圖形之魔方與反魔方
形態標號問題研究

On Graph Labeling Problems
of
Magic and Anti-magic Types

碩士論文

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On Graph Labeling Problems of Magic and Antimagic Types

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所提論文　On Graph Labeling Problems of Magic and Antimagic Types

（圖形之魔方與反魔方型態標號問題研究）

合於碩士班資格水準，業經本委員會評審通過，特此證明。

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中華民國　一〇二年　六月　十九日

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Abstract

Let $G = (V(G), E(G))$ be a finite simple graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. An **antimagic labeling** of G is a bijection from the set of edges to the set of integers $\{1, 2, \cdots, q\}$ such that the vertex sums are pairwise distinct, where the vertex sum at a vertex is the sum of labels of all edges incident to such vertex. A **vertex magic total labeling** is a bijection from $V(G) \cup E(G)$ to the set of integers $1, 2, \cdots, p+q$, with the property that, for every vertex u in $V(G)$, one has $f(u) + \sum_{uv \in E(G)} f(uv) = k$ for some constant k. On the other hand, for an undirected graph G, a **zero-sum flow** is an assignment of possibly repeated non-zero integers to the edges such that the sum of the values of all edges incident with each vertex is zero. In this thesis we study the above graph labeling problems of magic and antimagic types. In particular, we identify classes of graphs admitting antimagic labeling and vertex magic total labeling respectively, which generalize and extend previous results. We also consider zero-sum flow problems for the hexagonal graphs, for which infinite families of hexagonal grid graphs with small zero-sum flow numbers are presented.

Publication From This Thesis

[1] Tao-Ming Wang, Guang-Hui Zhang, **On Antimagic Labeling of Odd Regular Graphs**, the 23th International Workshop On Combinatorial Algorithms (IWOCA 2012, Krishnankoil, India), Lecture Notes in Computer Science (LNCS), 7643, pp. 162-168, 2012. (EI)

[2] Tao-Ming Wang, Guang-Hui Zhang, **Zero-Sum Flow Numbers of Hexagonal Grids**, presented in FAW-AAIM 2013, Dalian, China, June 26-28, Lecture Notes in Computer Science (LNCS) 7924, pp. 339-349, 2013. (EI)

[3] Tao-Ming Wang, Guang-Hui Zhang, **On Antimagic Labeling of Regular Graphs with Particular Factors**, to appear in a special issue of the Journal of Discrete Algorithms, 2013 (EI)

[4] Tao-Ming Wang, Guang-Hui Zhang, **A Note on E-super Vertex Magic Graphs**, submitted and under review, 2012.

[5] Tao-Ming Wang, Guang-Hui Zhang, **On Vertex Magic Total Labeling of Disjoint Union of Sun Graphs**, Utilitas Math. (SCI), accepted, 2013.

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Chapter 1

Introduction

A labeling of a graph is assigning labels to the vertices, edges or both vertices and edges. In most applications labels are positive (or nonnegative) integers, though in general real numbers could be used. In this thesis, we focus on edge labelings and total labelings(labelings on both vertices and edges). If the sums of labels of all edges incident with the vertex are all constant in certain sense, we call them **magic** labeling. And if the sums of labels of all edges incident with the vertices are pairwise distinct in certain sense, we call them **antimagic** labeling.

1.1 Magic Labeling

Let $G = (V(G), E(G))$ be a finite simple graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges, without isolated vertices or isolated edges. A vertex magic labeling is a bijection from $E(G)$ to the consecutive integers $1, 2, \cdots, q$, with the property that, for every vertex u in $V(G)$, one has $\sum\limits_{uv \in E(G)} f(uv) = k$ for some constant k.

Theorem 1.1.1. *[30] $K_{n,n}$ is magic for all $n \neq 2$.*

Theorem 1.1.2. *[30] If a bipartite graph G is decomposable into Hamilton cycles, then G is magic.*

Theorem 1.1.3. *[30] If a graph G is decomposable into two magic spanning subgraphs G_1 and G_2 is regular, then G is magic..*

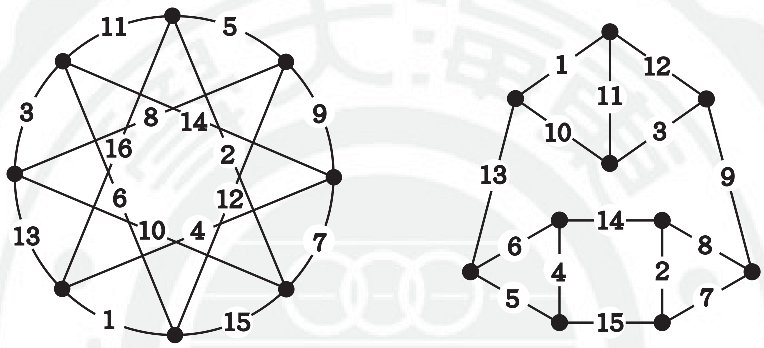

Figure 1.1.1: **vertex-magic labeling**

For more labelings of magic types, see Gallian's dynamic survey paper [15]. Note that in this thesis we deal with two types of magic labelings, namely vertex magic total labeling and zero-sum flows, which will discussed in later chapters.

1.2 Antimagic Labeling

An antimagic labeling of a finite simple undirected graph with q edges is a bijection from the set of edges to the set of integers $\{1, 2, \cdots, q\}$ such that the vertex sums are pairwise distinct, where the vertex sum at a vertex is the sum of labels of all edges incident to such vertex. A graph is called antimagic if it admits an antimagic labeling. It was conjectured by N. Hartsfield and G. Ringel in 1990 that all connected graphs besides K_2 are antimagic. Another weaker version of the conjecture is every regular graph is antimagic except

K_2. Both conjectures remain unsettled so far. Note that Cranston proved that all regular bipartite graphs are antimagic in 2009 (Regular bipartite graphs are antimagic, JGT, Vol. 60, Issue 3, pp. 173-182.).

Definition 1.2.1. For a graph $G = (V, E)$ with q edges and without any isolated vertex, an **antimagic** edge labeling is a bijection $f : E \to \{1, 2, \cdots, q\}$, such that the induced vertex sum $f^+ : V \to \mathbb{Z}^+$ given by $f^+(u) = \sum\{f(uv) : uv \in E\}$ is injective. A graph is called **antimagic** if it admits an antimagic labeling. If moreover for G the vertex sums form an arithmetic progression with initial term a and common difference d, we say G admits an (a, d)-**antimagic** labeling and G is (a, d)-**antimagic**.

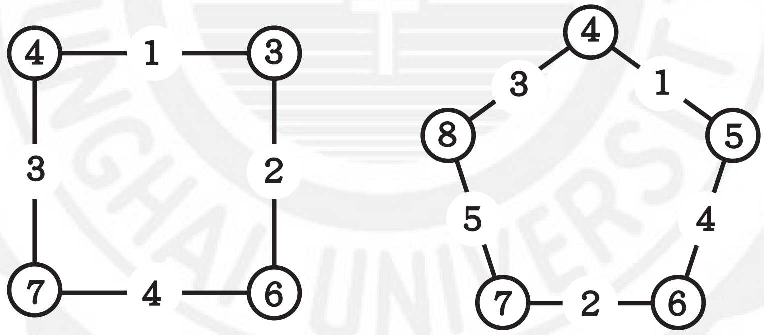

Figure 1.2.1: C_4 has **VAE** , C_5 has **(a,1)-VAE**

N. Hartsfield and G. Ringel showed that paths, cycles, complete graphs K_n $(n \geq 3)$ are antimagic. They conjectured that all connected graphs besides K_2 are antimagic, which remains open. In 2004 N. Alon et al [4] showed that the last conjecture is true for dense graphs using probabilistic method. They showed that all graphs with $n(\geq 4)$ vertices and minimum degree $\Omega(\log n)$ are antimagic. They also proved that if G is a graph with $n(\geq 4)$ vertices and the maximum degree $\Delta(G) \geq n-2$, then G is antimagic and all complete partite graphs except K_2 are antimagic. In 2005 D. Hefetz [19] proved that, among others, for $k \in \mathbb{Z}^+$ a graph G with 3^k vertices is antimagic if it admits a K_3-factor. In 2005, T.-M. Wang [38] studied antimagic labeling of sparse graphs, and showed that 2-regular graphs and moreover the toroidal grid graphs are antimagic. In 2008, T.-M. Wang and C.-C. Hsiao [39] showed various types of graph Cartesian product and lexicographic product (composition) are antimagic. Many various types of graphs have been shown to be antimagic [28, 6, 8, 10, 11, 19, 20, 46, 48] over the years. More variations of labelings of antimagic types, say (a, d)-antimagic labeling, edge-antimagic vertex labeling etc., can be referred to the dynamic survey by Gallian [15].

1.3 Applications

Some typical applications of labelings of magic types have been studied, mainly in network-related areas. Suppose it is required to assign addresses to the possible links in a communications network. It is required that the addresses are all different, and that the address of a link can be deduced from the identities of the two nodes linked, without the need of using a lookup table. This has been modeled using edge-magic labelings. Another application is in the construction of ruler models, which have been applied to the study of radar pulse codes. More details regarding these applications see [45].

Chapter 2

Antimagic Labeling of Odd Regular Graphs

Most of the contents in this chapter has been presented in the 23th International Workshop On Combinatorial Algorithms (IWOCA 2012, Krishnankoil, India. It was published as **On Antimagic Labeling of Odd Regular Graphs**, Lecture Notes in Computer Science (LNCS), 7643, pp. 162-168, 2012. Also another extended version has been published in **On Antimagic Labeling of Regular Graphs with Particular Factors**, Journal of Discrete Algorithms, 2013 (EI).

2.1 Introduction and Background

While Hartsfield-Ringel conjecture claims except K_2 all coneected simple graphs are antimagic, there is a weaker version conjectured that every regular graph except K_2 is antimagic. Among others, D. Cranston [12] proved that all regular bipartite graphs are antimagic in 2009. (Regular bipartite graphs are antimagic, Journal of Graph Theory, Vol. 60, Issue 3, pp. 173-182.) While some particular types of regular graphs have been shown to be antimagic, the conjecture for the antimagic-ness of regular graphs still remains unsettled till today. More recently we showed the antimagicness of certain classes of regular graphs with 1-factors and 2-factors, which contain examples such as all generalized Petersen graphs $P(n,k)$, certain Cayley graphs on $\mathbb{Z}_n$, and all powers of cycles. Also all Hamiltonian even regular graphs were shown antimagic. Very recently, Y. Liang and X. Zhu also showed that all cubic graphs are antimagic and Cartesian product related to regular graphs are antimagic[24, 25]. In this chapter, we show the existence of antimagic labeling for all even regular graphs with a 2-factor consisting of odd cycles only. For more conjectures and open problems on antimagic graphs and related type of graph labeling problems, readers are recommended to see the dynamic survey article of J. Gallian [14]. In this article, certain classes of regular graphs with particular type of 1-factors and 2-factors are shown antimagic. As a byproduct many well known examples are shown to be antimagic, such as all generalized Petersen graphs, all powers of cycles, and all even-regular circulant graphs (Cayley graphs of finite cyclic groups). Major results of this chapter can be summarized as follows among others:

- For odd regular graphs containing particular 3-factors:

  Theorem. All pseudo-prisms H are antimagic, where a pseudo-prism is a 3-regular graph which consists of the edge-disjoint union of a 1-factor and two $2k$-regular subgraphs of the same size. Moreover let K be any $2k$-factor. Then the odd $(2k+3)$-regular graph $G = H \oplus K$ is antimagic.

- For even regular graphs containing particular 2-factors:

  Theorem. Let G be a $2k$-regular, $k \geq 2$, Hamiltonian graph. Then G is antimagic.

  Theorem. Let G be a $2k$-regular graph $(k \geq 2)$ which contains a

2-factor F consisting of a vertex disjoint union of $(a,1)$-antimagic subfactors $H_1, H_2, \cdots, H_t$ of odd order, where $t \leq k$. Then G is antimagic.

Theorem. Let G be a $2k$-regular graph, $k \geq 2$. Assume that G contains a 2-factor $F = mC_n$ consisting of cycles of the same size. Then G is antimagic.

- For odd regular graphs containing particular odd claw factors, we show the following more general result regarding an odd graph (with odd degree vertices only):

Theorem. Let G be an odd graph formed by a mixed claw factor C and and an arbitrary 2-regular subfactor H over the pendant vertices of the odd claws in C, where C is consisting of vertex disjoint $K_{1,j}$'s for odd $j \geq 3$. Then G is antimagic, and also G remains antimagic after adding arbitrary $2k$-factors for $k \geq 1$. In particular if G is a $(2m+1)$-regular graph formed by an odd $K_{1,2m+1}$-factor and an arbitrary 2-regular subfactor over the pendant vertices of the odd claws, then G is antimagic.

2.2 Technical Preliminaries

In order to show the main results, we need the following facts. The first one is for assuring $(a,1)$-antimagic-ness while adding extra even factor to an $(a,1)$-antimagic graph, which was proved in [21] in 2006. We give the proof here for completeness:

Lemma 2.2.1. (J. Ivančo, A. Semaničová, 2006) *Assume H is a graph which arose from a graph G of p vertices and q edges by adding an arbitrary $2k$-factor. If G is $(a,1)$-antimagic, then H is $(a,1)$-antimagic, thus* *antimagic.*

Proof.

Without loss of generality let the $(a, 1)$-antimagic vertex sums for G be $a < a+1 < \cdots < a+p-1$ associated with the vertices $v_1, v_2, \cdots, v_p$ respectively while we labeling the edges $1, 2, \cdots, q$. By mathematical induction we need only to validate the situation while adding a 2-factor F to an $(a, 1)$-antimagic graph G. We proceed by assigning an orientation to the 2-factor F so that over each connected component (connected 2-cycle) the flow is either clockwise or counter-clockwise. Then we label over F by setting $f^{out}(w)$ and $f^{in}(w)$ respectively to be the outgoing edge label from the vertex $w \in V(G)$ and the incoming edge label to the vertex w, according to the given orientation. Precisely we give the labeling as follows:

$$f^{out}(v_i) = a + p + q - (a + i - 1)$$

for each $1 \le i \le p$. From the way $f^{out}(w)$ is defined, we see that the resulting vertex sum at the vertex v_i is

$$(a + i - 1) + f^{out}(v_i) + f^{in}(v_i) = a + p + q + f^{in}(v_i)$$

for each $1 \le i \le p$. Therefore the vertex sums are consecutive integers since the set of all outgoing edge labels is in one-to-one correspondence with the set of all incoming edge labels. Hence it is shown to be $(a, 1)$-antimagic and we are done.

