關於優美樹猜想之研究

Study of the Graceful Tree Conjecture

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東海大學應用數學研究所

研究生：楊振昌

指導教授：王道明

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Graduate: Cheng-Chang Yang

Adviser: Dr. Tao-Ming Wang

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Abstract

The well known Graceful Tree Conjecture(GTC) claimed that all trees are graceful, which still remains open until today. It was proved in 1999 by H. Broersma and C. Hoede that there is an equivalent conjecture for GTC that all trees containing a perfect matching is strongly graceful. In this thesis we verify by extending the above result that there exist infinitely many equivalent versions of the GTC. More precisely, for a fixed graceful tree T_k of order k, we show that for each $k \geq 2$, the conjecture that all trees containing a graceful T_k-factor is strongly T_k-graceful is equivalent to the conjecture that all trees are graceful. More applications are also included by way of identifying new classes of graceful graphs. In particular we verify infinitely many equivalent T_k-version conjectures of GTC for those trees of diameter no more than $2\lceil \frac{D(T_k)}{2} \rceil + 5$, where $D(T_k)$ is the diameter of T_k.

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Contents

Chapter 1

Introduction

1.1 Preliminary Background

Throughout this paper, by a graph we mean an undirected finite graph without multiple edges and loops. All terminologies and notations on graph theory not mentioned or defined here can be referred to the textbook by D. West [23]. Let G be a graph with q edges. A one-to-one function $f : V(G) \to \{0, 1, \cdots, q\}$ from the vertex set $V(G)$ (if any) is said to be a **graceful labeling** of G, if the absolute value $|f(u) - f(v)|$ is assigned to the edge uv as its label and the resulting edge labels are pairwise distinct. This is equivalent to requiring the set of induced edge labels is exactly $\{1, 2, \cdots, q\}$. A graph admitting such a graceful labeling is called a **graceful graph**. We focus on the graceful labeling of trees in this paper however. We reformulate the definition of graceful labeling for trees as follows:

Definition 1.1.1. A nontrivial tree T with p vertices and q edges is called **graceful** if there exists a bijection f from the vertex set $V(T)$ onto $\{1, 2, \cdots, p\}$ such that the induced edge labels are exactly $1, 2, \cdots, q$, where the induced edge label for an edge uv is the absolute value of the difference of two end vertex labels $|f(u) - f(v)|$.

Originally A. Rosa called a function f a β-valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \cdots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. S. W. Golomb subsequently [10] called such labelings graceful and this is now the popular term. A. Rosa introduced β-valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs. For example he defined an α-labeling (or α-valuation) as a graceful labeling with the additional property that there exists an integer k so that for each edge xy either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. (Other names for such labelings are balanced, interlaced, and strongly graceful.) Also, a graph with an α-labeling is necessarily bipartite and therefore can not contain a cycle of odd length.

In particular, β-valuations originated as a means of attacking the conjecture of Ringel [19] that K_{2n+1} can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with n edges. For this reason A. Rosa raised the **Graceful Tree Conjecture(GTC)** (which implies the conjecture of Ringel)that every tree is graceful [20], which is one of most challenging problems in graph theory and remains wide open until today.

Among the trees known to be graceful are: caterpillars (a caterpillar is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices; trees with diameter at most 5; symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree); rooted trees where the roots have odd degree and the lengths of the paths from the root to the leaves differ by at most one and all the internal vertices have the same parity; rooted trees with diameter D where every vertex has even degree except for one root and the leaves in level $\lfloor \frac{D}{2} \rfloor$; rooted trees with diameter D where every vertex has even degree except for one root and the leaves, which are in level $\lfloor \frac{D}{2} \rfloor$; rooted trees with diameter D where every vertex has even degree except for one root, the vertices in level $\lfloor \frac{D}{2} \rfloor - 1$, and the leaves which are in level $\lfloor \frac{D}{2} \rfloor$, etc. Among other results, in 1999

Broersma and Hoede [4] proved that the following conjecture is equivalent to the Graceful Tree Conjecture: Every tree containing a perfect matching is strongly graceful, where a tree T on n vertices is strongly graceful if T contains a perfect matching M and T admits a graceful labeling f such that $f(u) + f(v) = n + 1$ for every edge $uv \in M$. We generalize the above result to obtain infinitely many equivalent versions of GTC. In this article we make the following strongly T_k-graceful tree conjecture **(SGTC)** and prove that the **GTC** is equivalent to the **SGTC**.

