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On Edge Clique Partitions and Set Representations of Graphs

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On Edge Clique Partitions and Set Representations of Graphs

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Abstract

In 1966 Erdös et al. [8] proved that the edge set of any simple graph G with n vertices, no one of which is isolated vertex, can be partitioned using at most $\lfloor n^2/4 \rfloor$ cliques. A couple of tens of years behind McGuinness [12] proved that any greedy clique partition is such a partition.

A multifamily representation of a graph G is a family of sets each member of which represent a vertex in G and the intersection relation of two members of which represent the adjacency of the two corresponding vertices in G. Erdös et al. [8] suggested a one-one correspondence between multifamily representations and clique coverings of a graph G. In fact, if we define multifamily representation of a multigraph M to be a family of sets each member of which represent a vertex in M and the number of elements in the intersection of two members of which represent the number of edges between the two corresponding vertices in M, then there is also a one-one correspondence between multifamily representations and clique partitions of M.

In section 1 we will narrate this correspondence in full detail.

If a multiafmily representation of a multigraph M has its members pairwisely distinct, then it is designated as a representation of M.

In section 2 we turn the aforementioned correspondence between multiafmily representations and clique partitions of M to account to prove that any multigraph M with at most one edge between any two vertices of it can be represented by at most $\lfloor n^2/4 \rfloor$ elements and we can accomplish such a representation from any greedy clique partition by a straightward method based on this correspondence.

An antichain representation of a multigraph M is a representation of M with the sets in it without pairwise inclusion relation. An uniform represen-

tation of M is a representation of M with all sets in it of the same cardinality.

In sections 3, 4, we utilize some immediately available results in the theory of finite linear space to characterize all manners for forming a representation, antichain representation, or uniform representation of a complete multigraph M with at most one edge between any two vertices of it using the smallest number of elements.

In section 5 we make use of the results in sections 3, 4 to prove that there is only one manner for representing a diamond-free multigraph M with at most one edge between any two vertices of it using the smallest number of elements.

In section 6 we make use of the results in sections 3, 4 to characterize all manners to form a representation or antichain representation of the line graph of one simple graph using the smallest number of elements so that the representation sets of any two adjacent vertices overlap on exactly one element.

Prisner [14] proved that hereditary clique-Helly graphs are exactly hereditary maximal-clique irreducible graphs, whereas there is a graph that is clique-Helly but not maximal-clique irreducible and vice versa.

In section 7 we investigate which graphs are clique-Helly but not hereditary clique-Helly and which graphs are maximal-clique irreducible but not hereditary maximal-clique irreducible.

Keywords: Clique; Covering; Partition; Line graph; Intersection graphs; Uniquely intersectable graphs; Clique-Helly graphs; Helly property; Hereditary clique-Helly graphs

1 Introduction

By an *multigraph* $M = (V(M); q)$, we mean a set V of *vertices* along with a function q defined in the following way. For each unordered pair $\{u, v\} \subset V$, let $q(u, v)$ be the number of *parallel edges* joining u with v. If $q(u, v) \neq 0$, then we say that $\{u, v\}$ is an *edge* of M and $q(u, v)$ is called the *multiplicity* of the edge $\{u, v\}$. In this paper we consider only *finite, undirected, simple* multigraphs, where simple means that $q(u, v) \leq 1$ for every $\{u, v\} \subset V$ and $q(u, u) = 0$ for every $u \in V(M)$. We denote the set of edges of M by $E(M)$, that is, in this paper $E(M) = \{\{u, v\} : q(u, v) = 1, u \neq v\}$. For a vertex subset $S \subseteq V(M)$, $\langle S \rangle_V$ denotes the *subgraph induced by* S. For a vertex v in M, $d_M(v)$ or $d(v)$ denote the *degree* of v in M. Let $\mathcal{F} = \{S_1, ..., S_p\}$ be a family of distinct nonempty subsets of a set X. Then $\mathbf{S}(\mathcal{F})$ denotes the union of sets in $\mathcal{F}$. The *intersection multigraph* of $\mathcal{F}$, denoted $\Omega(\mathcal{F})$, is defined by $V(\Omega(\mathcal{F})) = \mathcal{F}$, with $|S_i \cap S_j| = q(S_i, S_j)$ whenever $i \neq j$. Of course, so long as we are involved in this paper, $|S_i \cap S_j|$ always equal either 0 or 1 for all $i \neq j$, as appointed above. We say that a multigraph M is intersection multigraph on $\mathcal{F}$ if there exist a family $\mathcal{F}$ such that $M \cong \Omega(\mathcal{F})$; in this case we also say that $\mathcal{F}$ is a *representation* of the multigraph M. The *intersection number*, denoted $\omega(M)$ [*multifamily intersection number*, denoted $\omega_{\mathrm{m}}(M)$], of a given multigraph M is the minimum cardinality of a set X such that M is intersection multigraph [*multifamily intersection multigraph*] on a family $\mathcal{F}$ consisting of distinct [not necessarily distinct] subsets of X. In this case we also say that $\mathcal{F}$ is a *minimum representation* [*multifamily representation*] of M. We also consider *intersection multigraph on antichain*, i.e. family with no set in it contained in some other set in it, *uniform family*, i.e. family with all sets in it having the same cardinality and distinct from each other, and *uniform multifamily*, i.e. family with all sets in it having the same cardinality and not necessarily distinct from each other. And similarly we can define *antichain, uniform, uniform multifamily intersection multigraph* on a family $\mathcal{F}$ and *intersection number*, denoted $\omega_{\mathrm{ai}}(M)$, $\omega_{\mathrm{u}}(M)$, and $\omega_{\mathrm{um}}(M)$, respectively.

Note that given a representation $\{S_v \mid v \in V(M)\}$ of M and a vertex subset $S \subseteq V(M)$, then $\{S_v \mid v \in S\}$ form a representation of $\langle S \rangle_V$. Thus we know that $\omega(M)$ is not less than $\omega(\langle S \rangle_V)$ for any $S \subseteq V(M)$. Similarly for $\omega_{\mathrm{m}}(M)$, $\omega_{\mathrm{ai}}(M)$, $\omega_{\mathrm{u}}(M)$, and $\omega_{\mathrm{um}}(M)$.

We say that M is *uniquely intersectable* (*ui*), if given a set X with $|X| = \omega(M)$ and any two families α, β of subsets of X such that α and β are both representations of M then β can be obtained from α by a permutation of elements of X. At this time we call α, β to be isomorphic.

Similarly we define *uniquely intersectable with respect to multifamily* (*uim*), *antichain* (*uia*), *uniform multifamily* (*uium*), and *uniform family* (*uiu*).

2 PARTITION EDGE SET BY CLIQUES

Given a multigraph $M = (V(M); q)$, $Q \subseteq V(M)$ is said to be a *clique* of M if every pair of distinct vertices u, v in Q has $q(u, v) \neq 0$. A *clique partition* of a multigraph is a set $\mathcal{Q}$ of cliques such that every pair of distinct vertices u, v in $V(M)$ simultaneously appear in exactly $q(u, v)$ cliques in $\mathcal{Q}$ and for each *isolated vertex*, that is, vertex with no edge incident to it, we need to use at least one *trivial clique*, that is, clique with only one vertex, in $\mathcal{Q}$ to cover it. The minimum cardinality of a clique partition of M is called the *clique partition number* of M, and is denoted by $cp(M)$. This number must exist as the edge set of M forms a clique partition for M. We refer to a clique partition of M with the cardinality $cp(M)$ as a *minimum clique partition* of

M.

Note that a clique partition $\mathcal{Q}$ of M give rise to a clique partition of $M-v$ by deleting the vertex v from each clique in $\mathcal{Q}$. Thus $cp(M)$ is not less than the clique partition number of any induced subgraph of M.

Erdös et al. [8] proved the following theorem.

THEOREM 2.1. *The edge set of any simple graph G with n vertices no one of which is isolated vertex can be partitioned using at most $\lfloor n^2/4 \rfloor$ triangles and edges, and that the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ gives equality.*

We somewhat modify their proof to prove the following theorem. We use $M^{(n)}$ to denote a multigraph M with n vertices.

THEOREM 2.2. *Any multigraph M with $n \geq 4$ vertices and at most one edge between any two vertices of it (perhaps with isolated vertices) can be partitioned with at most $\lfloor n^2/4 \rfloor$ cliques $Q_1, ..., Q_N$ such that for any two vertices u, v in M, we have*