□

However we may extend the above fact and obtain the following more general results:

Lemma 2.2.2. *Assume H is a graph of p vertices which arose from a graph G of p vertices by adding an arbitrary $2k$-factor. If the vertex sums of G by labeling edges $1, 2, \cdots, |E(G)|$ forms two non-overlapping consecutive sequences, say $a_1 < a_2 < \cdots < a_m < b_1 < b_2 < \cdots < b_n$. Then H is antimagic.*

Proof.

Let two non-overlapping consecutive sequences formed by the vertex sums via labeling edges $1, 2, \cdots, |E(G)|$ of G be $a_1 = a, a_2 = a+1, \cdots, a_m = a+m-1$ and $b_1 = b, b_2 = b+1, \cdots, b_n = b+n-1$ respectively, and assume $a_m < b_1$. Note that $m+n = |V(G)| = p$. We denote by $d = b_1 - a_m \geq 1$, and $D = b_n - a_1 + 1$ and note that $D = b_n - a_1 + 1 = b_1 + (n-1) - a_1 + 1 = a_m + d + (n-1) - a_1 + 1 = (m-1) + (n-1) + d + 1 = p + d - 1 \geq p$.

In order to apply the result in Lemma 5.2.4, we modify the $a_1, a_2, \cdots, a_m$ by $\widehat{a_i} = a_i + D$ for each $i = 1, \cdots, m$. Therefore $\widehat{a_i} = a_i + D = (a_1 + i - 1) + p + d - 1 = b_n + i$ for each i, thus $b_1, b_2, \cdots, b_n, \widehat{a_1}, \widehat{a_2}, \cdots, \widehat{a_m}$ are a sequence of consecutive integers. So G is temporarily $(a, 1)$-antimagic under the modified (fake) vertex sums. Then by Lemma 5.2.4, after adding an arbitrary $2k$-factor to G the resulting graph H still admits an $(a, 1)$-antimagic labeling and the vertex sums are pairwise distinct. By abusing the language we again denote the (fake) vertex sums $b_1, b_2, \cdots, b_n, \widehat{a_1}, \widehat{a_2}, \cdots, \widehat{a_m}$. Then it is clear that the original vertex sums $a_1, a_2, \cdots, a_m$ and $b_1, b_2, \cdots, b_n$ are pairwise distinct since $\widehat{a_i} = a_i + D$ for each i and note that $D \geq p$. Hence H is antimagic.

□

Even general we may have the following with similar discussion and mathematical induction:

Lemma 2.2.3. *Assume H is a graph of p vertices which arose from a graph G of p vertices by adding an arbitrary $2k$-factor. If the vertex sums of G by labeling edges $1, 2, \cdots, |E(G)|$ forms three non-overlapping consecutive sequences, say $a_1 < a_2 < \cdots < a_m < b_1 < b_2 < \cdots < b_n < c_1 < c_1 < \cdots < c_k$, and moreover either the gap $b_1 - a_m \geq p$ or the gap $c_1 - b_n \geq p$. Then H is antimagic.*

Lemma 2.2.4. *Assume H is a graph of p vertices which arose from a graph*

G of p vertices by adding an arbitrary $2k$-factor. If the vertex sums of G by labeling edges $1, 2, \cdots, |E(G)|$ forms t non-overlapping consecutive sequences, say $a_{1,1} < a_{1,2} < \cdots < a_{1,m_1} < a_{2,1} < a_{2,2} < \cdots < a_{2,m_2} < \cdots < a_{t,1} < a_{t,2} < \cdots < a_{t,m_t}$. Note that $\sum_{m_j} = p$ and each gap $a_{j+1,1} - a_{j,m_j} > p$ for every $1 \leq j \leq t-1$. Then H is antimagic.

2.3 Odd Regular Graphs With Particular 3-Factors

Note that there is a special class of 3-regular graphs which is called generalized Petersen graphs, for which we define as follows:

Definition 2.3.1. Let n, k be integers such that $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. The generalized Petersen graph $GP(n,k)$ is defined by $V(GP(n,k)) = \{u_i, v_i \mid 1 \leq i \leq n\}$, and $E(GP(n,k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \mid 1 \leq i \leq n\}$ where the subscripts are taken modulo n. (See **Figure 2.3.1**) We call $u_1, u_2, \cdots, u_n$ an outer cycle, and $v_1, v_2, \cdots, v_n$ an inner cycle.

In 2000, M. Miller and M. Bača studied antimagic labelings of arithmetic type for generalized Petersen graphs [28], which are referred as (a,d)-antimagic labelings. Note that (a,d)-antimagic labelings are requiring all vertex sums form an arithmetic progression, hence also antimagic. M. Miller and M. Bača showed (a,d)-antimagic-ness of $GP(n,2)$ for certain n, and also listed conjectures for other generalized Petersen graphs. In this section we show all generalized Petersen graphs are antimagic by proving a more general theorem regarding 3-regular graphs with a particular type of perfect matchings, which contain generalized Petersen graphs as special cases.

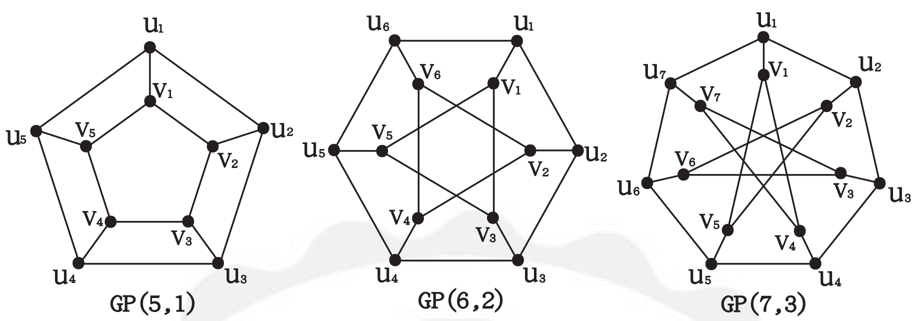

Figure 2.3.1: Examples of generalized Petersen graphs

Note that a r-factor of a graph is a r-regular spanning subgraph, and a 1-factor is a perfect matching. A factorization of a graph is a decomposition of the graph into union of factors so that the edge set is partitioned. Note that furthermore we call a r-regular subgraph of a factor to be a r-**subfactor**. In 2012 we have the following general result for antimagic-ness of 3-regular graphs and odd regular graphs [46], which was presented in the IWOCA 2012 conference held in India:

Theorem 2.3.2. *Let G be 3-regular with $2n$ vertices $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and $M = \{u_iv_i|\ 1 \le i \le n\}$ be a perfect matching of G. Assume additionally that $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ induce two 2-subfactors of the same order respectively. Then G is antimagic.*

Theorem 2.3.3. *Let $k \ge 1$ and let G be a $(2k+1)$-regular graph with $2n$ vertices $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and $M = \{u_iv_i|\ 1 \le i \le n\}$ be a perfect matching of G. Assume additionally that $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ induce two $2k$-regular subgraphs respectively. Then G is antimagic.*

Note that all generalized Petersen graphs $GP(n, k)$ with $V(GP(n, k)) =$

$\{u_i, v_i| \ 1 \leq i \leq n\}$ and $E(GP(n,k)) = \{u_iu_{i+1}, u_iv_i, v_iv_{i+k}| \ 1 \leq i \leq n\}$, are 3-regular with $2n$ vertices, $3n$ edges, and admitting perfect matchings $\{u_iv_i| \ 1 \leq i \leq n\}$. Obviously $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ induce two 2-regular subgraphs respectively. Therefore, as a byproduct of the above Theorem 2.3.2:

Corollary 2.3.4. *Every generalized Petersen graph $GP(n,k)$ is antimagic.*

Note that most recently Y.-C. Liang and X. Zhu showed that all cubic graphs are antimagic[24] (newly online in 2013). In the following we extend previous Theorem 2.3.2 and Theorem 2.3.3 to a more general situation for regular graphs of odd degree with particular 3-factor. First we state a well-known result we need here and also in later sections:

Theorem 2.3.5. (J. Petersen, 1891) *Let G be a $2r$-regular graph. Then there exists a 2-factor in G.*

Now we are in a position to prove the main result of this section:

Theorem 2.3.6. *Let G be an odd regular graph on $2n$ vertices $\{u_1, \cdots, u_n, v_1, \cdots, v_n\}$ with factorization $G = G' \bigoplus H$, where G' is an even-factor and H is a 3-factor consisting of the edge disjoint union of a 1-factor $M = \{u_iv_i| \ 1 \leq i \leq n\}$ and two 2-regular subgraphs H_1 and H_2 which are induced by $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_n\}$ respectively. Then G is antimagic.*

Proof. Let $H = M \bigoplus (H_1 \cup H_2)$, where H_1 and H_2 are two 2-regular subgraphs, each induced by n vertices, $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ respectively. On the other hand by Petersen's Theorem 4.1.2, G' can be factored as sum of 2-factors $F_1 \oplus F_2 \oplus \cdots \oplus F_k$.

Now we give an antimagic labeling f by the following steps. Note that G has $(2k+3)n$ edges. First we split all edge labels $1, 2, \cdots, (2k+3)n$ into

$2k+1$ groups as follows: $\{1,2,\cdots,n\},\{n+1,n+2,\cdots,2n\},\{2n+1,2n+2,\cdots,3n\},\cdots,\{(2k+2)n+1,(2k+2)n+2,\cdots,(2k+3)n\}$. Then we will put these groups of labels in order over the edges of H_1, M, H_2, $F_1,\cdots$, F_k respectively in below.

First we label the edges of M via $f(u_iv_i)=n+i$ for each $1\le i\le n$. Then labeling over edges of H_1 and H_2 as follows. Since H_1 and H_2 are 2-regular graphs, we assign an orientation so that over each connected component (connected 2-cycle) the flow is either clockwise or counter-clockwise. We label over H_1 and H_2 by setting $f^{out}(w)$ and $f^{in}(w)$ respectively to be the outgoing edge label from the vertex w and the incoming edge label to the vertex w, according to the given orientation. Precisely we give the labeling as follows:

$$f^{out}(u_i)=2n+1-(n+i),f^{out}(v_i)=4n+1-(n+i)$$

for each $1\le i\le n$. From the way $f^{out}(w)$ is defined, we see that over the 3-factor the partial vertex sums at $\{u_1,u_2,\cdots,u_n\}$ and $\{v_1,v_2,\cdots,v_n\}$ are uniquely determined by $f^{in}(w)$, and form consecutive integers respectively as $A_i=2n+1+i$ and $B_i=6n+1+i$ for $1\le i\le n$. Now we modify these two sequences of consecutive integers into one single sequence of consecutive integers, by letting $\widehat{B_i}=B_i-3n=3n+1+i$. Then we see the fake vertex sums $A_1,A_2,\cdots,A_n,\widehat{B_1},\widehat{B_2},\cdots,\widehat{B_n}$ are combined into one sequence of consecutive integers since $A_n+1=\widehat{B_1}$, that is, for the time being it is $(a,1)$-antimagic. We now may apply Lemma 5.2.4 to add the labeling of the rest of 2-factors $F_1,\cdots,F_k$ and keep the resulting (fake) vertex sums to be consecutive integers. By abusing language we still denote the vertex sums by A_i,B_i and $\widehat{B_i}$. Note that the order of these (fake) vertex sums might be different. After recovering the original vertex sums via $B_i=\widehat{B_i}+3n$ for $i=1$ to n, we claim the (true) vertex sums are pairwise distinct. To show the claim is true, we may first see $A_1=\sigma_1\le A_i\le\widehat{B_n}=\sigma_p$ and $A_1=\sigma_1\le\widehat{B_i}\le\widehat{B_n}=\sigma_p$ for each i. Then $\sigma_1+3n\le\widehat{B_i}+3n=B_i\le\sigma_p+3n$ for each i. Let $[a,b]$ be the set of integers $\{t\mid a\le t\le b\}$. We conclude that for each i, the vertex sums A_i belong to $[\sigma_1,\sigma_p]$, B_i belong to $[\sigma_1+3n,\sigma_p+3n]$, and note that $[\sigma_1,\sigma_p]\cap[\sigma_1+3n,\sigma_p+3n]=\phi$ since $3n>2n=\widehat{B_n}-A_1-1$.

Therefore the vertex sums are pairwise distinct and we are done.

□

To obtain more examples, we consider the circulant graphs as follows:

Definition 2.3.7. A **circulant graph** $CIR_n(S)$ with n vertices, with respect to $S \subset \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$, is a graph with the vertex set $V(CIR_n(S)) = \{0, 1, 2, \cdots, n-1\}$, and the edge set is formed by the following rule:

$$E(CIR_n(S)) = \{ij : \ i - j \equiv \pm s \pmod{n}, \ s \in S\}.$$

Note that the circulant graph $CIR_n(S)$ is also called a **Cayley graph of the finite cyclic group** $\mathbb{Z}_n$ generated by S.

For example, $CIR_n(\{a\}) \cong C_n$, the connected n-cycle, if $gcd(n, a) = 1$. Moreover $CIR_n(\{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}) \cong K_n$, the complete graphs, and $CIR_{2n}(\{1, n\}) \cong$ the n-Möbius ladder graphs. Note that $CIR_n(S)$ is odd-regular if n is even and $\frac{n}{2} \in S$, is even-regular otherwise. Let $S = \{a_1, a_2, ..., a_m\} \subseteq \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$, it is not hard to see that $CIR_n(S) = \bigoplus_{i=1}^{m} CIR_n(\{a_i\})$ is a factorization of circulant graphs with respect to one point sets $\{a_i\}$.

