Some (3+1)-Dimensional Vortex Solutions of the Gr(n,N) σ-Model

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Abstract

A class of vortex solutions of the Gr(n,N) Grassmannian σ-Model in (3+1) dimensions are presented. These solutions may be regarded as the generalization of the vortex solutions of the CP^N model [1]. The energy density of the vortices are related to the Noether charge and topological charge.

Keywords: Grassmannian sigma model, vortex solutions

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1 Introduction

The nonlinear σ-models in two dimensions are special interest because they bear many similarities to the nonabelian gauge theory in four dimensions and have a property of being an integrable system. However, the Exact topological soliton solutions are rare specialty in dimensions higher than two. The most well known examples are those provided by self-dual or BPS solutions like instantons and monopoles. In this paper we present the vortex-like solutions [1] [2] [3] [4] in (3+1)-dimensional for the $Gr(n,N)$ Grassmannian σ-Model [5] which is based on the homogeneous spaces

$$Gr(n,N) = \frac{SU(N)}{SU(n)\times SU(N-n)}. \tag{1}$$

Grassmannian sigma models are a generalization of CP^{N-1} sigma models [6]. They share some common features such as the Euler-Lagrange equations can be written in terms of projectors, infinite number of conserved quantities and the existence of multisoliton solutions etc.

We express elements $Gr(n,N)$ using the equivalent class of elements $g\in SU(N)$ as

$$[g] = \{g\Psi \mid \Psi = \begin{pmatrix} U_n & 0 \\ 0 & U_{N-n} \end{pmatrix}, U_n \in SU(n), U_{N-n}\in SU(N-n)\}, \tag{2}$$

and decompose $g\in SU(N)$ into submatrices X, Y

$$g = (\phi_1,...,\phi_N) = (X,Y), \quad X = (\phi_1,...,\phi_n), \quad Y = (\phi_{n+1},...,\phi_N) \tag{3}$$

where X is an $N\times n$ matrix and Y is an $N\times(N-n)$ matrix. Since the unitary condition $g^\dagger g = I_N$, i.e. $\phi_j^\dagger\phi_k = \delta_{jk}$, we find

$$X^\dagger X = I_n, \quad X^\dagger Y = 0, \quad Y^\dagger X = 0, \quad Y^\dagger Y = I_{N-n}. \tag{4}$$

Let x^0, x^1, x^2, x^3 be the standard Minkowski coordinates in R^4, with the metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{5}$$

In what follows we shall use the notation

$$z \equiv x^1 + ix^2, \qquad \bar{z} \equiv x^1 + ix^2, \qquad y_\pm \equiv x^3 \pm x^0. \tag{6}$$

2 Euler-Lagrange Equations

Let us assume that Ω is an open,connected and simply connected subset in R^4 with Minkowski metric (5). We define covariant derivative D_μ acting on maps $X:\Omega\to Gr(n,N)$ by

$$D_\mu X = \partial_\mu X - XX^\dagger\partial_\mu X, \qquad \mu = 0,1,2,3. \tag{7}$$

In the study of Grassmannian sigma models we are interested in maps $X:\Omega\to Gr(n,N)$ which are stationary points of the action functional

$$\mathcal{S} = \int_\Omega tr\{(D_\mu X)^\dagger(D^\mu X)\}\, d^4x. \tag{8}$$

The Lagrangian density can be further developed to get

$$\mathcal{L} = tr\{(D_\mu X)^\dagger(D^\mu X)\} = tr\{\partial^\mu X(\partial_\mu X)^\dagger\mathcal{P}\} = \frac{1}{2}tr\{\partial^\mu\mathcal{P}\partial_\mu\mathcal{P}\} \tag{9}$$

where

$$\mathcal{P} = I - XX^\dagger \tag{10}$$

is an orthogonal projector, i.e. $\mathcal{P}^2 = \mathcal{P}$, $\mathcal{P}^\dagger = \mathcal{P}$ satisfying $\mathcal{P}X = 0$, $X^\dagger \mathcal{P} = 0$.

The action (8) has the local (gauge) $SU(n)$ symmetry

$$X \to Xh, \qquad h \in \begin{pmatrix} SU(n) & 0 \\ 0 & I \end{pmatrix} \tag{11}$$

proving that the model doesn't depend on the choice of representatives X of the elements of $Gr(n, N)$. The action also invariant under the $SU(N)$ global symmetry transformation

$$X \to gX, \quad g \in SU(N). \tag{12}$$

These invariance properties are naturally reproduced on the level of Euler-Lagrange equations.

