Coherent Spin-Network States for Loop Quantum Gravity in Schwarzschild Space-time

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Abstract

In this paper we discuss the construction of the coherent spin-network states for loop quantum gravity. These states capture properties of curved space-time of the Schwarzschild metric on which they are peaked. For calculation purposes, we employ the heat-kernel method and the complex $SL(2,C)$ variables of the labeled states used in the spin foam setting.

Keywords: Loop Quantum Gravity, Coherent Spin-Network States

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1 Introduction

The loop representation for quantum gravity is a natural development from the Ashtekar's reformulation of general relativity [1][2] and Jacobson and Smolin's discovery [3] of a large class of solutions of the full set of quantum constraints equations. The theory is formulated in a background independent and essentially nonperturbative fashion. This theory is based on the canonical quantization of general relativity in terms of variables that are different from the standard metric variables. In terms of these variables general relativity can be cast into the form of background independent $SU(2)$ gauge theory partly analogous to $SU(2)$ Yang-Mills theory. A key ingredient in loop quantum gravity is semiclassical states.These semiclassical coherent states in the Hilbert space of quantum general relativity are states that are able to reproduce a given classical geometry in terms of their expectation values and peak on a prescribed space-time. The relation between quantum states and the classical theory is clarified by the construction of the coherent states. In some graviton calculations [4][5][6] , these states associated to a spin-network graph Γ are labeled by a spin j_l, an angle ξ_l per link l of the graph, and two unit vectors for the nodes of the link l. The graph Γ is dual to a simplicial decomposition of the spatial manifold, the unit vectors $\hat{n}_l$ are associated to the unit-normals of tetrahedra, the spin j_l is the average area of a face, and the simplicial extrinsic curvature is the angle ξ_l shared by the faces of tetrahedra.

In this paper, we discuss a class of coherent states on the background of Schwarzschild space-time. To construct the coherent states [7][8][9][10], we consider a spatial hypersurface Σ of the constant proper time of the Schwarzschild in the Lemaitre coordinates. Take a regular cellular decomposition of Δ_Σ and associated with it its dual graph. This decomposition provides us with a set of curves and surfaces to be used for the smearing process. Then compute the the holonomies h_l of the Ashtekar connection along links and fluxes X_l of the gravitational electric fields through the surface S_l dual to the link l.

The Hilbert space of the loop quantum gravity for each graph Γ is $\mathcal{H}_\Gamma = L^2(SU(2)^L/SU(2)^N)$, where L is the number of links of the graph and N the number of nodes, L^2 means the square-integrable functions. Holomorphic states in the space are functions of group elements h_l that are invariant under $SU(2)$ transformations at nodes,

$$\Psi(h_l) = \Psi(g_{s(l)}\, h_l\, g^{-1}_{t(l)}), \tag{1}$$

where $s(l)$ and $t(l)$ are the nodes that are source and target of the link l respectively. For general relativity, the classical configuration of the Ashtekar connection A and its conjugate momentum E, the gravitational electric field satisfy the fundamental brackets:

$$\begin{aligned} \{A^i_a(x), A^j_b(y)\} &= 0, \quad \{E^a_i(x), E^b_j(y)\} = 0 \\ \{A^i_a(x), E^b_j(y)\} &= 8\pi G\gamma\hbar\delta^i_j\delta^b_a\delta(x-y). \end{aligned} \tag{2}$$

The speed of light is set for one throughout this paper. The non-vanishing real number γ is the Barbero-Immirzi parameter. The cellular decomposition of the spatial hypersurface Σ provides a discretization of the manifold. The Ashtekar connection A is a $su(2)$-valued one form over links, while the E is a $su(2)$-valued densitized inverse triad over the dual surfaces. The connection is smeared along half-link l of the graph Γ, that is from the source node $s(l)$ to the point of intersection with the surface. The path-ordered exponential h_l is defined as

