve c in this variation such that $E(\gamma) = l(\gamma)^2 \leq l(c)^2 \leq E(c)$. Therefore, we know that c is the minimal geodesic. Which is a contradiction. Hence $l \leq \pi r$. ■

We mention the classic sphere theorem as following:

Theorem 3.1.2. *(classic sphere theorem)(M. Berger [4]; W. Klingenberg [34] 1960)* *Let M^n be a compact simply connected, Riemannian manifold, with sectional curvature K satisfing*

$$\frac{1}{4} < K \leq 1$$

Then M is homeomorphic to a sphere. If $K = \frac{1}{4}$, then M is isometric to a symmetric space.

To prove this theorem, we need the following two important lemmas.

Lemma 3.1.3. *[22] Let M^n be a compact, simply connected Riemannian manifold with sectional curvature K satisfing*

$$\frac{1}{4} < \delta \leq K \leq 1$$

and let $p, q \in M$ be such that $d(p,q) = diam(M)$. Then $M = B_\rho(p) \cup B_\rho(q)$, where $B_\rho(p) \subset M$ denotes the open geodesic ball of radius ρ and center $p \in M$ and ρ is such that $\frac{\pi}{2\sqrt{\delta}} < \rho < \pi$.

Proof. If not, suppose that there exists $\gamma \in M$ such that $d(p,r) \geq \rho$, $d(q,r) \geq \rho$. Without lose of generalization, we can assume that $d(p,r) \geq d(q,r) \geq \rho$.

There is a minimizing geodesic from q to γ intersects $\partial(B_\rho(q))$ in a point $q^{'} \notin B_\rho(p)$. On the other hand, by Bonnet-Myers theorem 3.1.1, $diam(M) \leq \frac{\pi}{\sqrt{\delta}} < 2\rho$. Therefore, if $q^{''}$ is a point of intersection of the minimizing geodesic from q to p with $\partial B_\rho(q)$, then $q^{''} \in B_\rho(p)$.

Since $\partial B_\rho(p)$ and $\partial B_\rho(q)$ are path connected (because $\partial B_\rho(p)$ and $\partial B_\rho(q)$ are homeomorphic to Euclidean ball), this implies $\partial B_\rho(p) \cap \partial B_\rho(q) \neq 0$, hence there exists $r_0 \in M$ such that $d(r_0, p) = d(r_0, q) = \rho$.

Next, we consider a minimizing geodesic λ joining p to r_0, by Berger's Lemma [4]. There exists a minimizing geodesic γ from p to q with $\langle \gamma^{'}(0), \lambda^{'}(0) \rangle \geq 0$. (see Figure

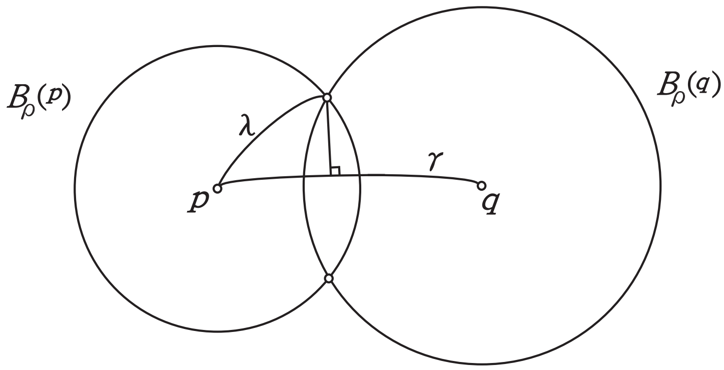

Figure 3.1.1:

3.1.1) Let s be a point of γ such that $d(p, s) = \rho$. From Rauch comparison theorem, the angle $\sphericalangle r_0 ps \leq \frac{\pi}{2}$ and $d(r_0, s) \leq \frac{\pi}{2\sqrt{\delta}}$.

Since $d(r_0, p) = d(r_0, q) = \rho$ and there exists a point s_0 of γ such that $d(r_0, s_0) < \rho$, the distance from r_0 to γ is realized by a point s_0 in the interior of γ. The minimizing geodesic from r_0 to s_0 is perpendicular to γ and

$$d(r_0, \gamma) = d(r_0, s_0) \leq \frac{\pi}{2\sqrt{\delta}}.$$

Since $d(p, q) = diam(M) \leq \frac{\pi}{\sqrt{\delta}}$, we have either $d(p, s_0) \leq \frac{\pi}{2\sqrt{\delta}}$ or $d(q, s_0) \leq \frac{\pi}{2\sqrt{\delta}}$. In either case, since $d(r_0, s_0) \leq \frac{\pi}{2\sqrt{\delta}}$ and $\sphericalangle ps_0r_0 = \frac{\pi}{2}$, we have $d(p, r_0) \leq \frac{\pi}{2\sqrt{\delta}} < \rho$, by applying Rauch comparison theorem 2.1.7. This contradicts the fact that $d(p, r_0) = \rho$. ■

Lemma 3.1.4. *[22] Under the conditions of Lemma 3.1.3, on each geodesic of length ρ starting from p there exists a unique point m such that*

$$d(p, m) = d(q, m) < \rho$$

Similarly, on each geodesic starting from q of length ρ there exists a unique point n equidistant from p and q.

Proof. First, we prove the existence. Let $\gamma(s)$ be a geodesic with $\gamma(0) = p$ and consider

$$f(s) = d(q, \gamma(s)) - d(p, \gamma(s)).$$

Then f is continuous and $f(0) = d(p, q) > 0$. Let s_0 be such that $\gamma(s_0)$ is a cut point of p along γ. Then $d(p, \gamma(s_0)) \geq \pi > \rho$. From Lemma 3.1.3, we have $d(q, \gamma(s_0)) < \rho$. Therefore,

$$f(s_0) = d(q, \gamma(s_0)) - d(p, \gamma(s_0)) < 0.$$

By the intermediate value theorem, there exists $s_1 \in (0, s_0)$ such that $f(s_1) = 0$. Choose $m = \gamma(s_1)$ such that $d(p, m) = d(q, m) < \rho$.