Taking into account the constraint

$$\mathcal{P}^2 - \mathcal{P} = 0, \tag{13}$$

the action (8) becomes

$$\mathcal{S} = \int \left(\frac{1}{2} tr\{\partial^\mu \mathcal{P} \partial_\mu \mathcal{P}\} + \lambda\, tr\{\mathcal{P}^2 - \mathcal{P}\} \right) d^4x. \tag{14}$$

Variating the action about $\mathcal{P}$ and using the boundary condition $\delta\mathcal{P} = 0$ on the surface integral, we get the equation of motion as

$$\partial^\mu \partial_\mu \mathcal{P} - \lambda\,(2\mathcal{P} - I) = 0. \tag{15}$$

Next, if we multiply this equation by $\mathcal{P}$ from the right and left separately, then we have two equations

$$\partial^\mu \partial_\mu \mathcal{P}\mathcal{P} - \lambda\,(2\mathcal{P} - I)\,\mathcal{P} = 0 \tag{16}$$
$$\mathcal{P}\partial^\mu \partial_\mu \mathcal{P} - \lambda\mathcal{P}\,(2\mathcal{P} - I) = 0. \tag{17}$$

Subtracting (16) from (17), we get the equation of motion in matrix form

$$[\partial^\mu \partial_\mu \mathcal{P}, \mathcal{P}] = 0 \tag{18}$$

or in the form of a conservation law

$$\partial_\mu [\partial^\mu \mathcal{P}, \mathcal{P}] = 0. \tag{19}$$

The conserved current density

$$J^\mu = [\partial^\mu \mathcal{P}, \mathcal{P}] \tag{20}$$

are invariant under local $SU(n)$ and global $SU(N-n)$ transformations.

The equation (18) can also be written as

$$\mathcal{P}\left(\partial^\mu \partial_\mu X - 2\partial_\mu X X^\dagger \partial^\mu X\right) = 0 \tag{21}$$

3 Vortex Solutions

To find the solutions of (18), one can parametrize ϕ_i in X of (3),

$$\phi_j = \begin{pmatrix} u_{1j} \\ \vdots \\ u_{Nj} \end{pmatrix}, \tag{22}$$

where

$$\phi_j^\dagger \phi_k = \sum_{A=1}^{N} u_{Aj}^* u_{Ak} = \delta_{jk}. \tag{23}$$

In terms of u_{Aj} the Lagrangian density (9) takes the form

$$\mathcal{L} = \sum_A \sum_j \partial^\mu u_{Aj}^* \, \partial_\mu u_{Aj} - \sum_A \sum_B \sum_k \sum_j u_{Aj} \, u_{Bk}^* \ \partial^\mu u_{Aj}^* \, \partial_\mu u_{Bk} \tag{24}$$

and the Euler-Lagrangian equation reads

$$\partial^\mu \partial_\mu u_{Aj} - 2 \left(\sum_B u_{Bk}^* \, \partial_\mu u_{Bk} \right) \partial^\mu u_{Aj} - u_{Aj} \partial^\mu \left(\sum_B u_{Bk}^* \, \partial_\mu u_{Bk} \right) = 0 \tag{25}$$

Any set of functions u_{Aj} that depend on the coordinates x^μ in special form

$$u_{Aj} = u_{Aj}(z, y_+) \tag{26}$$

is a solution of the equation (25). The Minkowski metric in the in the notation (6) becomes $ds^2 = dzd\bar{z} + dy_+ dy_-$. It then follows that (26) satisfies simultaneously $\partial^\mu \partial_\mu u_{Aj} = 0$ and $\partial_\mu u_{Bk} \partial^\mu u_{Aj} = 0$. Amongst the very many solutions of the type (26), there have been considered in the CP^N model in $3+1$ dimensions [ ]. In this paper we shall consider only some very special form, i.e.

$$u_{Aj}(z, y_+) = (z - \delta)^{n_{Aj}} \ (z + \delta)^{m_{Aj}} \ e^{ik_{Aj} y_+}, \tag{27}$$

where δ is a real constant and n_{Aj}, m_{Aj} are integers. The energy density of solutions (26) takes the form

$$\mathcal{H} = \sum_A \sum_B \sum_j \sum_k \left(\partial_{\bar{z}} u_{Aj}^\dagger \ (\Delta^2)_{AB,\,jk} \ \partial_z u_{Bk} + \partial_{y_+} u_{Aj}^\dagger (\Delta^2)_{AB,\,jk} \ \partial_{y_+} u_{Bk} \right), \tag{28}$$

where

$$(\Delta^2)_{AB,\,jk} \equiv \delta_{AB} \delta_{jk} - u_{Aj} u_{Bk}^*. \tag{29}$$

