Tunghai Science Vol. 15: 41-47
July, 2013

Classical r-matrices for the Elliptic Calogero Model in Harmonic Potential

Shyang-Ling Lou and **Yih-Shyan Su***
Deparment of Physics, Tunghai University
Taichung, 407, Taiwan

June 27, 2013

Abstract

For the classical elliptic Calogero system in an external harmonic potential it is shown that the Lax operator L^+L^- possesses a classical r-matrix structure. The relation of the r-matrix and the Yang-Baxter equation involving another dynamical matrix are discussed.

Keywords: elliptic Calogero model, r-matrix.

* Email: sllou@thu.edu.tw

1 Introduction

The Hamiltonian of the elliptic Calogero system [1] with N particles on a circle, interacting with potential $\sum_{\alpha<\beta}^{N}\wp\left(q_\alpha-q_\beta\right)$ and an external harmonic potential [3], reads

$$H=\frac{1}{2}\sum_{\alpha=1}^{N}\left(p_\alpha^2+q_\alpha^2\right)+\sum_{\alpha<\beta}^{N}\wp\left(q_{\alpha\beta}\right),\qquad q_{\alpha\beta}\equiv q_\alpha-q_\beta, \tag{1}$$

where $\wp(q)=\wp(q;\omega_1,\omega_2)$ is the Weierstrass elliptic function

$$\wp\left(q;\omega_1,\omega_2\right)=\sum_{m,n\in Z}\left(q+m\omega_1+n\omega_2\right)^{-2},\qquad \omega_2/\omega_1\notin R. \tag{2}$$

It is straightforward to show that the equations of motion are

$$\begin{aligned} \dot{q_\alpha} &= p_\alpha \qquad (3)\\ \dot{p_\alpha} &= -q_\alpha-\sum_{\alpha\neq\beta}\wp'(q_{\alpha\beta}). \qquad (4)\end{aligned}$$

Let us introduce the following $N\times N$ matrices:

$$\begin{aligned} X_\beta^\alpha(u) &= \delta_{\alpha\beta}q_\alpha, \qquad (5)\\ L_\beta^\alpha(u) &= p_\alpha\delta_{\alpha\beta}+i(1-\delta_{\alpha\beta})Q(q_{\alpha\beta},u), \qquad (6)\\ L^\pm &= L\pm iX, \qquad (7)\\ M_\beta^\alpha(u) &= -\delta_{\alpha\beta}\left(\sum_{\gamma\neq\alpha}\wp(q_{\alpha\gamma})-\wp(u)\right)-(1-\delta_{\alpha\beta})Q'(q_{\alpha\beta},u), \qquad (8)\end{aligned}$$

where u is the spectral parameter, $L_\beta^\alpha(u)$ in (6) is called the Krichever's L-operator [2] and

$$\begin{aligned} Q(q,u) &= \frac{\sigma(q-u)}{\sigma(q)\,\sigma(u)}\,exp(\zeta(u)q), \qquad (9)\\ \sigma(q) &= q\prod_{m,\,n\neq 0}\left(1-\frac{q}{\omega_{mn}}\right)exp\left[\frac{q}{\omega_{mn}}+\frac{1}{2}\left(\frac{q}{\omega_{mn}}\right)^2\right], \qquad (10)\\ \zeta(q) &= \frac{\sigma'(q)}{\sigma(q)}\qquad \wp(q)=-\zeta'(q), \qquad (11)\\ \omega_{mn} &= m\omega_1+n\omega_2 \qquad (12)\\ Q(q,u)\,Q(-q,u) &= \wp(u)-\wp(q). \qquad (13)\end{aligned}$$

Then the equations of motion of the system are equivalent to the following matrix equations

$$\begin{aligned} \dot{X}+i[M,\,X] &= L, \qquad (14)\\ \dot{L}+i[M,\,L] &= -X, \qquad (15)\end{aligned}$$

or equivalently

$$\dot{L}^\pm+i[M,\,L^\pm]=\pm iL^\pm. \tag{16}$$

It follows from (16) that

$$\partial_t(L^+L^-)+i[M,\,L^+L^-]=0. \tag{17}$$

Now L^+L^- and M are regarded as the Lax pair of this system. The Lax operator L^+L^- defines N integrals of motion,

$$\mathcal{I}_k(p,q) = \frac{1}{k}\,tr(L^+L^-)^k = \frac{1}{k}\,tr(L^-L^+)^k, \qquad k = 1, 2..., N. \tag{18}$$

As for $k = 1$,

$$\mathcal{I}_1 = tr(L^+L^-) \propto \ H \tag{19}$$

is the Hamiltonian (1) itself. The higher integrals of motion $\mathcal{I}_k$, $k = 2, 3...$ are in involution, i.e. have vanishing Poisson's bracket with each other. The existence of a high number of conserved quantities is the result of integrability of this system in the Jacobi-Liouville sense.

