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Lie 3-Algebra in Noncommutative Quantum Mechanics

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Abstract

The Lie 3-algebra in noncommutative quantum mechanics on the fuzzy sphere is studied. The coordinate and momentum operators derived from Moyal-Weyl product between two functions, for both position-position and momentum-momentum noncommutativity, form a realization of the Lie 3-algebra. By introducing a shift for magnetic vector potential, the Aharonov-Bohm effect on noncommutative space is discussed.

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1 Introduction

Since the discovery in string theory that the low-energy effective theory of a D-brane in the background of a NS-NS B field lives in a noncommutative space[1], many efforts have been paid to study noncommutative field theories and noncommutative quantum mechanics. In the study of D-branes in B-field background, it was found that analogous computation of correlation functions of open membranes was carried out in a large C-field background [2] and the interaction terms of M5-brane worldvolume action involving the interesting Nambu-Poisson structure considered long time ago by Nambu [3], as a generalization of the Poisson structure in the Hamiltonian formulation. The quantized version [4] of the Nambu-Poisson bracket takes the form

$$[x_i, x_j, x_k] = i\theta_{ijk}, \tag{1}$$

where the components of θ_{ijk} are constants and 3-bracket is given by Lie 3-bracket. A Lie 3-bracket is multilinear and antisymmetric under interchange of any pair of its components. Moreover, it satisfies the fundamental identity

$$[[f,g,h],k,l] = \; [[f,k,l],g,h] + \; [f,[g,k,l],h] + \; [f,g,[h,k,l]], \tag{2}$$

where f, g, h, k, l are any element of the algebra. The 3-bracket can be defined on ordinary operators as

$$[f,g,h] \equiv \; fgh \, + \, ghf \, + \, hfg \, - \, fhg \, - \, gfh \, - \, hgf. \tag{3}$$

This 3-bracket was originally introduced by Nambu [3] as a possible candidate of the quantization of the classical Nambu bracket $\{f, g, h\} \equiv \varepsilon^{ijk}\partial_i f \partial_i g \partial_i h$. So (3) is called the quantum Nambu bracket and geometry (1) the quantum Nambu geometry.

In this paper we will study the Lie 3-algebra which play roles in noncommutative quantum mechanics on the fuzzy sphere. The coordinate and momentum operators, $\hat{x}_i$ and $\hat{p}_i$, which relate the operators on noncommutative space and ordinary commutative space can be derived from Moyal-Weyl product between two functions for both position-position and momentum-momentum noncommutativity. These operators form a realization of the Lie 3-algebra. Then we will introduce a generalized Bopp's shift for magnetic vector potential and discuss the Aharonov-Bohm effect on noncommutative space.

2 Noncommutative Quantum Mechanics on the Fuzzy sphere

Let $H(x, p)$ be the Hamiltonian operator of the ordinary quantum system, then the stationary Schrödinger equation on noncommutative space is usually written as

$$H(x, p) * \Psi = E\Psi, \tag{4}$$

where the Moyal-Weyl (or star) product between two functions is defined by,

$$(f * g)(x) = e^{\frac{i}{2}\theta_{ij}\partial_{x_i}\partial_{x_j}} f(x_i)g(x_j) = \; e^{\frac{i}{2}\theta_{ij}\partial_{x_i}\partial_{x_j}} f(x_i)g(x_j) = \; f(x)g(x) + \; \frac{i}{2}\theta_{ij}\partial_i f \partial_j g \mid_{x_i = x_j}, \tag{5}$$

where $f(x)$ and $g(x)$ are two arbitrary functions. On noncommutative space the star product can be replaced by a Bopp's shift, i.e. the star product can be changed into ordinary product by replacing $H(x,p)$ with $H(\hat{x},\hat{p})$. Thus the Schrödinger equation can be written as,

$$H(\hat{x},\hat{p})\Psi = H(x_i - \frac{1}{2\hbar}\theta_{ij}p_j, p_i)\psi = E\Psi, \tag{6}$$

where x_i and p_i are coordinate and momentum operators in usual quantum mechanics.

Although in string theory [1] only the coordinates space exhibits a noncommutative structure, it is believed that the noncommutativity between momenta arises naturally as a consequence of noncommutativity between coordinates, as momenta are defined to be the partial derivatives with respect to the noncommutative coordinates.[5]

Now let us consider the case of both position-position and momentum-momentum noncommutativity. The star product is defined as

$$\begin{aligned}(f*g)(x,p)) &= e^{\frac{i}{2}\theta_{ij}\partial_{x_i}\partial_{x_j}+\frac{i}{2}\eta_{ij}\partial_{p_i}\partial_{p_j}}f(x,p)g(x,p)\\ &= f(x,p)g(x,p)+\frac{i}{2}\theta_{ij}\partial_{x_i}f\partial_{x_j}g\mid_{x_i=x_j}+\frac{i}{2}\eta_{ij}\partial_{p_i}f\partial_{p_j}g\mid_{p_i=p_j}.\end{aligned} \tag{7}$$

