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Study of Edge-Majority Indices of Equitable Signed Graphs

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Study of Edge-Majority Indices of Equitable Signed Graphs

by

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Abstract

Let G be a connected multi-graph with vertex set $V(G)$ and edge set $E(G)$. If there exists an edge labeling function $f : E(G) \rightarrow \{1, -1\}$, such that the difference of numbers of edges labeled 1 and -1 is at most one, then we call such f an **equitable labeling** and G an **equitable signed graph**. An equitable edge labeling induces a vertex labeling in the following way. For vertices incident with more 1-edges than (-1)-edges, we label them 1. For vertices incident with more (-1)-edges than 1-edges, we label them -1. For vertices incident with the same number of (-1)-edges and 1-edges, we label them 0. Then the **edge-majority index** is defined as the absolute difference of the number of 1-vertices and the number of (-1)-vertices with respect to an equitable edge labeling. The set of all possible edge-majority indices of G with respect to all possible equitable labelings is called the **edge-majority index set** of G. Given an equitable edge labeling f of a graph with all odd degrees(all even degrees), we show that all even numbers(all numbers) less than certain edge-majority index with respect to f may be realized by continuously switching edge labels.

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Contents

Chapter 1

Introduction

1.1 Definitions

All graphs $G(V, E)$ considered in this thesis are finite, undirected, connected with vertex set V and edge set E, with possibly parallel edges. An edge labeling of a graph is a function $f : E \to A \subseteq \mathbb{Z}$, from the edge set to the set of integers. Signed graphs are graphs with signatures over edges, which are represented by an edge labeling with 1 and -1 respectively. In this case of a signed graph, the edge labeling function is from E to $A = \{1, -1\}$.

Let $E(v)$ be the set of all incident edges of the vertex v. An edge labeling f induces a vertex labeling, via abusing the language, which can be treated as a function f from V to $\{1, 0, -1\}$ defined by

$$f(v) = \begin{cases} 1, & if \quad \Sigma_{e \in E(v)} f(e) \geq 1 \\ 0, & if \quad \Sigma_{e \in E(v)} f(e) = 0 \\ -1, & if \quad \Sigma_{e \in E(v)} f(e) \leq -1 \end{cases}$$

For a signed graph G, let $e_f(i)$ the cardinality of $\{e \in E(G) : f(e) = i\}$ where $i = 1, -1$. An edge labeling f is called ***equitable*** if $|e_f(1) - e_f(-1)| \leq 1$, and we say G is an ***equitable signed graph***. For an equitable labeling f, we denote $v_f(i) = |\{v \in V(G) : f(v) = i\}|$, where $i = 1, 0, -1$. We define $|v_f(1) - v_f(-1)|$ to be an ***edge-majority index*** of G with respect to f. The set of all edge-majority indices of G with respect to all possible equitable labeling is called the ***edge-majority index set*** of G, and is denoted by EMI(G).

1.2 Background

A labeling of a graph is called an edge labeling if it is a function from E to $A \subseteq \mathbb{Z}$. There are many different types of labelings such as graceful labelings[5], harmonious labeling, magic labelings, antimagic labelings, and graph coloring. Signed graphs is a graph in which each edge is assigned a positive or negative sign, were first introduced by F. Harary to handle a problem in social psychology[1].

The definitions in this article are the same in [3]. Equitable edge labeling are also called edge friendly labeling[4], and edge-majority index set may be called edge-balance index set in different articles like [6][7][8]. Even use the same definitions, they may focus on different things, for example, unlabeled vertices in[4] and 0-vertices are the same in definitions.

The following are some preliminary examples as in [2] and [3]:

Example 1.1. *The edge-majority index set* $\mathrm{EMI}(nK_2)$ *of* n *isolated* K_2 *is* $\{0\}$ *if* n *is even, and is* $\{2\}$ *if* n *is odd.*

Example 1.2. *The edge-majority index set* $\mathrm{EMI}(St(n))$ *of the star graph* $St(n)$ *with* n *pendant edges is* $\{0\}$ *if* n *is even, and is* $\{2\}$ *if* n *is odd.*

Example 1.3. *The edge-majority index set* $\mathrm{EMI}(C_n)$ *of the graph* C_n *with* n *pendant edges is* $\{0\}$ *if* n *is even, and is* $\{1\}$ *if* n *is odd.*

Example 1.4. *The edge-majority index set* $\mathrm{EMI}(P_n)$ *of paths* P_n *on* n *vertices is*

$$\mathrm{EMI}(P_n) = \begin{cases} \{2\}, & n = 2, \\ \{0\}, & n = 3, \\ \{1,2\}, & n = 4, \\ \{0,1\}, & n \geq 5 \text{ is odd,} \\ \{0,1,2\}, & n \geq 6 \text{ is even.} \end{cases}$$

In the following are some previous results of equitable edge labelings. A graph is called an **edge-balanced graph** if $|e_f(-1) - e_f(0)| \leq 1$ and $|v_f(-1) - v_f(0)| \leq 1$ and a graph is called **strongly edge-balanced graph** if $|e_f(-1) - e_f(0)| = 0$ and $|v_f(-1) - v_f(0)| = 0$.