When integrated over the x^1-x^2 plane the first term in (28) becomes proportional to the topological charge of the vortex solution as we will explain below. The second term in (28) is related to some Noether charges of the $Gr(n, N)$ model. To see this we note that $Gr(n, N)$ Lagrangian (9) is invariant under the $SU(N)$ global symmetry transformation. The parametrization of the fields in terms of the u fields given by (22) transform under the $SU(n)$ gauge symmetry. The Noether currents associated with these symmetries are given by

$$J_\mu^{(Aj)} = i \sum_{B=1}^{N} \sum_{i=1}^{n} \left(u_{Aj}^* (\Delta^2)_{AB,\,jk} \partial_\mu u_{Bk} - \partial_\mu u_{Bk}^* (\Delta^2)_{BA,\,kj} u_{Aj} \right). \tag{30}$$

The total energy density $\mathcal{H}$ is split into two parts,i.e.

$$\mathcal{H} = \mathcal{H}^{(1)} + \mathcal{H}^{(2)}, \tag{31}$$

where $\mathcal{H}^{(1)}$ is the first term in (28) and $\mathcal{H}^{(2)}$ is the second term in (28). One can show that

$$\sum_A \sum_B \sum_j \sum_k \partial_{y_+} u^\dagger_{Aj} (\Delta^2)_{AB,\,jk}\, \partial_{y_+} u_{Bk} = \sum_{A=1}^{N} \sum_{j=1}^{n} k_{Aj} J_0^{(Aj)}, \tag{32}$$

where k_{Aj} being the inverse of a wavelength. The Noether charge can be defined as

$$Q^{(Aj)}_{Noether} \equiv \int dx^1 dx^2 J_0^{(Aj)} \tag{33}$$

For a solution of the type (27) the first term of $\mathcal{H}$ reduces to

$$\mathcal{H}^{(1)} = \sum_A \sum_j \psi_{AA,\,jj} |u_{Aj}|^2 + \sum_A \sum_B \sum_j \sum_k (\psi_{AA,\,jj} - \psi_{AB,\,jk}) |u_{Aj}|^2 |u_{Bk}|^2, \tag{34}$$

where, if we define $w_\delta \equiv z - \delta$ and $w_{-\delta} \equiv z + \delta$, the function $\psi_{AB,\,jk}$ are given by

$$\psi_{AB,\,ij}(z,\bar{z}) \equiv \frac{n_{Aj}\, n_{Bk}}{|w_\delta|^2} + \frac{m_{Aj}\, m_{Bk}}{|w_{-\delta}|^2} + (n_{Aj}\, m_{Bk} + n_{Bk}\, m_{Aj}) \frac{\bar{w}_\delta w_{-\delta} + w_\delta \bar{w}_{-\delta}}{2|w_\delta|^2 |w_{-\delta}|^2}. \tag{35}$$

The second term of $\mathcal{H}$ reduces to

$$\mathcal{H}^{(2)} = \sum_A \sum_j k^2_{Aj} |u_{Aj}|^2 + \frac{1}{2} \sum_A \sum_B \sum_j \sum_k (k_{Aj} - k_{Bk})^2 |u_{Aj}|^2 |u_{Bk}|^2. \tag{36}$$

Therefore, the energy per unit length of the vortex solutions has the form

$$\mathcal{E} = \int dx^1 dx^2\, \mathcal{H}^{(1)} + \int dx^1 dx^2\, \mathcal{H}^{(2)} = \pi \left(Q_{top.} + \sum_A \sum_j k_{Aj} Q^{(Aj)}_{Noether} \right) \tag{37}$$

where $Q_{top.}$ is the topological charge.

4 Conclusions

In this paper we have shown that the $Gr(n,N)$ Grassmannian σ-Model in $(3+1)$ dimensions has many classical solutions. A class of vortex solutions are suggested. These solutions may be regarded as the generalization of vortex solutions, proposed by L.A.Ferreira et.al. [1], of the CP^N model in $(3+1)$ dimensions. The energy density are also related to topological charge and Noether charge as expected.

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在(3+1)維 Gr(n,N) σ -模型中的漩渦解

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摘 要

討論有一類在 Gr(n,N) σ -模型中(3+1)維的漩渦解。這些解是(3+1)維 CP^N σ -模型漩渦解的推廣。此漩渦解和 Noether 核和拓撲荷有關聯。

關鍵字:　Grassmannian sigma 模型, 漩渦解