$$h_l = Pexp \int_l A, \tag{3}$$

which is defined on half of the link l. The flux is defined as

$$E_l = \int_{S_l} Ad_h * E \tag{4}$$

Here Ad stands for the action of the adjoint representation of $SU(2)$ on Lie algebra elements. $*$ is the Hodge dual operator. The densitized inverse triad E is parallel-transported by the holonomy U. The holonomy h is computed along a path which is starts at the base-point σ_0 and ends at the integration point σ. The set of couples (h_l, E_l), one per each link of the graph, can be viewed as a point in a truncation of the phase space of General Relativity as captured by the graph Γ. The smeared Poisson algebra reads

$$\begin{aligned} \{h_l, h_{l'}\} &= 0, \quad \{E_l^i, E_{l'}^j\} = \delta_{ll'}\epsilon^{ijk}E_l^k \\ \{E_l^i, h_{l'}\} &= \delta_{ll'} 8\pi G\gamma\hbar\tau^i h_l. \end{aligned} \tag{5}$$

$\tau^i = i\sigma^i/2$ are $su(2)$ generators defined in terms of Pauli matrices σ^i The couple (h_l, E_l) can be identified with an element of $SL(2,C)$, the complexification of $SU(2)$, using the polar decomposition

$$H_l = \ h_l e^{X_l} \in SL(2,C) \tag{6}$$

where

$$X_l \equiv i\frac{\alpha_l E_l}{8\pi G\gamma\hbar}. \tag{7}$$

α_l in (7) called the heat-kernel time is a positive real number.

To construct the coherent states for quantum general relativity, we relies on the heat-kernel method. Apply the heat-kernel evolution to the Dirac delta distribution over the group,

$$K_{\alpha_l}(h,h') = \ e^{-\frac{\alpha_l}{2}\Delta_h}\ \delta(h,h'), \tag{8}$$

where the Laplace-Beltrami operator Δ_h on $SU(2)$ is defined with respect to the unique bi-invariant metric tensor. K_{α_l} is the heat-kernel on $SU(2)$, which can written explicitly as

$$K_{\alpha_l}(g) = \sum_j (2j+1)e^{-j(j+1)\frac{\alpha_l}{2}}\ TrD^{(j)}(g) \tag{9}$$

where $D^{(j)}$is the Wigner representation matrix of the representation j. $g = h^{-1}h' \in SL(2,C)$. Then applying the heat-kernel several copies of $SU(2)$ and considering the gauge-invariant projection of a product over the links of a graph Γ allows us to define the coherent spin-network states [11][12][13] as

$$\Psi_{H_l}(h_l) = \int \prod_n dg_n \prod_l K_{\alpha_l}(h_l,\ g_{s(l)} H_l\, g_{t(l)}^{-1}), \tag{10}$$

where we have a $SU(2)$ integration for each node n. The coherent spin-network states provide a Segal-Bargmann transform for loop quantum gravity, that has been lifted to spin foams. An element H_l of $SL(2,C)$ can be written as

$$H_l = g_{\hat{n}_{s(l)}} e^{(\eta_l + i\xi_l)\sigma_3/2} g_{\hat{n}_{t(l)}}^{-1}. \tag{11}$$

Freidel and Speziale discuss a compelling geometrical interpretation for the

$$(\hat{n}_{s(l)}, \hat{n}_{t(l)}, \xi_l, \eta_l) \tag{12}$$

labels defined on of each link by (11). For appropriate four-valent states representing a Regge 3-geometry with intrinsic and extrinsic curvature, the unit vectors $\hat{n}_{s(l)}$, $\hat{n}_{t(l)}$ are 3d normals to the triangles of the tetrahedra bounded by the triangles. η_l is the area of the triangle divided by $8\pi G\gamma\hbar$. ξ_l is a sum of two parts: the extrinsic curvature at the triangle and the 3d rotation due to the spin connection at the triangle. We will see how H_l, hence ξ_l, is determined from the Schwarzschild geometry.