Conjecture. Every nontrivial tree admitting a graceful T_k-factor is strongly T_k-graceful, where T_k is fixed tree of order $k \geq 2$ with graceful labeling.

The related terminology here, such as a graceful T_k-factor and strongly T_k-graceful-ness, will be defined in next chapter. We therefore obtain infinitely many equivalent versions of GTC, since T_k could be assigned to any given known graceful trees, say paths P_k, caterpillars, etc., in particular the case $T_k = P_2$ coincides with previous result of Broersma and Hoede. For more updated information about graceful graphs see the dynamic survey by J. Gallian [9].

1.2 Variants of β-Valuations by Rosa

In addition to graceful labeling, which Rosa called β-valuation, he also introduced three other kinds of valuations: α, σ, and ρ-valuation. In what follows we refer to them as labelings rather than valuations. We say that these four labelings are hierarchically related and we write that as $\alpha < \beta < \sigma < \rho$, where a-labeling is also a b-labeling if $a < b$. For example, a labeling that satisfies the requirements for α-labeling also qualifies as a β, σ and ρ-labeling, and every σ-labeling is also a ρ-labeling. Rosa defined these four labelings as follows [20]:

1. α-labeling of a graph G with n vertices and m edges is a one-to-one mapping f from the set of vertices of G to the set $\{0, 1, 2, \cdots, m\}$

such that all induced edge labels are distinct, where the induced edge label of the edge uv is $|f(u) - f(v)|$, and such that there exists a number $x \in \{0, 1, 2, \cdots, m\}$ such that for an arbitrary edge uv either $f(u) \leq x < f(v)$ or $f(v) \leq x < f(u)$.

2. β-labeling of a graph G with n vertices and m edges is a one-to-one mapping f from the set of vertices of G to the set $\{0, 1, 2, \cdots, m\}$ such that all induced edge labels are distinct, where the induced edge label of the edge uv is $|f(u) - f(v)|$. In 1972 S. W. Golomb independently introduced the same labeling and called it graceful.

3. σ-labeling of a graph G with n vertices and m edges is a one-to-one mapping f from the set of vertices of G to the set $\{0, 1, 2, \cdots, 2m\}$ such that all induced edge labels are distinct and in the range $\{0, 1, 2, \cdots, m\}$, where the induced edge label of the edge uv is $|f(u) - f(v)|$.

4. ρ-labeling of a graph G with n vertices and m edges is a one-to-one mapping f from the set of vertices of G to the set $\{0, 1, 2, \cdots, 2m\}$ such that the set of induced edge labels is $\{x_1, x_2, \cdots, x_m\}$, where $x_i = i$ or $x_i = 2m + 1 - i$.

1.3 Results of Broersma and Hoede

Broersma and Hoede proved in [4] that the conjecture that very tree containing a perfect matching is strongly graceful is equivalent to the Graceful Tree Conjecture. We recall their methods in this section.

Definition 1.3.1. The spike-tree $spik(T)$ of a tree T on n vertices is obtained by adding n new vertices to T along with n edges. The con-tree $con(T)$ of a tree T with a perfect matching M is obtained from T by contracting the edges of M.

The way Broersma and Hoede proved the equivalence of the two conjectures is by first using a graceful labeling of the con-tree of a tree with a perfect matching to show that a tree with a perfect matching is strongly graceful. This is done by finding vertex labels that sum to $n-1$ and have absolute difference as prescribed.