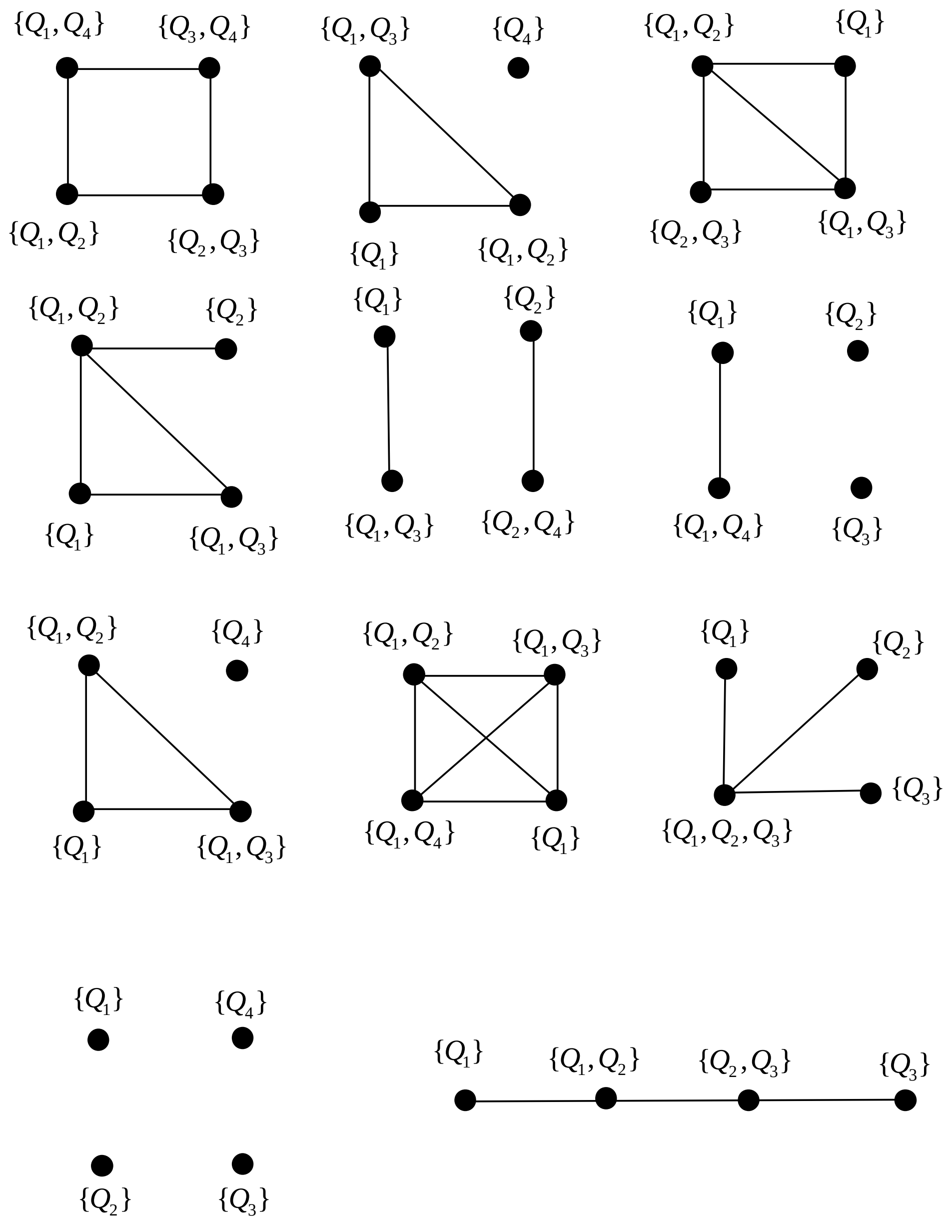
$$\begin{aligned} &\{Q_i \mid u \in Q_i \in \{Q_1, ..., Q_N\}\} \\ &\qquad \neq \quad \{Q_i \mid v \in Q_i \in \{Q_1, ..., Q_N\}\}. \end{aligned} \tag{1}$$

Further, in this partition we need only to use edges and triangles.

Proof. When $n = 4$, it is easy to draw all the 11 different multigraphs on 4 vertices, see figure 1, where the set attaching to each vertex u stand for

$$\{Q_i \mid u \in Q_i \in \{Q_1, ..., Q_n\}\}.$$

Thus for $n = 4$ our theorem hold.


$\{Q_1,Q_4\}$
$\{Q_3,Q_4\}$
$\{Q_1,Q_2\}$
$\{Q_2,Q_3\}$
$\{Q_1,Q_3\}$
$\{Q_4\}$
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$\{Q_1\}$
$\{Q_4\}$
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$\{Q_1,Q_2\}$
$\{Q_2,Q_3\}$
$\{Q_3\}$

Figure 1:

We proceed by induction from $n = 4$. First note that given any positive integer n,

$$\lfloor n^2/4 \rfloor = \lfloor (n-1)^2/4 \rfloor + \lfloor n/2 \rfloor.$$

Hence in the induction we should show that in going from $M^{(n-1)}$ to $M^{(n)}$ we need at most $\lfloor n/2 \rfloor$ more cliques.

In case that $M^{(n)}$ has a vertex of degree $\leq \lfloor n/2 \rfloor$, then at first we delete this vertex and all edges incident with it from $M^{(n)}$. Then by induction hypothesis we partition the resulting multigraph with at most $\lfloor (n-1)^2/4 \rfloor$ cliques as K_2 or K_3 with respect to (1). Then for going from $M^{(n-1)}$ to $M^{(n)}$ we need only to use the edges joining the deleted vertex to the other vertices of $M^{(n)}$ and then give rise to at most $\lfloor n/2 \rfloor$ more cliques as K_2. Clearly this resulting clique partition of $M^{(n)}$ still agree with the request of (1).

In the contrary case, every vertex of $M^{(n)}$ has degree $> \lfloor n/2 \rfloor$. Let x be the vertex with the minimum degree t, and set $t = \lfloor n/2 \rfloor + r$, where $r > 0$. Let x be adjacent to the vertices $y_1, ..., y_t$ and $M^{(t)}$ be the subgrapf of $M^{(n)}$ induced by $\{y_1, ..., y_t\}$.

We claim that $M^{(t)}$ has r edges no two of which have a common vertex. Assume that $M^{(t)}$ has only $r-1$ such edges (, the case that $M^{(t)}$ has less than $r-1$ such edges is similar), say

$$\{y_1, y_2\}, \{y_3, y_4\}, ..., \{y_{2r-3}, y_{2r-2}\}.$$

By $t = \lfloor n/2 \rfloor + r = d(x) \leq n-1$ (, since $M^{(n)}$ is without multiedges and loops), we know that $r \leq \lfloor n/2 \rfloor$ and thus $t \geq 2r$. Thus we can pick y_{2r-1} from $\{y_1, ..., y_t\}$.

By hypothesis, y_{2r-1} has degree $\geq \lfloor n/2 \rfloor + r$. But it can be adjacent to at most $2r-2$ of the vertices $y_1, ..., y_{2r-2}$ and to at most $n-t$ of the vertices not in $M^{(t)}$, hence the degree of y_{2r-1} is at most

$$\begin{aligned}(2r-2)+(n-t) &= (2r-2)+(n-(\lfloor n/2 \rfloor + r)) \\ &= (n-\lfloor n/2 \rfloor - 2) + r \\ &< \lfloor n/2 \rfloor + r.\end{aligned}$$

But $\lfloor n/2 \rfloor + r$ is the minimum degree. Hence y_{2r-1} is adjacent to some other vertex, say y_{2r}, in $M^{(t)}$ and

$$\{y_1, y_2\}, \{y_3, y_4\}, ..., \{y_{2r-3}, y_{2r-2}\}, \{y_{2r-1}, y_{2r}\}$$

is r edges in $M^{(t)}$ no two of which have a common vertex.

We remove these r edges from $M^{(n)} - x$. Partition the resulting multigraph with at most $\lfloor (n-1)^2/4 \rfloor$ cliques in terms of the request of (1).

Then the $\lfloor (n-1)^2/4 \rfloor$ cliques together with the triangles

$$\{x, y_1, y_2\}, \{x, y_3, y_4\}, ..., \{x, y_{2r-1}, y_{2r}\}$$

and the edges

$$\{x, y_k\}, \text{ where } 2r+1 \leq k \leq t,$$

form a clique partition, using at most

$$\begin{aligned}\lfloor (n-1)^2/4 \rfloor + r + (t-2r) \\ &= \lfloor (n-1)^2/4 \rfloor - r + (\lfloor n/2 \rfloor + r) \\ &= \lfloor n^2/4 \rfloor\end{aligned}$$

cliques.

Note that according to our convention in this paper, we need to use at least one trivial clique even for each isolated vertex for the clique partition of the multigraph $M^{(n)} - x$ with the r edges

$$\{y_1, y_2\}, \{y_3, y_4\}, ..., \{y_{2r-3}, y_{2r-2}\}, \{y_{2r-1}, y_{2r}\}$$

removed. Thus the resulting clique partition of $M^{(n)}$ given rise to above from the one of $M^{(n)} - x$ with the r edges removed must agree with the request of our theorem in the respect that for any two vertices u, v in $M^{(n)}$, we have

$$\begin{aligned}\{Q_i \mid u \in Q_i \in \{Q_1, ..., Q_N\}\} \\ &\neq \quad \{Q_i \mid v \in Q_i \in \{Q_1, ..., Q_N\}\}\end{aligned}$$

□

In the above theorem, we focus on the case that $n \geq 4$. For $n = 2$ [$n = 3$], clearly we need at least two [three] cliques for a clique partition agreeing with the request of (1).

We prove that the number $\lfloor n^2/4 \rfloor$ in theorem 2.2 cannot be replaced by any smaller number. Let $n = 2k$ or $2k + 1$. We consider the comlete bipartite multigraphs $M_{k,k}$ and $M_{k,k+1}$, which have $2k$ and $2k+1$ vertices in total, respectively. Clearly these two multigraphs have no triangle and their numbers of edges are

$$k^2 = \lfloor (2k)^2/4 \rfloor = \lfloor n^2/4 \rfloor, \text{ if } n = 2k,$$

and

$$k(k+1) = \lfloor (2k+1)^2/4 \rfloor = \lfloor n^2/4 \rfloor, \text{ if } n = 2k+1.$$

Hence the two multigraphs will always require $\lfloor n^2/4 \rfloor$ cliques for a clique partition.

Now we introduce a one-one correspondence between multifamily representations and clique partitions of a multigraph M as following.