Example 2.3.8. *Note that for odd $n \geq 5$, the circulant graphs $CIR_{2n}(\{a, b, n\})$, where $0 < a \neq b < n$ and $gcd(2n, a) = gcd(2n, b) = 2$, are examples of 5-regular graphs with perfect matchings, which satisfy the assumption in Theorem 2.3.3. Therefore $CIR_{2n}(\{a, b, n\})$ are antimagic. See Figure 2.3.2 for the example $CIR_{14}(\{4, 6, 7\})$.*

In a similar fashion, we may construct an infinite class of circulant graphs which represent the class of odd $(2r+1)$-regular graphs, for each $r \geq 2$, with perfect matchings, as stated in Theorem 2.3.3:

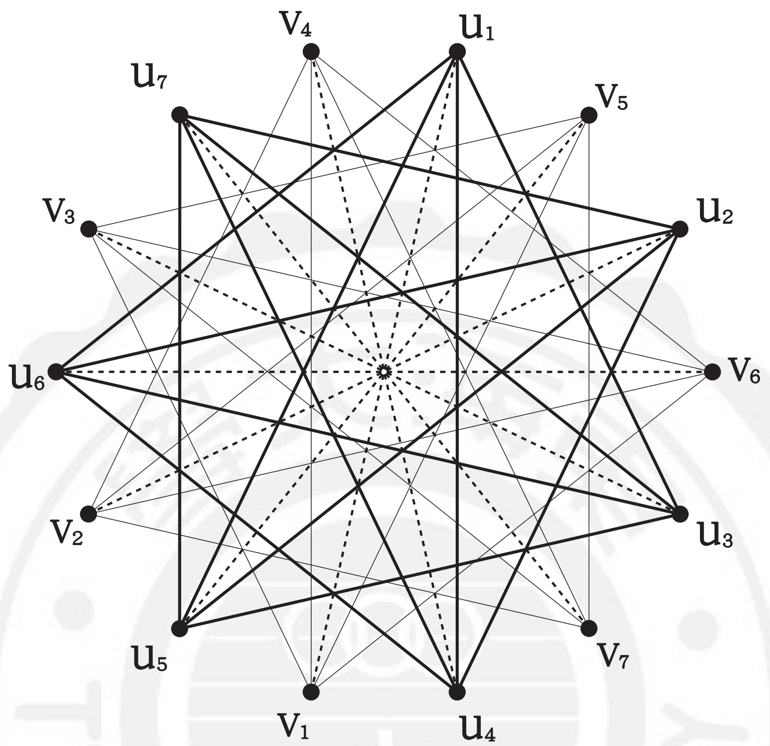

Figure 2.3.2: The Cayley graph $CIR_{14}(\{4, 6, 7\})$ of $\mathbb{Z}_{14}$

Example 2.3.9. *Let* $G = CIR_n(\{a_1, a_2, \cdots, a_m, \frac{n}{2}\})$ *be a circulant graph of even order* n*. By Theorem 2.3.3, it can be seen* G *is antimagic if* $\frac{n}{2}$ *is odd and* $gcd(n, a_i) = 2$ *for each* $1 \le i \le m$*.*

More general by Theorem 2.3.6 we have the following Corollary regarding the antimagic-ness of certain class of circulant graphs:

Corollary 2.3.10. *Let* $G = CIR_n(\{a_1, a_2, \cdots, a_m, \frac{n}{2}\})$ *be a circulant graph of even order* n*. Then* G *is antimagic if* $\frac{n}{2}$ *is odd and* $gcd(n, a_j) = 2$ *for some* $1 \le j \le m$*.*

2.4 Odd Regular Graphs With Odd Claw Factors

Definition 2.4.1. The complete bipartite graphs $K_{1,2m+1}$ is called an odd claw, where the integer $m \geq 1$. We call a spanning subgraph an odd claw factor if the edge set is partitioned into vertex disjoint union of isomorphic copies of odd claws $K_{1,2m+1}$ for fixed integer $m \geq 1$.

There is a well known cubic graph without any perfect matching as shown in the following Figure 2.4.1, given by J. Petersen as a related example for the fact that if a cubic graph is bridgeless then it admits a perfect matching.

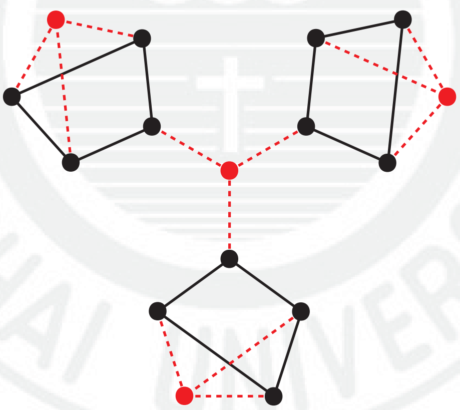

Figure 2.4.1: A Cubic Graph without any Perfect Matching but with a 3-Claw Factor

It is noticed the example can be treated as a factorization of one claw factor and one degenerate 2-factor and can be shown to be antimagic. We extend this fact to a more general situation in the following. We define first a spanning subgraph consisting of vertex disjoint $K_{1,3}$'s to be a **claw factor**, as seen in the Figure 2.4.1.

We start with the following:

Theorem 2.4.2. *Let G be a 3-regular graph with $4n$ vertices and $6n$ edges. Suppose G can be decomposed into the union of a claw factor C and a 2-regular subgraph induced by all pendant vertices of the claws in C. Then G is antimagic.*

Proof.

Note that G has the claw factor consisting of n vertex disjoint claws, which is named as $K(i)$ for $1 \leq i \leq n$. Let the center vertex of $K(i)$ of degree 3 be v_i for $1 \leq i \leq n$ and all other pendant vertices be u_j for $1 \leq j \leq 3n$.

We use $1, 2, \cdots, 3n$ to label the edges of 2-regular subgraph induced by all pendant vertices of the claws in C, and use the rest $3n+1, 3n+2, \cdots, 6n$ to label the edges of the claw factor C. Precisely we label the three edges of the claw $K(i)$ by $3n+i, 4n+i, 6n+1-i$ for $1 \leq i \leq n$. Therefore the vertex sum at the vertex v_i is $13n+i+1$ for $1 \leq i \leq n$, namely $13n+2, 13n+3, \cdots, 14n+1$.

On the other hand, in order to label the edges over $E(G)-E(C)$ properly, we put orientations over each connected cycle component either clockwise or counterclockwise. Then define the outgoing edge label at the vertex u_j by $f^{out}(u_j) = 6n + 1 - w(u_j)$ for $1 \leq j \leq 3n$, where $w(u_j)$ is the partial vertex sum while labeling the edges of claws in C, thus $w(u_j)$ ranges from $3n+1, 3n+2, \cdots$ to $6n$. Therefore the vertex sums over u_j are $6n+1+j$ for $1 \leq j \leq 3n$, namely $6n+2, 6n+3, \cdots, 9n+1$. Combined with the vertex sums over v_i for $1 \leq i \leq n$, that is $13n+2, 13n+3, \cdots, 14n+1$, we may see immediately that G is antimagic since the vertex sums are all distinct. □

By applying the above Theorem 2.4.2 we see the graphs like in Figure 2.4.1 are antimagic, and we also have the following examples:

Example 2.4.3. *The graph mK_{2n}, the disjoint union of m copies of K_{2n}'s, is antimagic, as shown in Figure 2.4.2.*

We may extend the above to the following more general situation since in previous proof we see the vertex sums $6n + 2, 6n + 3, \cdots, 9n + 1$ and

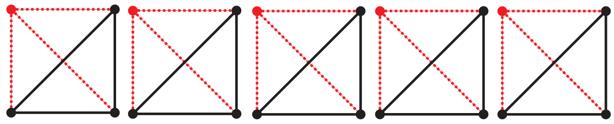

Figure 2.4.2: The disjoint union of 5 copies of K_4 is antimagic

$13n+2, 13n+3, \cdots, 14n+1$ are two groups of non-overlapping consecutive integers, by Lemma 2.2.2 after adding any arbitrary $2k$-factors the antimagicness is remained, therefore we have:

Theorem 2.4.4. *Let G be a 3-regular graph with $4n$ vertices and $6n$ edges. Suppose G can be decomposed into the union of a claw factor C and a 2-regular subgraph induced by all pendant vertices of the claws in C. Moreover let F be an arbitrary $2k$-factor. Then $G \oplus F$ is still antimagic.*

Moreover we may have the following result for odd regular graphs containing odd claw factors:

Theorem 2.4.5. *Let $m \geq 1$ and G be a $(2m+1)$-regular graph formed by an odd $K_{1,2m+1}$-factor and an arbitrary 2-regular subfactor over the pendant vertices of the odd claws. Then G is antimagic.*

In fact we are able to show the following more general situation for an odd graph (possibly non-regular) consisting of a mixed odd claw factor (that is a factor formed by possibly $K_{1,3}$'s, $K_{1,5}$'s, $K_{1,7}$'s, etc.) and a 2-subfactor over the pendant vertices of the claws, and therefore the above results are simply corollaries of this Theorem:

Theorem 2.4.6. *Let G be an odd graph formed by a mixed claw factor C and and an arbitrary 2-regular subfactor H over the pendant vertices of the* *odd claws in C, where C is consisting of vertex disjoint $K_{1,j}$'s for odd $j \geq 3$. Then G is antimagic, and also G remains antimagic after adding arbitrary $2k$-factors for $k \geq 1$.*

Proof.

Let G be the odd graph with p vertices. Assume that in C there are t_j odd $K_{1,j}$-factor for odd $j \geq 3$ and let $\sum_{j\geq 3} t_j = s$, where j is odd and t_j are non-negative integers. Therefore with similar notions, both G and C have $p = \sum_{j\geq 3}(j+1)\cdot t_j$ vertices and assume further that C has $L = \sum_{j\geq 3} j \cdot t_j$ edges, thus $L = p - s$. Note that H has $L = p - s$ vertices and $L = p - s$ edges, and we split all edges in G into two categories $E_1 = \{1, 2, \cdots, p - s\}$ and $E_2 = \{p - s + 1, p - s + 2, \cdots, p - s + L\}$. Now we are in a position to label the edges to verify the antimagic-ness of G with the following steps.

(Step 1): First we put the claws in order of non-increasing sizes as $K_{1,3}$, $K_{1,5}$ etc., then we may start labeling edges as follows: Labeling one edge (pick any) in each of these s odd claws (called the dominating edge of the claw) in order of non-increasing sizes, using the most centered edge labels within E_2 named the dominating edge labeling set $E_d = \{p - s + \frac{L-s}{2} + 1, p - s + \frac{L-s}{2} + 2, \cdots, p - s + \frac{L-s}{2} + s\}$.

(Step 2): Than within each claw there are an even number of edges left yet to be labeled, for which we use the rest of the labels in $E_2 - E_d$ by considering these labels in pairs with constant sum $2(p - s) + L + 1$. Let v_i be the center vertex for each of these claws and u_k^i's be the corresponding pendant vertices. Therefore the vertex sums at v_i for these claws is $\frac{j-1}{2} \cdot [2(p - s) + L + 1] + e(v_i u_k^i)$, where $e(v_i u_k^i)$ is the dominating edge picked in Step 1 for each claw. One see that the vertex sums for claws of the same size are sequences of consecutive integers, and any two such sequences have a gap larger than p. The reason is the gap must be a multiple of $2(p - s) + L + 1$ which is larger than $2(p - s) + 2s + 1 = 2p + 1$, thus larger than p.

(Step 3): The next stage is to calculate the vertex sums over the vertices of H, i.e. those pendant vertices of claws. Note that H is formed over the above vertices as a 2-subfactor in arbitrary way. Now We proceed by assigning an orientation to the 2-regular subgraph H so that over each connected component (connected 2-cycle) the flow is either clockwise or counter-clockwise. Let $e^{in}(u)$ and $e^{out}(u)$ be the incoming edge label and outgoing edge label respectively, and $e(uv)$ be the edge label assigned in Step 1 and Step 2. Than use the integers in $E_1 = \{1, 2, \cdots, p-s\}$ to label the edges of H by the following rule of constant sums:

$$e^{in}(u) + e(uv) = 1 + (p-s) + L.$$

Therefore the vertex sums at the pendant vertices of claws are $e^{in}(u)+e(uv)+e^{out}(u) = 1+(p-s)+L+e^{out}(u)$, thus they range over the set of consecutive integers $\{1+(p-s)+L+1, 1+(p-s)+L+2, \cdots, 1+(p-s)+L+p-s\}$.

(Step 4): Therefore G is antimagic by comparing the vertex sums at the center vertices of the claws and the vertex sums at the pendant vertices as follows. Note that in Step 2 we have the vertex sum at the center vertex v_i of the claw is $\frac{j-1}{2} \cdot [2(p-s)+L+1] + e(v_i u_k^i)$, which is larger than $2(p-s)+L+1+p-s+\frac{L-s}{2}+1$. We see $1+(p-s)+L+p-s$ is the largest possible vertex sum at the pendant vertices, and using $L = p-s$ one has $[2(p-s)+L+1+p-s+\frac{L-s}{2}+1] - [1+(p-s)+L+p-s] = \frac{3}{2} \cdot p - 2s + 1$, which is $> p$ since $p \geq 4s$.

(Step 5): With labeling edges in previous steps, we see the vertex sums are arranged into groups of consecutive integer sequence with gaps larger than p. By applying Lemma 2.2.4 we see G remains antimagic after adding arbitrary $2k$-factors for $k \geq 1$. $\square$

Note that the regular graphs of higher degree without any perfect matching as in Figure 2.4.3 are shown antimagic, in view of Theorem 2.4.6. Note that the example has no any perfect matching due to the well known Tutte's condition.

2.5 Concluding Remarks

In this chapter, we obtain antimagic labelings of regular graphs with particular types of 3-factors (odd regular graphs containing pseudo-prisms). Also the

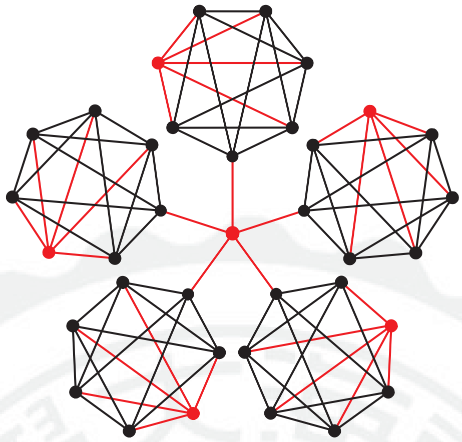

Figure 2.4.3: 5-Regular Graph without Perfect Matching but with a 5-Claw Factor

antimagic-ness is verified for the class of graphs with mixed odd claw factors, which contains certain odd regular graphs without any 1-factors. Hopefully these results may be helpful to resolve more general situations regrading the conjecture that every regular graph except K_2 is antimagic, or helpful to resolve the Hartsfields-Ringel conjecture that every connected graph except K_2 is antimagic. We add that in the stage of submission of this chapter, all 3-regular graphs are shown antimagic in [24].

Another remark is that most of the results in this chapter, the example graphs involved can be graphs with parallel edges. Therefore this leads to consider more general version of antimagic-ness for multi-graphs, which obviously should exclude the multiple K_2 defined by parallel edges on two vertices. Problems inspired from the antimagic-ness of simple graphs are obvious interesting to be explored over the cases of multi-graphs.

Chapter 3

Antimagic Labeling of Even Regular Graphs

3.1 Even Regular Hamiltonian Graphs

Note that a graph is called Hamiltonian if it contains a Hamiltonian cycle. We show in this section that Hamiltonian regular graphs of even degree are antimagic. First we note that in [21] the following result was obtained, which implies the antimagic-ness of even regular Hamiltonian graphs of odd order:

Theorem 3.1.1. *Let G be a $2k$-regular, $k \geq 2$, Hamiltonian graph of odd order n. Then G is $(a, 1)$-antimagic.*

As for Hamiltonian even regular graphs of even order, we have the following:

Theorem 3.1.2. *Let G be a $2k$-regular, $k \geq 2$, Hamiltonian graph of even order n. Then G is antimagic.*

Proof. We proceed with similar notations as in the previous Theorem where we showed that $2k$-regular Hamiltonian graphs of odd order are antimagic. We let G be a $2k$-regular Hamiltonian graph of even order n, and $G = F_1 \bigoplus F_2 \bigoplus \cdots \bigoplus F_k$, where F_i is a 2-factor for each $1 \leq i \leq k$. Moreover without loss of generality we assume F_1 is a Hamiltonian cycle of G.