In terms of these variables, the large η asymptotic behavior for the coherent spin-network states can be found

$$\Psi_{H_l}(h_l) \approx \sum_{j_l, i_l} \prod_l e^{-\frac{(j_l - j_l^0)^2}{2\sigma_l^2}} e^{-i\xi_l j_l} (\prod_n \Phi_{i_n}) \Psi_{j_l, i_l}(h_l) \tag{13}$$

The position of the peak j_l^0 is related to η_l by $(2j_l^0 + 1) = 2\eta_l/\alpha_l$, and the spread of the Gaussian around j_l^0 is governed by the parameter $\sigma_l = 1/\sqrt{\alpha_l}$. Φ_{i_n} is the coefficient for the expansion of the Livine-Speziale coherent interwiner [10] on a orthonormal basis labels by i_n, and carries the dependence on the unit vectors.

2 Schwarzschild Spacetime

The usual Schwarzschild metric has the following form:

$$ds^2 = -(1 - \frac{r_s}{r})\, dt^2 + \frac{dr^2}{(1 - \frac{r_s}{r})} + r^2 d\Omega^2. \tag{14}$$

$r_s = 2GM$. This coordinate chart only covers $r > r_s$. It would not be convenient to work with the t constant flat slices that the coherent states need to be defined on the spatial slice at a fixed time. One possible choice is the Lemaitre coordinate expression of the metric, where space-time is foliated by a set of proper-time observers. A transformation of the Schwarzschild coordinate system from $\{t, r\}$ to the new coordinates $\{\tau, R\}$,

$$d\tau = dt + \sqrt{\frac{r_s}{r}} \frac{1}{1 - \frac{r_s}{r}}\, dr \tag{15}$$

$$dR = dt + \sqrt{\frac{r}{r_s}} \frac{1}{1 - \frac{r_s}{r}}\, dr, \tag{16}$$

lead to the Lemaitre coordinate expression of the metric

$$ds^2 = -d\tau^2 + \frac{r_s}{\rho} dR^2 + \rho^2 d\Omega^2 \tag{17}$$

where

$$\rho = [\frac{3}{2}(R - \tau)]^{\frac{2}{3}}\, r_s^{\frac{1}{3}} \tag{18}$$

The trajectories R constant are time-like geodesics with τ the proper time along these geodesics. In Lemaitre coordinates there is no singularity at the gravitational radius, which instead corresponds to the point $\frac{3}{2}(R - \tau) = r_s$. However, there remains a genuine gravitational singularity at the centrum, where $R - \tau = 0$, which cannot be removed by a coordinate change. The Lemaitre coordinate system is synchronous, that is, the global time coordinate of the metric defines the proper time of co-moving observers. The constant τ slices are flat. The flat metric $d\rho^2 + \rho^2 d\Omega^2$ can be derived from (17), (18), and the relation

$$\rho' \equiv \frac{\partial \rho}{\partial R} = \sqrt{\frac{r_s}{\rho}}. \tag{19}$$

The constant τ slices across some finite range in time are enough for us to built the coherent states.

Consider a foliation of space-time in terms of space-like three dimensional surfaces Σ with induced metric q_{ab}. Define a triad in terms of the metric q_{ab} such that

$$q_{ab} = e^i_a e^j_b \delta_{ij} \tag{20}$$

where $i, j = 1, 2, 3$. From (17), we have the triad

$$e^1_R = \sqrt{\frac{r_s}{\rho}} = \rho', \qquad e^2_\theta = \rho, \qquad e^3_\phi = \rho\, sin\theta. \tag{21}$$

Using these variables we introduce the densitized triad

$$E^a_i = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e^j_b e^k_c, \tag{22}$$

the densitized triad for the Lemaitre coorinate expression of Schwarzschild Space-time are

$$E^R_1 = \rho^2 sin\theta, \qquad E^\theta_2 = \sqrt{r_s\rho}\, sin\theta \qquad E^\phi_3 = \sqrt{r_s\rho}. \tag{23}$$