Given a multigraph $M^{(n)} = (V(M); q)$, we at first construct a clique partition

$$\mathcal{Q} = \{Q_1, ..., Q_p\}$$

of it. Then with each clique Q_k we associate an element e_k and with each vertex v_α we associate a set $S_{\mathcal{Q}}(v_\alpha)$ of elements e_k, where

$$e_k \in S_{\mathcal{Q}}(v_\alpha) \Leftrightarrow v_\alpha \in Q_k,$$

i.e., $S_{\mathcal{Q}}(v_\alpha)$ is the collection of those elements for which the corresponding cliques contains v_α. Thus we obtain

$$\mathcal{F}(\mathcal{Q}) \equiv \{S_{\mathcal{Q}}(v) : v \in V(M)\}.$$

Then clearly

$$\mathbf{S}(\mathcal{F}(\mathcal{Q})) \equiv \bigcup_{v \in V(M)} S_{\mathcal{Q}}(v)$$

contains p elements. And

$$|S_{\mathcal{Q}}(v_\alpha) \cap S_{\mathcal{Q}}(v_\beta)| = q(v_\alpha, v_\beta),$$

since there is exactly $q(v_\alpha, v_\beta)$ cliques simultaneously containing the two vertices v_α, v_β. Thus we have constructed a multifamily representation

$$\mathcal{F}(\mathcal{Q}) = \{S_{\mathcal{Q}}(v) : v \in V(M)\}$$

from the clique partition $\mathcal{Q}$ of M, where

$$|\mathbf{S}(\mathcal{F}(\mathcal{Q}))| \equiv |\bigcup_{v \in V(M)} S_{\mathcal{Q}}(v)| = p = |\mathcal{Q}|.$$

Conversely, given a multifamily representation $\mathcal{F} = \{S_1, ...S_n\}$ of M with vertex set $V(M) = \{v_1, ..., v_n\}$, where S_α correspond to the set attaching to v_α, then we can also construct a clique partition of M by the following way.

Let

$$\mathbf{S}(\mathcal{F}) \equiv \bigcup_{\alpha=1}^{n} S_\alpha = \{e_1, ..., e_p\}.$$

For each fixed e_k in $\mathbf{S}(\mathcal{F})$ we form a clique $Q_{\mathcal{F}}(e_k)$ using those vertices v_α such that the set S_α attaching to it contains e_k. Clearly each $Q_{\mathcal{F}}(e_k)$ is indeed a clique of M. Thus we obtain

$$\mathcal{Q}(\mathcal{F}) = \{Q_{\mathcal{F}}(e_1), ..., Q_{\mathcal{F}}(e_p)\}.$$

And

$$q(v_\alpha, v_\beta) = |S_\alpha \cap S_\beta|$$

$$= \text{the number of cliques in } \mathcal{Q}(\mathcal{F}) \text{ simultaneously containing } v_\alpha, v_\beta,$$

since each element in S_α exactly represent a clique in $\mathcal{Q}(\mathcal{F})$ containing v_α. Thus we have constructed a clique partition $\mathcal{Q}(\mathcal{F})$ of M from the multifamily representation $\mathcal{F}$ of M, where

$$|\mathcal{Q}(\mathcal{F})| = p = |\bigcup_{\alpha=1}^{n} S_\alpha| \equiv |\mathbf{S}(\mathcal{F})|.$$

Thus we have established a one-one correspondence between multifamily representations and clique partitions of the multigraph M.

From above we know that $\omega_{\mathrm{m}}(M) = cp(M)$.

THEOREM 2.3. *Let M be a multigraph with at most one edge between any two vertices of it. Then $\omega_{m}(M) = cp(M)$.*

If we are given a multigraph $M^{(n)}$, then by theorem 2.2 we can obtain a clique partition $\mathcal{Q}$ with cardinality less than or equal to $\lfloor n^2/4 \rfloor$, agreeing with the request of (1). Then by the above method we can obtain a representation $\mathcal{F}(\mathcal{Q}) = \{S_{\mathcal{Q}}(v) : v \in V(M)\}$ of M consisting of distinct sets. Thus we obtain the following theorem.

THEOREM 2.4. *Let M be a multigraph with at most one edge between any two vertices of it. Then $\omega(M^{(n)}) \leq \lfloor n^2/4 \rfloor$.*

To see that $\lfloor n^2/4 \rfloor$ is the smallest number for which theorem 2.4 is true, we again consider the two complete bipartite multigraphs $M_{k,k}$ and $M_{k,k+1}$. Here each edge must give rise to at least one element, for if v_α and v_β are adjacent, then $S_\alpha \cap S_\beta$ contains some element $e_{\alpha\beta}$. But if this element were present in the set corresponding to any vertex, say v_γ, other than v_α and v_β, then $v_\alpha, v_\beta, v_\gamma$ would be vertices of a triangle in $M_{k,k}$ or $M_{k,k+1}$. But these two multigraphs contain no triangle. Hence each edge in $M_{k,k}$ and $M_{k,k+1}$ give rise to at least one new element. Hence any representation of $M_{k,k}$ and $M_{k,k+1}$ must use at least $\lfloor n^2/4 \rfloor$ elements.

One may not be contented with the above theorem and would rather ask that how to secure a representation of $M^{(n)}$ using at most $\lfloor n^2/4 \rfloor$ elements. In fact, McGuinness had regarded this problem in respect of the original theorem in Erdös et al. [8] having been the motive of our theorem 2.2. In McGuinness [12] he proved the following theorem.

THEOREM 2.5. *Every greedy clique partition of an n-vertex graph uses at most $\lfloor n^2/4 \rfloor$ cliques.*

In this theorem, the so-called "a greedy clique partition of a graph $G^{(n)}$" mean an ordered set $\mathbf{Q} = \{Q_1, ..., Q_m\}$ such that each Q_i is a maximal clique in $G - \bigcup_{j<i} E(Q_j)$, where $G - \bigcup_{j<i} E(Q_j)$ means the subgraph of G obtained by deleting all edges in the edge subet $\bigcup_{j<i} E(Q_j)$ while leaving all vertices in G preserved.

Here we also prove the following theorem, where for a representation $\mathcal{F}$ of M we referred to those elements in $\mathbf{S}(\mathcal{F})$ which appear in only one member of $\mathcal{F}$ as *monopolized elements.*

THEOREM 2.6. *Every representation $\mathcal{F}$ of $M^{(n)}$ with $n \geq 4$ derived from $\mathcal{F}(\boldsymbol{Q})$, where $\boldsymbol{Q}$ is any greedy clique partition of $M^{(n)}$, by successively attaching monopolized elements to the sets which repetitiously occur in $\mathcal{F}(\boldsymbol{Q})$, where note that by this method, provided that there are k sets in $\mathcal{F}(\boldsymbol{Q})$ being identical with each other, we need only $k-1$ monopolized elements rather than k, uses at most $\lfloor n^2/4 \rfloor$ elements.*

LEMMA 2.7. *For any clique partition $\mathcal{Q}$, $\mathcal{F}(\mathcal{Q}) = \{S_{\mathcal{Q}}(v) : v \in V(M)\}$ has two identical sets, say $S_{\mathcal{Q}}(u), S_{\mathcal{Q}}(v)$, in it only if the clique in $\mathcal{Q}$ simultaneously containing u, v is a maximal clique, say Q_{uv}, in M and has u, v as its monopolized elements, that is, u, v are in no clique of $\mathcal{Q}$ except Q_{uv}, implying that all vertices adjacent to u, v in M are just all vertices in $Q_{uv} - \{u, v\}$.*

Proof. If there is a clique Q' properly containing Q_{uv} in M, say vertex w being in Q' but not in Q_{uv}, then no clique in $\mathcal{Q}$ can simultaneously contain the three vertices u, v, w. Thus the clique in $\mathcal{Q}$ simultaneously containing u, w doesn't contain v and the clique in $\mathcal{Q}$ simultaneously containing v, w doesn't contain u, and therefore we must have $S_{\mathcal{Q}}(u) \neq S_{\mathcal{Q}}(v)$.

If u, say, belong to one clique Q'' in $\mathcal{Q}$ other than Q_{uv}, then there is a vertex, say u', adjacent to u and not in Q_{uv}. In case that u' is not adjacent to v we must have $S_{\mathcal{Q}}(u) \neq S_{\mathcal{Q}}(v)$. In case that u' is adjacent to v, then no clique in $\mathcal{Q}$ can simultaneously contain u, v, u'. Thus the clique in $\mathcal{Q}$ simultaneously containing u, u' doesn't contain v and the clique in $\mathcal{Q}$ simultaneously containing u', v doesn't contain u, and therefore we must have $S_{\mathcal{Q}}(u) \neq S_{\mathcal{Q}}(v)$. □

Proof of theorem 2.6:

Proof. We use induction on n.

When $n = 4$, it is an easy matter to draw all the 11 different graphs on four vertices and to check that every representation of any one of them derived from the method of theorem 2.6 uses at most $\lfloor n^2/4 \rfloor$ elements, see figure 1. As for $n = 5$, the number of nonisomorphic graphs is sufficiently large to make a reduction being desired. Note that $\lfloor 5^2/4 \rfloor - \lfloor 4^2/4 \rfloor = 6 - 4 = 2$ and therefore we have two new elements in proceeding from $n = 4$ to $n = 5$. If $M^{(5)}$ has one vertex with degree 2 or less, then we reduce $M^{(5)}$ to $M^{(4)}$ by deleting this vertex and all edges incident to it and then obtain one of the 11 graphs in figure 1. This vertex form a maximal clique in $M^{(5)}$ with some edge in $M^{(4)}$ only if $M^{(5)}$ is one of the 13 nonisomorphic graphs in figure 2, where hollow circle denote this vertex and dashed lines denote the edges incident to it. It is easy to check that every representation of any one of them derived from the method of theorem 2.6 uses at most $\lfloor 5^2/4 \rfloor = 6$ elements.