Theorem 1.1. *(Chen, Huang, Lee, and Liu [2])*
If G is a connected multigraph with an odd number of edges, then G is an edge-balanced graph if and only if G is neither $K_{1,2m+1}$ nor $K_2(2m+1)$, where $K_2(2m+1)$ is a multigraph with two vertices joined by $2n+1$ parallel edges.

Theorem 1.2. *(Chen, Huang, Lee, and Liu [2])*
If G is a multigraph, then G is an edge-balanced graph if and only if G is neither $K_{1,2m+1}$ nor $\bigcup_{i=1}^{2t+1} K_2(2n_i+1)$, where $n_i \geq 0$ and $t \geq 0$.

Theorem 1.3. *(Wang, Lin, Cozzens [3])*
Let G be a graph with at least $p \geq 4$ vertices, then the upper bound of $\mathrm{EMI}(G)$ is $p-2$.

Theorem 1.4. *(Wang, Lin, Cozzens [3])*
Let G be a cubic graph with p vertices with a perfect matching M. Then every even number strictly less than the upper bound $2\lceil \frac{3p}{4} \rceil - p$ can be realized as an edge-majority index of G. That is,

$$\{0, 2, 4, \cdots, 2\lceil \frac{3p}{4} \rceil - p - 2\} \subseteq \mathrm{EMI}(G).$$

Theorem 1.5. *(Chopra, Lee and Su [6])*
$EMI(W_n) = \{0, 2, ..., 2i, ..., n-2\}$ *for n is even,*
$EMI(W_n) = \{1, 3, ..., 2i+1, ..., n-2\} \bigcup \{0, 1, 2, ..., \frac{n-1}{2}\}$ *for n is odd.*

Theorem 1.6. *(Kropa, Lee and Raridan [4])*
For odd integers $n \geq 7$, there exists an edge-friendly labeling of K_n such that all the vertices are labeled (not 0-vertex).

1.3 Motivation

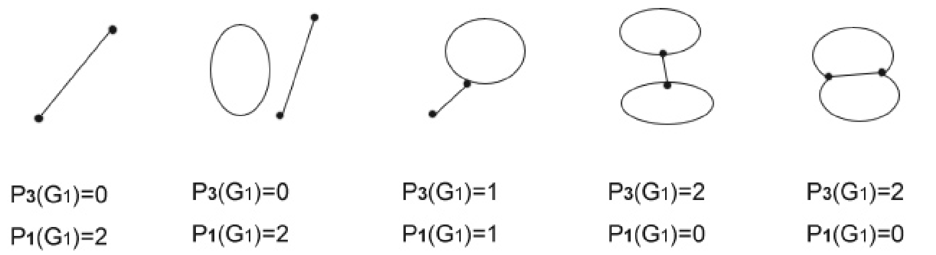
Let G_i be the edge induced subgraph of G obtained from G and an equitable labeling f by deleting $(-i)$-edges, where $i = 1, -1$. We denote $p_k(G_i)$ to be the number of vertices of degree k in G_i for $k = 0, 1, 2, 3$ and $i = 1, -1$.

In [3], we know that a cubic graph G with order p have a expected maximum number $2\lceil\frac{3p}{4}\rceil - p$ for EMI(G), which mean $max(EMI(G)) \leq 2\lceil\frac{3p}{4}\rceil - p$ and the equality holds if and only if

$$p_0(G_1) = p - \lceil\frac{3p}{4}\rceil, \quad p_1(G_1) = 0, \quad p_2(G_1) = \lceil\frac{3p}{4}\rceil, \quad p_3(G_1) = 0$$

if and only if G_1 is a disjoint union of cycles with possibly isolated vertices.

Then we want to know can every even number strictly less than the upper bound be realized as an edge-majority index of a cubic graph? If a cubic graph achieve the upper bound $2\lceil\frac{3p}{4}\rceil - p$, if only if G_1 is a disjoint union of cycles with possibly isolated vertices, pick any vertex v with $deg_{G_1}(v) = 2$,and there are vertices u_1, u_2 and u_3 which are adjacent to v, say vu_1 and vu_2 in G_1, delete one of those two from G_1 and let vu_3 in , then, we get one of those possibly edge labelings that the edge-majority index is $2\lceil\frac{3p}{4}\rceil - p - 2$ (no matter what shape G_1 is). So, if the edge-majority indices of a cubic graph can achieve a certain number, Does it achieve the lower one ?



By degree formula and the system of linear equations, we describe those relations between vertex degrees in G_1:

$$\begin{cases} p_1(G_1) + 2p_2(G_1) + 3p_3(G_1) = 2\lceil\frac{3p}{4}\rceil \\ p_2(G_1) + p_3(G_1) = \lceil\frac{3p}{4}\rceil - k \end{cases}$$

where $\begin{cases} k = 0, 1, \ldots, \lceil \frac{p}{4} \rceil - 1 & if \quad p\ mod 4 \equiv 2 \\ k = 0, 1, \ldots, \lceil \frac{p}{4} \rceil & if \quad p\ mod 4 \equiv 0 \end{cases}$

and $p_1(G_1), p_2(G_1), p_3(G_1) \in \mathbb{N}$.