Substituting two novel relations,

$$\theta_{ij}=\theta\,\varepsilon_{ijk}x_k, \qquad \eta_{ij}=\eta\,\varepsilon_{ijk}p_k, \quad (i,j,k=1,2,3), \tag{8}$$

into (7), replacing f by $\hat{x}_i$ or $\hat{p}_i$ and g by Ψ, we have

$$\hat{x}_i*\Psi = x_i\Psi+\frac{\theta}{2\hbar}\varepsilon_{ijk}x_jp_k\Psi \tag{9}$$

$$\hat{p}_i*\Psi = p_i\Psi+\frac{\eta}{2\hbar}\varepsilon_{ijk}x_jp_k\Psi. \tag{10}$$

θ and η are noncommutative parameters which can always be treated as a perturbation in ordinary quantum mechanics. There are many different bounds on the sort of parameters set by experiments[6]. θ and η are surely extremely small. The generalized Bopp's shift is obtained as

$$\hat{x}_i = x_i+\frac{\theta}{2\hbar}\varepsilon_{ijk}x_jp_k = x_i+\frac{\theta}{2\hbar}L_i \tag{11}$$

$$\hat{p}_i = p_i+\frac{\eta}{2\hbar}\varepsilon_{ijk}x_jp_k = p_i+\frac{\eta}{2\hbar}L_i, \tag{12}$$

where L_i is just the component of angular momentum in ordinary quantum mechanics. The commutation relation of $\hat{x}_i$ and $\hat{x}_j$ is

$$[\hat{x}_i,\hat{x}_j] = i\theta\,\varepsilon_{ijk}\hat{x}_k - \frac{i\theta^2}{4\hbar^2}\,\varepsilon_{ijk}L_k. \tag{13}$$

Other identities such as

$$[L_i,\hat{x}_j] = i\hbar\varepsilon_{ijk}\hat{x}_k \tag{14}$$

$$[L_i,\hat{p}_j] = i\hbar\varepsilon_{ijk}\hat{p}_k \tag{15}$$

are very similar to ordinary quantum mechanics. If we define a fuzzy sphere in terms of three non-commuting operators $\hat{x_1},\hat{x}_2,\hat{x}_3$ that satisfy

$$\hat{x}_1^2+\hat{x}_2^2+\hat{x}_3^2 = R^2\mathbf{1} \tag{16}$$

with **1** the unit operator of the algebra.

The quantum Nambu bracket of $\hat{x_i}, \hat{x_j}, \hat{x_k}$ is

$$\begin{aligned}
[\hat{x_i}, \hat{x_j}, \hat{x_k}] &= [\hat{x_i}, \hat{x_j}]\hat{x_k} + [\hat{x_j}, \hat{x_k}]\hat{x_i} + [\hat{x_k}, \hat{x_i}]\hat{x_j} \\
&= i\varepsilon_{ijk}\,\theta R^2\mathbf{1} - \mathrm{i}\varepsilon_{\mathrm{ijk}}\frac{\theta^3}{8\hbar^2}(\mathrm{L}_1^2 + \mathrm{L}_2^2 + \mathrm{L}_3^2) \\
&= i\varepsilon_{ijk}\theta R^2\{1 - \frac{\theta^2}{8R^2}l(l+1)\}\mathbf{1} \\
&= i\varepsilon_{ijk}\tilde{\theta}\mathbf{1}. \qquad (17)
\end{aligned}$$

$\tilde{\theta}$ is essentially a parameter independent of x_i and p_i. To check the fundamental identity (3) we need the identity

$$[\hat{x_i}, \hat{x_j}, \mathbf{1}] = [\hat{\mathrm{x}}_\mathrm{i}, \hat{\mathrm{x}}_\mathrm{j}]. \qquad (18)$$

So the noncommutative coordinate operators form a realization of the Lie 3-algebra. Similarly, If we define a fuzzy sphere in terms of three non-commuting operators $\hat{p_1}, \hat{p_2}, \hat{p_3}$ that satisfy

$$\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2 = \bar{R}^2\mathbf{1}. \qquad (19)$$

The quantum Nambu bracket of $\hat{p_i}, \hat{p_j}, \hat{p_k}$ is

$$\begin{aligned}
[\hat{p_i}, \hat{p_j}, \hat{p_k}] &= [\hat{p_i}, \hat{p_j}]\hat{p_k} + [\hat{p_j}, \hat{p_k}]\hat{p_i} + [\hat{p_k}, \hat{p_i}]\hat{p_j} \\
&= i\varepsilon_{ijk}\,\eta\bar{R}^2\mathbf{1} - \mathrm{i}\varepsilon_{\mathrm{ijk}}\frac{\eta^3}{8\hbar^2}(\mathrm{L}_1^2 + \mathrm{L}_2^2 + \mathrm{L}_3^2) \\
&= i\varepsilon_{ijk}\eta\bar{R}^2\{1 - \frac{\eta^2}{8\bar{R}^2}l(l+1)\}\mathbf{1} \\
&= i\varepsilon_{ijk}\tilde{\eta}\mathbf{1}. \qquad (20)
\end{aligned}$$