The other direction is proved by taking the spike-tree $Spik(T)$ of an arbitrary tree T. $Spik(T)$ is strongly graceful since it has a perfect matching, and has $2n$ vertices. Define the two sets matched by the perfect matching to be those containing odd and even labels. By giving labels to vertices of T according to half the even vertex label of the corresponding edge in the strong graceful labeling of $Spik(T)$, one produces a graceful labeling of T.

Another result that Broersma and Hoede obtained is the following:

Lemma 1.3.2. *Let T be a tree containing a perfect matching and let $con(T)$ be the con-tree of T. Then, T is strongly graceful if and only if the spike-tree $Spik(T)$ with con-tree T is strongly graceful.*

This result is that, in order to prove the Graceful Tree Conjecture, it is sufficient to show that every spike-tree is strongly graceful. However, as the authors observed, since a strongly graceful labeling of a spike-tree immediately yields a graceful labeling of its con- tree, this is not a substantial improvement. Moreover, the label 1 cannot be assigned to an arbitrary vertex of a spike-tree, the same obstacle that arises in attempts to prove the original Graceful Tree Conjecture.

Another result that Broersma and Hoede proved in [4] is the following:

Theorem 1.3.3. *Every tree containing a perfect matching and having a caterpillar as its con-tree is strongly graceful.*

Thus, trees meeting this hypothesis can be used to generate new strongly graceful trees with the original tree as their con-trees, by successively taking the spike-trees of a sequence of trees. This procedure generates what Broersma and Hoede called strongly graceful long-edged caterpillars.

Chapter 2

Main Results

2.1 Graceful Factors and Strongly Gracefulness

We first define related terminologies in order to describe our main results. Let $T_k = (V(T_k), E(T_k))$ be a fixed tree of order k with a given graceful labeling λ. Without loss of generality we may name the vertices by assuming $\lambda(v_j) = j$ for $v_j \in V(T_k)$ and $1 \leq j \leq k$, such that the differences $|\lambda(v_i) - \lambda(v_j)| = |i - j|$ are all distinct for $v_i v_j \in E(T_k)$. Assume that a tree T of order nk admits a T_k-factor, namely a spanning subgraph $T_k^* = T_k^1 \oplus T_k^2 \oplus \cdots \oplus T_k^n$, where $T_k^i \simeq T_k$ are vertex disjoint isomorphic copies of T_k for each $1 \leq i \leq n$. As in T_k we name the vertices in T_k^* in a similar fashion. Let $v_{ij} \in V(T_k^i)$ be the corresponding vertex of $v_j \in V(T_k)$ via the isomorphism of $T_k^i \cong T_k$ for $1 \leq i \leq n$ and $1 \leq j \leq k$. Note that by the notation $d(H_1, H_2) = min\ d(h_1, h_2)$ where $h_1 \in V(H_1)$ and $h_2 \in V(H_2)$, we denote the distance for two subgraphs H_1 and H_2 of G.

Definition 2.1.1. Let T_k be a fixed tree of order k with a given graceful labeling. We say a tree T of order nk admits a **graceful T_k-factor**, if it contains a T_k-factor as stated above, and also for each edge e in $E(T)-E(T_k^*)$ there exist T_k^s and T_k^t with $d(T_k^s, T_k^t) = 1$, where $1 \le s \ne t \le n$, and a unique j, $1 \le j \le k$, such that $e = v_{sj}v_{tj}$.

Remark. Note that in order for a tree T of order nk to admit a graceful T_k-factor, one simply requires that for each of the extra $n-1$ edges in $E(T) - E(T_k^*)$, their end vertices should be in the corresponding positions of both copies of T_k^i's.