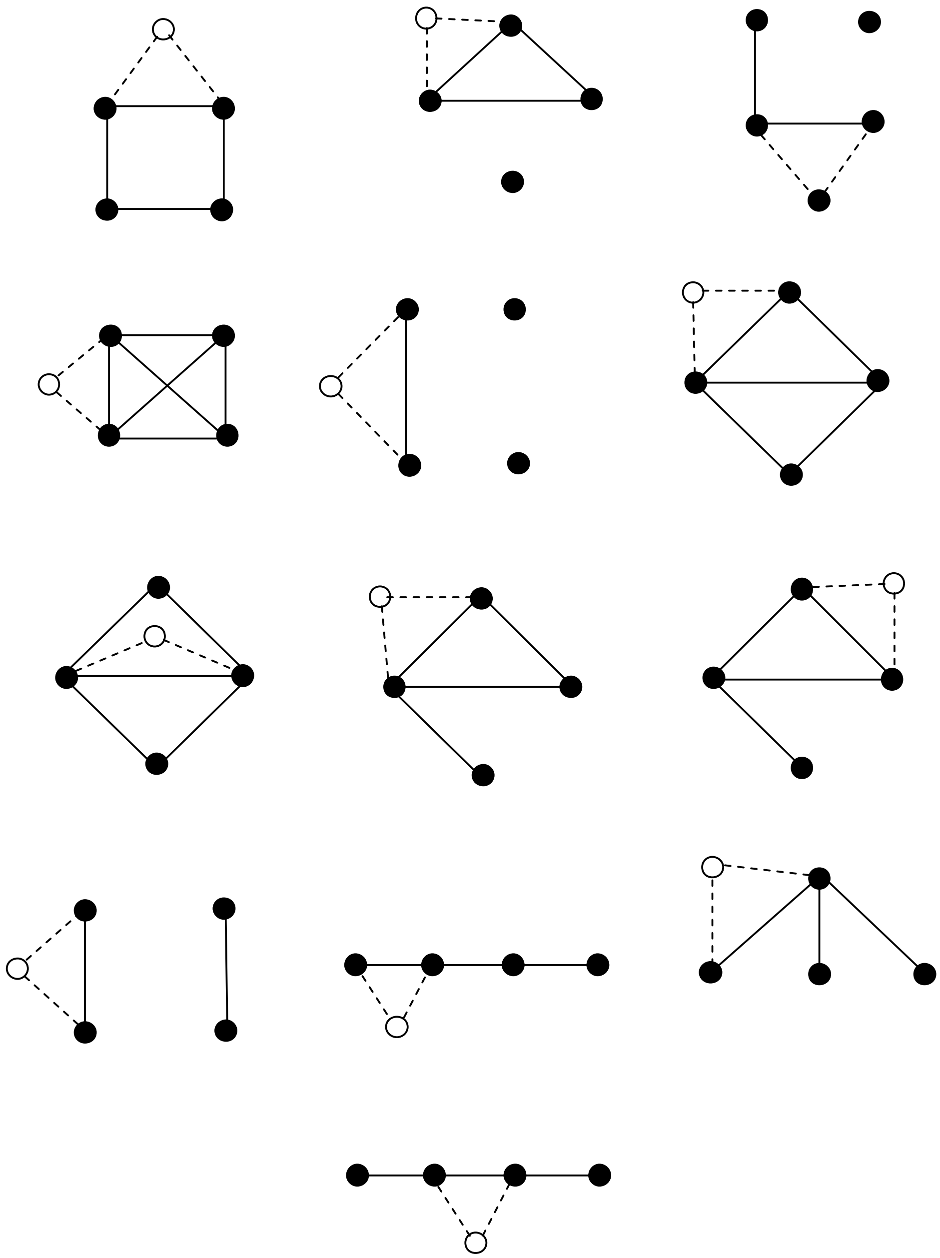


Figure 2:

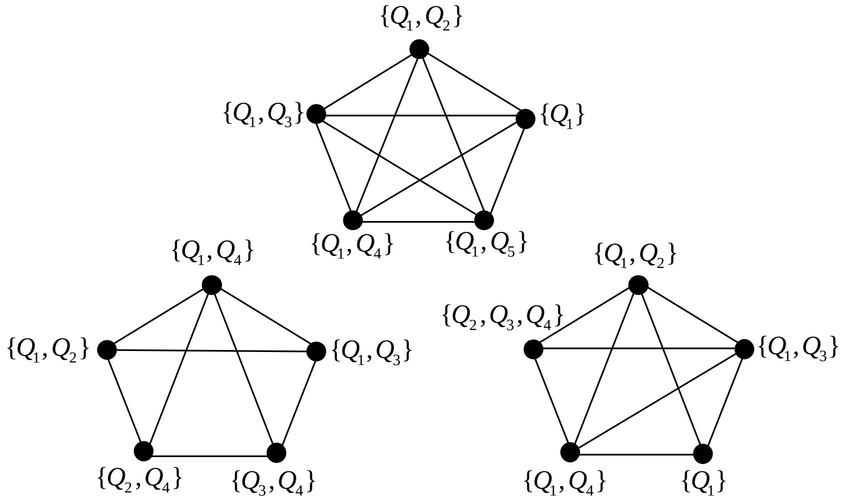


Figure 3:

As for the case that there is no maximal clique in $M^{(5)}$ simultaneously containing this vertex and some edge in $M^{(4)}$, then in any greedy clique partition of $M^{(5)}$ we must use all the edges incident to this vertex as members of this greedy clique partition. Thus in this case in order to obtain any representation derived from the method of theorem 2.6, we can at first form a representation of $M^{(4)}$ by the method of theorem 2.6 and then go back to $M^{(5)}$ using the available two new elements to represent at most two edges incident to this vertex and thus we can affirm that in this case all representations of $M^{(5)}$ derived from the method of theorem 2.6 uses at most $\lfloor 5^2/4 \rfloor = 6$ elements.

As for the case that there is no edge in $M^{(5)}$ incident to this vertex, we can at first form a representation of $M^{(4)}$ by the method of theorem 2.6 and then go back to $M^{(5)}$ using one new monopolized element on this vertex.

Due to above, now we need to consider only those multigraphs on five vertices for which every vertex has degree greater than or equal to 3. There are only three such graphs and these are easy to discuss, see figure 3.

Thus we have proved the theorem for $n = 4, 5$.

Now let $\mathcal{F}$ be a representation of $M^{(n)}$ with $n \geq 6$ derived from $\mathcal{F}(\mathbf{Q})$, where $\mathbf{Q} = \{Q_1, ..., Q_m\}$ is a greedy clique partition of $M^{(n)}$, by the method of theorem 2.6. Note that deleting Q_j from the set $\mathbf{Q}$ leaves a greedy clique partition of $M - E(Q_j)$.

In case that every Q_j has at least three edges, we have $m \leq \binom{n}{2}/3 < n^2/6$. Assume for the time being that every Q_i has exactly three edges, that is, is exactly a triangle. Now if every triangle in $\mathbf{Q}$ has at most one of its three vertices of degree 2, then by lemma 2.7 we needn't use any monopolized element in the method of theorem 2.6 for this greedy clique partition. If there is a triangle in $\mathbf{Q}$ with at least two of its three vertices of degree 2, then recalling that $M^{(n)}$ have at least six vertices, two vertices of degree 2 in this triangle make m be less than or equal to $\left(\binom{n}{2}/3\right) - 2 < (n^2/6) - 2$. Thus despite that we maybe need two more monopolized elements for this triangle, yet in the same time we also have two less cliques (as K_3) in $\mathbf{Q}$. Besides, if there is a clique of cardinality $3 + r$ where $r > 0$ in $\mathbf{Q}$, then despite that we maybe need r more monopolized elements for this clique, yet in the same time by the fact that $\binom{3+r}{2} \geq 3(r+1)$, we also have r less cliques (as K_3)in $\mathbf{Q}$, where note that $\binom{3+r}{2}$ is the number of edges in a clique of cardinality $3 + r$ and $3(r + 1)$ is the total number of edges in $r + 1$ triangles. In fact, we may need rather $r + 1$ or $r + 2$ than r more monopolized elements for this clique of cardinality $3 + r$. By lemma 2.7, we need use $r + 2$ more monopolized elements for this clique only when either this clique is a isolated clique or $M^{(n)}$ is itself a clique. For the latter case, in the method of theorem 2.6 we use n elements to represent $M^{(n)}$ and note that $n^2/6 \geq n$ for $n \geq 6$. As for the former case, we lose all the edges joining this isolated clique to all the vertices not on this isolated clique, therefore we lose at least 5 edges from the calculated $\binom{n}{2}$ edges and hence further lose at least two cliques from the calculated $n^2/6$ cliques (as K_3). Besides, by lemma 2.7, we need use $r+1$ more monopolized elements for this clique only when this clique has exactly $r+2$ vertices of degree $(3+r)-1$. In this case, this clique has a vertex v adjacent to one vertex, say v', not in this clique, and all vertices in this cliques other than v are not adjacent to v'. Therefore in $M^{(n)}$ we have $r+2 \geq 3$ less edges than complete graph K_n, and thus we have still one less triangle in $\mathbf{Q}$. Now we have brought to the conclusion that in case that each Q_j has at least three edges, we never use more than $n^2/6$ elements in the method of theorem 2.6 in order to form a representation of $M^{(n)}$.

Now we have secured a justification for assuming that some Q_j is an edge xy.

