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Tachyons in Five-Dimensional Spacetime

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Abstract

We consider tachyons in five-dimensional spacetime with the canonical metric and the conformally-flat metric respectively. The four-dimensional effective cosmological constant Λ is calculated. The modified Newton's gravitational force law is different from that of slower-than-light particles. The vacuum instability is also discussed.

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1 Introduction

Recently it has been argued that tachyons (faster-than-light particles) might be as a possible source of dark energy in cosmological models[1]. In addition, tachyons may have a role in gravitational Cherenkov radiation [2], elementary-particle interactions and brane dynamics [3]. In the theory of special relativity, objects having a relative velocity greater than light speed imply the violation of causality. Whereas in general relativity, velocity is a local notion, so velocity calculated using comoving coordinates does not have a simple relation to velocity calculated locally. Rules that apply to relative velocities in special relativity may not apply to relative velocities in comoving coordinates which are often describes in terms of the expansion between distant galaxies. So in general relativity might allow the space between distant galaxies to expand in a way that they have a velocity exceeds the speed of light. Another phenomenon predicted by general relativity is the travelable wormhole. Traveler moving through the wormhole would not locally move faster than light which travels through the wormhole alongside them, but they would be able to reach their destination faster than light traveling outside the wormhole.

In this paper we investigate tachyons in five-dimensional spacetime with the canonical metric and the conformally-flat metric respectively. To get notation right, let us start from the special relativity and consider the motion of a particle described by the four-vector coordinate $x^\mu(\tau)$, $\mu = 0, 1, 2, 3$ and τ is the proper time. For a free particle moving in a straight line with velocity $\mathbf{v}$ one has,

$$x^\mu(\tau) = (\gamma\tau, \gamma\mathbf{v}\tau). \tag{1}$$

Defining $\dot{x^\mu} = dx^\mu/d\tau$, we calculate

$$\dot{x^\mu}\dot{x_\mu} = \eta_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = \gamma^2(1 - v^2) = \mathcal{Q}, \tag{2}$$

where the signature of the Minkowski space metric $\eta_{\mu\nu}$ is $(+,-,-,-)$ and $\gamma \equiv 1/\sqrt{|1 - v^2|}$ is always a real number. The speed of light is $c \equiv 1$. The value of $\mathcal{Q}$ is $+1$ for normal (slower-than-light) particles, 0 for light, and -1 for tachyons.

For the case of general relativity[4], the invariant can also be written similarly as

$$g_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = \mathcal{Q}. \tag{3}$$

The Schwarzschild metric, for example, is

$$\mathcal{Q}ds^2 = (1 - \frac{r_s}{r})\, dt^2 - \frac{dr^2}{(1 - \frac{r_s}{r})} - r^2 d\theta^2 - r^2 sin^2\theta\, d\phi^2, \tag{4}$$

where the Schwarzschild radius $r_s = 2GM$. From the geodesic equation in this metric one can obtain

$$(1 - \frac{r_s}{r})\,\frac{dt}{ds} = K, \qquad r^2\,\frac{d\phi}{ds} = h, \qquad (1 - \frac{r_s}{r})^{-1}\,\{(\frac{dr}{ds})^2 - K^2\} + \frac{h^2}{r^2} = -\mathcal{Q}, \tag{5}$$

where K and h are constants of integration. For normal particles ($\mathcal{Q} = +1$), these will give the Kepler orbits for $r > r_s$, along with some relativistic corrections. For tachyons ($\mathcal{Q} = -1$), these do not give localized orbits but scattering states.

2 Five-Dimensional Spacetime

The five-dimensional line element including the scalar field Φ and the electromagnetic gauge potential A_μ has the usual form [5]

$$d\bar{s}^2 = ds^2 + \varepsilon\Phi^2(dx^4 + A_\mu dx^\mu)^2, \tag{6}$$

where $\varepsilon = \pm 1$, both are allowed by the mathematics, determines whether the extra dimension is spacelike or timelike. The four-dimensional interval is $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$.