In the following we define the strongly T_k-graceful-ness for a tree T with a graceful T_k-factor. We denote the graceful T_k-factor of T by $T_k^* = T_k^1 \oplus T_k^2 \oplus \cdots \oplus T_k^n$. Since trees are bipartite, we may assume the bi-partitions of $V(T_k) = A \cup B$ and $V(T_k^i) = A_i \cup B_i$ for $1 \le i \le n$, where $A_i = \{v_{ij} \in V(T_k^i) : \ v_j \in A\}$ contains corresponding vertices with that in A, and $B_i = \{v_{ij} \in V(T_k^i) : \ v_j \in B\}$ contains corresponding vertices with that in B. Obviously $|A_i| = |A|$ and $|B_i| = |B|$ for each i.

Definition 2.1.2. Let T_k be a fixed tree of order k with a given graceful labeling and T be a tree of order nk with a graceful T_k-factor, where the graceful T_k-factor of T is denoted as above by $T_k^* = T_k^1 \oplus T_k^2 \oplus \cdots \oplus T_k^n$. Also let the bi-partition of $V(T_k^i) = A_i \cup B_i$ for $1 \le i \le n$ be as above. Then we say a bijection f (vertex labeling) from the vertex set $V(T)$ onto $\{1, 2, \cdots, |V(T)|\}$ is a **strongly T_k-graceful labeling** of T if (1) f satisfies the following conditions:

$$f(v_{ij}) = j + (i-1)k, \text{ if } v_{ij} \in A_i,$$

$$f(v_{ij}) = j + (n-i)k, \text{ if } v_{ij} \in B_i.$$

for $v_{ij} \in V(T_k^i) = A_i \cup B_i$, where $1 \le i \le n$ and $1 \le j \le k$. And (2) f is a graceful labeling.

Remark. Note that $\bigcup_{i=1}^{n}(A_i \cup B_i)$ is exactly $\{1, 2, \cdots, nk\}$.

Definition 2.1.3. We say a tree T' of order n is a **contraction tree** of the tree T of order nk admitting a graceful T_k-factor T_k^*, if it can be obtained by using the T_k^i copies as vertices of T', and two copies of T_k^s and T_k^t are adjacent if $d(T_k^s, T_k^t) = 1$ for $1 \le s, t \le n$.

Remark. Note that we contract edges in one copy of T_k^i to one vertex for each i to form the contraction tree. Also the edges of the contraction tree T' are in one-to-one correspondence with the edges in $E(T) - E(T_k^*)$.

2.2 Infinitely Many Equivalences

We then have the following result:

Theorem 2.2.1. *Let T be a tree of order nk admitting a graceful T_k-factor for a given graceful tree T_k of order k. If the contraction tree T' of the tree T is graceful, T is strongly T_k-graceful.*

Proof.

With notations defined above, let f be a vertex labeling of T such that $f(v_{ij}) = \lambda(v_j) + (i-1)k = j + (i-1)k$, if $v_{ij} \in A_i$, and $f(v_{ij}) = \lambda(v_j) + (n-$

$i)k = j + (n-i)k$, if $v_{ij} \in B_i$ for $1 \le i \le n$ and $1 \le j \le k$. To show f is strongly T_k-graceful, it suffices to show that f is graceful.

Let the graceful labeling of T' be λ', and identify $u_i \in V(T')$ with T_k^i via $\lambda'(u_i) = i$ for each $1 \le i \le n$. For an edge $u_i u_j \in E(T')$ is in one-to-one correspondence with some unique edge $v_{i,m}v_{j,m} \in E(T) - E(T_k^*)$ where $v_{i,m} \in V(T_k^i)$ and $v_{j,m} \in V(T_k^j)$. Note that $v_{i,m}$ and $v_{j,m}$ are either simultaneously in A_i and A_j respectively, or simultaneously in B_i and B_j respectively. Thus for edges in $E(T) - E(T_k^*)$, the induced edge labels $|f(v_{i,m}) - f(v_{j,m})| = k|i-j|$ are distinct, since $|\lambda'(u_i) - \lambda'(u_j)| = |i-j|$ are distinct for edges $u_iu_j \in E(T')$. Also note that particularly the induced edge labels $|f(v_{i,m}) - f(v_{j,m})|$ in $E(T) - E(T_k^*)$ are multiples of k.