In case that $d(x) = d(y) = 1$, we can at first form a representation of $M^{(n)} - x - y$ by the method of theorem 2.6 using at most $\lfloor (n-2)^2/4 \rfloor$ elements, and then use two new elements for the isolated edge xy to form a representation for $M^{(n)}$ using at most $\lfloor n^2/4 \rfloor$ elements. Thus in this case every representation of $M^{(n)}$ derived from $\mathcal{F}(\mathbf{Q})$, where $\mathbf{Q}$ is any greedy clique partition of $M^{(n)}$, by the method of theorem 2.6 uses at most $\lfloor n^2/4 \rfloor$ elements.

As for the case that one of x, y has degree more than one, in any representation of $M^{(n)}$ derived from the method of theorem 2.6 we can't use monopolized element on x or y. Now let R consist of the members of $\mathbf{Q} - \{Q_j\}$ that are incident to x, and S consist of those incident to y. Then the set

$$\mathbf{Q}' \equiv \mathbf{Q} - (R \cup S \cup \{Q_j\})$$

is a greedy clique partition of

$$M' \equiv (M^{(n)} - x - y) - \bigcup_{Q_i \in R \text{ or } S} E(Q_i),$$

except possibly leaving some isolated vertices in M' uncovered by any members of $\mathbf{Q}'$. Recall that under the present case, in $\mathcal{F}$ we never use monopolize element on x, y. Now only if we can prove that every monopolized elementd in $\mathbf{S}(\mathcal{F})$ is always necessary for deriving a representation of M' from $\mathcal{F}(\mathbf{Q}')$ by the method of theorem 2.6, then by induction hypothesis we can prove that

$$|\mathcal{Q}(\mathcal{F}) - (R \cup S \cup \{Q_j\})| \leq \lfloor (n-2)^2/4 \rfloor. \tag{2}$$

If in $\mathcal{F}$ we had used one monopolized element on some vertex v not belonging to any member of $R \cup S$, then in $\mathcal{F}(\mathbf{Q})$ $S_{\mathbf{Q}}(v)$ must be identical with some $S_{\mathbf{Q}}(u)$ where u is also a vertex not belonging to any member of $R \cup S$. Since both u and v don't belong to any member of $R \cup S$, in $\mathcal{F}(\mathcal{Q}')$ $S_{\mathbf{Q}'}(u) = S_{\mathbf{Q}'}(v)$. Thus this monopolized element is necessary for deriving a representation of M' from $\mathcal{F}(\mathbf{Q}')$ by the method of theorem 2.6.

If in $\mathcal{F}$ we had used one monopolized element on some vertex v belonging to one member, say Q_v, of $R \cup S$, then in $\mathcal{F}(\mathbf{Q})$ $S_{\mathbf{Q}}(v)$ must be identical with some $S_{\mathbf{Q}}(u)$ where u is also a vertex belonging to Q_v. Now by lemma 2.7 v must have all its neighbors in Q_v. Thus v is an isolated vertex in M'. Thus this monopolized element is necessary for deriving a representation of M' from $\mathcal{F}(\mathbf{Q}')$ by the method of theorem 2.6.

Thus we have proved (2).

Now it suffices to prove that

$$|R \cup S| \leq n-2,$$

since

$$n-2 \leq \lfloor n^2/4 \rfloor - \lfloor (n-2)^2/4 \rfloor - 1.$$

We prove this by choosing distinct vertices in $V(M)-\{x,y\}$ from the vertex sets of the members of $R \cup S$. Note that since each edge is covered exactly once in a clique partition, each $v \notin \{x,y\}$ appears once in R if v is adjacent to x and once in S if v is adjacent to y. Consieder $Q_1 \in R$. If Q_1 contains a vertex v not adjacent to y, then we choose such a v for Q_1. If all vertices in Q_1 are adjacent to y, then we choose for Q_1 a vertex $v \in Q_1$ such that vy belongs to the first member of $\mathbf{Q}$, say Q_2, that contains both y and some vertex of Q_1. Note that Q_2 is the only member of S containing v.

Now we have two cases, that is, either that Q_1 precedes xy in $\mathbf{Q}$ or that xy precedes Q_1 in $\mathbf{Q}$. For the first case, since Q_1 and xy are maximal when chosen, Q_2 must precedes Q_1 in $\mathbf{Q}$ for otherwise from the aforementioned hypothesis that all vertices in Q_1 are adjacent to y and Q_1 precedes xy in $\mathbf{Q}$, Q_1 should have contained y and hence xy. For the second case, since xy is maximal when chosen, one of Q_1, Q_2 precedes xy or otherwise xy should have contained v. Thus in this case Q_2 precedes Q_1 in $\mathbf{Q}$. Note that in both above cases, we have that Q_2 precedes both of Q_1, xy in $\mathbf{Q}$.

For the members of S, similarly as above choose vertices by reversing the roles of x and y.

In above we have shown that if v belongs to some $Q_1 \in R$ and to some $Q_2 \in S$, and v is chosen for one of them, then the one for which it is chosen occurs after the other one in the ordered set $\mathbf{Q}$. Hence no vertex is chosen twice. Thus we conclude that

$$|R \cup S| \leq n-2$$

. □

3 Clique partition of complete multigraph K_n and finite linear space.

A *[finite] linear sapce* $\Gamma = (\mathcal{P}, \mathcal{L})$ is a system consisting of a [finite] point set $\mathcal{P}$ of n *points* and a line set $\mathcal{L}$ of *lines* satisfying the following axioms.

(L1) Any line has at least two points.

(L2) Two points are on precisely one line.

(L3) Any line has at most $n-1$ points.

If a space satisfy **(L1)** and **(L2)** but not **(L3)**, then clearly this space contain a unique line. This type of spaces is referred to as *trivial linear space.*

Suppose that $n \geq 3$. Let $\mathcal{Q}$ be a clique partition of K_n such that each member of $\mathcal{Q}$ has at least 2 and no more than $n-1$ vertices. Let $\Gamma(\mathcal{Q})$ be the system whose set of points is the vertex set of K_n, and whose lines are the members of $\mathcal{Q}$. Incidence is defined as following. A points v is incident with a line Q if v is a vertex of Q. Then $\Gamma(\mathcal{Q})$ is a finite linear space. Conversely, if Γ is a finite linear space on n points, then there is a clique partition $\mathcal{Q}$ of K_n such that $\Gamma = \Gamma(\mathcal{Q})$, where each member of $\mathcal{Q}$ has at least 2 and no more than $n-1$ vertices.

Thus there is a one-one correspondence between all clique partitions of K_n by cliques with cardinality at least 2 and at most $n-1$ and all finite linear spaces with n points.

A *projective plane* is a finite linear spaces Π satisfying the following two axioms.

(P1) Any two distinct lines have a point in common.

(P2) There are four points, no three of which are on a common line.

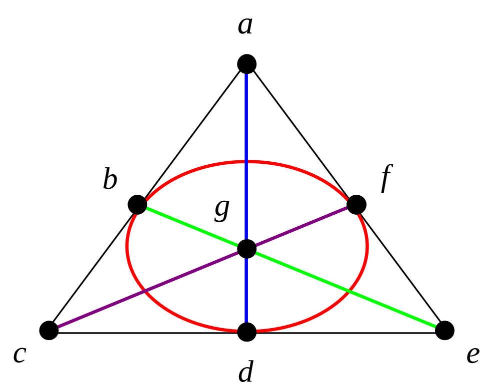


Figure 4:

Suppose that Π is a projective plane with a finite number n of points and a finite number l of lines. Then it is probative that for some $k \geq 2$, $n = l = k^2 + k + 1$, and Π has point and line regularity $k + 1$, where each point is on exactly $k + 1$ lines and each line contains exactly $k + 1$ points. We call such a number k the order of the projective plane. Besides, any two lines in a projective plane intersect on a common point, or paraphrased into terms of clique partition, any two cliques intersect on a common vertex.

One can prove that the smallest projective plane has order $k = 2$. It is well-known as the *Fano Plane*, as illustrated in figure 4, where the segments on $\{a, b, c\}, \{c, d, e\}, \{a, f, e\}, \{a, g, d\}, \{b, g, e\}, \{f, g, c\}, \{b, d, f\}$ respectively stand for seven lines.

It is not the case that one could construct a projective plane of order k for any $k \geq 2$. It is known that there are unique projective planes of orders $2, 3, 4, 5, 7,$ and 8 while there is no one of order 6 and 10, as revealed in [ ]. There are at least 4 non-isomorphic projective planes of order 9, but no one as yet can know the exactly number.

As for the projective planes of other orders, what have been known as yet is just the following.

We can construct many projective planes by using a vector space over a skew-field.

THEOREM 3.1. *Let F be a skew-field and denote by V a 3-dimensional vector space over F. Define the structure $P = P(V)$ as follows. The* ***points*** *of P are the 1-dimensional subspaces of V. The* ***lines*** *of P are the 2-dimensional subspaces of V. The point p is on the line L if the corresponding 2-dimensional subspace lies in the corresponding 1-dimensional subspace. Then P is a projective plane.*

It is a well-known fact that if k is a prime number or prime power, that is, $k = p^n$ for p a prime and n a positive integer, then there exists a field with k elements and thus by theorem 3.1 we can construct a projective plane of order k.

In 1948, de Bruijn and Erdös [6] proved a theorem about linear space which we paraphrase in terms of clique partition as follows.

THEOREM 3.2. *If $\mathcal{Q}$ with $|\mathcal{Q}| > 1$ is a clique partition of K_n with $n \geq 3$, no one of whose members is a trivial clique, that is, the clique consisting of one single vertex, then $|\mathcal{Q}| \geq n$, with equality if and only if*

(a) *$\mathcal{Q}$ consists of one clique on $n-1$ vertices and $n-1$ copies of K_2,*

or

(b) *The finite linear space corresponding to $\mathcal{Q}$ is a projective plane.*

The linear spaces corresponding to the class of clique partitions in (a) are traditionally referred to as *near-pencil* in finite linear space theory. What is the harm of somewhat abusing the terminology for both clique partition and finite linear space? Afterwards we shall do so.