We start labeling as before, on the Hamiltonian cycle F_1 we give $f(v_i v_{i+1}) = \frac{i+1}{2}$ for i odd, and $f(v_i v_{i+1}) = \frac{i}{2} + \frac{n}{2}$ for i even. Note that by labeling this way, we have the unique conflict of the pair of vertex sums $f_1^+(v_1) = f_1^+(v_{\frac{n}{2}+1}) = n+1$. However we may resolve the conflict here for the time being, by adding extra $\frac{n}{2}$ to $f_1^+(v_1)$ to make it an arithmetic sequence of common difference 1, and we denote the fake vertex sum to be $\widehat{f^+}$.

Then as before we may label G recursively, and get an arithmetic progression of common difference 1 for the vertex sums $\frac{(2k^2-1)n}{2} + r + i$ for $i = 1, \cdots, n$. In particular note that $\widehat{f^+}(v_1) = \frac{(2k^2-1)n}{2} + k + 1 = f^+(v_1) + \frac{n}{2}$. We claim that one may keep the antimagic-ness of the graph, while removing the extra $\frac{n}{2}$ we added previously from $\widehat{f^+}(v_1)$, by switching certain edge labels to get $f^+(v_1)$ back. We split the situation into the following cases:

Case 1: $\frac{(2k^2-1)n}{2} + k + 1 \leq \widehat{f^+}(v_1) < \frac{(2k^2-1)n}{2} + k + 1 + \frac{n}{2}$. Then $f^+(v_1) < \frac{(2k^2-1)n}{2} + k + 1$, and together with other vertex sums, it is seen that f is antimagic.

Case 2: $\frac{(2k^2-1)n}{2} + k + 1 + \frac{n}{2} \leq \widehat{f^+}(v_1) \leq \frac{(2k^2-1)n}{2} + k + 1 + n$. Then in this case $\frac{(2k^2-1)n}{2} + k + 1 \leq f^+(v_1) \leq \frac{(2k^2-1)n}{2} + k + 1 + \frac{n}{2}$, there is some conflict happened as $f^+(v_1) = f^+(v_r)$ for some r. Note that $f(v_1 v_2) = 1$, and we switch the edge labels of $v_1 v_2$ and some edge $v_a v_b$ of the second 2-factor F_2, where the edge $v_a v_b$ is incident with v_r. There are two possibilities:

Sub-case 2.1: $v_r \neq v_2$. Therefore in this case $v_1 v_2$ and $v_a v_b$ are disjoint. Then we switch the edge labels of them as follows, the resulting labeling is antimagic. First note that $f(v_a v_b) \geq n + 1$, and thus $f^+(v_1)$ and $f^+(v_2)$ increase simultaneously by at least n. Then both $f^+(v_1)$ and $f^+(v_2)$ are distinct, and $> \frac{(2k^2-1)n}{2} + k + n$. On the other hand, both $f^+(v_a)$ and $f^+(v_b)$ decrease simultaneously by at least n. Also both are distinct, and $< \frac{(2k^2-1)n}{2} +$

$k+1$. Therefore the antimagic-ness is assured together with other vertex sums.

Sub-case 2.2: $v_r = v_2$. Then we switch the edge labels of v_1v_2 and v_av_b, note that in this case v_2 is either v_a or v_b, say $v_2 = v_a$. It can be seen that $f^+(v_2)$ is unchanged. Note that similarly $f^+(v_1) > \frac{(2k^2-1)n}{2} + k + n$ and $f^+(v_b) < \frac{(2k^2-1)n}{2} + k + 1$, thus the resulting new labeling is antimagic. □

In the following we provide with more examples about antimagic labelings of regular graphs. The first class of examples come from the power of cycles C_n^k, which are even regular graphs.

Example 3.1.3. **(Powers of Cycles C_n^k)** *In 2011, M. Lee, C. Lin, and W. Tsai [23] proved that in particular that the square of cycles C_n^2 with odd order are antimagic, and further conjectured that all powers of cycles of any order are also antimagic. It is not hard to see that their results and moreover the conjecture follow from the results in this section, since all powers of cycles C_n^k are clearly Hamiltonian and even regular.*

Also we have the following examples from circulant graphs.

Example 3.1.4. *$CIR_n(S)$ is antimagic if there exists an element s with $gcd(n, s) = 1$ such that $s \in S$, since it contains a Hamiltonian cycle $CIR_n(\{s\})$.*

3.2 Even Regular Graphs With Particular 2-Factors

In this section we generalize the results for the antimagic-ness of Hamiltonian even regular graphs of odd order as in Theorem 3.1.1. We start with the following Lemma:

Lemma 3.2.1. *Let G be a 2-regular graph of order p. Assume that G has a bijective edge labeling from $E(G)$ to the consecutive integers $a, a+1, \cdots, a+(p-1)$ such that the vertex sums are consecutive integers. Then*

1. *p is odd;*

2. *The consecutive vertex sums are $A_i = 2a + \frac{p-1}{2} + i - 1$ for $1 \leq i \leq p$.*

Main result is as follows:

Theorem 3.2.2. *Let G be a $2k$-regular graph with p vertices, $k \geq 2$. Assume that G contains a 2-factor consisting of $(a,1)$-antimagic 2-subfactors $H_1, H_2, \cdots, H_t$ of odd order, where t is a fixed positive integer at most k. Then G is antimagic.*

Proof.

Since G is a $2k$-regular graph with p vertices, one label edges using consecutive integers $1, 2, \cdots, pk$. By Petersen's Theorem 4.1.2, G can be factored as sum of 2-factors $G = F_1 \oplus F_2 \oplus \cdots \oplus F_k$. Without loss of generality, may assume $F_1 = H_1 \sqcup H_2 \sqcup \cdots \sqcup H_t$, a disjoint union of 2-regular $(a,1)$-antimagic subgraphs $H_1, H_2, \cdots, H_t$. Assume H_i has m_i vertices (hence has m_i edges), for each $1 \leq i \leq t$. Thus $m_1 + m_2 + \cdots + m_t = p$. Also note that by Lemma 4.1.3, the integers m_i is odd for each $1 \leq i \leq t$.

We first label $F_1 = H_1 \sqcup H_2 \sqcup \cdots \sqcup H_t$. Since $H_1, \cdots, H_t$ are $(a,1)$-antimagic, by applying Lemma 4.1.3 we may have consecutive partial vertex sums via labeling them respectively using $\{1, 2, ..., m_1\}$ for H_1, $\{p+m_1+1, p+m_1+2, \cdots, p+m_1+m_2\}$ for H_2, $\{2p+m_1+m_2+1, 2p+m_1+m_2+2, \cdots, 2p+m_1+m_2+m_3\}$ for H_3, $\cdots$, $\{(t-1)p + \sum_{i=1}^{t-1} m_i + 1, (t-1)p + \sum_{i=1}^{t-1} m_i + 2, \cdots, (t-1)p + \sum_{i=1}^{t-1} m_i + m_t\}$ for H_t. Let A_j^i be the j-th partial vertex sum

$(1 \le j \le m_i)$ over H_i, $1 \le i \le t$. Also let $D_i = A_1^{i+1} - A_{m_i}^i - 1$, where $1 \le i \le t-1$. Note that by Lemma 4.1.3, $D_i = A_1^{i+1} - A_{m_i}^i - 1 = \frac{1}{2}(m_i + m_{i+1}) + 2p > p$ for $i = 1$ to $t-1$. To make the partial vertex sums to be a sequence of consecutive integers for the time being, one may modify the partial vertex sums A_j^i to be the fake partial vertex sums $\widehat{A_j^i}$ by the following translations: $\widehat{A_j^{i+1}} = A_j^{i+1} - D_i - D_{i-1} - \cdots - D_1$, for $i = 1$ to $t-1$. Note that $\widehat{A_j^1} = A_j^1$, that is A_j^1 is fixed for each j. Therefore the fake partial vertex sums $\widehat{A_j^i}$ form a sequence of consecutive integers $\widehat{A_1^1} = \sigma_1 < \sigma_2 < \cdots < \sigma_p = \sigma_1 + (p-1)$.

We now may apply Lemma 5.2.4 to add the labeling of the rest of 2-factors $F_2, \cdots, F_k$ and keep the resulting vertex sums to be consecutive integers. By abusing language we still denote the vertex sums by $\widehat{A_j^i}$. After recovering the original vertex sums via $A_j^{i+1} = \widehat{A_j^{i+1}} + D_i + D_{i-1} + \cdots + D_1$ for $i = 1$ to $t-1$ and $A_j^1 = \widehat{A_j^1}$, again we still abuse language and call them A_j^{i+1}. We claim the original vertex sums A_j^i are pairwise distinct.

To show the claim is true, we may first see $\sigma_1 \le \widehat{A_j^1} = A_j^1 \le \sigma_p$ for each j. Secondly $\sigma_1 + D_1 \le \widehat{A_j^2} + D_1 = A_j^2 \le \sigma_p + D_1$ for each j. Similarly $\sigma_1 + D_1 + D_2 \le \widehat{A_j^3} + D_1 + D_2 = A_j^3 \le \sigma_p + D_1 + D_2$. Then we proceed until $\sigma_1 + D_1 + D_2 + \cdots + D_{t-1} \le \widehat{A_j^t} + D_1 + D_2 + \cdots + D_{t-1} = A_j^t \le \sigma_p + D_1 + D_2 + \cdots + D_{t-1}$ for each j. Let $[a, b]$ be the set of integers $\{t \mid a \le t \le b\}$. We conclude that for each j, the vertex sums A_j^1 belong to $[\sigma_1, \sigma_p]$, A_j^2 belong to $[\sigma_1 + D_1, \sigma_p + D_1]$, A_j^3 belong to $[\sigma_1 + D_1 + D_2, \sigma_p + D_1 + D_2]$, $\cdots$, until A_j^t belong to $[\sigma_1 + D_1 + D_2 + \cdots + D_{t-1}, \sigma_p + D_1 + D_2 + \cdots + D_{t-1}]$. Note that $[\sigma_1, \sigma_p] \cap [\sigma_1 + D_1, \sigma_p + D_1] \cap [\sigma_1 + D_1 + D_2, \sigma_p + D_1 + D_2] \cap \cdots \cap [\sigma_1 + D_1 + D_2 + \cdots + D_{t-1}, \sigma_p + D_1 + D_2 + \cdots + D_{t-1}] = \phi$ since $D_i > p$ for each $i = 1, \cdots, t-1$. (See Figure 3.2.1) Therefore A_j^i are pairwise distinct for all i, j and we are done.

$\square$

Corollary 3.2.3. *Let G be a $2k$-regular graph ($k \ge 2$) which contains a 2-factor consisting of at most k odd cycles only. Then G is antimagic.*

Followed from the above general Theorem 3.2.2 we may have a lot more

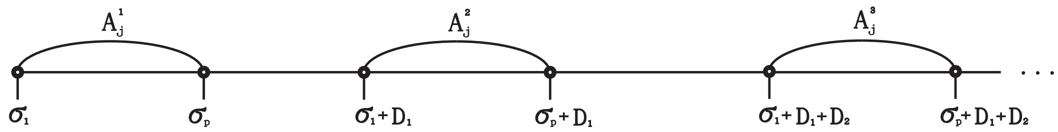

Figure 3.2.1: Translation of Ranges

examples of circulant graphs (Cayley graphs of finite cyclic groups), not just limited to the case in the above Corollary 3.2.3. To see this we first need a lemma saying that the disjoint union of an odd number of odd cycles is $(a,1)$-antimagic. In 2003 V. Swaminathan and P. Jeyanthi already showed the following result in [37], however the labeling way they gave in the article contains errors and can not be properly checked. Therefore here we rewrite a proof for completeness:

Lemma 3.2.4. (V. Swaminathan and P. Jeyanthi, 2003) *Let G be mC_n, which is the vertex disjoint union of m copies of connected n-cycles C_n. Then G is $(a,1)$-antimagic if and only if m and n are both odd.*

Proof.

We prove the necessary part first. Since mC_n is $(a,1)$-antimagic, may suppose the vertex sums form an arithmetic progression with initial term a and common difference 1. Hence

$$2(1+2+\cdots+mn) = a+(a+1)+\cdots+(a+mn-1)$$

Therefore $a = \frac{3}{2}+\frac{1}{2}mn$, thus mn must be odd, and m and n must be both odd.

Conversely, for m, n odd, we show that mC_n is $(a,1)$-antimagic. Sort and name the vertices of these n-cycles clockwise as v_j^i, which indicate the j-th vertex in the i-th cycle, where $1 \le i \le m$ and $1 \le j \le n$. Denote the corresponding edge $v_j^i v_{j+1}^i = e_j^i$ for $1 \le i \le m$ and $1 \le j \le n-1$, and $v_n^i v_1^i = e_j^i$ for $1 \le i \le m$ and $j = n$.

Then we define the edge labeling f as follows:

$$f(e^i_j)=\begin{cases} i, & 1\le i\le m, j=1\\ m(\frac{j}{2})+i+\frac{1}{2}, & 1\le i\le \frac{m-1}{2}, 3\le j\le n, \text{ for odd } j.\\ m(\frac{j}{2}-1)+i+\frac{1}{2}, & \frac{m+1}{2}\le i\le m, 3\le j\le n, \text{ for odd } j.\\ m(\frac{n+j+1}{2})-2i+1, & 1\le i\le \frac{m-1}{2}, 2\le j\le n-3, \text{ for even } j.\\ m(\frac{n+j+3}{2})-2i+1, & \frac{m+1}{2}\le i\le m, 2\le j\le n-3, \text{ for even } j.\\ m(n-1)+i, & 1\le i\le m, j=n-1. \end{cases}$$

Then it may be checked that the above labeling induces the desired $(a,1)$-antimagic labeling.

□

Corollary 3.2.5. *All circulant graphs G of odd order (hence even regular) are antimagic.*

Proof.

Any 2-factor of G is consisting of an odd number of odd cycles, which is $(a,1)$-antimagic by Lemma 4.3.1. From the above Theorem 3.2.2 it follows G is antimagic.

□

Corollary 3.2.6. *Let $G=CIR_n(S)$ be a circulant graph of even order n. Suppose there exists an element $a\in S\subseteq\{1,2,\cdots,\frac{n}{2}\}$ and $\frac{n}{gcd(n,a)}$ is odd. Moreover $\frac{n}{2}$ is not in S. Then G is antimagic.*

Proof.