On the other hand, we consider the remaining edge labels for edges in $E(T_k^*)$. Assume $v_{i,a}v_{i,b} \in E(T_k^i)$ and $v_{j,c}v_{j,d} \in E(T_k^j)$, where $v_{i,a} \in A_i$, $v_{i,b} \in B_i$, $v_{j,c} \in A_j$, and $v_{j,d} \in B_j$, for $1 \le i,j \le n$ and $1 \le a,b,c,d \le k$. We prove in the following that the induced edge labels are all distinct over the edges in T_k^*. Assume $(i,a,b) \ne (j,c,d)$. Suppose on the contrary that $|f(v_{i,a}) - f(v_{i,b})| = |f(v_{j,c}) - f(v_{j,d})|$. Then there are two cases:

Case 1: $(n-2i+1)k + (b-a) = (n-2j+1)k + (d-c)$.
$\Rightarrow 2(j-i)k = -(b-a) + (d-c)$.
$\Rightarrow 2(j-i)k = -(b-a)+(d-c) = 0$ (since $-2(k-1) \le -(b-a)+(d-c) \le 2(k-1)$.)
$\Rightarrow i = j$, $a = c$, $b = d$ since $T_k \cong T_k^i \cong T_k^j$ is graceful. A contradiction.

Case 2: $(n-2i+1)k + (b-a) = -[(n-2j+1)k + (d-c)]$.
$\Rightarrow 2(n-j-i+1)k = -[(b-a)+(d-c)]$.
$\Rightarrow 2(n-j-i+1)k = -[(b-a)+(d-c)] = 0$ (since $-2(k-1) \le -[(b-a)+(d-c)] \le 2(k-1)$.)
However $(b-a)+(d-c) = 0$ is impossible, since if $a = c$ and $b = d$, then $2(b-a) = 0$, a contradiction because $v_a \in A$ and $v_b \in B$. Otherwise $a \ne c$ or $b \ne d$, that is v_av_b and v_cv_d are two distinct edges in T_k, hence

$|a-b| \neq |c-d|$ because T_k is graceful. Therefore $(b-a)+(d-c)$ cannot be 0.

Therefore $|f(v_{i,a})-f(v_{i,b})| \neq |f(v_{j,c})-f(v_{j,d})|$ whenever $(i,a,b) \neq (j,c,d)$, that is, the induced edge labels are all distinct for edges in $E(T_k^*)$. Furthermore, it is not hard to see that these induced edge labels are not multiples of k, since $|f(v_{i,a}) - f(v_{i,b})| = |(a-b) + (2i-n-1)k|$ and $-(k-1) \leq (a-b) \leq (k-1)$. Combining all cases above, we see that all induced edge labels are all distinct, that is, f is a graceful labeling, hence a strongly T_k-graceful labeling. □

Conversely we have the following:

Theorem 2.2.2. *Let T be a tree of order nk admitting a graceful T_k-factor for a given graceful tree T_k of order k. If T is strongly T_k-graceful, then the contraction tree T' of the tree T is graceful.*

Proof. Let f be the strongly T_k-graceful labeling of T defined as above. Identify T_k^i with a vertex u_i in T' for each $1 \leq i \leq n$ and assign the label i to the vertex u_i in T', via $\lambda(u_i) = i$. As above for an edge $u_iu_j \in E(T')$ is in one-to-one correspondence with some unique edge $v_{i,m}v_{j,m} \in E(T) - E(T_k^*)$ where $v_{i,m} \in V(T_k^i)$ and $v_{j,m} \in V(T_k^j)$. Thus the induced edge label $|f(v_{i,m}) - f(v_{j,m})| = k|i-j|$ are all distinct since f is strongly T_k-graceful. Therefore $|i-j| = |\lambda(u_i) - \lambda(u_j)|$ are all distinct, hence λ is graceful. □