We prove uniqueness by contradiction. Suppose that there exist $m_1 \neq m_2$ and both equidistant from p to q. We can assume that m_1 is between p and m_2. Then

$$d(q, m_2) = d(p, m_2) = d(p, m_1) + d(m_1, m_2) = d(q, m_1) + d(m_1, m_2).$$

Hence $q \in \gamma$ and $m_1 \neq m_2$. Therefore $p = q$, which contradicts the hypothesis. ■

Finally, we prove the classic sphere theorem as following:

Proof of classic sphere theorem. Let $p, q \in M$ such that diam(M)=$d(p, q)$. Let D_1 and D_2 be subsets of M determined by all geodesic segments $\overline{pm}$ and $\overline{qn}$ respectively, where the points m and n are given by Lemma 3.1.4. By continuity of m and n, we have D_1 and D_2 are closed subset of M. To prove this theorem, we follow the following steps:

step (1) Prove that $D_1 \cup D_2 = M$ and $\partial D_1 = \partial D_2 = D_1 \cap D_2$.

step (2) Construct the mapping $\varphi : S^n \to M^n$ by

$$\begin{cases} \varphi(\gamma(s)) = \exp_p\Big(s\frac{2}{\pi}d(p, m)\big(i \circ r'(0)\big)\Big), & 0 < s \leq \frac{\pi}{2} \\ \varphi(\gamma(s)) = \exp_q\Big(\big(2 - \frac{2s}{\pi}\big)d(q, m)V\Big), & \frac{\pi}{2} \leq s < \pi \end{cases}$$

step (3) Check that $\varphi_1 : N^o \to D_1^o$ is bijective, $\varphi_2 : S^o \to D_2^o$ is bijective and $\varphi_3 : E \to \partial D_1 = \partial D_2 = D_1 \cap D_2$, where N^o is intrinsic of northern hemisphere N and S^o is intrinsic of southern hemisphere S.

proof of step (1):

First, we know that $D_1 \cup D_2 \subset M$, so we will show that $D_1 \cup D_2 \supset M$. Let $r \in M$. By Lemma 3.1.3 either $d(p, r) < \rho$ or $d(q, r) < \rho$. We only consider the first case, since the second is analogous.

Suppose $d(p,r) < \rho$. Since $d(p, C_m(p)) \geq \pi > \rho$, there exists a unique minimizing geodesic γ passing through p and r. Then $d(p,m) = d(p,r) + d(r,m)$. By Lemma 3.1.4 there exists a unique point m on γ such that $d(p,m) = d(q,m) < \rho$. Hence

(a) If $d(p,r) < d(q,r)$, then $r \in \overline{pm}$ (i.e. $r \in D_1$).

(b) If $d(p,r) = d(q,r)$, by uniqueness, we have $r = m$ (i.e. $r \in \partial D_1$ or $r \in \partial D_2$).

(c) If $d(p,r) > d(q,r)$, then $d(q,r) < \rho$ and hence $r \in \overline{qn}$ (i.e. $r \in D_2$).

Therefore $r \in D_1 \cup D_2$.

If $r \in D_1 \cap D_2$, then $d(p,r) = d(q,r)$ and hence $r = m = n$ (i.e. $r \in \partial D_1(\partial D_2)$). This complete the proof of step (1).

proof of step (2):

Fixed a point $N \in S^n$ associate p, and to its antipodal point $S \in S^n$ associate q. Choose a linear isometry mapping $i : T_N S^n \to T_P M$. For each point e of the equator E of S^n relative to the north pole N, consider the geodesic $\gamma(s)$ of S^n, $0 \leq s \leq \pi$, given by $\gamma(0) = N$, $\gamma(\frac{\pi}{2}) = e$. Let m be the point given by Lemma 3.1.4 on the geodesic of M which pass through p with $i(r'(0))$. Define:

$$\begin{cases} \varphi(\gamma(s)) = \exp_p\Big(s\frac{2}{\pi}d(p,m)\big(i \circ r'(0)\big)\Big), & 0 < s \leq \frac{\pi}{2} \\ \varphi(\gamma(s)) = \exp_q\Big(\big(2 - \frac{2s}{\pi}\big)d(q,m)V\Big), & \frac{\pi}{2} \leq s < \pi \end{cases}$$

where V is the unit tangent vector at q of the unique minimizing geodesic from q to m.

proof of step (3):

Clearly, it is easy to check $\varphi_1 : N^o \to D_1^o$ is bijective, $\varphi_2 : S^o \to D_2^o$ is bijective and $\varphi_3 : E \to \partial D_1 = \partial D_2 = D_1 \cap D_2$, where N^o is intrinsic of northern hemisphere N and S^o is intrinsic of southern hemisphere S.

By step (2) and step (3), the mapping $\varphi : S^n \to M$ is homeomorphism. The proof is complete. ■

3.2 The diameter sphere theorem

In this section, we shall discuss the diameter sphere theorem proposed by Grove and Shiohama. The argument presented here relies on the variation theory for geodesics and the Morse theory. The statement is as following:

Theorem 3.2.1. *(diameter sphere theorem)(Grove, Shiohama [27] 1977)*
Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$ with sectional curvature $K \geq 1$ and diam$(M) > \frac{\pi}{2}$. Then M is homeomorphic to S^n.

Before the proof of the diameter sphere theorem, we need the following two lemmas:

Lemma 3.2.2. *[16] Let (M, g) be a complete Riemannian manifold and let q be a point in M. Suppose that $\gamma : (-\varepsilon, 0] \to M$ is a smooth path satisfying $d(\gamma(s), q) \geq d(\gamma(0), q) + \mu s$, $\forall s \in (-\varepsilon, 0]$. Then there exists a vector $v \in T_{\gamma(0)}(M)$ such that $\exp_{\gamma(0)}(v) = q$, $|v| = d(\gamma(0), q)$, and $< \gamma^{'}(0), v > \geq -\mu|v|$.*