2 r-matrix structure

As shown in [4][5][6], for the commutativity of the spectral invariants $trL(u)^n$ of the Krichever's L-operator it is necessary and sufficient that the Poisson bracket $\{L^{\alpha_1}_{\beta_1}(u), L^{\alpha_2}_{\beta_2}(v\}$ could be represented in the commutator form

$$\begin{aligned}\{L^{\alpha_1}_{\beta_1}(u), L^{\alpha_2}_{\beta_2}(v\} \;\; &= \;\; \sum_{\gamma_1\gamma_2}\{\, r^{\alpha_1\alpha_2}_{\gamma_1\beta_2}(u,v)\; L^{\gamma_1}_{\beta_1}(u) - L^{\alpha_1}_{\gamma_1}(u)\; r^{\gamma_1\alpha_2}_{\beta_1\beta_2}(u,v) \\ &\qquad - r^{\alpha_2\alpha_1}_{\gamma_2\beta_1}(v,u)\; L^{\gamma_2}_{\beta_2}(v) + L^{\alpha_2}_{\gamma_2}(v)\; r^{\gamma_2\alpha_1}_{\beta_2\beta_1}(v,u)\}\end{aligned} \tag{20}$$

or,using the notation [7]

$$L^{(1)} \equiv L \otimes \mathbb{I}, \quad L^{(2)} \equiv \mathbb{I} \otimes L \tag{21}$$

as

$$\{L^{(1)}(u), L^{(2)}(v)\} = [r^{(12)}, L^{(1)}(u)] - [r^{(21)}, L^{(2)}(v)] \tag{22}$$

where $r^{(12)}$ is an $N^2 \times N^2$ matrix and $r^{(21)}$ is

$$r^{(21)}(u,v) \equiv \mathbb{P}\, r^{(12)}(u,v)\mathbb{P}, \tag{23}$$

$\mathbb{P}$ being the permutation, $\mathbb{P}x \otimes y = y \otimes x$.

For the Krichever's L-operator, in terms of the basic matrices $e_{\alpha\beta}$

$$(e_{\alpha\beta})_{\alpha'\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'} \tag{24}$$

expressed by

$$L(u) = \sum_{\alpha=1}^{N} p_\alpha e_{\alpha\alpha} + i \sum_{\beta\neq\alpha} Q(q_{\alpha\beta}, u)\, e_{\alpha\beta}, \tag{25}$$

the identity (22) holds with the matrix $r^{(12)}$,

$$r^{(12)}(u,v) = a\sum_{\alpha=1}^{N} e_{\alpha\alpha} \otimes e_{\alpha\alpha} + \sum_{\alpha\neq\beta} c_{\alpha\beta}(e_{\alpha\beta} \otimes e_{\beta\alpha}) + \sum_{\alpha\neq\beta} d_{\alpha\beta}(e_{\alpha\alpha} \otimes e_{\alpha\beta} + e_{\beta\beta} \otimes e_{\alpha\beta}) \tag{26}$$

where

$$a = r^{\alpha\alpha}_{\alpha\alpha} = -\zeta(u - v) - \zeta(v), \qquad c_{\alpha\beta} = r^{\alpha\beta}_{\beta\alpha} = -Q(q_{\alpha\beta}, u - v), \tag{27}$$

$$d_{\alpha\beta} = r^{\alpha\alpha}_{\alpha\beta} = r^{\beta\alpha}_{\beta\beta} = -\frac{1}{2}Q(q_{\alpha\beta}, v). \tag{28}$$

Now for the type of ours described by (1), we have to define

$$L^{+(1)} \equiv L^{+} \otimes \mathbb{I}, \quad L^{+(2)} \equiv \mathbb{I} \otimes L^{+} \tag{29}$$

$$L^{-(1)} \equiv L^{-} \otimes \mathbb{I}, \quad L^{-(2)} \equiv \mathbb{I} \otimes L^{-} \tag{30}$$