Now let us incorporate tachyons into (6) and the line element can be rewritten as

$$d\bar{s}^2 = \mathcal{Q}\; g_{\alpha\beta}(x^\gamma, l)dx^\mu dx^\nu + \varepsilon\Phi^2(x^\gamma, l)dl^2. \tag{7}$$

In this we have set the electromagnetic potential A_μ to zero, but the remaining degree of coordinate freedom are preserved. $l \equiv x^4$ is the fifth coordinate of the spacetime. Then the components of the five-dimensional Ricci tensor for metric (7) are

$$\begin{aligned}
\bar{R}_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{\Phi}\Phi_{,\alpha;\beta} + \frac{\varepsilon\mathcal{Q}}{2\Phi^2}\{\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} \\
&\quad + g^{\lambda\mu}g_{\alpha\lambda,4}\; g_{\beta\mu,4} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu,4}\; g_{\alpha\beta,4}\} && (8)\\
\bar{R}_{4\alpha} &= \frac{\Phi}{2}\{(\frac{g^{\beta\lambda}g_{\alpha\lambda,4}}{\Phi})_{,\beta} - (\frac{g^{\mu\nu}g_{\mu\nu,4}}{\Phi})_{,\alpha}\} \\
&\quad + \frac{1}{4}g^{\mu\beta}g_{\mu\beta,\lambda}\, g^{\lambda\sigma}g_{\sigma\alpha,4} + \frac{1}{4}g^{\sigma\lambda}{}_{,\alpha}g_{\sigma\lambda,4} && (9)\\
\bar{R}_{44} &= -\varepsilon\mathcal{Q}\,\Phi\Box\Phi - \frac{1}{2}(g^{\lambda\beta}g_{\lambda\beta,4})_{,4} \\
&\quad + \frac{1}{2\Phi}\Phi_{,4}g^{\lambda\beta}g_{\lambda\beta,4} + \frac{1}{4}g^{\mu\sigma}{}_{,4}g_{\mu\sigma,4}. && (10)
\end{aligned}$$

Here a comma denotes the ordinary partial derivative, a semicolon denotes four-dimensional covariant derivative, $\Box\Phi \equiv g^{\mu\nu}\Phi_{,\mu;\nu}$.

The vacuum Einstein's equations in five-dimensional spacetime, $\bar{R}_{AB} = 0$, can be used to construct an induced four-dimensional Ricci tensor $R_{\alpha\beta}$, the scalar curvature tensor R and the equation of a scalar field Φ. When the scalar component $\bar{R}_{44}$ is set to zero in accordance with the equation (10), one will get a wave-type equation of Φ with a source induced by the fifth dimension,

$$\Box\Phi = \frac{\varepsilon\mathcal{Q}}{2\Phi}\,(\frac{\Phi_{,4}g^{\lambda\beta}g_{\lambda\beta,4}}{\Phi} - \frac{1}{2}g^{\mu\sigma}{}_{,4}g_{\mu\sigma,4} - g^{\mu\sigma}g_{\mu\sigma,44}) \tag{11}$$

When the vector components $\bar{R}_{4\alpha}$ of (9) are set to zero, one will obtain a set of conservation laws which resemble those found in electromagnetism and many other field theories. They read

$$\{\frac{1}{2\Phi}(g^{\beta\sigma}g_{\sigma\alpha,4} - \delta^\alpha_\beta g^{\mu\nu}g_{\mu\nu,4})\}_{;\beta} = 0 \tag{12}$$

As for setting $\bar{R}_{\alpha\beta} = 0$ from (8), we will obtain the induced Ricci tensor,

$$R_{\alpha\beta} = \frac{1}{\Phi}\Phi_{,\alpha;\beta} - \frac{\varepsilon\mathcal{Q}}{2\Phi^2}\,\{\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + g^{\lambda\mu}g_{\alpha\lambda,4}\,g_{\beta\mu,4} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu,4}\,g_{\alpha\beta,4}\}. \tag{13}$$

Contracting $R_{\alpha\beta}$ in (13) and then combining with (11), the scalar curvature tensor is

$$R = \frac{\varepsilon\mathcal{Q}}{4\Phi^2}\,\{g^{\mu\nu}{}_{,4}\,g_{\mu\nu,4} + (g^{\mu\nu}g_{\mu\nu,4})^2\} \tag{14}$$

3 Cosmological Constant and Modified Newton's Law

Let us now focus on a special case called the five-dimensional canoical metric. [6] The line element is given by

$$\begin{aligned} d\bar{s}^2 &= \frac{l^2}{L^2}ds^2 - dl^2 && (15)\\ &= \mathcal{Q}\,\frac{l^2}{L^2}g_{\alpha\beta}(x^\gamma)dx^\mu dx^\nu - dl^2 && (16)\end{aligned}$$

where L is a constant length introduced for the consistency of physical dimensions. The induced Ricci tensor $R_{\alpha\beta}$ and scalar curvature tensor R can be calculated from (13) and (14) by setting $\varepsilon = -1$, $\Phi = 1$ and making a replacement

$$g_{\alpha\beta}(x^\gamma, l) \to \frac{l^2}{L^2}g_{\alpha\beta}(x^\gamma), \qquad g^{\alpha\beta}(x^\gamma, l) \to \frac{L^2}{l^2}g^{\alpha\beta}(x^\gamma). \tag{17}$$