Actually, when $k \to \lceil \frac{p}{4} \rceil (or \lceil \frac{p}{4} \rceil - 1)$, combinations of $(p_1(G_1), p_2(G_1), p_3(G_1))$ cannot be minor. For example, let $p = 10$.

When $k = 0$, the edge-majority index is 6.

$$\begin{cases} x + 2y + 3z = 16 \\ y + z = 8 \end{cases} \Rightarrow \begin{cases} y = 8 \\ x + z = 0 \end{cases} \Rightarrow (x, y, z) = (0, 8, 0)$$

When $k = 1$, edge-majority index is 4.

$$\begin{cases} x + 2y + 3z = 16 \\ y + z = 7 \end{cases} \Rightarrow \begin{cases} x + z = 2 \\ y + z = 7 \end{cases} \Rightarrow (x, y, z) = (0, 5, 2), (1, 6, 1), (2, 7, 0)$$

When $k = 2$, edge-majority index is 2
$\Rightarrow (x, y, z) = (0, 2, 4), (1, 3, 3), (2, 4, 2), (3, 5, 1)$.

When $k = 3$, edge-majority index is 0
$\Rightarrow (x, y, z) = (1, 0, 5), (2, 1, 4), (3, 2, 3), (4, 3, 2), (6, 5, 0)$.

It seems the minor the edge-majority index is, the more combinations of $(p_1(G_1), p_2(G_1), p_3(G_1))$ are. Then it should be easier (greater chances) to find then the bigger number of edge-majority index?

By intuition and naive method, for any simple graph G if we want to find its highest edge-majority index, it seem like a game using half of the edges to make as many as possible vertices our own, so we probably first look at those vertices that degree relatively smaller than most other vertices (since they much easy to take over) , and see if those vertices connected to each other by using as less edges as it can. on the other hand, If we randomly label those edges in G, it look like it is less possible to get its potential highest edge-majority index since it require more conditions but more likely to get a smaller edge-majority index. So for a graph, can we translate "finding $EMI(G)$" into "finding its highest edge-majority index"?

Chapter 2

Edge-Majority Indices of Odd Graphs

For an odd graph G whose vertices are all odd, we want to show that if we find an equitable labeling function f such that $x \in EMI(G)$, then by switch of labels, create a new labeling function such that $x - 2 \in EMI(G)$. Then, by continue the process, create a series of new labeling function that make $\{0, 2, \ldots, x-2, x\} \subseteq EMI(G)$.

2.1 Basics

Note that by interchanging the edge labels 1 and -1, we may assume without loss of generality that $v_f(1) \geqslant v_f(-1)$ with respect to f. Since we will focus on those positive and negative edges connect with each individual vertex in G, let $deg(v^+)$ the number of positive edge connected to v and $deg(v^-)$ the number of negative edge connected to v.

Claim 2.1. *Graph G with order n and all vertex degrees are odd. If $v_f(1) > \frac{n}{2}$ and $deg(v_i^+) - deg(v_i^-) = -1$ for all v_i that $f(v_i) = -1$, then $v_f(1) = \frac{n}{2} + 1$ and $e_f(1) - 1 = e_f(-1)$.*

Proof.

By degree formula, n is even since all vertex degrees are odd.

Let $v_f(1) = k$. For a labeling function f of a graph G, which all vertex degrees are odd, since v can not be 0 when degree of v is odd, let $\{v_1, v_2, \ldots, v_k\} = V^+$ the vertex set of vertices labeling with 1, $\{v_{k+1}, \ldots, v_n\} = V^-$ the vertex set of vertices labeling with -1.

Know $deg(v_i^+) - deg(v_i^-) = -1$ for all $i = k+1, \ldots n$, then $\sum_{i=k+1}^{n} deg(v_i^+) - \sum_{i=k+1}^{n} deg(v_i^-) = \sum_{i=k+1}^{n}(deg(v_i^+) - deg(v_i^-)) = k - n$.
Since $f(v_i) = 1$ for $i = 1, ..., k$, then $\sum_{i=1}^{k} deg(v_i^+) - \sum_{i=1}^{k} deg(v_i^-) \geq k...(*)$

Know $\sum_{i=1}^{k} deg(v_i^+) - \sum_{i=1}^{k} deg(v_i^-) = \sum_{i=1}^{n} deg(v_i^+) - \sum_{i=1}^{n} deg(v_i^-) - (\sum_{i=k+1}^{n} deg(v_i^+) - \sum_{i=k+1}^{n} deg(v_i^-)) = 2e_f(1) - 2e_f(-1) + n - k < 2e_f(1) - 2e_f(-1) + k...(**)$

So we can see that if $e_f(1) - 1 \neq e_f(-1)$ then $(*) \rightarrow\leftarrow (**)$. Therefore $e_f(1) - 1 = e_f(-1)$. From $(*)$, $2 + (n - k) \geq k$, hence $k \leq \frac{n}{2} + 1$. But by assumption, $k = v_f(1) > \frac{n}{2}$, so $v_f(1) = \frac{n}{2} + 1$.