Note that since $\frac{n}{2}$ is not in S and n is even, $G=CIR_n(S)$ is even regular, say $2k$-regular and $k\ge 2$. Also $CIR_n(\{a\})$ is a 2-factor consisting of $gcd(n,a)$ cycles with the same odd order $\frac{n}{gcd(n,a)}$, which means an even number of odd cycles. We see that one may treat the 2-factor $CIR_n(\{a\})$ as $H_1\sqcup H_2$ where H_1 and H_2 are both 2-regular subgraphs consisting of an odd number of odd cycles. Hence H_1 and H_2 are $(a, 1)$-antimagic by Lemma 4.3.1. Therefore by the above Theorem 3.2.2, it follows G is antimagic.

□

In the following we have another criterion for testing the antimagic-ness for an even regular graph with a 2-factor consisting of cycles of the same size:

Theorem 3.2.7. *Let G be a $2k$-regular graph. Assume that G contains a 2-factor $F = mC_n$ consisting of cycles of the same size. Then G is antimagic.*

Proof.

In case m, n both odd, by Lemma 4.3.1 and Lemma 5.2.4, one see immediately that such graph G is $(a, 1)$-antimagic, hence antimagic. In case m even and n odd, one may treat the case as two groups of cycles, one is $m-1$ (odd) C_n's and the other one is a single odd cycle C_n. By applying Lemma 4.3.1 one see the first group of cycles is $(a, 1)$-antimagic, and then using the lager edge labels over the single C_n one obtain another $(a', 1)$-antimagic labeling, which does not overlap with the former group of $(a, 1)$-antimagic vertex sums. Therefore we may apply Lemma 2.2.2 to get G is antimagic. One may get the remaining cases in similar fashions, and the details are left to the interested readers.

□

Corollary 3.2.8. *All even regular circulant graphs are antimagic.*

Chapter 4

Edge-Super Magic Labeling

A vertex magic total labeling is a bijection from $V(G) \cup E(G)$ to the consecutive integers $1, 2, \cdots, p+q$, with the property that, for every vertex u in $V(G)$, one has $f(u) + \sum_{uv \in E(G)} f(uv) = k$ for some constant k. Such a labeling is called E-super vertex magic if $f(E(G)) = \{1, 2, \cdots, q\}$. A graph G is called Edge-Super(E-super for short)x x vertex magic if it admits a E-super vertex magic labeling. Most recently [29] G. Marimuthua and M. Balakrishnan ("E-super vertex magic labelings of graphs", Discrete Applied Math. 160, 2012, pp. 1766-1774) studied some basic properties of such labelings and established E-super vertex magic labeling of some families of graphs. In this chapter we extend their results and more examples are also provided. Note that the results were written as a paper **A Note on E-super Vertex Magic Graphs**, which was submitted to a journal in 2012 and under review.

4.1 Introduction and Background

Over the past few decades many kinds of graph labelings have been studied intensively, and an excellent survey of graph labeling can be found in Gallian's chapter [15]. In 1963, Sedláček [32] introduced the concept of magic labeling. Suppose that G is a graph with q edges and one shall say that G is magic if the edges of G can be labeled by the numbers $1, 2, \cdots, q$ so that the sum of labels of all the edges incident with any vertex is constant. In 2002 MacDougall et al. [26] introduced the concept of vertex magic total labeling as follows. If G is a finite simple undirected graph with p vertices and q edges, then a vertex magic total labeling is a bijection f from $V(G) \cup E(G)$ to the integers $1, 2, \cdots, p+q$ with the property that for every $u \in V(G)$, the sum $f(u) + \sum_{uv \in E(G)} f(uv)$ is constant. They studied the basic properties of vertex magic graphs and showed some families of graphs having a vertex magic total labeling. MacDougall et al. [27] further introduced the concept of super vertex magic total labeling. They call a vertex magic total labeling is super if $f(V(G)) = \{1, 2, \cdots, p\}$. Swaminathan and Jeyanthi [37] introduced a concept with the name super vertex magic labeling, but with different definition. They call a vertex magic total labeling is super if $f(E(G)) = \{1, 2, \cdots, q\}$. More recently Marimuthua and Balakrishnan [29] studied some basic properties of such labelings and established E-super vertex magic labeling of some families of graphs. Here we define it formally:

Definition 4.1.1. Let G be a finite simple graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. A **vertex magic total labeling** is a bijection from $V(G) \cup E(G)$ to the consecutive integers $1, 2, \cdots, p+q$, with the property that, for every vertex u in $V(G)$, one has $f(u) + \sum_{uv \in E(G)} f(uv) = k$ for some constant k. Such a labeling is called **E-super vertex magic** if $f(E(G)) = \{1, 2, \cdots, q\}$. A graph G is called **E-super vertex magic** if it admits a E-super vertex magic labeling.

In this chapter, we generalize some of previous results in [29]. More examples regarding the E-super vertex magicness of regular graphs, such as circulant graphs, are also provided. We first state a well-known result we need later:

Theorem 4.1.2. (J. Petersen, 1891 [31]) *Let G be a $2r$-regular graph. Then there exists a 2-factor in G.*

Notice that after removing edges of the 2-factor by the Petersen Theorem, we will get an even regular graph again and again. Thus an even regular graph has a 2-factorization. Also we need another fact as pointed out in [37]:

Theorem 4.1.3. *Let G be a graph and g be a bijection from $E(G)$ onto $\{1, 2, \cdots, |E(G)|\}$. Then g can be extended to an E-super vertex magic labeling of G if and only if $\{w(u) = \sum_{uv \in E(G)} g(uv) \mid u \in V(G)\}$ consists of $|V(G)|$ consecutive integers.*

This allows us to use the edge labeling instead to be the tool studying the E-super vertex magic total labeling throughout this chapter.

4.2 Even Regular Hamiltonian Graphs of Odd Order

Note that a graph is called Hamiltonian if it contains a Hamiltonian cycle. We show in this section that Hamiltonian even regular graphs of odd order are E-super vertex magic. In Theorem 3.5. of [29] the following result is proved: Let G be a (p, q) graph of odd order. If G can be decomposed into two Hamilton cycles, then G is E-super vertex magic. Here we generalize to the following form:

Theorem 4.2.1. *Let G be a $2r$-regular, $r \geq 2$, Hamiltonian graph of odd order n. Then G is E-super vertex magic.*

Proof.

By Petersen's Theorem 4.1.2, $G = F_1 \bigoplus F_2 \bigoplus \cdots \bigoplus F_r$, where F_i is a 2-factor for each $1 \leq i \leq r$. Without loss of generality, we assume F_1 is the Hamiltonian cycle which G has, and let $V(F_1) = \{v_1, v_2, \cdots, v_n\}$ and $E(F_1) = \{v_1v_2, v_2v_3, \cdots, v_{n-1}v_n\}$.

Now we give an edge labeling f by the following steps. Note that G has rn edges. First we split all edge labels $1, 2, \cdots, rn$ into r groups as follows: $\{1, 2, \cdots, n\}, \{n+1, n+2, \cdots, 2n\}, \cdots\cdots \{(r-1)n+1, (r-1)n+2, \cdots, rn\}$. Then we will put these groups of labels in order over the edges of F_1, F_2, $\cdots$, F_r, respectively. Similarly we may define G recursively as before, namely $G_k = F_1 \bigoplus F_2 \bigoplus \cdots \bigoplus F_k$ for $1 \leq k \leq r$. We label G recursively in below.

Since F_j are 2-factors for each $1 \leq j \leq r$, we assign an orientation so that over each connected component (connected 2-cycle) the flow direction is either clockwise or counter-clockwise. We set $f_k^{out}(w)$ and $f_k^{in}(w)$ respectively, for each $1 \leq k \leq r$, to be the outgoing edge label over the 2-factor F_k from the vertex w and the incoming edge label to the vertex w according to the given orientation. On the other hand, we denote $f^+(w)$ to be the induced vertex sum at the vertex w, and we use $f_k^+(w)$ to stand for the partial vertex sum at w for G_k for each $1 \leq k \leq r$. Then we may start labeling recursively over G_1, G_2, $\cdots$, $G_r = G$. Precisely we give the labeling in the following steps:

Step 1: For $G_1 = F_1$, first by $f(v_iv_{i+1}) = \frac{i+1}{2}$ for i odd, and $f(v_iv_{i+1}) = \frac{i}{2} + \frac{n+1}{2}$ for i even. Thus the partial vertex sum $f_1^+(v_i) = i + \frac{n+1}{2}$, $1 \leq i \leq n$, which form an arithmetic progression of common difference 1.

Step 2: For G_2, G_3, ..., G_r we proceed recursively as follows: For $2 \leq k \leq r$, over F_k we set $f_k^{out}(v_i) = \frac{(2k^2-2k+1)n+2k-1}{2} - f_{k-1}^+(v_i)$ for each $1 \leq i \leq n$.

Therefore $f_k^+(v_i) = f_k^{in}(v_i) + f_{k-1}^+(v_i) + f_k^{out}(v_i) = \frac{(2k^2-2k+1)n+2k-1}{2} + f_k^{in}(v_i)$. Also note that $f_k^{out}(v_i) = f_k^{in}(v_i)$ for a unique j, where $1 \leq j \neq i \leq n$. Therefore $f_k^+(v_i) = \frac{(2k^2-1)n+2k+1}{2} + (i-1)$ for $1 \leq i \leq n$, which form an arithmetic progression of common difference 1.

Therefore one obtain an edge labeling where the induced vertex sums are consecutive integers, and hence an E-super vertex magic labeling by Theorem 4.1.3.

□

In fact with similar proof technique as above, it is not hard to see that one may furthermore generalize the above to the following:

Theorem 4.2.2. *Let G be decomposed into the sum of two spanning subgraphs $G_1 \oplus G_2$, where G_1 is E-super vertex magic and G_2 is regular of even degree. Then G is E-super vertex magic.*

Note that the above Theorem 4.2.2 generalizes Theorem 3.7. in [29]: If a graph G can be decomposed into two E-super vertex magic spanning subgraphs G_1 and G_2 where G_2 is regular, then G is E-super vertex magic. Also we remark that just recently similar techniques were employed in [46] to deal with another graph labelling problem.

4.3 Even Regular Graphs of Odd Order

In 2003 V. Swaminathan and P. Jeyanthi showed the following result in [37] as pointed out by G. Marimuthu and M. Balakrishnan in [29]:

Theorem 4.3.1. *Let G be mC_n, which is the disjoint union of m copies of C_n. Then G is E-super vertex magic if and only if m and n are both odd.*

Then we have the following result which can be used to create many examples of E-super vertex magic regular graphs and also can be treated as a natural generalization of Theorem 4.2.1:

Theorem 4.3.2. *Let G be a even-regular graph of odd order which contains a 2-factor consisting of an odd number of odd cycles. Then G is E-super* *vertex magic.*

Proof.

Let G be a regular graph of even degree with odd number of vertices. Then by Petersen's Theorem 4.1.2, G may be decomposed into sums of 2-factors, say $G = F_1 \bigoplus F_2 \bigoplus \cdots \bigoplus F_r$, where F_i is a 2-factor for each $1 \leq i \leq r$. Since G is of odd order, any F_i consists of an odd number of odd connected cycles. Therefore by Theorem 4.3.1 and Theorem 4.2.2 we may see G is E-super vertex magic. $\square$

To obtain more examples, we consider the circulant graphs as follows:

Definition 4.3.3. A circulant graph $CIR_n(S)$ with n vertices, with respect to $S \subset \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$, is a graph with the vertex set $V(CIR_n(S)) = \{0, 1, 2, \cdots, n-1\}$, and the edge set is formed by the following rule:

$$E(CIR_n(S)) = \{ij :\ i - j \equiv \pm s \ \ (mod\ n),\ s \in S\}.$$

Note that the circulant graph $CIR_n(S)$ is also called a Cayley graph of the finite cyclic group $\mathbb{Z}_n$ generated by S.

For example, $CIR_n(\{a\}) \cong C_n$, the connected n-cycle, if $gcd(n, a) = 1$. Moreover $CIR_n(\{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}) \cong K_n$, the complete graphs, and $CIR_{2n}(\{1, n\}) \cong$ the n-Möbius ladder graphs. Note that $CIR_n(S)$ is odd-regular if n is even and $\frac{n}{2} \in S$, is even-regular otherwise.

Therefore directly from Theorem 4.3.2, we have the following class of circulant graphs which is E-super vertex magic:

Theorem 4.3.4. *Let G be a circulant graph with odd order. Then G is E-super vertex magic.*

4.4 Conclusion Remark

More properties of E-super vertex magic labelling can be explored based upon results in this chapter and previous work. In particular among others one can study the E-super vertex magic labelling for odd regular graphs and general regular graphs, contrast to results here for even regular graphs.

Chapter 5

Vertex Magic Total Labeling

Let $G = (V(G), E(G))$ be a finite simple graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges, without isolated vertices or isolated edges. A vertex magic total labeling is a bijection from $V(G) \cup E(G)$ to the consecutive integers $1, 2, \cdots, p+q$, with the property that, for every vertex u in $V(G)$, one has $f(u) + \sum_{uv \in E(G)} f(uv) = k$ for some constant k. In 2004 MacDougall et al. [26] first introduced the concept of vertex magic total labeling and studied their properties. In 2006 Slamin et al. [35] studied such vertex magic total labeling of disconnected graphs. In this paper we study the properties of such vertex magic total labeling for various graph classes. Among others we settle a conjecture mentioned in [35], which claimed the existence of the vertex magic total labeling of disjoint union of multiple copies of distinct sun graphs, where the sun graph is the corona product of a cycle with a point. We furthermore provide with an infinite class of graphs admitting such labelings based upon adding arbitrary $4k$-regular factors to the above disjoint union of sun graphs. Note that the results we obtain in this paper could be extended to those pseudo-graphs with multiple edges or loops. Note that the results in this chapter has been accepted as a regular journal paper **On Vertex Magic Total Labeling of Disjoint Union of Sun Graphs**, Utilitas Math., 2013.

5.1 Introduction and Background

Unless otherwise stated all graphs in this chapter are finite simple, undirected, possibly disconnected, but without any isolated vertex or any isolated edge. Over the past few decades many kinds of graph labelings have been studied intensively, and an excellent survey of graph labeling can be found in Gallian's paper [15]. In 1963, Sedl*áč*ek [32] introduced the concept of magic labeling. Suppose that G is a graph with q edges and one shall say that G is magic if the edges of G can be labeled by the numbers $1, 2, \cdots, q$ so that the sum of labels of all the edges incident with any vertex is constant. In 2004 MacDougall et al. [26] introduced the concept of vertex magic total labeling, studied the basic properties of vertex magic graphs, and showed some families of graphs having a vertex magic total labeling.