Proof. Since (M, g) is complete, by Hopf-Rinow theorem 2.1.9, there exists a vector $v \in T_{\gamma(0)}(M)$ such that $\exp_{\gamma(0)}(v) = q$ and $|v| = d(\gamma(0), q)$. If $v = 0$, then we are done. Hence it is suffices to consider the case $v \neq 0$. Define a smooth map $\alpha : [0, 1] \times (-\varepsilon, 0] \to M$ such that $\alpha(0, s) = \gamma(s)$, $\alpha(1, s) = q$ for all $s \in (-\varepsilon, 0]$ and $\alpha(t, 0) = \exp_{\gamma(0)}(tv)$ for all $t \in [0, 1]$. Then fix s and let t change. We obtain

$$L(\alpha(s)) \geq d(\gamma(s), q) \geq d(\gamma(0), q) + \mu s$$

for all $s \in (-\varepsilon, 0]$. By the formula for the first variation of arc-length 2.1.10, thus

$$-\frac{1}{|v|} < \gamma^{'}(0), v >= \frac{d}{ds} L(\alpha(s)) \mid_{s=0} \leq \mu.$$

Hence we obtain $< \gamma^{'}(0), v > \leq |v|\mu$. This proof is complete. ■

Lemma 3.2.3. *[16] Let (M, g) be a complete Riemannian manifold and let q be a point in M. Suppose that $\gamma : [0, \varepsilon) \to M$ is a smooth path satisfying $d(\gamma(s), q) \leq d(\gamma(0), q) + \mu s$, $\forall s \in [0, \varepsilon)$. Then there exists a vector $v \in T_{\gamma(0)}(M)$ such that $\exp_{\gamma(0)}(v) = q$, $|v| = d(\gamma(0), q)$, and $< \gamma^{'}(0), v > \geq -\mu|v|$.*

Proof. First, define

$$s_k = inf\{s \in [0, \varepsilon) \mid d(\gamma(s), q) \leq d(\gamma(0), q) + (\mu + \frac{1}{k})s - \frac{1}{k^2}\}$$

where k large enough. Clearly, $s_k \in (0, \frac{1}{k}]$. Moreover, we have

$$d(\gamma(s), q) \geq d(\gamma(s_k), q) + (\mu + \frac{1}{k})(s - s_k)$$

for all $s \in [0, s_k]$. By Lemma 3.2.2 there exists a $v_k \in T_{\gamma(s_k)}M$ such that $exp_{\gamma(s_k)}(v_k) = q$, $|v_k| = d(\gamma(s_k), q)$, and $< \gamma^{'}(s_k), v_k >\geq -(\mu + \frac{1}{k})|v_k|$. This proof is complete, when we choose the limit $k \to \infty$. ■

We prove the diameter sphere theorem as following:

Proof of diameter sphere theorem. We claim that M is $(n-1) - connected$ and prove it by contradiction. Suppose this false. There exists an integer $k \in \{1, \ldots, n-1\}$ such that $\pi_k(M) \neq 0$. Let us fix two points $p, q \in M$ such that $d(p, q) = diam(M) > \frac{\pi}{2}$. Then we follow the following steps:

step (1) Prove that if $\gamma : [0, 1] \to M$ is a geodesic satisfying $\gamma(0) = \gamma(1) = p$, then γ has Morse index at least $n - 1$

step (2) Show that there exists a geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0) = \gamma(1) = p$ and $ind(\gamma) < k$, where $ind(\gamma)$ denotes the index of γ

proof of step (1):

By assumption, we have $d(\gamma(s), q) \leq d(\gamma(0), q)$ for all $s \in [0, 1]$. By Lemma 3.2.3, there exists $v \in T_pM$ such that $\exp_p = q$, $|v| = d(p, q)$, and $< \gamma^{'}(0), v >\geq 0$.

Next, we claim that $L(\gamma) \leq \pi$. To prove this, we argue by contradiction. If $L(\gamma) \leq \pi$, then the Toponogov's theorem 2.1.8, implies that

$$\cos(d(\gamma(1), q)) \geq \cos(L(\gamma)) \cos(d(\gamma(0), q)) + \sin(L(\gamma)) \sin(d(\gamma(0), q)) \cos(\sphericalangle(\gamma^{'}(0), v)).$$

By assumption, we have $L(\gamma) \in (0, \pi]$ and $d(\gamma(0), q) \in (\frac{\pi}{2}, \pi]$. Moreover, the inequality

$\sphericalangle(\gamma^{'}(0), v) \geq 0$ implies $cos(\sphericalangle(\gamma^{'}(0), v)) \geq 0$, we obtain

$$\cos(d(\gamma(1), q)) \geq \cos(L(\gamma)) \cos(d(\gamma(0), q)).$$

Hence, we get $\cos(d(\gamma(1), q)) \neq \cos(d(\gamma(0), q))$, which contradicts to the fact that $\gamma(0) = \gamma(1) = p$. Consequently, we have $L(\gamma) > \pi$.

Let $\Bbbk$ be the space of all vector fields of the form $V(s) = sin(\pi s)X(s)$, where X is a parallel vector field along γ satisfying $< \gamma^{'}(s), X(s) >= 0$ for all $s \in [0, 1]$. Then

$$D_{\frac{d}{ds}} D_{\frac{d}{ds}} V(s) = -\pi^2 \sin(\pi s) X(s) = -\pi^2 V(s).$$

Let I denote the index form associated with the second variation of arclength. Then

$$\begin{aligned}
I(V, V) &= \int_0^1 \Big(|D_{\frac{d}{ds}} V(s)|^2 - R\big(\gamma^{'}(s), V(s), \gamma^{'}(s), V(s)\big)\Big) ds \\
&= \int_0^1 \Big(\pi^2 |V(s)|^2 - R\big(\gamma^{'}(s), \sin(\pi s) X(s), \gamma^{'}(s), \sin(\pi s) X(s)\big)\Big) ds \\
&= \int_0^1 \Big(\pi^2 |V(s)|^2 - (\sin(\pi s))^2 R\big(\gamma^{'}(s), X(s), \gamma^{'}(s), X(s)\big)\Big) ds \\
&\leq \int_0^1 \pi^2 |V(s)|^2 - \big(L(\gamma)\big)^2 |V(s)|^2 ds \\
&= \big(\pi^2 - L(\gamma)^2\big) \int_0^1 |V(s)|^2 ds
\end{aligned}$$

where $V(s) \in \Bbbk$. Since $L(\gamma) > \pi$ implies $-L(\gamma)^2 < -\pi^2$, therefore

$$I(V, V) \leq \big(\pi^2 - L(\gamma)^2\big) \int_0^1 |V(s)|^2 ds < 0.$$

Hence the restriction of I to the vector space $\Bbbk$ is negative definite. This implies $ind(\gamma) \geq \dim \Bbbk = n - 1$.