□

As a corollary of the above Claim, while the edge-majority index is not 0(that is $v_f(1) > \frac{n}{2}$), if $v_f(1) \neq \frac{n}{2} + 1$ or $e_f(1) = e_f(-1)$, there must exist an (-1)-vertex v_s such that $deg(v_i^+) - deg(v_i^-) \leq -3$.

Actually, More generality we have

deg	$v_1 \dots v_k$	$v_{k+1} \dots v_m$
$2e_f(1)$	$\sum_{i=1}^{k} deg(v_i^+)$	$\sum_{i=k+1}^{m} deg(v_i^+)$
$2e_f(-1)$	$\sum_{i=1}^{k} deg(v_i^-)$	$\sum_{i=k+1}^{m} deg(v_i^-)$

By Handshaking lemma,
$\sum_{i=1}^{k} deg(v_i^+)+\sum_{i=k+1}^{m} deg(v_i^+) = 2e_f(1)$ and $\sum_{i=1}^{k} deg(v_i^-)+\sum_{i=k+1}^{m} deg(v_i^-) = 2e_f(-1)$.

It hold for all graph with equitable edge labeling function f with $V^+ = \{v_1, v_2, \dots, v_k\}$ the vertex set of vertices labeling with 1, $V^- = \{v_{k+1}, \dots v_m\}$ the vertex set of vertices labeling with -1, and $V^0 = \{v_{m+1}, \dots v_n\}$ the vertex set of vertices labeling with 0.

So if we know $\sum_{i=1}^{k} deg(v_i^+)-\sum_{i=1}^{k} deg(v_i^-)$, then we also know$\sum_{i=k+1}^{m} deg(v_i^+)-\sum_{i=k+1}^{m} deg(v_i^-) = 2e_f(1) - 2e_f(-1) - (\sum_{i=1}^{k} deg(v_i^+) - \sum_{i=1}^{k} deg(v_i^-))$, vice versa.

2.2 Main Result

Define $D_f(v) = deg(v^+) - deg(v^-)$ with respect to f a function from V to $D(V) \subseteq \mathbb{Z}$. When $deg(v)$ is odd for all v, $D(V) = \{\ldots -3, -1, 1, 3\ldots\}$. $D(v) > 0 (< 0)$ if and only if $v \in V^+ (v \in V^-)$.

Theorem 2.1. *For a multi-graph G that all vertex degrees are odd and order n. If there exists an edge-equitable labeling f such that $v_f(1) > \frac{n}{2} + |e_f(1) - e_f(-1)|$, then there exists an edge-equitable labeling g such that $v_g(1) = v_f(1) - 1$. That is, for G size even, if $x \geq 2 \in EMI(G)$, then $x-2 \in EMI(G)$; for G size odd, if $x \geq 4 \in EMI(G)$, then $x-2 \in EMI(G)$.*

Proof.

Let G as in the statement of the theorem with edge-equitable labeling f and $v_f(1) = k$. and $\{v_1, v_2, \ldots, v_k\} = V^+$ the vertex set of vertices labeling with 1, $\{v_{k+1}, \ldots v_n\} = V^-$ the vertex set of vertices labeling with -1.

For an equitable edge labeling f, we choose one 1-edge and one (-1)-edge, and exchanging the labels, since we didn't change the number of 1 and -1 labels, the new edge labeling is still an equitable labeling, and called it f_1 , take this edge labeling f_1, we swap a pair of 1 and -1 labels of edges again, called it f_2, and so on, we get a series of equitable edge labeling $(f_1, f_2, f_3 \ldots\ldots)$, and there is a series $(\sum_{v_i \in V^+} D_{f_1}(v_i), \sum_{v_i \in V^+} D_{f_2}(v_i), \sum_{v_i \in V^+} D_{f_3}(v_i) \ldots\ldots)$ respect to each edge labeling.
we want to make sure that

$$\sum_{v_i \in V^+} D_{f_j}(v_i) < \sum_{v_i \in V^+} D_{f_k}(v_i)$$

and

$$\sum_{v_i \in V^-} D_{f_j}(v_i) > \sum_{v_i \in V^-} D_{f_k}(v_i)$$

for some k and j that $j > k$.
because for the worst case, if $D(v) = 1$ for all $v \in V^+$ respect to edge labeling

f_i, and $\sum_{v_i \in V^+} D_{f_j}(v_i) < \sum_{v_i \in V^+} D_{f_i}(v_i)$ for some $j > i$, then $e_{f_j}(1) < e_{f_i}(1)$.

Suppose V^- is an empty set, then $\sum_{i=1}^{n} D(v_i) = \sum_{i=1}^{k} D(v_i) = 2e_f(1) - 2e_f(-1)$, and $D(v) > 0$ for every $v \in V$ since v is 1-vertices, therefore, $n = 2$ and $e_f(1) = e_f(-1) + 1$ when G is an odd graph (or $n = 1$ and $e_f(1) = e_f(-1) + 1$ when G is an even graph) , and $EMI(G) = 2$ for such graph G', such graph G is not in our discussion. Hence, there must be an (-1)-vertices.