Using these definition, the inverse metric q^{ab} can be related to the densitized triad as follows

$$qq^{ab} = E^a_i E^b_j \delta^{ij}. \tag{24}$$

The constant τ slices are flat, and there are no intrinsic curvature. The extrinsic curvatures given by $K_{ab} = \frac{1}{2}\partial_\tau q_{ab}$ are

$$K_{RR} = \frac{1}{2}\rho^{-\frac{5}{2}}(r_s)^{\frac{3}{2}} = -\rho'\rho'', \tag{25}$$

$$K_{\theta\theta} = -\rho^{\frac{1}{2}}(r_s)^{\frac{1}{2}} = -\rho\rho', \tag{26}$$

$$K_{\phi\phi} = -\rho^{\frac{1}{2}}(r_s)^{\frac{1}{2}} sin^2\theta = -\rho\rho' sin^2\theta, \tag{27}$$

Let us define

$$K^i_a = \frac{1}{\sqrt{det(E)}} K_{ab}E^b_j\delta^{ij} \tag{28}$$

, then we have

$$K^1_R = \frac{r_s}{2\rho^2} = -\rho'', \tag{29}$$

$$K^2_\theta = -\rho^{-\frac{1}{2}}(r_s)^{\frac{1}{2}} = -\rho', \tag{30}$$

$$K^3_\phi = -\rho^{-\frac{1}{2}}(r_s)^{\frac{1}{2}} sin\theta = -\rho' sin\theta, \tag{31}$$

and get

$$K^i_a = -\frac{\partial}{\partial R} e^i_a. \tag{32}$$

3 Holonomies and Fluxes

We can then introduce the Ashtekar-Barbero connection, in this constant-proper-time gauge,

$$A^i_a = \gamma K^i_a = c\, e^i_a \tag{33}$$

with

$$c = -\gamma\frac{\partial}{\partial R}, \tag{34}$$

where γ is any non-vanishing real number called Immirzi parameter. The holonomy of the connection A along the link l for the cellular decomposition is the path-ordered exponential

$$h_l(A) = \mathbb{P}exp \int A = \sum_{m=0}^{\infty} I_m, \tag{35}$$

where the m-th integral has the form

$$I_m = \int_0^L ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{m-1}} ds_m \dot{\zeta}(s_1) \cdots \dot{\zeta}(s_m) A(s_1) \cdots A(s_m). \tag{36}$$

Here we have used an explicit parametrization of the geodesic $\zeta(s)$ in terms of the proper distance along the link l and L is the proper length of the link. The components of the vector

$$n^i \equiv e^i_a \dot{\zeta}^a \tag{37}$$

are conserved quantities, i.e. constant along spatial geodesics. The holonomy of the connection

$$h_l(A) = \sum_{m=0}^{\infty} \frac{1}{m!} (cL\vec{n} \cdot \vec{\tau})^m = e^{\frac{i}{2} cL\vec{n}\cdot\vec{\sigma}}. \tag{38}$$

For the cellular decomposition of the constant proper time surface the four links emanate from a node in isotropic directions, the four unit vectors defined in (37) are such that $\vec{n}_l \cdot \vec{n}_{l'} = cos^{-1}(-\frac{1}{4})$ for $l \neq l'$, and $\vec{n}_l = -\vec{n}_{l^{-1}}$. Moreover the length of a full link, a link goes from the source node $s(l)$ to the target node $t(l)$ of the geodesic link l, is

$$L = 2\Theta, \qquad \Theta = cos^{-1}(\frac{-1}{4}). \tag{39}$$

With the standard embedding of $SU(2)$ in $\mathbb{R}^4$, the two nodes $N_{s(l)}$, $N_{t(l)}$, viewed as vectors in $\mathbb{R}^4$, have the scalar product $N_{s(l)} \cdot N_{t(l)} = cos\Theta$. From the other side, the geodesic, embedded in $\mathbb{R}^4$, has the form $N(s) = (cos\frac{s}{2}, sin\frac{s}{2}, 0, 0)$, and then

$$N(s) \cdot N(0) = cos\frac{s}{2} = cos\Theta. \tag{40}$$

We find the value $s = 2\Theta$ for the geodesic length. Thus One has the holonomy of the Ashtekar-Barbero connection along half-link,