Theorem 3.2 characterizes all those finite linear spaces on n points having exactly n lines, as being the two classes of linear spaces described in the above theorem.

4 Various intersection numbers of complete multigraph.

For any complete multigraph K_n with $n \geq 3$, we can always construct a representation of it by the following method.

Adopt an element, say e_1 common to the representation sets of all vertices. Then attach elements $e_2, ..., e_{n-1}$ to some $n-1$ vertices of the n vertices, respectively. On the other hand, there cannot exist a representation $\mathcal{F}$ of K_n with $|\mathbf{S}(\mathcal{F})| \leq n-1$, for otherwise we can at first delete all elements in $\mathbf{S}(\mathcal{F})$ that appear in the representation set of only one vertex, which we would referred to as *monopolized element* in the rest of this paper, from the representation sets of all vertices and say the resulting representation $\mathcal{F}'$. Then $\mathcal{F}'$ does be a multifamily representation of K_n, since monopolized elements have nothing to do with multifamily representation of a multigraph. Now we take $\mathcal{Q}(\mathcal{F}')$. Note that $|\mathcal{Q}(\mathcal{F}')| \leq n-1$. Clearly $\mathcal{Q}(\mathcal{F}')$ is a clique partition of K_n containing no trivial clique. By theorem 3.2 and the fact that $|\mathcal{Q}(\mathcal{F}')| \leq n-1$, we know that $|\mathcal{Q}(\mathcal{F}')| = 1$, that is, $\mathcal{Q}(\mathcal{F}')$ consists of only one clique, containing all n vertices of K_n. But clearly we cannot recover $\mathcal{F}$ from $\mathcal{F}(\mathcal{Q}(\mathcal{F}')) = \mathcal{F}'$ by adding monopolized elements to the members of $\mathcal{F}'$, since $\mathcal{F}$ is a representation of K_n with $|\mathbf{S}(\mathcal{F})| \leq n-1$, a contradiction. From above we know that $\omega(K_n) = n$.

Now we investigate the uniqueness of K_n's representation. Assume a representation $\mathcal{F}$ of K_n with $|\mathbf{S}(\mathcal{F})| = n$. Delete all monopolized elements in $\mathbf{S}(\mathcal{F})$ from the representation sets of all vertices, say the resulting representation $\mathcal{F}'$, and then take $\mathcal{Q}(\mathcal{F}')$. Now $|\mathcal{Q}(\mathcal{F}')| \leq n$. Clearly $\mathcal{Q}(\mathcal{F}')$ is a clique partition of K_n containing no trivial clique. By theorem 3.2 and $|\mathcal{Q}(\mathcal{F}')| \leq n$, we know that $\mathcal{Q}(\mathcal{F}')$ consists of only one clique, or is a near-pencil or projective plane. If $\mathcal{Q}(\mathcal{F}')$ is a near-pencil or projective plane, then $|\mathcal{Q}(\mathcal{F}')| = n$ and thus it is clear that in these two cases we had never deleted any monopolized element from the representation set of any vertex when we proceed from $\mathcal{F}$ to $\mathcal{F}'$. Thus in these two cases, the original representation $\mathcal{F}$ is just $\mathcal{F}$(near-pencil) or $\mathcal{F}$(projective plane). (Note that here we use the two terminologies "near-pencil" and "projective plane" to stand for their corresponding clique partition, respectively.) Clearly these two representations indeed have their constituting sets pairwise-distinct.

For the remaining case, $\mathcal{Q}(\mathcal{F}')$ consists of only one clique. Thus in this case we must had deleted $n-1$ monopolized elements in proceeding from $\mathcal{F}$ to $\mathcal{F}'$. And clearly all constituting sets of $\mathcal{F}$ has a common element, say e_1, and some $n-1$ constituting sets of $\mathcal{F}$ have monopolized elements, say $e_2, ..., e_{n-1}$, respectively.

Thus in above we have proved that every complete multigraph K_n with $n \geq 3$ has intersection number n and has three manners for forming its minimum representations. Note that the practicability of the one manner derived from projective plane depends on whether or not $n = k^2 + k + 1$ for some $k \geq 2$ and there exists projective plane of order k.

Then we investigate the minimum antichain representations of K_n with $n \geq 3$. Because $\mathcal{F}$(near-pencil) itself is a antichain representation of K_n making use of n elements, we know that $\omega_{\text{ai}}(K_n) \leq n$. Assuming an antichain representation $\mathcal{F}$ of K_n with $|\mathbf{S}(\mathcal{F})| \leq n$. Delete all monopolized elements in $\mathbf{S}(\mathcal{F})$ from the representation set of all vertices, say the resulting representation $\mathcal{F}'$, and then take $\mathcal{Q}(\mathcal{F}')$. Now $|\mathcal{Q}(\mathcal{F}')| \leq n$ and $\mathcal{Q}(\mathcal{F}')$ is a clique partition of K_n with no trivial clique. By theorem 3.2, we know that $\mathcal{Q}(\mathcal{F}')$ have only one member, or is a near-pencil, or projective plane. Clearly $\mathcal{F}$(near-pencil) and $\mathcal{F}$(projective plane) are both antichain representation. As for the remaining case that $\mathcal{Q}(\mathcal{F}')$ have only one member, we cannot recover $\mathcal{F}$ from $\mathcal{F}(\mathcal{Q}(\mathcal{F}')) = \mathcal{F}'$ by adding monopolized elements to the members of $\mathcal{F}'$, since $\mathcal{F}$ is an antichain representation of K_n with $|\mathbf{S}(\mathcal{F})| \leq n$.

Thus we have proved that every complete multigraph K_n with $n \geq 3$ has antichain intersection number n and has two manners for forming its minimum antichain representations with the one manner derived from projective plane being provisory upon the existence of projective plane of appropriate order.

It is clear that $\omega_{\text{um}}(K_n) = 1$ for all n.

As for the investigations of the minimum uniform representations of K_n with $n \geq 3$, we shall refer to the following theorem due to Bridges [5].

THEOREM 4.1. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a finite linear space with $n \neq 5$ points and l lines. Then $l = n + 1$ if and only if Γ is a projective plane with one point removed from $\mathcal{P}$ and every line of $\mathcal{L}$. As for the case of $n = 5$, see figure 5.*

Assume an uniform representation $\mathcal{F}$ of K_n with $|\mathbf{S}(\mathcal{F})| \leq n$. Delete all monopolized elements in $\mathbf{S}(\mathcal{F})$ from the representation sets of all vertices, say the resulting representation $\mathcal{F}'$, and then take $\mathcal{Q}(\mathcal{F}')$. Now $|\mathcal{Q}(\mathcal{F}')| \leq n$ and $\mathcal{Q}(\mathcal{F}')$ is a clique partition of K_n with no trivial clique. By Theorem 3.2, we know that $\mathcal{Q}(\mathcal{F}')$ have only one member, or is a near-pencil, or a projective plane. Clearly $\mathcal{F}$(projective plane) is an uniform representation in its own right, while we cannot recover an uniform representation $\mathcal{F}$, with $|\mathbf{S}(\mathcal{F})| \leq n$, of K_n with $n \geq 3$ from $\mathcal{F}$(near-pencil) by adding monopolized elements to

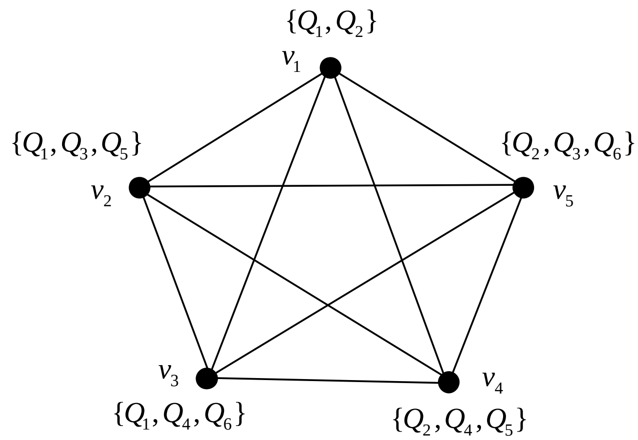


Figure 5:

the members of it except possibly $n = 3$. And for the remaining case, that is, $\mathcal{Q}(\mathcal{F}')$ has only one clique, we also cannot recover $\mathcal{F}$ from $\mathcal{F}(\mathcal{Q}(\mathcal{F}'))$.

Thus whenever $n \geq 4$, we have that $\omega_{\text{U}}(K_n) = n$ and K_n is uiu if and only if $n = k^2 + k + 1$ for some $k \geq 2$ and there exists projective plane of order k.

In case that $4 \leq n \neq k^2 + k + 1$ or there exists no projective plane of order k, since we can always form an uniform representation $\mathcal{F}$ of K_n with $|\mathbf{S}(\mathcal{F})| = n+1$ by at first adopting an element common to the representation set of all vertices and then for the representation set of each vertex attaching a monopolized element to it. Thus for this case we have $\omega_{\text{U}}(K_n) = n + 1$. Now given an uniform representation $\mathcal{F}$ of K_n with $|\mathbf{S}(\mathcal{F})| \leq n + 1$, at first we delete all monopolized elements of $\mathbf{S}(\mathcal{F})$ from the representation set of each vertex resulting in another representation, say $\mathcal{F}'$, and then take $\mathcal{Q}(\mathcal{F}')$. Now $|\mathcal{Q}(\mathcal{F}')| \leq n + 1$. By theorem 3.2 and 4.3, (note that we have assumed that $n \geq 4$ and there exists no projective plane of appropriate order) and the fact that we cannot recover $\mathcal{F}$ from $\mathcal{F}$(near-pencil with $n \geq 4$ vertices) or $\mathcal{F}$(the clique partition as in figure 5), we know that $\mathcal{Q}(\mathcal{F}')$ either consists of only one clique, or is a projective plane with one vertex deleted. The corresponding representation of the latter is an uniform representation in its own right and we can easily recover $\mathcal{F}$ from the corresponding representation set of the former by returning monopolized element to each member of it.