By claim 2.1, We choose a (-1)-edge connect to v_s which $D(v_s) \leq -3$, and a 1-edge connect to v_x which $v_x \in V^+$.
For all situation we may have:

- Case. 01

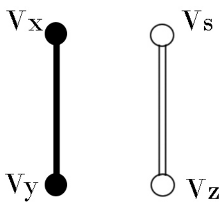


Figure 1: Case.01

If $v_y \in V^+$ and connect v_x with 1-edge, and $v_z \in V^-$ and connect v_s with (-1)-edge, switches the labels of edge v_xv_y and edge v_zv_s.
Then the new labeling f_1 compared to the old edge labeling f, $\sum_{v_i \in V^+} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i) - 4$ and $\sum_{v_i \in V^-} D_{f_1}(v_i) = \sum_{v_i \in V^-} D_f(v_i) + 4$.

Note that there may be a vertex v is 1-vertex ((-1)-vertex) with respect to edge labeling f, after that, v is (-1)-vertex (1-vertex) with respect to edge labeling f_1. If there exit a vertex v that $D_f(v)D_{f_1}(v) = -1$, $\sum_{v_i \in V^+} D_{f_1}(v_i) =$

$\sum_{v_i \in V^+} D_f(v_i) - 4 + 1$ and $\sum_{v_i \in V^-} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i) + 4 - 1$.

And those vertices that may change label by switches the labels of edge in this case are $v_x, v_y \in V^+$, and $v_z \in V^-$.

- Case. 02

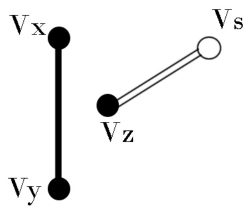


Figure 2: Case.02

If $v_y \in V^+$ and connect v_x with 1-edge, and $v_z \in V^+$ and connect v_s with (-1)-edge, switches the labels of edge $v_x v_y$ and edge $v_z v_s$.
Then the new labeling f_1 compared to the old edge labeling f, $\sum_{v_i \in V^+} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i) - 2$ and $\sum_{v_i \in V^-} D_{f_1}(v_i) = \sum_{v_i \in V^-} D_f(v_i) + 2$.

And those vertices that may change label by switches the labels of edge in this case are $v_x, v_y \in V^+$.

- Case. 03

If $v_y \in V^+$ and connect v_x with 1-edge, and v_x connect v_s with (-1)-edge, switches the labels of edge $v_x v_y$ and edge $v_x v_s$.
Then the new labeling f_1 compared to the old edge labeling f, $\sum_{v_i \in V^+} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i) - 2$ and $\sum_{v_i \in V^-} D_{f_1}(v_i) = \sum_{v_i \in V^-} D_f(v_i) + 2$.

And those vertices that may change label by switches the labels of edge in this case are $v_y \in V^+$.

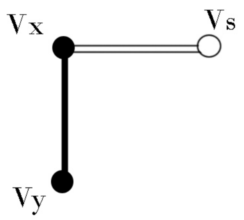


Figure 3: Case.03

- Case. 04

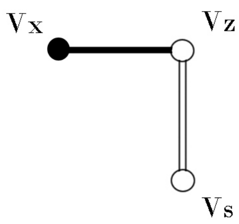


Figure 4: Case.04

If $v_z \in V^-$ and connect v_x with 1-edge, and v_z connect v_s with (-1)-edge, switches the labels of edge $v_x v_z$ and edge $v_z v_s$.
Then the new labeling f_1 compared to the old edge labeling f, $\sum_{v_i \in V^+} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i) - 2$ and $\sum_{v_i \in V^-} D_{f_1}(v_i) = \sum_{v_i \in V^-} D_f(v_i) + 2$.

And those vertices that may change label by switches the labels of edge in this case are $v_x \in V^+$.

- Case. 05

If $v_y \in V^-$ and connect v_x with 1-edge, and $v_z \in V^-$ and connect v_s with (-1)-edge, switches the labels of edge $v_x v_y$ and edge $v_z v_s$.

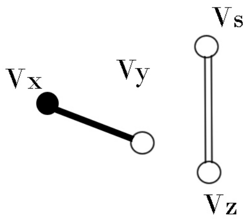


Figure 5: Case.05

Then the new labeling f_1 compared to the old edge labeling f, $\sum_{v_i \in V^+} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i) - 2$ and $\sum_{v_i \in V^-} D_{f_1}(v_i) = \sum_{v_i \in V^-} D_f(v_i) + 2$.

And those vertices that may change label by switches the labels of edge in this case are $v_x \in V^+$ and $v_z \in V^-$.

- Case. 06

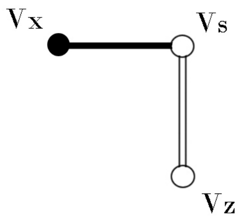


Figure 6: Case.06

If v_s connect v_x with 1-edge, and $v_z \in V^-$ and connect v_s with (-1)-edge, switches the labels of edge $v_x v_s$ and edge $v_z v_s$.
Then the new labeling f_1 compared to the old edge labeling f, $\sum_{v_i \in V^+} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i) - 2$ and $\sum_{v_i \in V^-} D_{f_1}(v_i) = \sum_{v_i \in V^-} D_f(v_i) + 2$.

And those vertices that may change label by switches the labels of edge in this case are $v_x \in V^+$ and $v_z \in V^-$.