Here we are in a position to state our main result:

Theorem 2.2.3. *The Graceful Tree Conjecture* **GTC** *is equivalent to the Strongly Graceful Tree Conjecture* **SGTC** *for $k \geq 2$.*

Proof.
(**GTC** $\Rightarrow$ **SGTC**)
Assume T is an arbitrary tree with a graceful T_k-factor. Then its contraction

tree T' is a tree, which is graceful by **GTC**. Therefore by Theorem 2.2.1, we see T is strongly T_k-graceful.

(**SGTC** $\Rightarrow$ **GTC**)

Assume T is an arbitrary tree. Consider the extension $\widetilde{T}$ of the tree T by attaching one copy of T_k at each vertex of T on the same corresponding positions, say attaching the first (with the vertex order determined by the graceful-ness) vertex of T_k if one will, which makes $\widetilde{T}$ to be a tree with a graceful T_k-factor, and its contraction tree is T. By **SGTC** the extension $\widetilde{T}$ admits a strongly T_k-graceful labeling f, thus the contraction tree T is also graceful by Theorem 2.2.2. □

Example 2.2.4. When $T_k = P_2$, the above Theorem reduces to previous result of Broersma and Hoede [4], which is the equivalence of the GTC and the conjecture that every tree with a perfect matching is strongly graceful.

Chapter 3

Applications

3.1 Graceful m-Distance Trees

In this section we make use of above results to identify new classes of strongly T_k-graceful trees, hence graceful graphs. First we need the following definition:

Definition 3.1.1. Let m be a non-negative integer. A tree T is called an m**-distance tree** if it becomes a path after at least m recursive steps of leaf removal, where one step of leaf removal for a tree T means removing all leaves from T.

Remark. Therefore a 0-distance tree is a path, a 1-distance tree is a caterpillar (not a path), and a 2-distance tree is a lobster (neither a caterpillar nor a path).

Lemma 3.1.2. *Let T_k be a graceful tree with diameter $D(T_k)$, and T be a m-distance tree with a graceful T_k-factor. Let T' be the contraction m'-distance tree of T . Then $m \geq \lceil \frac{D(T_k)}{2} \rceil + m'$.*

Proof. Note that it takes at least $\lceil \frac{D(T_k)}{2} \rceil + 1$ steps of removing leaves from T, in order to remove one leaf of T'. Then we see that for T with a graceful T_k-factor, one needs at least $\lceil \frac{D(T_k)}{2} \rceil + 1 + (m' - 1) = \lceil \frac{D(T_k)}{2} \rceil + m'$ steps of removing leaves to make T become a path. Then $m \geq \lceil \frac{D(T_k)}{2} \rceil + m'$. □

Theorem 3.1.3. *Let T_k be a graceful tree with diameter $D(T_k)$, and T be an m-distance tree with a graceful T_k-factor. If T is an m-distance tree for $m \leq \lceil \frac{D(T_k)}{2} \rceil + 1$, then T is strongly T_k-graceful.*

Proof. It suffices to show that the contraction tree T' of the m-distance tree T is either 0 or 1-distance tree, since paths or caterpillars are known graceful, then by Theorem 2.2.1 we are done. Therefore we assume that the contraction tree T' is a m'-distance tree for $m' \geq 2$. By Lemma 3.1.2, one needs at least $\lceil \frac{D(T_k)}{2} \rceil + m' \geq \lceil \frac{D(T_k)}{2} \rceil + 2$ steps of removing leaves to make T become a path. This is a contradiction since T is an m-distance tree for $m \leq \lceil \frac{D(T_k)}{2} \rceil + 1$. Therefore we are done. □

Example 3.1.4. When $T_k = P_2$, and note also that $D(P_2) = 1$, the above theorem implies that all lobsters with a perfect matching is graceful, a result previously shown in [17].