Definition 5.1.1. Let G be a finite simple graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. A **vertex magic total labeling** is a bijection from $V(G) \cup E(G)$ to the consecutive integers $1, 2, \cdots, p + q$, with the property that, for every vertex u in $V(G)$, one has $f(u) + \sum_{uv \in E(G)} f(uv) = k$ for some constant k. The constant k is called the **magic constant**. Moreover G is called **vertex-magic** if it admits a vertex magic total labeling.

Since the introduction of this notion, there have been several results on vertex magic total labeling of particular classes of graphs. For example, MacDougall et al. [26] proved that cycle C_n for $n \geq 3$, path P_n for $n \geq 2$, complete graph K_n for odd n, complete bipartite graph $K_{n,n}$ for $n > 1$, have vertex magic total labelings. Bača, Miller and Slamin [7] proved that for $n \geq 3$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, every generalized Petersen graph $P(n, m)$ has a vertex-magic total labeling with the magic constant $k = 9n + 2$, $k = 10n + 2$, and $k = 11n + 2$. In 2007 [17] Gray studied such labelling for regular graphs. The complete survey of the known results on vertex magic total labeling of graphs can be found in [15], and also in other references [17, 29, 35, 45].

Most of the known results are concerning on vertex magic total labeling of connected graphs. For the case of disconnected graph, Wallis [45] proved the following theorem.

Theorem. Suppose G is regular graph of degree r which has a vertex magic total labeling. (i) If r is even, then tG is vertex magic whenever t is an odd positive integer. (ii) If r is odd, then tG is vertex magic for every positive integer t.

In 2006 Slamin et al. [35] studied such vertex magic total labeling of disconnected graphs and made a conjecture that there is a vertex-magic total labeling of the disjoint union of non-isomorphic suns, where the graph **sun** is the corona product of a cycle with a point. That is it was conjectured that the vertex-magic total labeling exists for the corona product of an arbitrary 2-regular graph with a point. The **corona product** of G_1 and G_2, defined by Frucht and Harary [13] is the graph which is the disjoint union of one copy of G_1 and $V(G_1)$ copies of G_2 in which each vertex of the copy of G_1 is connected to all vertices of a separate copy of G_2. Please see in the following Figure 5.1.1 for an example of corona product of a 2-regular graph with one point, i.e. a disjoint union of sun graphs.

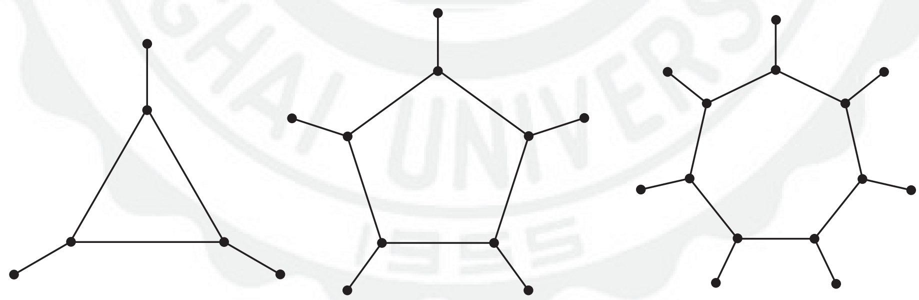

Figure 5.1.1: Example of Disjoint Union of Sun Graphs

In this chapter we verify the conjecture completely and provide with more examples by way of showing the following:

Theorem. There exists a vertex-magic total labeling for the disjoint union of t not necessarily isomorphic suns $S_{m_1} \cup S_{m_2} \cup \cdots \cup S_{m_t}$, for any positive integer t. Moreover let $G = (S_{m_1} \cup S_{m_2} \cup \cdots \cup S_{m_t}) \oplus H$ be the graph consisting of the disjoint union of t suns and an arbitrary $4k$-factor H. Then G also admits a vertex-magic total labeling.

We notice that the method employed here is also valid for those graphs with multiple edges and loops. Therefore we verify moreover that the vertex-magic total labelings exist for the corona product of an arbitrary 2-regular pseudo-graphs with one point. More examples and open problems will be provided in the concluding remark.

5.2 Main Results

The main result of this note is the following theorem, which verifies the conjecture made by Slamin et al. in 2006 [35]:

Theorem 5.2.1. *There exists a vertex-magic total labeling for the disjoint union of t (not necessarily isomorphic) sun graphs $S_{m_1} \cup S_{m_2} \cup \cdots \cup S_{m_t}$, for any positive integer t.*

Proof. Let G be the graph $S_{m_1} \cup S_{m_2} \cup \cdots \cup S_{m_t}$, and let $\sum_{i=1}^{t} m_i = n$. Hence the number of vertices is $2n$ and the number of edges is also $2n$. Assume that the pendant vertices are $u_1, u_2, \cdots, u_n$ and the vertices over the cycle are $v_1, v_2, \cdots, v_n$, so that $u_i v_i \in E(G)$ for each $1 \leq i \leq n$. Then we start doing the total labeling f to the vertices $u_1, u_2, \cdots, u_n$ using integers $2n+2, 2n+4, \cdots, 4n$ and to the edges $u_i v_i$ by $6n+1-f(u_i)$ for each $1 \leq i \leq n$. Therefore the weights over the vertices $u_1, u_2, \cdots, u_n$ are $6n+1$, which is the magic constant.

Now we do the labeling over the vertices and edges of the cycle as follows. First we assign an orientation on the cycles so that over each connected component the flow is either clockwise or counter-clockwise. Let e_i^{out} and e_i^{in} respectively to be the outgoing edge from the vertex v_i and the incoming edge to the vertex v_i, according to the given orientation. Precisely we give the labeling of e_i^{in} as follows:

$$f(e_i^{in}) = 4n - f(u_iv_i)$$

for each $1 \leq i \leq n$. Note that the range for the the edge labels of e_i^{in} is $\{1, 3, 5, \cdots, 2n-1\}$.

On the other hand, notice that there is a one-to-one correspondence between the edge labels of e_i^{out} and the edge labels of e_i^{in}. We may define the labeling of v_i as follows:

$$f(v_i) = 6n + 1 - (f(e_i^{out}) + f(e_i^{in}) + f(u_iv_i))$$

for each $1 \leq i \leq n$. Therefore $f(v_i)$ uses the rest of the labels $2, 4, 6, \cdots, 2n$ since $f(e_i^{in}) + f(u_iv_i) = 4n$ for each i, and $f(v_i) = 2n + 1 - f(e_i^{out})$. It is seen that $f(v_i)$ is uniquely determined by the label of e_i^{out}, hence uniquely determined by the label of e_i^{in}. The weights over these vertices $v_1, v_2, \cdots, v_n$ are also $6n+1$. Thus we have the desired vertex-magic total labeling.

$\square$

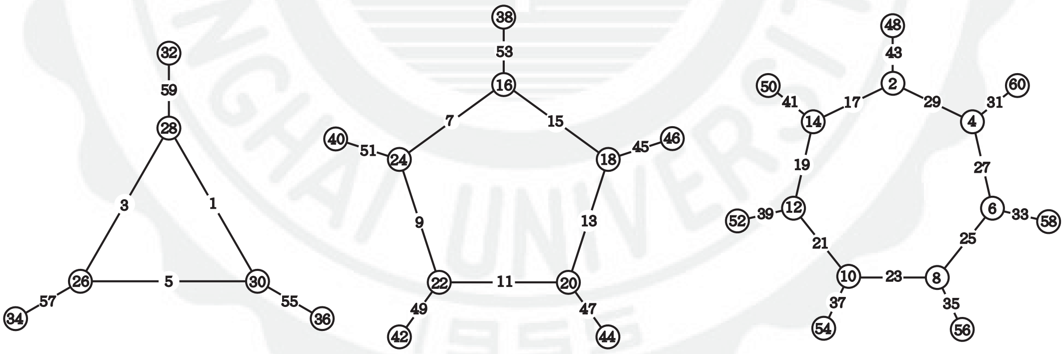

Figure 5.2.1: The Vertex-Magic Labeling of Disjoint Union of Three Suns

In order to obtain some technical lemmas for more examples of vertex-magic graphs, we define another type of related edge labeling:

Definition 5.2.2. For a graph $G = (V, E)$ with q edges and without any isolated vertex, an **antimagic** edge labeling is a bijection $f : E \to \{1, 2, \cdots, q\}$, such that the induced vertex sum $f^+ : V \to \mathbb{Z}^+$ given by $f^+(u) = \sum\{f(uv) : uv \in E\}$ is injective. A graph is called **antimagic** if it admits an antimagic labeling. If moreover for G the vertex sums form an arithmetic progression with initial term a and common difference d, we say G admits an (a, d)-**antimagic** labeling and G is (a, d)-**antimagic**.

Definition 5.2.3. We say for convenience that, a graph G admits (a, d)-**antimagic** vertex sums, if under certain edge labeling for G the associated vertex sums form an arithmetic progression of initial term a and common difference d.

Note that we have the following fact for assuring $(a, 1)$-antimagic-ness while adding extra even factor to an $(a, 1)$-antimagic graph, which was proved in [21] in 2006:

Lemma 5.2.4. (J. Ivančo, A. Semaničová, 2006) *Assume H is a graph which arose from a graph G of p vertices and q edges by adding an arbitrary $2k$-factor. If G is $(a, 1)$-antimagic, then H is $(a, 1)$-antimagic.*

We extend the above result as follows for assuring (a, d)-antimagic-ness while adding extra even factor to an (a, d)-antimagic graph:

Lemma 5.2.5. *Assume the graph G admits (a, d)-antimagic vertex sums via certain edge labeling. After adding an arbitrary 2-factor H by labeling the edges of H with another arithmetic progression with common difference d, the new graph $G \oplus H$ still admits (a, d)-antimagic vertex sums, i.e. an arithmetic progression vertex sums with common difference d.*

Proof. Let the (a,d)-antimagic vertex sums of G be $a, a+d, \cdots, a+(|V(G)|-1)d$. Also let the edge labels for H be $b, b+d, \cdots, b+(|V(G)|-1)d$. Then we we assign an orientation on the cycles of H so that over each connected component the flow is either clockwise or counter-clockwise. Let e_i^{out} and e_i^{in} respectively to be the edge from the vertex v_i and the incoming edge to the vertex v_i, according to the given orientation. Assume the original vertex sum at v_i is $w(v_i)$ for each i. We see the resulting vertex sum for $G \oplus H$ at the vertex v_i is $W(v_i) = w(v_i) + e_i^{in}$. We define

$$e_i^{in} = a + b + (|V(G)| - 1)d - w(v_i)$$

for each $1 \leq i \leq |V(G)|$. Since at the vertex v_i there is a one-to-one correspondence between the outgoing edge labels of e_i^{out} and the incoming edge labels of e_i^{in}, we see the resulting vertex sums $W(v_i)$ form an arithmetic progression vertex sums with common difference d.

□

Lemma 5.2.6. *Let the graph G admit (a,d)-antimagic vertex sums under certain edge labeling. After adding an arbitrary $2k$-factor H where k is a multiple of d, the new graph $G \oplus H$ still admits (a,d)-antimagic vertex sums. In particular, adding an arbitrary $2k$-factor to an (a,d)-antimagic graph still keeps the (a,d)-antimagic-ness.*

Proof. Let the graph G have p vertices and q edges. Then we see over the $2k$-factor H one can label the edges with $q+1, q+2, \cdots, q+kp$. Since $d|k$, let $k = td$. Now we split the above edge labels into t categories as follows:

$$q+1, \cdots\cdots, q+dp,$$

$$q+dp+1, \cdots\cdots, q+2dp,$$

$$q+2dp+1, \cdots\cdots, q+3dp,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$q+(t-1)dp+1,\cdots\cdots,q+tdp.$$

Again we split each of the t categories into d sub-categories, such that the (i,j)-th sub-category is the j-th sub-category of the i-th category. Note that the (i,j)-th sub-category is an arithmetic progression with initial term $q+(i-1)dp+j$ and common difference d for $1\le i\le t$ and $1\le j\le d$. Therefore from Lemma 5.2.5 one may check the graph $G\oplus H$ still admits (a,d)-antimagic vertex sums.

□

Therefore in case $d=1$ the above more general result goes back to the previous one in [21]. With the above lemmas we may push one step forward and provide with an infinite class of examples of the vertex magic total labeling by adding arbitrary $4k$-factors:

Theorem 5.2.7. *Let $G=(S_{m_1}\cup S_{m_2}\cup\cdots\cup S_{m_t})\oplus H$ be the graph consisting of the disjoint union of t (not necessarily isomorphic) suns and an arbitrary $4k$-factor H. Then G admits a vertex-magic total labeling.*

Proof. As previously in Theorem 5.2.1 let $\sum_{i=1}^{t} m_i=n$. Then the number of vertices for the graph $S_{m_1}\cup S_{m_2}\cup\cdots\cup S_{m_t}$ is $2n$ and the number of edges is also $2n$. In the proof of Theorem 5.2.1 we see for $S_{m_1}\cup S_{m_2}\cup\cdots\cup S_{m_t}$ one has the vertex-magic total labeling with magic constant $6n+1$ and the labelings used over vertices are consecutive even numbers $2,4,\cdots,2n,2n+2,\cdots,4n$. We remove these consecutive even numbers, and we see that with the remaining edge labels the graph admits $(a,2)$-antimagic vertex sums. Then by Lemma 5.2.6 after adding any arbitrary $4k$-factor H one still keep the $(a,2)$-antimagic-ness for vertex sums of edge labels. Finally according to the $(a,2)$-antimagic-ness for vertex sums over these vertices, we put back the consecutive even numbers $2,4,\cdots,2n,2n+2,\cdots,4n$ over all the vertices in reverse ascending order to make the constant sum, then the desired vertex-magic total labeling for the resulting graph $(S_{m_1} \cup S_{m_2} \cup \cdots \cup S_{m_t}) \oplus H$ is obtained.

□

5.3 Conclusion Remark

More properties of vertex magic total labellings for disconnected graphs can be explored based upon results in this paper and previous work. In particular one may further study such labelling for the corona products and other types of products of graphs, as we have the result for corona product of a 2-regular graph with a null graph (which is the disjoint union of suns) in this note.

Another remark is that the results in this note also valid for pseudo-graphs, that is, graphs with possibly multiple edges and loops. Please see the following Figure 5.3.1 for example. Note that the edge labels over the loops have to be counted twice while calculating the weight of the given vertex. Therefore via adding arbitrary $4k$-factors, the resulting vertex-magic graphs could be graphs with multiple edges and loops.