To prove step (2), we need the following some tools. We say that a critical point $p \in M$ for a smooth function $f : M \to \mathbb{R}$ has *index* $\geq m$ if the Hessian of f is negative definite on a $m-$dimensional subspace in T_pM. Define

$$\Omega_{A,B}(M) = \{\gamma : [0,1] \to M | \gamma(0) \in A, \gamma(1) \in B\}$$

where γ is a geodesic on M. If $A, B \subset M$ are compact, then the energy functional

$$E : \Omega_{A,B}(M) \to [0, \infty)$$

is reasonably nice in the sense that it behaves like a proper smooth function on a manifold. If in addition A and B are submanifolds then the variational fields for variations in $\Omega_{A,B}(M)$ consist of fields along the curve that are tangent to A and B at the endpoint. Therefore, critical points are naturally identified with geodesic that are perpendicular to A and B at the endpoints. We say that the index of such a geodesic $\geq k$ if there is a $k-$dimensional space of fields along the geodesic such that the second variation of the these fields is negative.

proof of step (2):

Logically, the step (2) is equivalent to "*Let $A \subset M$ be a compact $submanifold. If$ every geodesic in $\Omega_{A,A}(M)$ has index $\geq k$, then $A \subset M$ is $k-connected$.*"

Identify $A = E^{-1}(0)$ and use the above as a guide for what should happen. This shows that $A \subset \Omega_{A,A}(M)$ is $(k-1)-$connected. Next we note that

$$\pi_l(\Omega_{A,A}(M), A) = \pi_{l+1}(M, A).$$

This gives the result.

By step (1) and step (2), this is contradiction. Hence, we have $\pi_k(M) = 0$, and M is $(n-1)-connected$. Which implies that M is a homotopy sphere. Therefore, by result of Freedman[23] and Smale[42] that M is homeomorphic to S^n. ■

3.3 The sphere theorem of Ricci curvature comparison

In this section, we shall prove some fundamental results for manifolds with lower Ricci curvature bounds.

Let $r(x) = d(x, p)$ be the distance function.

Theorem 3.3.1. *(the Ricci comparison sphere theorem)(S.Y. Cheng [20] 1975)*
If (M, g) is a complete Riemannian manifold with $Ric \geq (n-1)k > 0$ and $diam(M) = \frac{\pi}{\sqrt{k}}$, then M is isometric to S_k^n.

To prove this theorem, we need the following two important lemmas.

Lemma 3.3.2. *(the Ricci comparison result) [38]*
Suppose that (M, g) has $Ric \geq (n-1)k$ for some $k \in \mathbb{R}$. Then

$$\triangle r \leq (n-1)\frac{\sin_k'(r)}{\sin_k(r)},$$

$$dvol \leq dvol_k,$$

where $dvol_k$ is the volume form in constant sectional curvature k.

Lemma 3.3.3. *(relative volume comparison [38])*
Suppose (M, g) is a complete Riemannian manifold with $Ric \geq (n-1)k$. Then

$$r \to \frac{volB(p, r)}{v(n, k, r)}$$

is a nonincreasing function and its limit is 1 as $r \to 0$, where $v(n, k, r)$ denotes the volume of a ball of radius r in the constant curvature space form S_k^n.

Next, we present the proof of Ricci comparison sphere theorem.

Proof of Ricci comparison sphere theorem. Fix $p, q \in M$ such that $d(p, q) = \frac{\pi}{\sqrt{k}}$. Define $r(x) = d(x, p)$, $\tilde{r}(x) = d(x, q)$ and $\sin_k(r) = \frac{\sin(\sqrt{k}r)}{\sqrt{k}}$. We claim that

step (1) $r + \tilde{r} = d(p, x) + d(x, q) = d(p, q) = \frac{\pi}{\sqrt{k}}, x \in M$.

step (2) $r, \tilde{r}$ are smooth on $M - \{p, q\}$.

step (3) $Hessr = (\frac{\sin_k'}{\sin_k})ds_{n-1}^2$.

step (4) $g = dr^2 + \sin_k^2 ds_{n-1}^2$

We know that step (3) implies step (4) and that step (4) implies M must be isometric to S_k^n.

proof of step (1):

We prove it by contradiction. The triangle inequality shows that

$$\frac{\pi}{\sqrt{k}} = d(p,q) \leq d(p,x) + d(x,q).$$

Hence it suffices to show that $d(p,q) \geq d(p,x) + d(x,q)$ by contradiction. Suppose that $d(p,q) < d(p,x) + d(x,q)$. we can find $\varepsilon > 0$ such that (see Figure 3.3.1)

$$\begin{aligned} 2\varepsilon + \frac{\pi}{\sqrt{k}} &= 2\varepsilon + d(p,q) \\ &= 2\varepsilon + d(p,p^{'}) + d(q^{'},q) \\ &= d(p,p^{'}) + d(p^{'},x) + d(x,q^{'}) + d(q^{'},q) \\ &= d(p,x) + d(x,q). \end{aligned}$$

When $r_1 \leq d(p,x), r_2 \leq d(q,x)$ and $r_1 + r_2 = \frac{\pi}{\sqrt{k}}$, the metric balls $B(p,r_1)$, $B(q,r_2)$ and $B(x,\varepsilon)$ are pairwise disjoint. Thus,

$$\begin{aligned} 1 = \frac{volM}{volM} &\geq \frac{volB(x,\varepsilon) + volB(p,r_1) + volB(q,r_2)}{volM} \\ &\geq \frac{v(n,k,\varepsilon)}{v(n,k,\frac{\pi}{\sqrt{k}})} + \frac{v(n,k,r_1)}{v(n,k,\frac{\pi}{\sqrt{k}})} + \frac{v(n,k,r_2)}{v(n,k,\frac{\pi}{\sqrt{k}})} \\ &= \frac{v(n,k,\varepsilon)}{v(n,k,\frac{\pi}{\sqrt{k}})} + 1, \end{aligned}$$

which is a contradiction.