Thus we have proved that for $n = 3$ K_n has uniform intersection number n and has only one manner to form its minimum uniform representations; for $n \geq 4$ such that $n = k^2 + k + 1$ for some $k \geq 2$ and there exists projective plane of order k, K_n has uniform intersection number n and has only one manner to form its minimum uniform representations; for $n \geq 4$ such that $n = k^2 + k$ for some $k \geq 2$ and there exists projective plane of order k, K_n has uniform intersection number $n + 1$ and has two manners to form its minimum uniform representations; and for $n \geq 4$ such that $n \neq k^2 + k + 1$ and $n \neq k^2 + k$ for any $k \geq 2$, K_n has uniform intersection number $n + 1$ and has only one manner to form its minimum uniform representations.

5 The intersection number of diamond-free multigraph.

We call a multigraph *H-free* if it has no induced subgraph isomorphic to H. We call the multigraph obtained by deleting an edge from K_4 a *diamond*.

For two multigraphs M and M', an *isomorphism* from M to M' is a bijection f that maps $V(M)$ to $V(M')$ and $E(M)$ to $E(M')$ so that each edge of M, say with endpoints u and v, is mapped to an edge with endpoints $f(u)$ and $f(v)$. An *automorphism* of M is an isomorphism from M to M. Two clique partitions $\{Q_1, ..., Q_n\}$ and $\{Q'_1, ..., Q'_n\}$ of a multigraph M are said to be isomorphic if there exists an automorphism A of M and a permutation π on $\{1, ..., n\}$ such that $A(Q_i) = Q'_{\pi(i)}$ for $i = 1, ..., n$.

Since we study only multigraphs without loop and with at most one edge between any two vertices of it, our technicality for isomorphism may be reduced to the following. An isomorphism from a simple multigraph M to a simple multigraph M' is a bijection $f : V(M) \to V(M')$ such that $uv \in E(M) \Leftrightarrow f(u)f(v) \in E(M')$.

It is possible to prove that given two isomorphic multifamily representations of M, then the two clique partitions corresponding to them are also isomorphic and vice versa, but the formal proof is too scholastic and tedious in nature to be worth being included here. By the above we have the following theorem.

THEOREM 5.1. *A multigraph M is uim if and only if it has an unique minimum clique partition upto isomorphism.*

One can easily see the following proposition, where a *maximal clique* means a clique not properly contained in another one.

PROPOSITION 5.2. *A multigraph M is diamond-free if and only if every edge of M is in exactly one maximal clique. Further if M is diamond-free then there exists unique one minimum clique partition of M, consisting of all the maximal cliques of M.*

By theorem 5.1 and proposition 5.2 we have the following theorem.

THEOREM 5.3. *Every diamond-free multigraph is uim.*

Next we investigate the uniqueness of representation of diamond-free multigraphs in the following theorem.

THEOREM 5.4. *Every connected diamond-free multigraph $M^{(n)}$ other than K_n is ui.*

Proof. We say that a vertex is a *monopolized vertex* of a maximal clique if this vertex appear in no other maximal clique.

It is clear that two monopolized vertices separetely belonging to two distinct maximal cliques can't be adjacent for otherwise the edge connecting these two monopolized vertices induce some maximal clique, simultaneously containing two monopolized vertices separately belonging to two distinct maximal cliques, a contradiction with the definition of monopolized vertex. Thus for any representation of M, the two sets attached to two monopolized vertices separately belonging to two distinct maximal cliques are disjoint. Besides, the subgraph, say, M' induced by the subset of $V(M)$ consisting of all the monopolized vertices is a disjoint union of complete multigraphs with all vertices of each connected component exactly being all monopolized vertices of some maximal clique.

Let $Q_1, ..., Q_p$ be all maximal cliques of M, and each Q_i contain q_i vertices, m_i of which are monopolized vertices. Recall that we had reasoned out that a complete multigraph K_n have intersection number n and there exists three manners to form its representation before. Thus this induced subgraph has intersection number $\sum_{i=1}^{p} m_i$ and for each connected component of it we have three manners to form representation.

We call a vertex belonging to more than one maximal clique a *shared vertex.*

We first try not to use more elements than $\sum_{i=1}^{p} m_i$ to form a representation for M. Assuming that such a representation $\mathcal{F}$ exists, then under $\mathcal{F}$ each connected component of M' has three possible manners for representation. Assume that under $\mathcal{F}$ there is one connected component, say, K in M' using the projective plane or near-pencil manner for forming its representation. We say that K has its vertices as monopolized vertices of maximal clique Q_k.

(Note that projective plane and near-pencil have a common property, that is, any two lines intersect on a common point , or paraphrased into terms of clique partition, any two cliques intersect on a common vertex, and recall that in the method by which we construct a correspondence between multifamily representation and clique partition, an element in multifamily representation correspond to a clique in clique partition.)

At this time if Q_k has one vertex other than any vertex of K, that is, Q_k has one shared vertex, then we aren't allowed to use more than one element from $\mathbf{S}(\mathcal{F})$ in order to make this shared vertex be adjacent to all vertices of K for if we use two elements, say e_1, e_2, then the representation set of the monopolized vertex on which the two cliques corresponding to e_1, e_2 intersect also contain e_1, e_2. On the other hand, clearly one element can't afford to make this shared vertex be adjacent to all vertices of K when the representation of K is derived from a projective plane or near-pencil. It is clear that each maximal clique of M has shared vertices in it unless M itself is K_n or M is not connected. Thus in this case we fail in the trial.

As for the case that under $\mathcal{F}$ there is no connected component of M' using the projective plane or near-pencil manner for forming its representation? In this case, each connected component, say with m_i vertices, of M' has its representation derived from the clique partition with one member as K_{m_i} and $m_i - 1$ members as trivial clique. Now clearly in each maximal clique, to make one shared vertex be adjacent to all monopolized vertices we can only use the element of $\mathbf{S}(\mathcal{F})$ corresponding to the member as K_{m_i} in the clique partition of the subgraph induced by all monopolized vertices of this maximal clique.

After we do so, two vertices not simultaneously belonging to any maximal clique in M must have obtained distinct representation sets. On the other hand, for two vertices simultaneously belonging to some maximal clique, we have three cases as following. In case that these two vertices are both monopolized, we had distingished their representation sets by trivial cliques. In case that these two vertices are both shared, since two maximal cliques in M never share more than one vertex, there must be a maximal clique in M that occupy one of the two vertices but not the other and hence the representation sets of these two vertices must be distinct. In case that one of the two is monopolized and the other is shared, the shared vertex has at least one element in its representation set that is not in the representation set of the monopolized vertex. Thus we have succeeded in the trial and in the same time we have also proved that M is ui. □

6 Intersection number and antichain intersection number of line graph.

The *line graph* of a graph G, which we assume to be finite, undirected and simple in this paper, written $L(G)$, is the graph whose vertices are the edges of G, with its two vertices adjacent if and only if the two edges in G corresponding to these two vertices have a common endpoint in G.

For each vertex v of G, the set e_v consisting of all edges in G containing v induces a maximal clique in $L(G)$. This is one of the only two types of maximal cliques in $L(G)$, while the rest of maximal cliques is induced by triangles in G. Besides, any edge $ef \in E(L(G))$ with $e = uv$ and $f = vw$ being two edges in G can only be contained in either a clique induced by e, f possibly together with some edges in G with v as endpoint or the clique induced by the triangle uvw in G (, if u is adjacent to w). Clearly the set $P = \{e_v : v \in G, d(v) \geq 2\}$ is a clique partition of $L(G)$ which we will

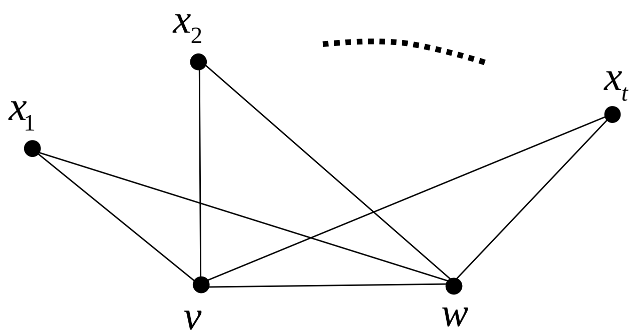


Figure 6:

call the *canonical clique partition* of $L(G)$. Note that each vertex of $L(G)$ is contained in exactly two cliques in P.

Let G be a graph. A *wing* in G is a triangle with the property that exactly two of its vertices have degree two in G, while a *3-wing* is a wing with the vertex in it having degree greater than two having degree exactly three. Besides, we define a *star* in G to be a collection of edges in G which intersect on a common vertex. Note that a star need not consist of all edges incident with some vertex, but only a subcollection of those edges. We will use the notation S_v^i to indicate a star with i edges, centered at v. The *join* of simple graphs G and H, denoted $G \vee H$, is the graph obtained from the vertex-disjoint union $G + H$ by adding all the edges $\{xy : x \in V(G), y \in V(H)\}$. We denote the graph as in figure 6 W_t, $t \geq 2$.