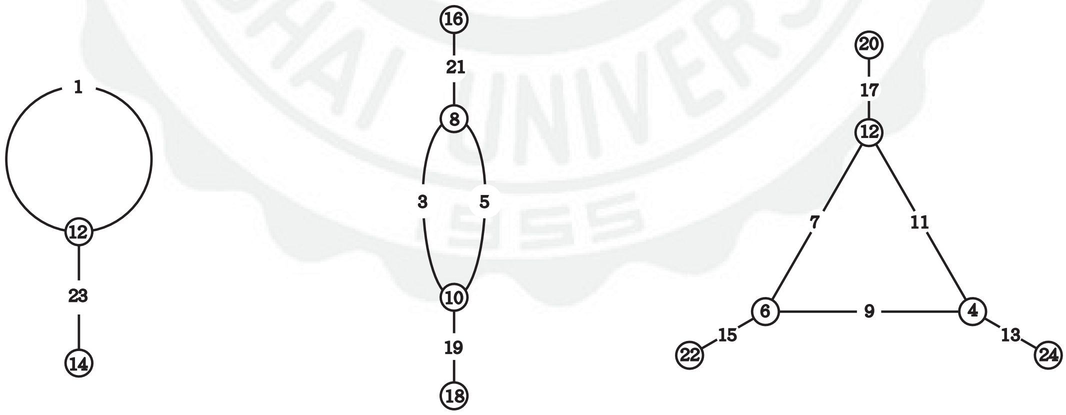

Figure 5.3.1: A Vertex-Magic Labeling of Pseudo-Suns with Magic Constant 37.

We also conjecture that the Theorem 5.2.7 is also valid for adding an arbitrary $2k$-factor instead of a $4k$-factor:

Conjecture. Let $G = (S_{m_1} \cup S_{m_2} \cup \cdots \cup S_{m_t}) \oplus H$ be the graph consisting of the disjoint union of t (not necessarily isomorphic) suns and an arbitrary $2k$-factor H. Then G admits a vertex-magic total labeling.

Chapter 6

Zero-Sum Flows

As an analogous concept of nowhere-zero flows for directed and bi-directed graphs, we consider zero-sum flows for undirected graphs in this article. For an undirected graph G, a **zero-sum k-flow** is an assignment of non-zero integers whose absolute values less than k to the edges, such that the sum of the values of all edges incident with each vertex is zero. Furthermore we generalize the notion via considering a combinatorial optimization problem, which is to calculate the **zero-sum minimum flow number** of a graph G, namely, the least integer k for which G may admit a zero-sum k-flow. The **Zero-Sum 6-Flow Conjecture** was raised by Akbari et al. in 2009: If a graph with a zero-sum flow, it admits a zero-sum 6-flow. It turns out that this conjecture was proved to be equivalent to the classical Bouchet 6-flow conjecture for bi-directed flows. We study zero-sum minimum flow numbers of graphs induced from plane tiling by regular hexagons in an arbitrary way, namely, the **hexagonal grid graphs**. In particular we are able to verify the Zero-Sum 6-Flow Conjecture for the class of hexagonal grid graphs by determining the zero-sum flow number of any non-trivial hexagonal grid graph is 3 or 4. We further use the concept of dual graphs to specify classes of infinite families of hexagonal grid graphs with minimum flow numbers 3 and 4 respectively. Further open problems are included. Note that the results in this chapter has been presented in the international conference FAW-AAIM 2013, Dalian, China, June 26-28, and already is published as **Zero-Sum Flow Numbers of Hexagonal Grids**, Lecture Notes in Computer Science (LNCS) 7924, pp. 339-349, 2013. (EI)

6.1 Background and Motivation

Let G be a directed graph. A **nowhere-zero flow** on G is an assignment of non-zero integers to each edge such that for every vertex the Kirchhoff current law holds, that is, the sum of the values of incoming edges is equal to the sum of the values of outgoing edges. A **nowhere-zero k-flow** is a nowhere-zero flow using edge labels with maximum absolute value $k-1$. Note that for a directed graph, admitting nowhere-zero flows is independent of the choice of the orientation, therefore one may consider such concept over the underlying undirected graph. A celebrated conjecture of Tutte in 1954 says that every bridgeless graph has a nowhere-zero 5-flow. F. Jaeger showed in 1979 that every bridgeless graph has a nowhere-zero 8-flow[22], and P. Seymour proved that every bridgeless graph has a nowhere-zero-6-flow[33] in 1981. However the original Tutte's conjecture remains open. There is a more general concept of a nowhere-zero flow that uses bidirected edges instead of directed ones, first systematically developed by Bouchet[9] in 1983. Bouchet raised the conjecture that every bidirected graph with a nowhere-zero integer flow has a nowhere-zero 6-flow, which is still unsettled.

Recently another analogous nowhere-zero flow concept has been studied, as a special case of bi-directed one, over the undirected graphs by S. Akbari et al.[1, 2] in 2009 and 2010, which is defined as follows:

Definition 6.1.1. For an undirected graph G, a **zero-sum flow** is an assignment of non-zero integers to the edges such that the sum of the values of all edges incident with each vertex is zero. A **zero-sum k-flow** is a zero-sum flow whose values are integers with absolute value less than k.

S. Akbari et al. raised a conjecture (called **Zero-Sum 6-Flow Conjecture**) for zero-sum flows similar to the Tutte's 5-flow Conjecture for nowhere-zero flows as follows: If G is a graph with a zero-sum flow, then G admits a zero-sum 6-flow. It was proved in 2010 by Akbari et al. [1] that the above Zero-Sum 6-Flow Conjecture is equivalent to the Bouchet's 6-Flow Conjecture for bidirected graphs, and the existence of zero-sum 7-flows for regular graphs were also obtained. Based upon the results, they raised another weaker conjecture for regular graphs: If G is a r-regular graph with $r \geq 3$, then G admits a zero-sum 5-flow.

In literature a more general concept **minimum flow number**, which is defined as the least integer k for which a graph may admit a k-flow, has been studied for both directed graphs and bidirected graphs. We extend the concept in 2011 to the undirected graphs and call it the zero-sum minimum flow number [43]:

Definition 6.1.2. Let G be a undirected graph. The **zero-sum minimum flow number** $F(G)$ is defined as the least number of k for which G may admit a zero-sum k-flow. $F(G) = \infty$ if no such k exists.

In particular we obtain a characterization of graphs with flow number 2, and also a characterization of 3-regular graphs with flow number 3 among other results[42]. Note that the related result were presented in the FAW 2012 conference by the first author in Beijing. We introduce the basic properties and previous results of the zero-sum minimum flow numbers in later section. On the other hand, it is well known that grids are extremely useful in all areas of computer science. One of the main usage, for example, is as the discrete approximation to a continuous domain or surface. Numerous algorithms in computer graphics, numerical analysis, computational geometry, robotics and other fields are based on grid computations.

It is known that there are only three possible types of regular tessellations, which are tilings made up of squares, equilateral triangles, and hexagons. We consider and study the minimum flow numbers of graphs induced from plane tiling by regular polygons in an arbitrary way. Formally, a square grid, or a **square grid graph** is induced by an arbitrary finite subset of the infinite integer lattice grid $\mathbb{Z} \times \mathbb{Z}$. The vertices of a square grid are the lattice points, and the edges connect the points which are at unit distance from each other. The infinite grid $\mathbb{Z} \times \mathbb{Z}$ may be viewed as the set of vertices of a regular tiling of the plane with unit squares. Another type is with equilateral triangles, which defines an infinite triangular grid in a similar way. A **triangular grid graph** is a graph induced by an arbitrary finite subset of the infinite triangular grid. One more type of plane tiling is with regular hexagons which defines an infinite hexagonal grid, and the graph induced by an arbitrary finite subset of the infinite hexagonal grid is called a **hexagonal grid graph**. (See Figure 6.1.1) A hexagonal grid graph is also named a **honeycomb graph** in literature. We pay attention to hexagonal grid graphs in this article.

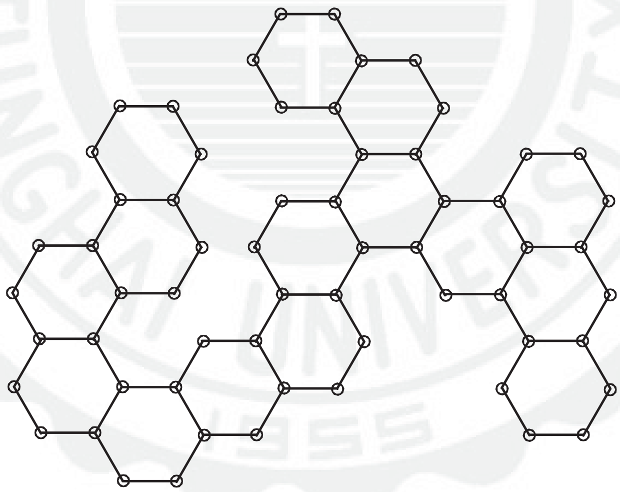

Figure 6.1.1: Example of a Hexagonal Grid Graph

Note that Akbari. et al. showed that in [2] if **Zero-Sum 6-Flow Conjecture** is true for $(2,3)$-graphs (in which every vertex is of degree 2 or 3), then it is true for any graph. Henceforth the study can be reduced to $(2,3)$-graphs. It is clear non-trivial hexagonal grid graphs are a special class of $(2,3)$-graphs. Therefore in this paper we focus the study over the zero-sum flow numbers for hexagonal grid graphs. In particular we are able to verify the Zero-Sum 6-Flow Conjecture for the class of hexagonal grid graphs by determining the zero-sum flow number of any non-trivial hexagonal grid graph is 3 or 4. We further use the concept of dual graphs to specify classes of infinite families of hexagonal grid graphs with minimum flow numbers 3 and 4 respectively.

6.2 Preliminaries of Zero-Sum Flow Numbers

In 2011 [43] we generalize the notion zero-sum flows by considering a combinatorial optimization problem, which is to find the **zero-sum minimum flow number** of a graph G, namely the least number of k for which G may admit a zero-sum k-flow. Obviously the zero-sum minimum flow numbers provide with more detailed information regarding zero-sum flows. For example, we may restate the previously mentioned Zero-Sum Conjecture as follow: Suppose a undirected graph G has a zero-sum flow, then $F(G) \leq 6$. We showed in [42] some general properties of small minimum flow numbers, so that the calculation of zero-sum minimum flow numbers becomes easier and efficient. In particular we obtained the following pretty useful technical lemma for the characterization of graphs with minimum flow number 2 which is used frequently in this paper, and we provide with a proof for completeness here:

Lemma 6.2.1. (T.-M. Wang and S.-W. Hu, [42]) *A graph G has zero-sum minimum flow number $F(G) = 2$ if and only if G is Eulerian with even size (even number of edges) in each component.*

Proof.

Without loss of generality, we may assume G is connected. We start showing the necessary part. Since a graph G has flow index $F(G) = 2$ meaning it admits a zero-sum 2-flow, thus the edge function $f(e) \in \{1, -1\}$.

For each vertex $v \in V(G)$, the number of incident edges labeled 1 must equal to the number of incident edges labeled -1. Note that both numbers are equal to $\frac{1}{2}deg(v)$, therefore $deg(v)$ must be even, and G is Eulerian. On the other hand, the number of all 1-edges (or (-1)-edges) in G is $\frac{1}{2}\sum_{v\in V(G)}(\frac{1}{2}deg(v)) = \frac{1}{2}|E(G)|$ which is an integer, so $|E(G)|$ are even. Conversely, to show the sufficiency we label the edges in an Euler tour of G by 1 and -1 alternatively. Then every vertex is incident with the same number of 1-edges and (-1)-edges, including the starting(ending) vertex, since the number of edges is even. Therefore it is a zero-sum 2-flow in G. □

In [42] we also calculate the zero-sum flow numbers of regular graphs, which is closely related to the zero-sum 5-flow conjecture for regular graphs. Recently it is known that the zero-sum 5-flow conjecture for regular graphs was nearly completely resolved by S. Akbari and other authors [3], except the case for 5-regular graphs. We study the zero-sum flows more recently and obtain certain results toward to these conjectures. Among other results we show that in [42] that every bridgeless 5-regular graph G admits a 5-flow, which strengthens the zero-sum 5-flow conjecture for regular graphs.

In next section we calculate the zero-sum minimum flow numbers for various types of graphs induced from the plane tiling by hexagons.

6.3 Zero-Sum Flow Numbers of Hexagonal Grid Graphs

It is well known for the notion of the dual graph $D(G)$ of a plane graph G for a fixed plane drawing representation of G embedded in a sphere or the plane. Note that generally the dual graph of a hexagonal grid is (a partial subgraph of) a triangular grid. See for example Figure 6.3.1.

Now we set up fundamental symbols for the trivial regular hexagon labeled ± 1 and ± 2 over the edges as in Figure 6.3.2. The symbol $\mathcal{I}$ stands

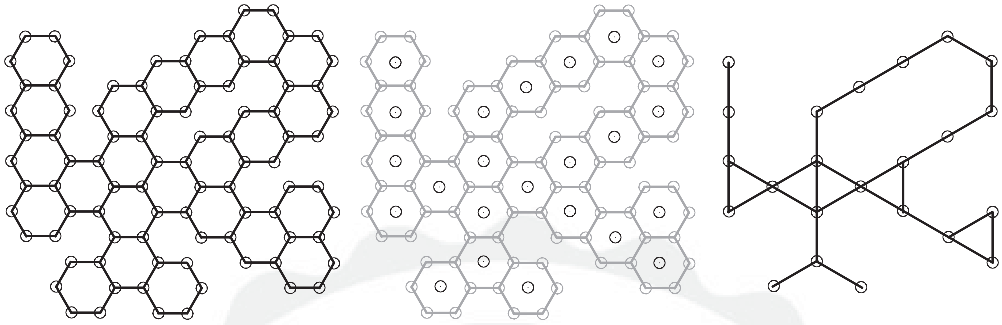

Figure 6.3.1: A Hexagonal Grid Graph with its Dual Graph

for the trivial regular hexagon edge-labeled 1 and -1 consecutively with zero-sums. $-\mathcal{I}$ and $\pm 2\mathcal{I}$ stand for the ones with zero-sums using labels of $\mathcal{I}$ multiplied by -1 and ± 2 respectively. Note that in figures below, the weight of the overlapping edge for any two neighboring fundamental symbols are summed up from both patterns.

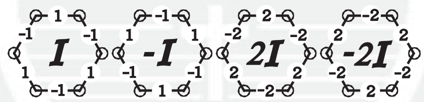

Figure 6.3.2: Fundamental Hexagons with Zero-Sum 2-Flows and 3-Flows

Note that also the zero-sum minimum flow number of the trivial regular hexagon is 2. The following gives the optimal upper bound for the minimum flow number of any finite non-trivial hexagonal grid graph:

Theorem 6.3.1. *The infinite hexagonal grid graph $\widetilde{H}$ admits a zero-sum 3-flow and $F(\widetilde{H}) = 3$. Moreover let H be any finite non-trivial hexagonal grid graph. Then $F(H) = 3$ or 4.*

Proof.