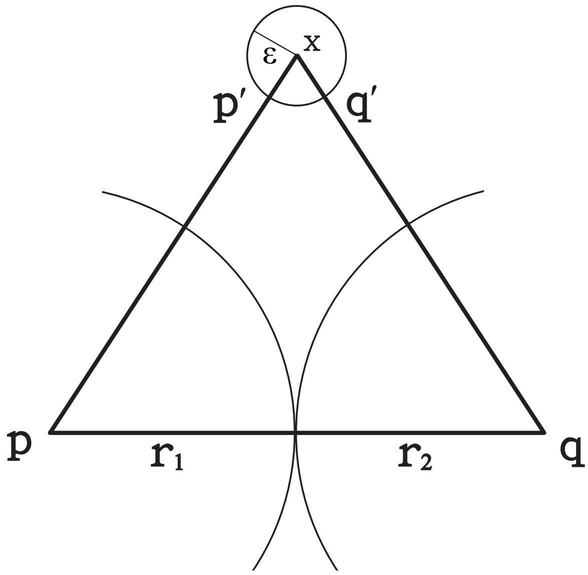

Figure 3.3.1:

proof of step (2):

If $x \in M - \{p, q\}$, then x can be joined to both p and q by segments σ_1, σ_2. The previous statement says that if we put these two segments together, then we get a segment from p to q through x. Such a segment must be smooth, and thus σ_1 and σ_2 are both subsegments of a larger segment. This implies from our characterization of when distance functions are smooth that both r and $\tilde{r}$ are smooth at $x \in M - \{p, q\}$.

proof of step (3):

We have $r(x) + \tilde{r}(x) = d(p, q) = \frac{\pi}{\sqrt{k}}$, thus $\triangle r = -\triangle \tilde{r}$. On the other hand, by Lemma 3.3.2

$$\begin{aligned}
(n-1)\frac{\sin_k'\big(r(x)\big)}{\sin_k\big(r(x)\big)} &\geq \triangle r(x) \\
&= -\triangle \tilde{r}(x) \\
&\geq -(n-1)\frac{\sin_k'\big(\tilde{r}(x)\big)}{\sin_k\big(\tilde{r}(x)\big)} \\
&= -(n-1)\frac{\sin_k'\big(\frac{\pi}{\sqrt{k}} - r(x)\big)}{\sin_k\big(\frac{\pi}{\sqrt{k}} - r(x)\big)} \\
&= (n-1)\frac{\sin_k'\big(r(x)\big)}{\sin_k\big(r(x)\big)}.
\end{aligned}$$

This implies,

$$\triangle r(x) = (n-1)\frac{\sin_k'\big(r(x)\big)}{\sin_k\big(r(x)\big)}.$$

Hence,

$$Hessr = \frac{\triangle r}{n-1} g_r = \frac{\sin_k'}{\sin_k} g_r.$$

$\blacksquare$

The conditions in theorem 3.3.1 require lower bounds for the Ricci curvature and the diameter of (M, g). It is natural to ask whether these assumptions can be replaced by lower bounds for the Ricci curvature and volume of (M, g). An important result is as following:

Theorem 3.3.4. *(J. Cheeger, T. Colding [19] 1997)*
For each integer $n \geq 2$, there exists a real number $\psi(n) \in (0,1)$ with the following property: if (M, g) is a compact Riemannian manifold of dimension n with $Ric \geq (n-1)g$ and $vol(M, g) \geq (1-\psi(n))vol(S^n(1))$, then M is diffeomorphic to S^n.

3.4 The sphere theorem with positive curvature

In this section, we shall introduce the sphere theorem of the positively (sectional) curved manifolds. We denote a compact positively curved manifolds by CPCM. First, we will introduce the *Weinstein and Synge theorem*.

Theorem 3.4.1. *(Weinstein and Synge theorem [22])*
Let f be an isometry of a compact oriented Riemannian manifold M^n. Suppose that M has positive sectional curvature and that f preserves the orientation of M if n is even, and reverses it if n is odd. Then f has a fixed point, i.e., there exists $p \in M$ with $f(p) = p$.

We are going to introduce the *Synge's theorem* as following.

Theorem 3.4.2. *(Synge theorem [22])*
Let M^n be a compact manifold with positive sectional curvature.

(a) If M^n is orientable and n is even, then M is simply connected.

(b) If n is odd, then M^n is orientable.

Proof.

(a) Let $\pi : \tilde{M} \to M$ be the universal covering of M. Introduce on $\tilde{M}$ the covering metric, and orient $\tilde{M}$ in such a way that π preserves the orientation. Because M is compact and has positive curvature, we must have $K \geq \delta > 0$. From the fact that π is a local isometry, the same curvature condition holds on $\tilde{M}$. Since $\tilde{M}$ is complete, so $\tilde{M}$ is compact. Let $k : \tilde{M} \to \tilde{M}$ be a covering transformation of $\tilde{M}$, that is, $\pi(k) = \pi$. Then k is an isometry of $\tilde{M}$, from the way that we oriented $\tilde{M}$, preserves the orientation. Because n is even, we can use the theorem ro conclude that k has a fixed point. But a covering transformation which has a fixed point is the identity. It follows that the group of covering transformations of $\tilde{M}$ reduces to the identity. Therefore M is simply connected.

(b) We prove it by contradiction. Suppose that M is not orientable, and consider the orientable double cover $\overline{M}$ of M. Where $\overline{M} = \{(p, O_p) | p \in M, O_p \in \mathcal{O}_p\}$ and

$O_p \in \mathcal{O}_p$ will be called an *orientation*. We introduce on $\overline{M}$ the covering metric. Since $\overline{M}$ is the double cover of a compact manifold, $\overline{M}$ is compact. Let k be a covering transformation of $\overline{M}$, $k \neq id$. Because M is not orientable, k is an isometry which reverses the orientation of $\overline{M}$. Since n is odd, we can apply the Weinstein Synge theorem 3.4.1 which guarantees that k has a fixed point. Therefore $k = id$, which is a contradiction. ∎

Synge's theorem 3.4.2 asserts that an even dimensional, orientable CPCM is simply connected. This theorem together with the topological classification of compact surfaces implies that a 2-dimensional, orientable CPCM is homeomorphic to S^2. Three dimensional CPCM's have been determined by Hamilton [29]; they are diffeomorphic to space forms. Hence we consider only 4-dimensional CPCM.