The Poisson algebra of these operators are related to commutation relations of the r-matrix,

$$\begin{aligned} \{L^{+(1)}(u), L^{+(2)}(v)\} &= [r^{(12)}, L^{+(1)}(u)] - [r^{(21)}, L^{+(2)}(v)] && (31)\\ \{L^{-(1)}(u), L^{-(2)}(v)\} &= [r^{(12)}, L^{-(1)}(u)] - [r^{(21)}, L^{-(2)}(v)] && (32)\\ \{L^{+(1)}(u), L^{-(2)}(v)\} &= [r^{(12)}, L^{+(1)}(u)] - [r^{(21)}, L^{-(2)}(v)] + i\Pi, && (33) \end{aligned}$$

where

$$\Pi \equiv \sum_{\alpha\neq\beta} e_{\alpha\beta} \otimes e_{\beta\alpha} + \sum_{\gamma} e_{\gamma\gamma} \otimes e_{\gamma\gamma}. \tag{34}$$

From (31) (32) it immediately follows that the invariants $tr(L^{\pm})^k$ Poisson-commute when having the same $\pm$ gradation. From (33) one can deduce, not so straightforwardly, that the conserved quantities $tr(L^+L^-)^k$ Poisson-commute. More precisely one has

$$\begin{aligned} \{L^{+(1)}L^{-(1)}, L^{+(2)}L^{-(2)}\} = \; & [r^{(12)}L^{-(2)} + L^{+(2)}r^{(12)}, L^{+(1)}L^{-(1)}] \\ & -[L^{+(1)}r^{(21)} + r^{(21)}L^{-(1)}, L^{+(2)}L^{-(2)}] \\ & + i\, L^{+(2)}\Pi L^{-(1)} - i\, L^{+(1)}\Pi L^{-(2)}. \end{aligned} \tag{35}$$

Incidently the r-matrix structure in the first two terms of (35) is an example of a second Poisson structure obtained from the first structure by a Sklyanin-type bracket [10]. Hence in computing the brackets $\{tr(L^{+(1)}L^{-(1)})^n, tr(L^{+(2)}L^{-(2)})^m\}$ the contribution from the commutators vanish and one is left with

$$\begin{aligned} \{tr(L^{+(1)}L^{-(1)})^n, tr(L^{+(2)}L^{-(2)})^m\} = \; & +iL^{-(1)}(L^{+(1)}L^{-(1)})^{n-1}(L^{+(2)}L^{-(2)})^{m-1}L^{+(2)}\Pi \\ & -i(L^{+(1)}L^{-(1)})^{n-1}L^{+(1)}L^{-(2)}(L^{+(2)}L^{-(2)})^{m-1}\Pi \end{aligned} \tag{36}$$

The r.h.s. of (36) will change sign when $L^{+(1)} \leftrightarrow L^{-(1)}$ and $L^{+(2)} \leftrightarrow L^{-(2)}$. However, $tr(L^{+(1)}L^{-(1)})^n = tr(L^{-(1)}L^{+(1)})^n$ and $tr(L^{+(2)}L^{-(2)})^m = tr(L^{-(2)}L^{+(2)})^m$ by cyclicity and therefore the l.h.s. of (36) does not change sign. It follows that both sides must vanish. Hence the conserved quantities $tr(L^+L^-)^n$ are Poisson-commute quantities and the potential in (1) is integrable in the sense of Liouville.

3 Yang-Baxter equation

It is known that it is sufficient for a purely numeric r matrix to satisfy the Yang-Baxter equation [7],

$$[r^{(12)}, r^{(13)}] + [r^{(12)}, r^{(23)}] + [r^{(13)}, r^{(23)}] = 0, \tag{37}$$

if the Poisson bracket defined by (22) satisfies the Jacobi identity. The generalization of (37) for the r matrices, related to the system (1) but without an external harmonic potential, comes from the Jacobi identity

$$\{\{L^{(1)}, L^{(2)}\}, L^{(3)}\} + \{\{L^{(2)}, L^{(3)}\}, L^{(1)}\} + \{\{L^{(3)}, L^{(1)}\}, L^{(2)}\} = 0 \tag{38}$$

where

$$L^{(1)} \equiv L \otimes \mathbb{I} \otimes \mathbb{I}, \quad L^{(2)} \equiv \mathbb{I} \otimes L \otimes \mathbb{I}, \quad L^{(3)} \equiv \mathbb{I} \otimes \mathbb{I} \otimes L. \tag{39}$$