3.2 Strongly Graceful Trees with Bounded Diameters

In 2009 B. Yao et al. [25] showed that all trees admitting a perfect matching (that is admitting a graceful P_2-factor) of diameter $D \leq 5$ are strongly graceful. We improve the result with the following:

Theorem 3.2.1. *All trees admitting a perfect matching of diameter $D \leq 7$ is strongly graceful.*

In fact we prove a more general situation as follows, which verifies the Strongly T_k-Graceful Tree Conjecture (SGTC) for such trees with diameter no more than a bound determined by the diameter of T_k:

Lemma 3.2.2. *Let T be a tree with a graceful T_k-factor, and with diameter $D(T)$. Let T' be the contraction tree of T with diameter $D(T')$. Then $D(T) \leq 2\lceil \frac{D(T_k)}{2} \rceil + D(T')$.*

Proof. There exists a path P of length $D(T')$ in T'. Consider the pull back of the path P in T, its length is at least $2\lceil\frac{D(T_k)}{2}\rceil + D(T')$, where the part $2\lceil\frac{D(T_k)}{2}\rceil$ is contributed by looking at the pull back of the two end vertices (two copies of T_k's) of the path P. Then $D(T) \leq 2\lceil\frac{D(T_k)}{2}\rceil + D(T')$. □

Theorem 3.2.3. *Let T be a tree with a graceful T_k-factor, and with diameter $D(T)$. Then T is strongly T_k-graceful if $D(T) \leq 2\lceil\frac{D(T_k)}{2}\rceil + 5$.*

Proof.

It suffices to show by Theorem 2.2.1 that the diameter of the contraction tree T' satisfies $D(T') \leq 5$ since all trees of diameter no more than 5 is graceful as proved in [12]. Suppose that $D(T') \geq 6$, then by Lemma 3.2.2 $D(T) \geq 2\lceil\frac{D(T_k)}{2}\rceil + 6$, a contradiction. □

Remark. Note that in case of $T_k = P_2$, the diameter $D(P_2) = 1$. Thus Theorem 3.2.3 reduces to Theorem 3.2.1, which generalizes previous result in [25].

3.3 α-Labeling and α-Factor

It is known that α-labeling is a stronger graceful labeling with additional balanced property. We see that an **α-labeling** of a tree T of order k is a graceful labeling f with the additional property that there exists an integer m, $1 \leq m \leq k$, so that for the bipartition of $V(T) = A \cup B$ one has that $f(A) = \{1, 2, \cdots, m\}$ and $f(B) = \{m+1, \cdots, k\}$.

Definition 3.3.1. A tree T is **equitable** if for the bipartition of $V(T) = A \cup B$ one has that $||A| - |B|| \leq 1$.

We call a graceful T_k-factor to be an α**-factor** if T_k is admitting an α-labeling. Then one has the following result to tell when a tree with an α-factor admits an α-labeling:

Theorem 3.3.2. *Let T_k be a fixed tree of order k admitting an α-labeling. Assume that T is a tree of order nk with a graceful T_k-factor (an α-factor), and its contraction tree T' of order n admits an α-labeling. Let f be the associated strongly T_k-graceful labeling. Then T' is equitable if and only if f is an α-labeling.*

Proof. With notations defined above, let f be a vertex labeling of T such that $f(v_{ij}) = \lambda(v_j) + (i-1)k = j + (i-1)k$, if $v_{ij} \in A_i$, and $f(v_{ij}) = \lambda(v_j) + (n-i)k = j + (n-i)k$, if $v_{ij} \in B_i$ for $1 \leq i \leq n$ and $1 \leq j \leq k$. Note that the bi-partition of $V(T_k^i)$ is $A_i \cup B_i$. Also let the graceful labeling of T' be λ', and identify $u_i \in V(T')$ with T_k^i via $\lambda'(u_i) = i$ for each $1 \leq i \leq n$.