It is known that the existence of a nontrivial Killing vector field on a compact Riemannian manifold M is equivalent to the existence of a nontrivial S^1-action on M. Let $F(S^1, M)$ be the fixed point set of such an S^1-action on M. An S^1-Riemannian manifold is a Riemannian manifold with a given isometric S^1-action and denoted (S^1, M).

Let $y \in M$ be an isolated fixed point. Let $\pi : S^1 \times M \to M$ be the canonical surjection. The local geometry of M near a point $\pi^{-1}(y) \in S^1 \times M$ is determined by the geometry of the local representation at $y \in M$. This representation is equivalent to

$$\phi_{k,l} : S^1 \times \mathbb{C}^2 \to \mathbb{C}^2; \quad \phi_{k,l}[e^{i\theta}(z_1, z_2)] = (e^{ik\theta} z_1, e^{il\theta} z_2),$$

where $z_1, z_2 \in \mathbb{C}$ and $k, l \in \mathbb{Z}$ with $g.c.d(k, l) = 1$. Let $S^3(1) \subseteq \mathbb{C}^2$ be the unit sphere and let $d : S^3(1) \times S^3(1) \to \mathbb{R}$ be given by $d(v, w) = \angle(v, w)$ =the angle between v and w. Let (X_{kl}, d_{kl}) be the orbit space of $(\phi_{kl}, S^3(1), d)$ with orbital distance metric d_{kl}.

Lemma 3.4.3. *[31] If x_1, x_2, x_3 are arbitrary points in X_{kl}, then*

$$d_{kl}(x_1, x_2) + d_{kl}(x_2, x_3) + d_{kl}(x_3, x_1) \leq \pi.$$

Proof. The two great circles in $S^3(1)$ given by $z_1 = 0$ and $z_2 = 0$ are orbits of $\phi_{k,l}$ for all k, l with $g.c.d(k, l) = 1$. Let $\tilde{X}_{k,l} = X_{k,l} - \{z_1 = 0, z_2 = 0\}$. $\tilde{X}_{k,l}$ consists of principal orbits, so we give it the Riemannian submersion metric coming form the canonical Riemannian metric in $S^3(1)$. We will be using the fact that this Riemannian submersion metric induces the distance function $d_{k,l}$ on $\tilde{X}_{k,l}$.

In the special case where $k = l = 1$, the projection $\pi : S^3(1) \to X_{1,1}$ is the Hopf fibration and it is easily checked that $X_{1,1}$ is isometric to a $\mathbb{CP}^1$ with diameter $\frac{\pi}{2}$, (i.e., $X_{1,1}$ is isometric to $S^2(\frac{1}{2}) \subseteq \mathbb{R}^3$.) Hence the inequality $d_{1,1}(x_1, x_2)+d_{1,1}(x_2, x_3)+d_{1,1}(x_3, x_1) \leq \pi$ is obvious.

We now fix $(k, l) \neq (1, 1)$. The isometric T^2-action

$$T^2 \times S^3(1) \to S^3(1), \quad (e^{i\theta_1}, e^{i\theta_2})(z_1, z_2) \Rightarrow (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$$

induce an isometric S^1-action on the Riemannian manifold $\tilde{X}_{k,l}$. $\tilde{X}_{k,l}$ is a connected noncomplete surface of revolution with diameter $\frac{\pi}{2}$, so it admits a coordinate system

$$(r, \theta) : (0, \frac{\pi}{2}) \times S^1 \to \tilde{X}_{k,l}$$

such that the metric in these coordinate is

$$ds^2 = dr^2 + (f(r))^2 d\theta^2$$

where $d\theta$ is standard $1-$form on S^1. We can arrange that the latitude circle $r = c$ corresponds to the orbit space of the torus $T^2(c) = T^2(\cos c, \sin c) \subseteq S^3(1)$. Hence

$$2\pi f(c)(\text{the length of a } \phi_{k,l} \text{ orbit in } T^2(c)) = 4\pi \cos c \sin c.$$

The orbits of $\phi_{k,l}$ all have length $\geq 2\pi$, so $f(c) \leq \cos c \sin c = \frac{1}{2} \sin 2c$. Hence there is a length nonincreasing bijection of $\tilde{X}_{1,1}$ onto $\tilde{X}_{k,l}$ with same coordinates in $(0, \frac{\pi}{2}) \times S^1$. The inequality

$$d_{k,l}(x_1, x_2) + d_{k,l}(x_2, x_3) + d_{k,l}(x_3, x_1) \leq \pi$$

for $x_1, x_2, x_3 \in \tilde{X}_{k,l}$ now follows from the corresponding inequality already proved for

$k = l = 1$. This proof is complete. ■

Let $l_{ij} = dist(p_i, p_j)$ and let $C_{ij} = \{\gamma : [0, l_{ij}] \to M | \gamma$ is a minimizing geodesic segment from p_i to $p_j\}$, $1 \leq i, j \leq 4$. For each triple $1 \leq i, j, k \leq 4$, set

$$\alpha_{ijk} = min\{\angle(\gamma_j'(0), \gamma_k'(0)) | \gamma_j \in C_{ij}, \gamma_k \in C_{ik}\}.$$

Lemma 3.4.4. *[31] For each triple of distinct integers $1 \leq i, j, k \leq 4$,*

$$\alpha_{ijk} + \alpha_{kij} + \alpha_{jki} > \pi.$$

Proof. Assume that $(i, j, k) = (1, 2, 3)$. Set $\frac{1}{R^2} = \delta$ =minimum of sectional curvature of M. Choose x_1', x_2', x_3' on $S^2(R)$ such that the spherical triangle $\triangle(x_1', x_2', x_3')$ has l_{12}, l_{23}, l_{31} be its three lengths. Applying Toponogov's theorem 2.1.8 to an arbitrary $\gamma_{12} \in C_{12}, \gamma_{23} \in C_{23}, \gamma_{13} \in C_{13}$, as its three side, one gets

$$\angle(\gamma_{12}'(0), \gamma_{13}'(0)) \geq \angle(\overline{x_1'x_2'}, \overline{x_1'x_3'}),$$

and hence, by the definition of α_{123}, that $\alpha_{123} \geq \angle(\overline{x_1'x_3'}, \overline{x_1'x_3'})$. Therefore $\alpha_{123} + \alpha_{312} + \alpha_{231} \geq \sum \triangle(x_1', x_2', x_3') > \pi$. ■

We prove the sphere theorem with positive curvature as following:

Theorem 3.4.5. *(Wu-Yi Hsiang, B. Kleiner [31] 1989)*

Let M be 4-dimensional orientable CPCM. If M has a nontrivial Killing vector field, then M is homeomorphic to S^4 or $\mathbb{CP}^2$.