Then we have

$$R_{\alpha\beta} = \frac{-3\mathcal{Q}}{L^2}\;g_{\alpha\beta}(x^\gamma), \qquad R = \frac{-12\mathcal{Q}}{l^2}, \tag{18}$$

and the Einstein tensor

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}\;g_{\alpha\beta}(x^\gamma, l)R = \frac{3\mathcal{Q}}{L^2}g_{\alpha\beta}(x^\gamma). \tag{19}$$

The induced cosmological constant Λ in four-dimensional spacetime can be read from $R_{\alpha\beta} = \Lambda\; g_{\alpha\beta}(x^\gamma)$ or $G_{\alpha\beta} = -\Lambda\; g_{\alpha\beta}(x^\gamma)$,

$$\Lambda = \frac{-3\mathcal{Q}}{L^2}. \tag{20}$$

Another special but physically instructive case we want to look at is the conformally-flat metric,

$$d\bar{s}^2 = \mathcal{Q}\frac{l^2}{L^2}\{f(x^\gamma, l)\;\eta_{\alpha\beta}\;dx^\alpha dx^\beta\} - dl^2. \tag{21}$$

Here $\eta_{\alpha\beta}$ is the metric for flat Minkowski space and the signature of $\eta_{\alpha\beta}$ is also $(+,-,-,-)$. We are particularly interested in the $l-$ dependence of $f(x^\gamma, l)$.

Setting $\varepsilon = -1$, $\Phi = 1$ and making a replacement

$$g_{\alpha\beta}(x^\gamma, l) \to \frac{l^2}{L^2} f(x^\gamma, l)\eta_{\alpha\beta}, \qquad g^{\alpha\beta}(x^\gamma, l) \to \frac{L^2}{l^2} f^{-1}(x^\gamma, l)\eta^{\alpha\beta}, \tag{22}$$

The equation of scalar field (11), i.e. $\bar{R}_{44} = 0$, becomes

$$2U_{,4} + U^2 = 0 \tag{23}$$

where

$$U \equiv \frac{f_{,4}}{f} + \frac{2}{l}. \tag{24}$$

The solution for the conformal factor f is

$$f(x^\gamma, l) = \; [1 - \frac{l_0(x^\gamma)}{l}]^2 \; k(x^\gamma), \tag{25}$$

where $l_0(x^\gamma)$ is an arbitrary length function of integration, $k(x^\gamma)$ is an arbitrary dimensionless function. $f(x^\gamma, l)$ involves both arbitrary functions. However, substituing (22) into (12), the vector component of the field equation, one will get

$$(\frac{f_{,4}}{f})_{,\alpha} = 0 \tag{26}$$

and this implies l_0 is indeed a constant. Now the conformal factor becomes

$$f(x^\gamma, l) = \; [1 - \frac{l_0}{l}]^2 \; k(x^\gamma), \tag{27}$$

which involves only one arbitrary function $k(x^\gamma)$. Using this conformal factor, the induced Ricci tensor (13) becomes

$$R_{\alpha\beta} = \frac{-3}{L^2} \frac{\mathcal{Q} \, l^2}{(l - l_0)^2} \; g_{\alpha\beta}, \tag{28}$$

where $g_{\alpha\beta} = f(x^\gamma, l)\eta_{\alpha\beta}$. So an effective cosmological constant is given by

$$\Lambda = \frac{-3\mathcal{Q}}{L^2} \; (\frac{l}{l - l_0})^2. \tag{29}$$

Furthermore, let us assume a null 5-dimensional path and rewrite the line element (15) as

$$d\bar{s}^2 = [\frac{l^2}{L^2} - \; (\frac{dl}{ds})^2] \; ds^2 = 0. \tag{30}$$

Since a massive particle in spacetime has $ds^2 \neq 0$, the velocity in fifth dimension is given by

$$(\frac{dl}{ds})^2 = (\frac{l}{L})^2. \tag{31}$$

Thus the solution can be obtained as

$$l = l_0\, e^{\pm s/L}, \tag{32}$$

where it is natural for us to locate the big bang at the zero point of proper time and to choose $l = l_0$ when $s = 0$. The cosmological constant now yields

$$\Lambda = \frac{-3}{L^2}\,\frac{\mathcal{Q}}{(1 - e^{\mp s/L})^2}. \tag{33}$$

In the case of upper minus sign, Λ decays from an unbounded value at the big bang ($s = 0$) to its asymptotic value of $-3\mathcal{Q}/L^2$ ($s \to \infty$). Whereas in the case of lower plus sign, Λ decays from an unbounded value at the big bang ($s = 0$) and approach zero ($s \to \infty$). For normal (slower-than-light) particles, from astrophysical data, the case of upper minus sign is the one corresponds to our universe.