Let the bi-partitions of $V(T_k)$ and $V(T')$ be $A \cup B$ and $A' \cup B'$ respectively. Since T_k and T' both admit an α-labeling, we assume λ and λ' be their graceful labelings respectively, and also there are two constants k_1 and k_2 such that $\{\lambda(u) |\ u \in A\} = \{1, 2, \cdots, k_1\}$, $\{\lambda(u) |\ u \in B\} = \{k_1 + 1, k_1 + 2, \cdots, k_1 + k\}$, and $\{\lambda'(u) |\ u \in A'\} = \{1, 2, \cdots, k_2\}$, $\{\lambda'(u) |\ u \in B'\} = \{k_2+1, k_2+2, \cdots, n\}$ respectively. Without loss of generality, one may assume $|A'| \geq |B'|$.

With notations defined here and that mentioned before, we see $\lambda'(u_i) = i \leq k_2$ for $u_i \in A'$ corresponding to T_k^i, hence

$$f(v_{ij}) = j + (i - 1)k \leq k_1 + (k_2 - 1)k, \text{if } v_{ij} \in A_i, \tag{3.3.1}$$

$$f(v_{ij}) = j + (n - i)k \geq (k_1 + 1) + (n - k_2)k, \text{if } v_{ij} \in B_i. \tag{3.3.2}$$

Also we see $\lambda'(u_i) = i \geq k_2 + 1$ for $u_i \in B'$ corresponding to T_k^i, hence

$$f(v_{ij}) = j + (i - 1)k \geq k_2 k + 1, \text{if } v_{ij} \in A_i, \tag{3.3.3}$$

$$f(v_{ij}) = j + (n - i)k \leq (n - k_2)k, \text{if } v_{ij} \in B_i. \tag{3.3.4}$$

From above inequalities (3.3.1), (3.3.2), (3.3.3), and (3.3.4), we have the following:

T' is equitable

$\Longleftrightarrow 0 \leq |A'| - |B'| \leq 1.$

$\Longleftrightarrow 0 \leq k_2 - (n - k_2) \leq 1.$

$\Longleftrightarrow n - k_2 \geq k_2 - 1$ and $k_2 \geq n - k_2$.

$\Longleftrightarrow max\{k_1 + (k_2 - 1)k, (n - k_2)k\} < min\{(k_1 + 1) + (n - k_2)k, k_2k + 1\}$.

$\Longleftrightarrow max\{f(v_{ij})|\ v_{ij} \in A^*\} < min\{f(v_{ij})|\ v_{ij} \in B^*\}$.

$\Longleftrightarrow T$ admits an α-labeling.

where $A^* = \{v_{ij}|\ (v_{ij} \in A_i) \wedge (u_i \in A')$, or $(v_{ij} \in B_i) \wedge (u_i \in B'), 1 \leq i \leq n, 1 \leq j \leq k\}$ and $B^* = \{v_{ij}|\ (v_{ij} \in B_i) \wedge (u_i \in A')$, or $(v_{ij} \in A_i) \wedge (u_i \in B'), 1 \leq i \leq n, 1 \leq j \leq k\}$. □

Chapter 4

Concluding Remarks

4.1 Summary of Results

In this thesis we give infinitely many equivalent versions of the Graceful Tree Conjecture. Precisely, with a fixed graceful tree T_k one may define the graceful T_k-factor. Then we prove that a tree with a graceful T_k-factor is strongly T_k-graceful if and only if its contraction tree is graceful. Using the above main result it is easy to identify new classes of graceful graphs.

4.2 Further Studies

It would be interesting to explore and identify more related concepts and relationships among them. For example, it is nice trying to figure out the specific situations when a strongly T_k-graceful graph admits the α-valuations, σ-valuations, ρ-valuations, which are hierarchically related to the graceful labelings (β-valuations). Also like many authors studied variants of the graceful labelings in literature, one may also study similar variants for strongly T_k-graceful labelings.

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