Proof. Let M be a 4-dimensional orientable CPCM. Then by Synge's theorem 3.4.1 M is simply connected. We will use the orbital geometry of the given S^1-action to prove that $\chi(M)$ at most 3. It follows directly from the work of Freedman [23] that M is homeomorphic to either S^4 or $\mathbb{CP}^2$. By Wu-Yi Hsiang and Bruce Kleiner's paper [31], we have $\chi(M) = \chi(F)$ and

$$F = \begin{cases} \chi(M) \quad isolated \quad point. \\ or \quad S^2 \cup \{(\chi(M) - 2) \quad isolated \quad points.\} \end{cases}$$

Hence the proof of the theorem reduces to proving that F consists of at most three isolated points or S^2 plus at most one more isolated point. We will divide the proof into two cases and we will prove each case by contradiction.

Case 1, $dimF = 2$. Suppose that $F = S^2$ plus at least two isolated fixed points. Let p, q be two isolated fixed points and let γ be a minimizing geodesic in M joining p to q. Let η be a minimizing geodesic segment from S^2 to $S^1(\gamma)$, the S^1 orbit of γ; hence length(η)=d(S^2,$S^1(\gamma)$), and η has endpoints $A \in S^2$ and $B \in S^1(\gamma)$. We will claim the second variation $E''(0) < 0$.

Suppose B lies in the interior of γ. We consider the minimizing geodesic segment η from S^2 to p such that $\eta(0) = p'$ and $\eta(l) = p$. Let $e_1(t)$ be a unit parallel field along η and $e_1(0)$ be its tangent vector. Let $\beta(s), s \in (-\varepsilon, \varepsilon)$, be geodesic in S^2 such that $\beta(0) = p'$ and $\beta'(0) = e_1(0)$. Let $h(s,t)$ be variation of η given by

$$h(s,t) = \exp_{\eta(t)}(se_1(t)), \quad s \in (-\varepsilon, \varepsilon), \quad t \in [0, l].$$

since $h(s, 0) = \beta(0)$, then $h(s, l) = \exp_p(se_1(l)) = p$. Therefore

$$\frac{\partial h}{\partial s}(0,t) = V(t) = \frac{\partial}{\partial s}\exp_{\eta(t)}(se_1(t))|_{s=0} = e_1(t),$$

hence $\frac{D^2V}{dt^2} = 0$. Using the second variation formula 2.1.10 and the fact that $\frac{\partial h}{\partial s}(0,t) = e_1(t)$, we obtain

$$\begin{aligned}
\frac{1}{2}E''(0) &= -\int_0^l g(V(t), \frac{D^2V}{dt^2} + R(\frac{d\eta}{dt}, V(t))\frac{d\eta}{dt})dt - g(\frac{D}{ds}\frac{\partial h}{\partial s}, \frac{d\eta}{dt})(0,0) \\
&+ g(\frac{D}{ds}\frac{\partial h}{\partial s}, \frac{d\eta}{dt})(0,l) - g(V(0), \frac{DV}{dt}(0)) + g(V(l), \frac{DV}{dt}(l)) \\
&= -\int_0^l g(e_1(t), R(\frac{d\eta}{dt}, e_1(t))\frac{d\eta}{dt})dt \\
&= -\int_0^l K(e_1(t), \frac{d\eta}{dt})dt.
\end{aligned}$$

Because $K(e_1(t), \frac{d\eta}{dt})$ is positive,

$$\frac{1}{2}E''(0) < 0,$$

and therefore there exists a local minimum, so length(η)>d(S^2,$S^1(\eta)$). This contradicts the assumption that length(η)=d(S^2,$S^1(\eta)$).

Suppose $B = p$. Apply same argument. The second variation formula can now be applied to the geodesic segment η. It is shown that length(η)>d(S^2,$S^1(\gamma)$). This contradicts the assumption that length(η)=d(S^2,$S^1(\gamma)$). The same argument rule out $B = q$. Hence F can contain at most one isolated fixed point in addition to the S^2.

Case 2, $dimF = 0$. Suppose that F contains at least four isolated points, $P_i, 1 \leq i \leq 4$. By lemma3.4.4, It follows easily that

$$\sum_{1\leq i\leq 4}\sum_{1\leq j<k\leq 4} \alpha_{ijk} > 4\pi \quad j,k \neq i.$$

But, on the other hand, by lemma 3.4.3 it is easily seen that

$$\sum_{1\leq j<k\leq 4} \alpha_{ijk} \leq \pi \quad j,k \neq i$$

for each $1 \leq i \leq 4$, which gives a contradiction. Hence F contains at most three isolated points when $dimF = 0$. This completes the proof of the theorem. ■

3.5 Other sphere theorems

In this section, we first introduce some curvature conditions. A Riemannian manifold M is said to have *positive isotropic curvature* if

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0$$

for all points $p \in M$ and all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_pM$.

We say that M has *nonnegative isotropic curvature* if

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 0$$

for all points $p \in M$ and all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_pM$.

Next, we shall define another curvature condition. A Riemannian manifold (M, g) is said to be *weakly $\delta -$ pinched in the pointwise sense* if $0 \leq \delta K(\pi_1) \leq K(\pi_2)$ for all point $p \in M$ and all two-dimensional planes $\pi_1, \pi_2 \subset T_PM$. If the strict inequality holds, we say that (M, g) is *strictly $\delta -$ pinched in the pointwise sense*. Similarly, M is said to be *weakly $\delta -$ pinched in the global sense* if the sectional curvature of M satisfies $\delta \leq K \leq 1$. If the strict inequality holds, we say that M is *strictly $\delta -$ pinched in the global sense*.