• Case. 07.1

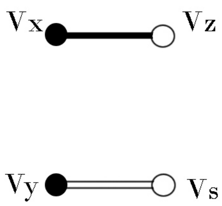


Figure 7: one of cases of case.07

If $v_z \in V^-$ and connect v_x with 1-edge, and $v_y \in V^+$ and connect v_s with (-1)-edge.

Case. 07.2

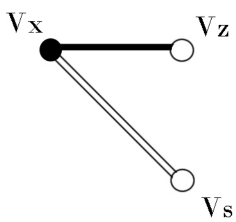


Figure 8: one of cases of case.07

If $v_z \in V^-$ connect v_x with 1-edge, and v_x connect v_s with (-1)-edge.

Case. 07.3

If v_s connect v_x with 1-edge, and $v_y \in V^+$ and connect v_s with (-1)-edge.

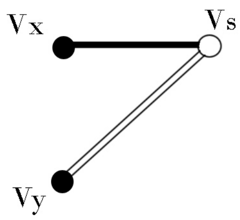


Figure 9: one of cases of case.07

Case. 07.4

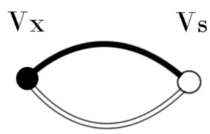


Figure 10: one of cases of case.07

If v_s connect v_x with 1-edge, and v_x connect v_s with (-1)-edge.

switches the labels of those two edges.
Then the new labeling f_1 compared to the old edge labeling f, $\sum_{v_i \in V^+} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i)$ and $\sum_{v_i \in V^-} D_{f_1}(v_i) = \sum_{v_i \in V^-} D_f(v_i)$.

For any graph, if it only have such pair edges in this cases, then every (-1)-edge connect with 1-vertex is connect to (-1)-vertex, and every 1-edge connect with -1-vertex is connect to -1-vertex, then $\sum_{v_i \in V^+} D(v_i) \leq 0$ and $\sum_{v_i \in V^-} D(v_i) \geq 0$, there is no such graph.

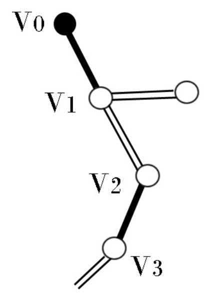
Note that, in case.05 and case.06, when $D_f(v_x) = 1$, $D_f(v_z) = -1$ in those two cases ($D_f(v_y) \in V^-$), $\sum_{v_i \in V^+} D(v_i)$ and $\sum_{v_i \in V^-} D(v_i)$ remain the same even after switches the labels of edges, and $f_1(v_x) = -1$, $f_1(v_z) = 1$. Which mean it doesn't change much thing but the labels of two vertices, call this situation statement.A. So We want to show that it won't be on this statement.A forever.

Suppose any 1-edge in graph G are like case in case 05 and case 06 ,that is, there is no $v \in V^+$ that $D(v) \geq 3$, and there is no (1)-edge that its (1)-endpoints both are $D(v) = 1$, in addition, there is no (-1)-edge that its (-1)-endpoints both are $D(v) \leq -3$. So, all $v \in V^+$ is $D(v) = 1$ and its 1-edge connect to a (-1)-vertices.
Fix any $v_0 \in V^+$, its 1-edge connect to a (-1)-vertices v_1, since $D_f(v_1) \leq 0$, the v_1 have to connect with (-1)-edge, and the edge connect to vertices $v_2, v_{2_2} \ldots$:
If there is a v_{2_j} that $D(v_{2_j}) \leq -3$ for some j, we can switches the labels of edge v_0v_1 and edge $v_1v_{1_j}$, then $v_0 \in V^-$ and $\sum_{v_i \in V^+} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i) - 1$, done; Otherwise, $D_f(v_{2_i}) \geq -2$ for all i.

It can't be all v_{2_i} are $D_f(v_{2_i}) > 0$ for all $v \in V^+$, since if so, $G = K_2(n)$; and It can't be all v_{2_i} are $D_f(v_{2_i}) = -1$ and $deg(v_{2_i}) = 1$ for all $v \in V^+$, since if so, $G = St(n)$; Choose one of v_{2_i} that $0 \geq D_f(v_{2_i}) \geq -1$ and $deg(v_{2_i}) \geq 2$ for some j, name it v_2, switches the labels of edge v_0v_1 and edge v_1v_2, then $v_0 \in V^-$, $v_2 \in V^+$ and $\sum_{v_i \in V^+} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i)$.

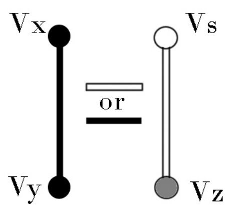
Since $ded(v_2) \neq 1$ and $0 \geq D_f(v_2) \geq -1$, there v_2 have to connect with (-1)-edge, and the edge connect to vertices $v_{3_1}, v_{3_2} \ldots$: If there exist a $v_{3_j} \neq v_{3-2}$ that $ded(v_2) = 1$ for some j, switches back the labels of edge $v_2v_{3_j}$ and (-1)-edge v_sv for some $v \in V^-$, then $v_2, v_{3_j} \in V^-$, $v \in V^+$ and $\sum_{v_i \in V^+} D_{f_2}(v_i) = \sum_{v_i \in V^+} D_{f_1}(v_i) - 1$ Otherwise, $D_f(v_{3_i}) \leq 0$ for all i, Choose one of v_{3_i} and name it v_3, and there is a (-1)-edge connect to a v_3, do the same thing above.