$$h_l(A) = e^{c\Theta\vec{n}_l\cdot\vec{\tau}}. \tag{41}$$

The computation of fluxes is more tricky as it relies on the definition (4). The flux $E_l(S) = E^i_l(S)\tau^i$ depends on the surfaces as well as on the holonomies along a path. $E_l(S)$ is the flux across the oriented surface S_l punctured by the link l. Now we can take a family of geodesics joining the intersection point σ_0 with the generic point σ of the integration on the surface. One has

$$E^i_l(S) = \int_S n^i \, det\, h\, d^2\sigma \tag{42}$$

where $h_{ab}, a, b = 1, 2$, is the metric induced on the surface from q_{ab}, and n^i is the components of a unit vector given by

$$n^i = \frac{N^i}{\sqrt{N^i N^i}} \tag{43}$$

with

$$N^i(\sigma) = R^{ij} e^{aj}(\sigma_0) n_a(\sigma). \tag{44}$$

The rotation matrix R^{ij} is the holonomy in the adjoint representation of the $SU(2)$, that acts on the internal indices. It performs the parallel transport of the triad from the base point σ_0 to the point of integration σ, along a geodesic path. Since we are averaging n^i around the barycenter σ_0, we have

$$E_l^i(S) = |E_l(S)|\, n^i(\sigma_0) \tag{45}$$

where $|E_l(S)|$ denotes the modulus of the flux, whose time dependence is $|E_l(S)| \propto \rho(\tau)^2$. We have

$$E_l(S) = |E_l|\, \vec{n_l} \cdot \vec{\tau}. \tag{46}$$

It is important to remark that the unit flux $\vec{n_l} = \vec{E_l}/|E_l|$ in the last equation coincide with the unit direction of the Ashtekar-Barbero holonomy h_l in(41). This is because the orientation of the link l and the surface S_l coincides.

4 Coherent Spin-Network States

We can now define the coherent spin-network states for for Loop Quantum Gravity in Schwarzschild Space-time as the one labeled by $SL(2, C)$ variables on links as defined by the smearing process in (6), (7). We apply the decomposition (11) to obtain:

$$H_l = g_{\vec{n}_l}\, e^{c\Theta\tau_3 + i|X|\tau_3}\, g_{\vec{n}_l}^{-1}. \tag{47}$$

By the asymptotic formula (13), the asymptotic behavior of the coherent spin-network for large $|X| \propto \rho(\tau)^2$ can be found. The Schwarzschild Space-time coherent state is

$$\Psi_{H_l}(j_l, i_n) \approx \prod_l e^{-\frac{(j_l - j_l^0)^2}{2\sigma^2}}\, e^{ic\Theta j_l} (\prod_n \Phi_{i_n}) \tag{48}$$

with

$$j_l^0 = \frac{|E_l|}{8\pi G\hbar\gamma}, \tag{49}$$

$$\Theta = cos^{-1}(-\frac{1}{4}). \tag{50}$$

Remarkably, these coefficients are similar to those ones used in order to define correlation functions over flat space in the Spin Foam setting []. Moreover, in the Spin Foam setting, the angle $\Theta = cos^{-1}(-\frac{1}{4})$ is interpreted as a 4-dimensional dihedral angle between two tetrahedra lying in the boundary of an equilateral, flat 4-simplex.

5 Some Remarks

We provided a class of coherent spin-network states for Loop Quantum Gravity which are peaked around the Schwarzschild Space-time. We can compare the result with the standard boundary states of Spin Foam vertex amplitudes. The applications of such a class of coherent states in the context of cosmological interest could open new perspectives within the semiclassical analysis of the Spin Foam dynamics, as a possible development of a cosmological perturbation theory. We hope the simple coherent states discussed here could shed some light on the relationship with the full quantum cosmological.

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環量子重力理論在舒瓦茲時空中的同調自旋網絡態

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摘 要

在此篇論文中，我們討論在環量子重力理論中，建構同調自旋網絡態。這些態捕獲了，彎曲時空舒瓦茲度量張量的峰值處。為了方便計算，我們使用熱核方法，以及在自旋泡沫背景理論中，所用的複 SL(2,C)變數。

關鍵字: 環量子重力理論, 同調自旋網絡態