Now, we are going to introduce the compact, simply connected Riemannian manifold with positive isotropic curvature. M. Micallef and J.D. Moore obtained the following result:

Theorem 3.5.1. *(M. Micallef, J.D. Moore [35] 1988)*

Let M be a compact simply connected Riemannian manifold with positive isotropic curvature. Then M is a homotopy sphere and hence M is homeomorphic to S^n.

Next, we describe sufficient conditions for the vanishing of the second Betti number. M. Berger [5] proved that the second Betti number of a manifold with pointwise $\frac{1}{4}$-pinched sectional curvatures is equal to 0. In even dimensions, the same result holds under the weaker assumption that M has positive isotropic curvature:

Theorem 3.5.2. *(M. Michallef, M. Wang [36] 1993)*

Let M be a compact Riemannian manifold of dimension $n \geq 4$. Suppose that n is even and M has positive isotropic curvature. Then the second Betti number of M vanishes.

In odd dimensions, The following result was established by M.Berger.

Theorem 3.5.3. *(M. Berger [5] 1960)*

Let M be a compact Riemannian manifold of dimension $n \geq 5$. Suppose that n is odd and M has pointwise $\frac{n-3}{4n-9}$-pinched sectional curvatures. Then the second Betti number of M vanishes.

Finally, we mention a result concerning the topology of four-manifolds with positive sectional curvature.

Theorem 3.5.4. *(W. Seaman [41], M. Ville [44] 1989)*

Let (M, g) be a compact, orientable Riemannian manifold of dimension 4 which is δ-pinched in the global sense ($\delta \sim 0.188$). Then (M, g) is homeomorphic to S^4 or $\mathbb{CP}^2$.

Table 3.1: The figure of Topological Sphere Theorem

	M^n :Compact, simply connected	Mathematician
$\frac{1}{4}$ − pinched sectional curvature	M^n is homeomorphic to S^n	1960, M. Berger & W.Klingberg
positive isotropic curvature	1. M^n is homotopic to S^n 2. $n \geq 4$, M^n is homeomorphic to S^n	1988, M. Micallef & J.D. Moore

	M^n :Compact, $n \geq 4$	M^nCompact, $n \geq 5$	Mathematician
positive isotropic curvature	$b_2 = 0$ ($n \in$ even)		1993, M. Micallef & M. Wang
$\frac{n-3}{4n-9}$−pinched sectional curvature		$b_2 = 0$ ($n \in$ odd)	1960, M. Berger

	M^n :Compact,orientable	Mathematician
δ−pinched in the global sense($\delta \sim 0.188$)	M^n is homeomorphic to S^4 or $\mathbb{CP}^2$	1989, W. Seaman & M. Ville
1. positive sectional curvature 2. nontrivial Killing field	M^n is homeomorphic to S^4 or $\mathbb{CP}^2$	1989, Wu-Yi Hsiang & B. Kleniner

	M^n :Compact	Mathematician
1. sectional curvature$K \geq 1$ 2. $diam(M) > \frac{\pi}{2}$	M^n is homeomorphic to S^n	1975, S.Y. Cheng
1. $Ric \geqq (n-1)g$ 2. $vol(M,g) \geq (1-\delta(n)vol(S^n(1))$	M^n is diffeomorphic to S^n	1997, J. Cheeger & T. Colding

Chapter 4

Conclusion

The sphere theorem is determined by the topology of the manifold with curvature condition. For example, in 2-dimensional, we have the $Gauss - Bonnet\ theorem$.

We introduce the classic sphere theorem in Riemannian geometry. The first natural question is if it is possible to replace "homeomorphic" by "diffeomorphic" in the statement of the sphere theorem. Observe that the homeomorphism of the sphere theorem is obtain by "glueing" two discs along their boundaries. Such a construction may lead to a differentiable structure on M distinct from the usual structure of the sphere. Therefore, the proof of the sphere theorem presented here is not sufficient to establish a diffeomorphism.

So, we introduce another sphere theorem. The hypothesis of "$\frac{1}{4}-$pinched" is replaced by a hypothesis on the diameter: $If\ M\ is\ compact,\ K \geq 1,\ and\ diam(M) > \frac{\pi}{2}\ then\ M\ is\ homeomorphic\ to\ a\ sphere.$ The case $diam(M) = \frac{\pi}{2}$ (where the theorem is false, as shown by the example of real projective space) was essentially classified by Gromoll and Grove [25].

From Myer's diameter estimate 3.1.1, it is natural to ask what happens if the diameter attains it maximal value. Hence we introduce the Ricci comparison theorem.

We introduce the 4-dimensional orientable CPCM with nontrivial Killing vector field. It is homeomorphic to S^4 or $\mathbb{CP}^2$. Therefore it is natural to ask the following question:

Question 1. $A\ 4-dimensional\ CPCM\ with\ a\ nontrivial\ Killing\ vector\ field\ should\ be\ diffeomorphic\ to\ S^4, \mathbb{RP}^4,\ or\ \mathbb{CP}^2\ ?$

Question 2. *A compact, simply connected, nonnegatively curved* $4-manifold$ *with a nontrivial Killing vector field should be diffeomorphic to either* $S^4, \mathbb{CP}^2, \mathbb{CP}^2 \sharp \pm \mathbb{CP}^2$, *or* $S^2 \times S^2$?

Of course, it is possible that these theorems would remain true without the assumption on infinitesimal symmetry, but then their proofs would require completely new ideas and techniques.

Furthermore, a much-studied problem in Riemannian Geometry is to classify all Einstein manifolds satisfying a suitable curvature condition. This question was first studied by M.Berger [6], [7], in the 1960s. Berger showed that if (M, g) is a compact Einstein manifold of dimension n which is strictly $\frac{3n}{7n-4}-$pinched in the global sense, then (M, g) has constant sectional curvature. In 1974, S. Tachibana [43] proved that any compact Einstein manifold with positive curvature operator has constant sectional curvature. Furthermore, Tachibana showed that a compact Einstein manifold manifold with nonnegative curvature operator is locally symmetric. Other results in this direction were obtained M.Gursky and C.LeBrun [28] and D.Yang [45].

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