Suppose no matter how we switches the labels of edges like above, situation remain in statement.A, that mean for any $+-+-+-\ldots$ path star from



$v \in V^+$, $v_s \neq v_i$ where i is even, therefore, if there is a vertex v that connect to v_s with (-1)-edge, then $D(v) \in V^-$ and $deg(v) = 1$, or $D(v) \in V^+$. therefore, $D(v_s) \geq -1$ since $|e_f(1) - e_f(-1)| = 1$, contradiction.

Note that, in case.01 and case.02, when $D_f(v_x) = 1$, $D_f(v_y) = 1$ and $D_f(v_z) \leq -3$ in case.01, $D_f(v_x) = 1$ and $D_f(v_y) = 1$ in case.02, the new edge labeling g will make v_x, v_y change the labeling, which mean $v_{f_{i+1}}(1) - 2 = v_{f_i}(1)$ for some i (but you can consider it if $v_{f_{j+1}}(1) + 1 = v_{f_j}(1)$ for $j < i$). So we may have to change the plane.



If there is other choices then choose it, since G is an connected graph, there must be a edge between two side.
If the edge is label with 1, switches the labels with edge $v_s v_z$ $\sum_{v_i \in V^+} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i) - 1$, and only one 1-vertex change its label to -1.
If the edge is label with -1, switches the labels with edge $v_x v_y$ $\sum_{v_i \in V^+} D_{f_1}(v_i) =$

$\sum_{v_i \in V^+} D_f(v_i) - 1$, and only one 1-vertex change its label to -1.

Hence, We can avoid the situation that creating too many (-1)-vertices in one time.

As long as there exist v_s such that $D(v_s) \leq -3$, the switch of labels process will not stop. Decreasing $\sum_{v_i \in V^+} D_{f_j}(v_i)$ and increasing $\sum_{v_i \in V^-} D_{f_j}(v_i)$ while j is increasing every time by witches the labels. So We switch the labels of edges of (-1)-edge connect with v_s and 1-edge connect with a 1-vertex until $v_{f_i}(1) = v_f(1) - 1$ for some i

□

Corollary 2.1. *For a connected graph G neither isomorphic to $K_{1,2m+1}$ nor $K_2(2m+1)$ and all vertex degrees are odd. If $x \geq 2 \in EMI(G)$, then $\{0, \ldots, x-4, x-2, x\} \subseteq EMI(G)$.*

Proof.

Let G as in the statement of the theorem. By theorem2.1, theorem1.1, for $|E|$ is odd, $|e_f(1) - e_f(-1)| = 1$ for all edge friendly labeling f, and there is a labeling such that $|v_f(-1) - v_f(0)| \leq 1$, so $0 \in EMI(G)$.

□

Example 2.1. *For $G = W_n$ and n is even, we see that if the wheels are labeling 1 and the axles labeling -1, the edge-majority index is $n-2$, then $\{0, 2, ..., n-2\} \subseteq EMI(W_n)$ by Corollary 2.1.*

Actually, $EMI(W_n) = \{0, 2, ..., 2i, ..., n-2\}$ for n is even[6].

Chapter 3

Edge-Majority Indices of Even Graphs

3.1 Basics

For a graph G that all vertices in G is even degrees, we want to show that if we find an equitable labeling function f such that $x \in EMI(G)$, then $\{0, \ldots, x-2, x-1, x\} \subseteq EMI(G)$. It use the same methods in Section 2.

Let $|V| = m$. For a labeling function f of a graph G, let $\{v_1, v_2, \ldots, v_k\} = V^+$ the vertex set of vertices labeling with 1, $\{v_{k+1}, \ldots, v_n\} = V^-$ the vertex set of vertices labeling with -1, and $V^0 = \{v_{n+1}, \ldots v_m\}$ the vertex set of vertices labeling with 0.

Claim 3.1. *Graph G with order m and all vertex degrees are even. If $v_f(1) \geq \lfloor\frac{n+|e_f(1)-e_f(-1)|}{2}\rfloor$ and $D(v_i) = -2$ for all v_i that $f(v_i) = -1$, then $v_f(1) = \lfloor\frac{n+|e_f(1)-e_f(-1)|}{2}\rfloor$, where $n = v_f(1) + v_f(-1)$*

Proof.

Let $v_f(1) = k$. For a labeling function f of a graph G, which all vertex degrees are even, let $\{v_1, v_2, \ldots, v_k\} = V^+$ the vertex set of vertices labeling with 1, $\{v_{k+1}, \ldots, v_n\} = V^-$ the vertex set of vertices labeling with -1, and $V^0 = \{v_{n+1}, \ldots v_m\}$ the vertex set of vertices labeling with 0.