From the viewpoint of general relativity, the cosmological constant has associated with the energy density $\frac{\Lambda c^4}{8\pi G}$. The cosmological constant (29), $\Lambda = \frac{-3\mathcal{Q}}{L^2}\,(\frac{l}{l-l_0})^2$, may be identified the divergence at $l = l_0$ with the big bang. Let us take derivatives of (29) to obtain

$$d\Lambda = \mathcal{Q}(\frac{6}{L^2})(l - l_0)^{-3} l l_0\, dl. \tag{34}$$

We are mainly interested in the region near $l = l_0$. Putting $dl = l - l_0$ and using $l = l_0 e^{s/L}$ as in (32), one will get

$$d\Lambda\; ds^2 = 6. \tag{35}$$

This implies vacuum instability, [7] since $d\Lambda \to \infty$ for $ds \to 0$.

To investigate the physics further, we assume that the 5-dimensional path may be null. The 4-dimensional part of the geodesic equation is

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\,\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = f^\mu, \tag{36}$$

where f^μ can be identified as the force (per unit mass) associated with 5-dimensional spacetime. To evaluate f^μ, we can differentiate

$$g_{\alpha\beta}(x^\gamma, l)\, u^\alpha u^\beta = \mathcal{Q} \qquad (u^\alpha = \frac{dx^\alpha}{ds}) \tag{37}$$

with respect to s. Using symmetries under the exchange of α and β to introduce the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$, there comes

$$2g_{\alpha\mu}u^\alpha\,(\frac{du^\mu}{ds} + \Gamma^\mu_{\beta\gamma}u^\beta u^\gamma) + \frac{\partial g_{\alpha\beta}}{\partial l}\,\frac{dl}{ds}\,u^\alpha u^\beta = 0. \tag{38}$$

Due to the motion of 4-dimensional frame with respect to the fifth dimension is parallel to the 4-velocity u^μ , one can substitute

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\beta\gamma}u^\beta u^\gamma = \beta u^\mu \tag{39}$$

into (36) and determine the constant β , then find

$$f^\mu = -\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial l}\frac{dl}{ds}\,u^\alpha u^\beta u^\mu. \tag{40}$$

After substituting the conformally-flat metric tensor, $g_{\alpha\beta} = (1-\frac{l_0}{l})^2\,k(x^\gamma)\,\eta_{\alpha\beta}$ into (38), the result is

$$\begin{aligned} f^\mu &= -\mathcal{Q}\,\frac{l_0}{l(l-l_0)}\frac{dl}{ds}\frac{dx^\mu}{ds} && (41)\\ &= -\frac{\mathcal{Q}}{L}\frac{dx^\mu}{ds}\frac{1}{(e^{s/L}-1)}. && (42)\end{aligned}$$

We reintroduce conventional units for the speed of light c and the gravitational constant G. Adding the velocity-dependent extra force $f = -\mathcal{Q}(vc/L)(e^{s/L}-1)^{-1}$ to the usual Newton's law, putting $s = ct$ in the extra force, then the Newton's law is modified so that the force per unit mass becomes

$$F = -\frac{GM}{r^2} - \mathcal{Q}(\frac{vc}{L})\frac{1}{(e^{s/L}-1)}, \tag{43}$$

where $M = M(r)$ is the mass interior to radius r of for a system with spherical symmetry. The gravitational gravitational constant is modified to be

$$\tilde{G} = G\,\{1 + \frac{\mathcal{Q}vr^2}{GM}(\frac{c}{L})\frac{1}{(e^{ct/L}-1)}\}. \tag{44}$$

4 Concluding Remarks

We have considered tachyons in five-dimensional spacetime with the canonical metric and the conformally-flat metric respectively. The four-dimensional effective cosmological constant Λ and the modified Newton's gravitational force law obtained from tachyons and normal (slower-than-light) particles are different in many places by just a sign factor $\mathcal{Q}$. This means that any experiment claimed that the effects are measured for normal particles associated with 5-dimensional spacetime might be measured for the case of tachyons. However, it is also possible that some effects associated with 5-dimensional spacetime cannot be measured just due to the contributions of normal particles and tachyons are canceled each other.

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五維時空的超光速粒子

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摘 要

我們考慮在五維時空的超光速粒子，其度量張量分別爲正則和共型平坦。計算四維等效的宇宙常數 Λ。修正的牛頓萬有引力公式不同於低於光速的粒子。並討論真空不穩定性。

關鍵字：超光速粒子、五維時空