Using (22), then we obtain the equality [8][9]

$$[R^{(123)}, L^{(1)}] + [R^{(231)}, L^{(2)}] + [R^{(312)}, L^{(3)}] = 0, \tag{40}$$

where

$$R^{(123)} \equiv r^{(123)} - \{r^{(13)}, L^{(2)}\} + \{r^{(12)}, L^{(3)}\}, \tag{41}$$

$r^{(123)}$ being the left-hand-side of (37). $R^{(123)}$ satisfy (40) with the form [6]

$$R^{(123)} = [X^{(123)}, L^{(2)}] - [X^{(312)}, L^{(3)}], \tag{42}$$

where

$$\begin{aligned} X^{(123)} &\equiv X^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_1\beta_1}(u,v,w) & (43)\\ &= -i\sum_{\alpha\neq\beta} Q(q_{\alpha\beta}, w)\{-\frac{5}{8}e_{\alpha\alpha}\otimes e_{\alpha\alpha}\otimes e_{\alpha\beta} + \frac{1}{8}e_{\beta\beta}\otimes e_{\beta\beta}\otimes e_{\alpha\beta} \\ &\quad + \frac{1}{4}e_{\alpha\alpha}\otimes e_{\beta\beta}\otimes e_{\alpha\beta} + \frac{1}{4}e_{\beta\beta}\otimes e_{\alpha\alpha}\otimes e_{\alpha\beta}\}. & (44) \end{aligned}$$

Thus we have the following generalization of the Yang-Baxter equation for the r matrices

$$\begin{aligned} [r^{(12)}, r^{(13)}] &+ [r^{(12)}, r^{(23)}] + [r^{(13)}, r^{(23)}] \\ &- \{r^{(13)}, L^{(2)}\} + \{r^{(12)}, L^{(3)}\} \\ &- [X^{(123)}, L^{(2)}] + [X^{(312)}, L^{(3)}] = 0. \end{aligned} \tag{45}$$

Now the system (1) involving an external harmonic potential, the Lax operator L^+L^- will satisfy the Jacobi identity

$$\begin{aligned} &\{\{L^{+(1)}L^{-(1)}, L^{+(2)}L^{-(2)}\}, L^{+(3)}L^{-(3)}\} \\ +\ &\{\{L^{+(2)}L^{-(2)}, L^{+(3)}L^{-(3)}\}, L^{+(1)}L^{-(1)}\} \\ +\ &\{\{L^{+(3)}L^{-(3)}, L^{+(1)}L^{-(1)}\}, L^{+(2)}L^{-(2)}\} = 0. \end{aligned} \tag{46}$$

It is interesting to obtain the generalized Yang-Baxter equation by using (31)∼(33)to satisfy the equality

$$[\widetilde{R}^{(123)}, L^{+(1)}L^{-(1)}] + [\widetilde{R}^{(231)}, L^{+(2)}L^{-(2)}] + [\widetilde{R}^{(312)}, L^{+(3)}L^{-(3)}] = 0 \tag{47}$$

with the ansatz

$$\widetilde{R}^{(123)} = [\widetilde{X}^{123}, L^{+(2)}L^{-(2)}] - [\widetilde{X}^{(312)}, L^{+(3)}L^{-(3)}]. \tag{48}$$

However, to solve $\widetilde{X}^{(123)}$ is an intricate problem. We hope it may be obtained in the future.

4 Concluding Remarks

We have shown that the Lax operator L^+L^- possesses a classical r-matrix structure for the classical elliptic Calogero system in an external harmonic potential and discussed the relation of the r-matrix and the Yang-Baxter equation. There are many questions still waiting for an investigation. It is expected that some properties of the higher r matrices will hold. So far almost nothing is known about the quantum version of this model. .

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在外加簡諧位能橢圓 Calogero 模型中的 r 矩陣

婁祥麟　　蘇懿賢

摘 要

考慮外加簡諧位能，古典橢圓 Calogero 模型中，探討其 Lax 算符具有 r 矩陣的結構。並討論此 r 矩陣和 Yang-Baxter 方程式的關係。

關鍵字: 橢圓 Caloger 模型, r 矩陣