Know $D(v_i) = -2$ for all $i = k+1, \ldots n$, then $\sum_{i=k+1}^{n} deg(v_i^+) - \sum_{i=k+1}^{n} deg(v_i^-) = \sum_{i=k+1}^{n}(D(v_i)) = -2(n-k)$.
Since $f(v_i) = 1$ for $i = 1, ..., k$ and degrees are even, then $\sum_{i=1}^{k} deg(v_i^+) - \sum_{i=1}^{k} deg(v_i^-) = \sum_{i=1}^{k} D(v_i) \geq 2k$

Know $\sum_{i=1}^{k}(D(v_i)) + \sum_{i=k+1}^{n} D(v_i) = \sum_{i=1}^{k}(D(v_i)) + \sum_{i=k+1}^{n} D(v_i) + \sum_{i=n+1}^{m} D(v_i) = 2e_f(1) - 2e_f(-1)$, then $e_f(1) - e_f(-1) \geq 2k - n$. Therefore $k \leq \frac{n+e_f(1)-e_f(-1)}{2}$.

Know $k \geq \lfloor \frac{n+|e_f(1)-e_f(-1)|}{2} \rfloor$. So, $k = \lfloor \frac{n+|e_f(1)-e_f(-1)|}{2} \rfloor$.

$\square$

3.2 Main Result

Consequently, for G size even, when $k > \lfloor \frac{n}{2} \rfloor$, there exists an vertex v_s such that $D(v_s) \leq -4$; for G size odd, when $k > \lfloor \frac{n+1}{2} \rfloor$, there exists an vertex v_s such that $D(v_s) \leq -4$;where $n = v_f(1) + v_f(-1)$

Theorem 3.1. *For a multi-graph G that all vertex degrees are even and order m. If there exists an edge-equitable labeling f such that $v_f(1) > \lfloor \frac{n+|e_f(1)-e_f(-1)|}{2} \rfloor$ where $n = v_f(1) + v_f(-1)$, and edge-majority index equal to x with respect to f, then there exists an edge-equitable labeling g such that edge-majority index equal to $x - 1$.*

Proof.

Let G with order m vertices as in the statement of the theorem with edge-equitable labeling f, $\{v_1, v_2, \ldots, v_k\} = V^+$ the vertex set of vertices labeling with 1, $\{v_{k+1}, \ldots, v_n\} = V^-$ the vertex set of vertices labeling with -1, and $V^0 = \{v_{m+1}, \ldots v_n\}$ the vertex set of vertices labeling with 0.
Define $D_f(v) = deg(v^+) - deg(v^-)$ with respect to f. When $deg(v)$ is even, $D(v) = \ldots - 4, -2, 0, 2, 4 \ldots$.

We switch of labels of (-1)-edge incident to a vertex in $V^{-,0} = V^- + V^0$ and (1)-edge incident to a vertex in V^+ like proof in Theorem2.1 until $v_{f_i}(1) - v_{f_i}(-1) = v_f(1) - v_f(-1) - 1$ for some i.

Note that there may be a vertex v change its label during exchanging the labels of edges. If $v \in V^+$ change its label, which mean $D(v)_f = 2$ respect to the old edge labeling f, and $D(v)_{f_1} = 0$ respect to the old edge labeling f_1, So We don't have to do anything else, Same for $v \in V^-$ change its label to 0.
If $v \in V^0$ change its label, which mean $D(v)_f = 0$ respect to the old edge labeling f, and $D(v)_{f_1} = 2$ respect to the old edge labeling f_1, So $\sum_{v_i \in V^+} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i) - C + 2$ and $\sum_{v_i \in V^-} D_{f_1}(v_i) = \sum_{v_i \in V^+} D_f(v_i) + C - 2$, where $C = 4$ or 2 depend on which cases (which two edges).
For any Case 01 to case06, $\sum_{v_i \in V^+} D_{f_j}(v_i)$ is decreasing and $\sum_{v_i \in V^-} D_{f_j}(v_i)$ is increasing every time we witches the labels of edges.

We know that $v_f(1) - v_f(-1) = |V| - v_f(0) - 2v_f(-1)$ for any edge-equitable labeling f of G, so there is a edge-equitable labeling g such that $v_g(1) - v_g(-1) = |V| - v_g(0) - 2v_g(-1) = |V| - [v_f(0) + 1] - 2[v_f(-1) - 1]$ or $v_g(1) - v_g(-1) = |V| - [v_f(0) + 1] - 2v_f(-1)$, so $x - 1 \in EMI(G)$.

□

Corollary 3.1. *For a simple connected graph G with all vertex degrees are even. For G size even, if $x \geq 1 \in EMI(G)$, then $\{0, \dots, x-1, x\} \subseteq EMI(G)$; For G size odd, if $x \geq 2 \in EMI(G)$, then $\{1, \dots, x-1, x\} \subseteq EMI(G)$.*

Chapter 4

Conclusion and Further Studies

In this thesis we consider edge-majority index set of all odd graphs and all even graphs. For a graph G that is neither an odd graph nor an even graph, since no information for which vertices will change labels during exchanging the labels of edges, so we do not know which vertices will be even or odd, and this is directly related to the edge-majority index. Therefore it is not easy to calculate the edge-majority index set of general graphs.

There are a lot of problems left for further exploration. For example, what are other obvious applications of the notion of edge-majority index and the generalized edge-majority index?

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