The noncommutative momentum operators also form a realization of the Lie 3-algebra. As for the commutation relation of $\hat{x_i}$ and $\hat{p_j}$,

$$\begin{aligned}
[\hat{x_i}, \hat{p_j}] &= [x_i + \frac{\theta}{2\hbar}L_i, p_j + \frac{\eta}{2\hbar}L_i] \\
&= i\hbar\,\delta_{ij} + \frac{i}{2}\,\theta\varepsilon_{ijk}\hat{p}_k + \frac{i}{2}\,\eta\varepsilon_{ijk}\hat{x}_k - \frac{i}{4\hbar}\,\theta\eta\varepsilon_{ijk}L_k, \qquad (21)
\end{aligned}$$

where $[\hat{x_1}, \hat{p_1}] = i\hbar$, $[\hat{x_2}, \hat{p_2}] = i\hbar$, $[\hat{x_3}, \hat{p_3}] = i\hbar$ are similar to ordinary quantum mechanics.

The annihilation and creation operators in the noncommutative space may be defined as

$$\hat{a}_i = \frac{1}{\sqrt{2\hbar}}(\hat{x}_i + i\hat{p}_i), \qquad \hat{a}_i^\dagger = \frac{1}{\sqrt{2\hbar}}(\hat{x}_i - i\hat{p}_i). \qquad (22)$$

The commutation relation of $\hat{a}_i$ and $\hat{a}_j^\dagger$ becomes

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} + i\frac{\theta}{2\hbar}\varepsilon_{ijk}\hat{x}_k + i\frac{\eta}{2\hbar}\varepsilon_{ijk}\hat{p}_k - i\frac{\theta^2}{8\hbar^2}\varepsilon_{ijk}L_k - i\frac{\eta^2}{8\hbar^2}\varepsilon_{ijk}L_k, \qquad (23)$$

where $[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = [\hat{a}_3, \hat{a}_3^\dagger] = 1$ are similar to the ordinary quantum mechanics.

3 The Aharonov-Bohm Effect

When magnetic is involved [7] [8], the Schrödinger equation (1) becomes

$$H(x_i, p_i, A_i) * \Psi = E\Psi, \tag{24}$$

The generalized Bopp's shift of the magnetic vector potential A_i, calculated by (7),

$$\hat{A}_i * \Psi = A_i\Psi + \frac{i}{2}\,\theta\varepsilon_{lmk}x_k\,\partial_l A_i\partial_m\Psi, \tag{25}$$

can be given as follows

$$\begin{aligned}\hat{A}_i &= A_i - \frac{\theta}{2\hbar}\varepsilon_{lmk}x_k p_m\partial_l A_i \\ &= A_i + \frac{\theta}{2\hbar}L_l\partial_l A_i. \end{aligned} \tag{26}$$

Thus the Schrödinger equation (19) in the presence of magnetic field becomes

$$H(x_i + \frac{\theta}{2\hbar}L_i,\, p_i + \frac{\eta}{2\hbar}L_i,\, A_i + \frac{\theta}{2\hbar}L_l\partial_l A_i)\Psi = E\Psi \tag{27}$$

Let us consider a particle of mass m and charge q moving in a magnetic field with magnetic vector potential A_i, the Schrödinger equation is

$$\hat{H}\Psi = \frac{1}{2m}(\hat{p}_i - q\hat{A}_i)^2\Psi \tag{28}$$

$$= \frac{1}{2m}(p_i + \frac{\eta}{2\hbar}L_i - qA_i - \frac{q}{2\hbar}\,\theta L_l\partial_l A_i)^2\Psi = E\Psi \tag{29}$$

The solution for (24) reads

$$\Psi = \Psi_0\, exp\,[iq\int_{x_0}^{x}(A_i - \frac{\eta}{2\hbar}L_i + \frac{q}{2\hbar}\,\theta L_l\partial_l A_i)dx_i], \tag{30}$$

where Ψ_0 is the solution of (24) when A_i, θ, η vanish. The phase term of (25) is the so called Aharonov-Bohm phase. If we consider a charged particle pass through a double slits, then the integral reus from the source x_0 through one of the two slits to the point x of the screen. Thus the total phase shift for this Aharonov-Bohm effect is

$$\Delta\Phi = \delta\Phi_0 + \delta\Phi^{NC} \tag{31}$$

$$= iq\oint A_i dx_i + \frac{iq}{2\hbar}\oint(-\eta L_i + \theta L_l\partial_l A_i)dx_i \tag{32}$$

4 Concluding Remarks

In this article we study the Lie 3-algebra in noncommutative quantum mechanics on the fuzzy sphere. The coordinate and momentum operators form a realization of the Lie 3-algebra. The introduction of the generalized Bopp's shift, for the coordinate operator, momentum operator and the magnetic vector potential, may be useful to study the gauge fields or string theories. More examples about the Lie 3-algebra in noncommutative space are interesting.

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不可交換性量子力學中的三重李代數

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摘 要

研究在模糊球上不可交換性量子力學中的三重李代數。由 Moyal-Weyl 乘積所得出的座標和動量算符，形成一三重李代數的表現。引入一磁向量位移用來討論 Aharonov-Bohm 效應。

關鍵字: 不可交換性量子力學