Note that one obtains a zero-sum flow of the whole figure while patching together sub-figures with zero-sums in an arbitrary way of union. See Figure 6.3.3 and note that the weight of the overlapping edge for any two neighboring fundamental symbols are summed up from both patterns. Therefore one has a zero-sum 3-flow for the infinite hexagonal grid graph $\widetilde{H}$ using the fundamental figures $\pm\mathcal{I}$. On the other hand, it is impossible for $\widetilde{H}$ to admit a 2-flow due to the existence of odd degree vertices. Thus $F(\widetilde{H}) = 3$.

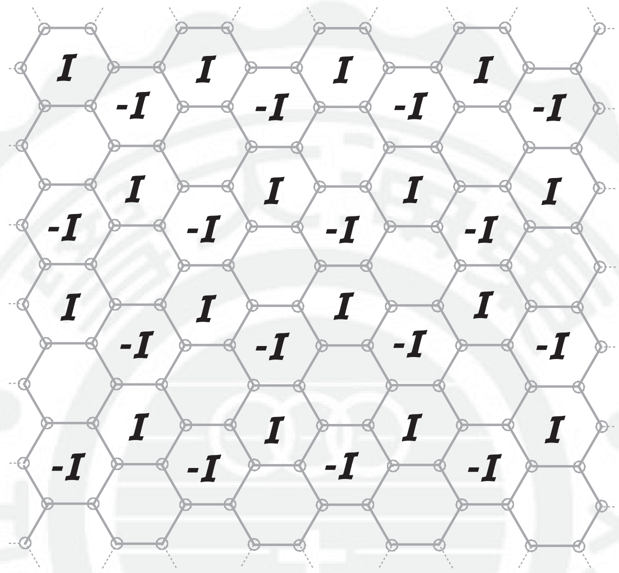

Figure 6.3.3: A 3-Flow of the Infinite Hexagonal Grid $\widetilde{H}$

As for any finite non-trivial hexagonal grid graph, we obtain the bounds for the flow numbers via the labeling of the infinite hexagonal grid. It is not hard to check as in Figure 6.3.4 one has a zero-sum 4-flow for the infinite hexagonal grid graph $\widetilde{H}$, using the fundamental figures $\pm\mathcal{I}$ and $\pm 2\mathcal{I}$. Note that again the weight of the overlapping edge for any two neighboring fundamental symbols are summed up from both patterns.

Note that then any finite non-trivial hexagonal grid graph H may be treated a piece of finite sub-figure cut from the infinite hexagonal grid $\widetilde{H}$. Therefore, H admits a zero-sum 4-flow using exactly the same edge labels induced from those of $\widetilde{H}$ (see Figure 6.3.5). Thus by Lemma 6.2.1 the minimum flow numbers are 3 or 4 except that the trivial regular hexagon has flow number 2 as indicated in $\mathcal{I}$.

□

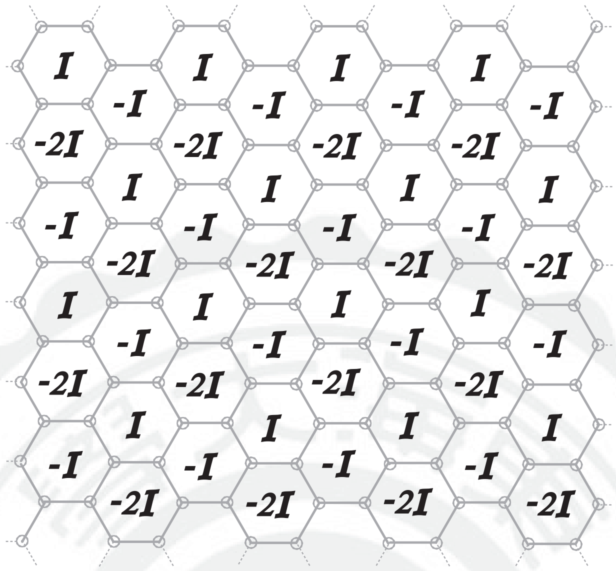

Figure 6.3.4: A 4-Flow of the Infinite Hexagonal Grid $\widetilde{H}$

We also determine various classes of infinite families of hexagonal grid graphs with flow numbers 3 and 4 respectively. First we start with classes of flow numbers 3:

Theorem 6.3.2. *Let G be a non-trivial hexagonal grid graph with the dual graph $D(G)$ to be bipartite. Then $F(G) = 3$.*

Proof.

Note that if the dual graph $D(G)$ is bipartite, it is 2-colorable. Then using $\pm\mathcal{I}$ as two colors to put over the vertices of the dual graph. We see G admits a 3-flow with edge labeling this way and again by Lemma 6.2.1 the zero-sum flow number is 3.

$\square$

Therefore we may easily have the following examples of flow numbers 3 since their dual graphs are trees, thus bipartite:

Theorem 6.3.3. *Let G be a non-trivial hexagonal grid graph with the dual*

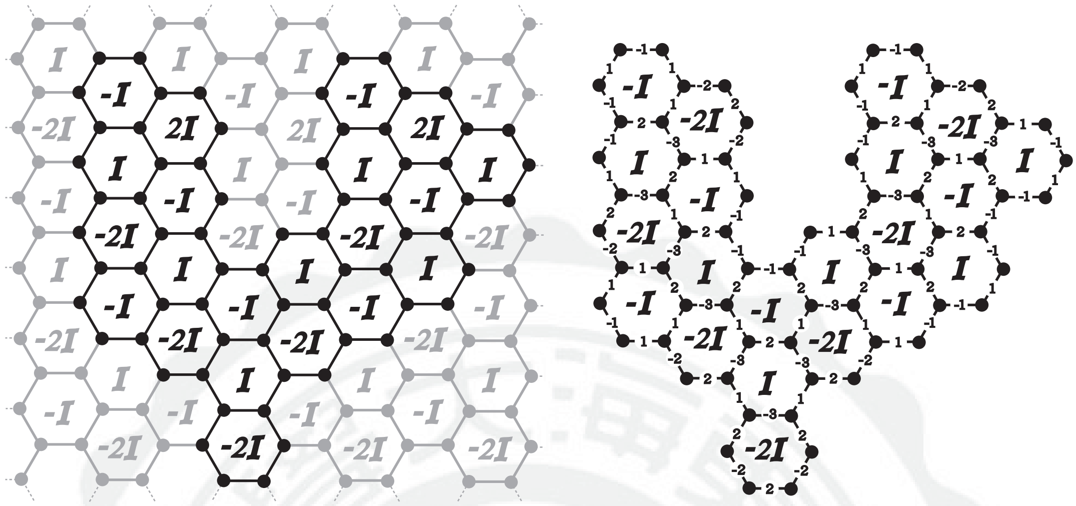

Figure 6.3.5: A 4-Flow of Arbitrary Finite Hexagonal Grid Induced from $\widetilde{H}$

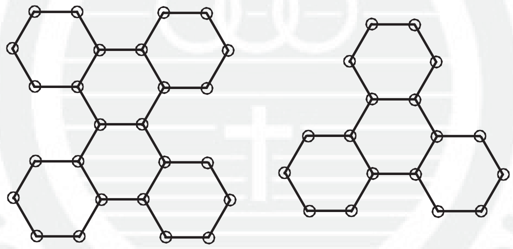

Figure 6.3.6: Examples of Hexagonal Grids with Flow Number 3

graph $D(G)$ consisting of multiple W_6 copies, for which one W_6 shares at most one edge with another copy of W_6. (see Figure 6.3.7) Then $F(G) = 3$.

Proof.

Note that if the dual graph $D(G)$ consists of W_6 copies for which one sharing at most one edge with another, one may fix it into a hexagonal grid graph by dropping the central vertex of each copy of W_6 (see Figure 6.3.7 for an example to reduce the dual graph). It is clear that the resulting reduced dual graph is bipartite. Hence by Theorem 6.3.2 we see $F(G) = 3$.

□

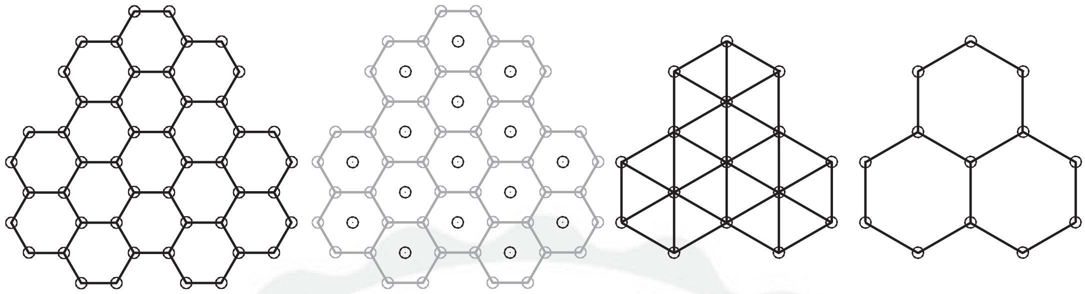

Figure 6.3.7: Example of Hexagonal Grid with Dual Graph Multiple W_6 Copies

The following are examples of classes of infinitely many hexagonal grids with flow number 4:

Theorem 6.3.4. *Let G be a hexagonal grid graph with the dual graph $D(G)$ which contains a triangle with one degree 2 vertex (see Figure 6.3.8). Then $F(G) = 4$.*

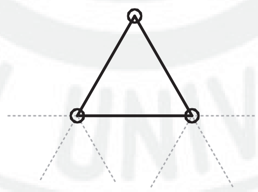

Figure 6.3.8: Dual graph $D(G)$ contains a triangle with one degree 2 vertex

Proof.

Assume G admits a zero-sum 3-flow, which allows only labels $\pm 1, \pm 2$. See Figure 6.3.9 without loss of generality may assume $a = 1$ or 2. In both cases through detailed calculation one will reach contradictions for c and d, for either $c + d = 0$ or $c + d = \pm 3$. Therefore $F(G) = 4$.

$\square$

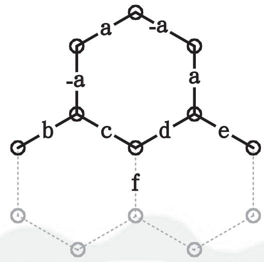

Figure 6.3.9: Example of G whose dual contains a triangle with one degree 2 vertex

Theorem 6.3.5. *Let G be a hexagonal grid graph with the dual graph $D(G)$. Suppose that $D(G)$ contains a kite with one degree 4 vertex or an antenna triangle with one degree 3 vertex (as in the Figure 6.3.10). Then $F(G) = 4$.*

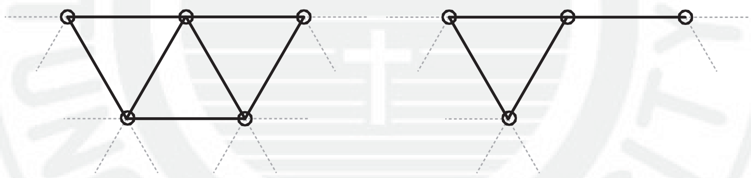

Figure 6.3.10: Dual graph $D(G)$ contains a Kite or an Antenna Triangle

Proof.

Assume G admits a zero-sum 3-flow. The common figure of a hexagonal grid graph containing a kite or an antenna triangle as its dual graph can be seen in the Figure 6.3.11. Then without loss of generality we may assume $a = 1$ or 2. In both cases through detailed calculation one will reach contradiction for g and h, for either $g + h = 0$ or $g + h = \pm 3$. Therefore $F(G) = 4$.

□

As corollary one may determine the flow numbers of regular hexagonal cluster grids H_n, which are the graphs in Figure 6.3.12. Note that H_n contains diagrams in Figure 6.3.11 for each $n \geq 3$. Thus by the Theorem 6.3.5 we

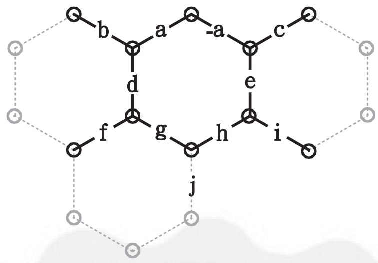

Figure 6.3.11: Hexagonal Grid with Kite or Antenna Triangle as its Dual Graph

have:

Corollary 6.3.6. *The minimum flow number of the regular hexagonal cluster grid H_n of n layers are as follows:*

$$F(H_n) = \begin{cases} 2, & n = 1. \\ 3, & n = 2. \\ 4, & n \geq 3. \end{cases}$$

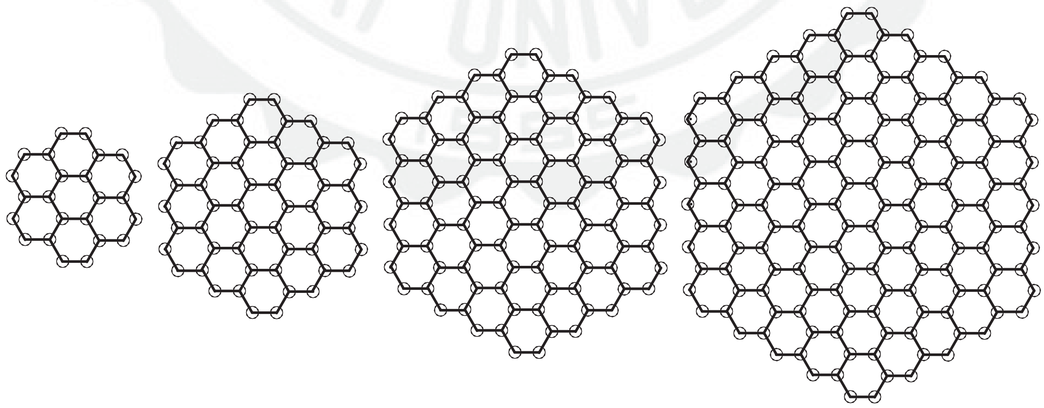

Figure 6.3.12: Hexagonal Cluster H_2, H_3, H_4, H_5

6.4 Concluding Remark and Open Problems

In this chapter we are able to determine that the zero-sum flow number of any non-trivial hexagonal grid graph is 3 or 4. We further find classes of infinite families of hexagonal grid graphs with minimum flow numbers 3 and 4 respectively. We also calculate as corollaries the zero-sum minimum flow numbers of infinite families of regular hexagonal grids.

However while one may calculate the zero-sum flow numbers of above classes of hexagonal grids, it is interesting to characterize completely the classes of non-trivial hexagonal graphs with zero-sum flow numbers 3 and 4 respectively. The zero-sum flow numbers of square grid graphs are not hard to calculate, while the complete characterizations of triangular grid graphs with various flow numbers and their optimal bounds are relatively nice open problems worth to work on.

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