S. McGuinness and Rolf REES [11] proved the following theorem.

THEOREM 6.1. *Let G be a connected graph, and $G \neq K_3, K_4, (K_2 + K_2 + K_2) \vee K_1$ (or $3K_2 \vee K_1$ in abbreviation), or W_t, $t \geq 2$. Let $V_2(G)$ denote the set of vertices in G with degree at least two , and let w_3 denote the number of 3-wing in G. Then $cp(L(G)) = |V_2(G)|$ and there are exactly 2^{w_3} distinct minimum clique partitions of $L(G)$.*

A cursory illustration of the above theorem here would be advantageous

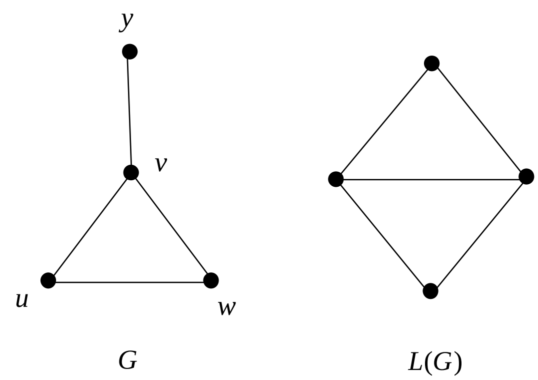


Figure 7:

for our further study. Note that the above theorem wouldn't concern itself with "isomorphism", that is, it would regard two clique partitions to be distinct if the cliques used by the two clique partitions don't derive from the same stars and triangles in G. For illustration, to attain a minimum clique partition of the line graph of the graph G in figure 7, we have two "distinct manners", one by the upper triangle and the inferior two edges in $L(G)$, that is, by the three stars in G centered at u, v, w, whereas the other by the inferior triangle and the upper two edges in $L(G)$, that is, by the 3-wing uvw and the two stars $\{vw, vy\}$, $\{vu, vy\}$ in G.

We will follow this criterion when deciding whether or not two clique partitions are the same. The above theorem clarify the fact that to attain a minimum clique partition of $L(G)$, where note that G is the class of graphs aforementioned in the above theorem, no triangle in $L(G)$ induced by a triangle in G other than 3-wing can be used. And the adopting in a clique partition of $L(G)$ of any triangle induced by one 3-wing in G can also yield a minimum clique partition other than the unique other minimum clique partition, called the canonical one, which consists of all maximal cliques of $L(G)$ induced by one maximal star in G. Thus each 3-wing in G, refer to figure 7,

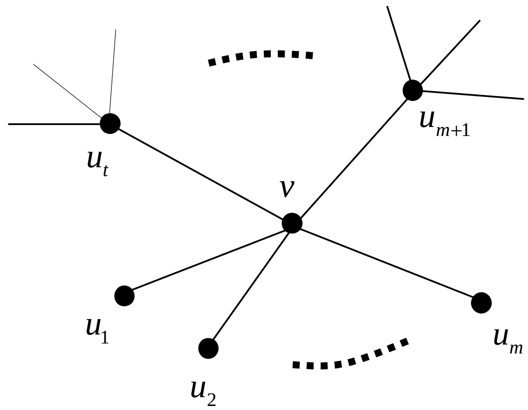


Figure 8:

yield two distinct clique partitions of $L(G)$, one adopting the upper triangle and the inferior two edges in the right graph of figure 7, while the other adopting the inferior triangle and the upper two edges; and therefore as the aforementioned by the above theorem G has exactly 2^{w_3} distinct minimum clique partitions.

Regarding $L(G)$ as a multigraph, subsequentally we investigate the intersection number of the line graph $L(G)$ of a connected simple graph $G \neq K_3, K_4, 3K_2 \vee K_1$, or $W_t, t \geq 2$.

At first we consider the following question. When do a minimum clique partition, say $\mathcal{Q}$, of $L(G)$ has two vertices obtaining the same representation set after we take $\mathcal{F}(\mathcal{Q})$? Clearly if such two vertices, say e_1, e_2, exist, then their two corresponding edges in G, say vu_1, vu_2, inetrsect and either $d(u_1) = d(u_2) = 1$ or vu_1u_2 is a wing in G with $d(u_1) = d(u_2) = 2$.

For the former case see figure 8, where for the sake of generality we suppose that $u_1, ..., u_m$ are vertices in G with degree one and $u_{m+1}, ..., u_t$ with degree at least 2. Immediately after we ask the question whether or not we can represent the complete subgraph K_m in $L(G)$, refer to figure 8, with vertex set $\{vu_1, ..., vu_m\}$ by exactly m elements in some minimum representation of

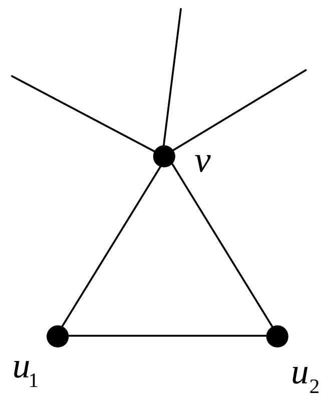


Figure 9:

$L(G)$. (Note that it is impossible to represent it by $m-1$ elements.) Assuming that we can, then this K_m's representation can correspond to three types of clique partitions, say the corresponding clique partition being $\mathcal{Q}$, that is, near-pencil, projective plane, or K_m together with $m-1$ trivial cliques. Note that projective plane and near-pencil have a common property, that is, any two lines intersect on a common point , or paraphrased into terms of clique partition, any two cliques intersect on a common vertex, and recall that in the method by which we construct a correspondence between multifamily representation and clique partition, an element in multifamily representation correspond to a clique in clique partition. Thus for the former two cases, to make $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$, we shouldn't rely on more than one element in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$, since for any two elements, the vertex on which the two cliques respectively corresponding to them intersect has its representation set comprising them. Nor should we use one. (Unless G itself is a star, that is, $t = m$.) But for the third case we can use the element in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ corresponding to the clique K_m (, and note that this is the unique approach if we would like not to use new elements).

On the other hand, for the case that vu_1u_2 is a wing in G with $d(u_1) =$

$d(u_2) = 2$, refer to figure 9. In this case, whether or not vu_1u_2 is a 3-wing in G, that is, whether or not the adopting of the triangle in $L(G)$ induced by the triangle vu_1u_2 in G can occur in one minimum clique partition of $L(G)$, we can't have $S_{\mathcal{Q}}(vu_1) = S_{\mathcal{Q}}(vu_2)$. We summarize the above by the following theorem.

THEOREM 6.2. *Let G be a connected simple graph, and $G \neq K_3, K_4, 3K_2 \vee K_1$, or $W_t, t \geq 2$. In addition, we suppose that G is not a star. Let $V_2(G)$ denote the set of vertices in G with degree at least two, and let w_3 denote the number of 3-wing in G. And let $u_1^{(i)}, ..., u_{m_i}^{(i)}$ be all vertices in G of degree one and adjacent to v_i with $d(v_i) > 1$. We suppose that there are k vertices with its degree more than one in G in total which are adjacent to some vertex of degree one, i.e., $1 \leq i \leq k$. Then when regarding $L(G)$ as a multigraph, $\omega(L(G)) = |V_2(G)| + \sum_{i=1}^{k}(m_i - 1)$ and there are exactly 2^{w_3} distinct minimum representations of $L(G)$.*

The case that $G = K_3$ or a star is an easy affair. For $G = 3K_2 \vee K_1$, S. McGuinness and Rolf REES [11] have shown that $L(G)$ admits exactly three distinct minimum clique partitions, and with a little direct inspection we see that these three partitions correspond to three distinct minimum (antichain) representations, respectively. (As a matter of fact, two of the three are isomorphic.)

As for $G = K_4$, it is easily verified that there are exactly two distinct but in fact isomorphic clique partitions, one by all the cliques in $L(G)$ induced by some maximal star in G, while the other by all the triangles in $L(G)$ induced by some triangle in G, and with a little direct inspection we see that these two partitions correspond to two distinct minimum (antichain) representations, respectively.

As for $G = W_t, t \geq 2$, S. McGuinness and Rolf REES [11] have shown that $L(G)$ has exactly two distinct minimum clique partitions, and with a little direct inspection we see that these two partitions correspond to two distinct minimum (antichain) representations, respectively.

Regarding $L(G)$ as a multigraph, next we consider the antichain intersection number of the line graph $L(G)$, where G is connected simple and $\neq K_3, K_4, 3K_2 \vee K_1, \text{or } W_t, t \geq 2$. At first, we consider the question that when do a minimum clique partition, say $\mathcal{Q}$, of $L(G)$ has two vertices the two corresponding representation sets for which after we take $\mathcal{F}(\mathcal{Q})$ would have one in it contained in the other in it. Clearly, the two edges in G, say e_1, e_2, corresponding to such two vertices must intersect, say $e_1 = vu_1, e_2 = vu_2$, and one of u_1, u_2, say u_1 throughout the rest of this paper, has no neighbor other than v, u_2.

We at first consider exclusively the case that vu_1u_2 form a triangle in G. Now $d(u_1) = 2$. If only we have never made use of the clique in $L(G)$ induced by the triangle vu_1u_2 in G in a clique partition, say $\mathcal{Q}$, of $L(G)$, we utterly needn't to worry about the inclusion relation between the two representation sets $S_{\mathcal{Q}}(e_1), S_{\mathcal{Q}}(e_2)$. Thus what we need to consider is mere the case that there exists a minimum clique partition of $L(G)$ making use of the triangle in $L(G)$ induced by the triangle vu_1u_2 in G, i.e., that the triangle vu_1u_2 is a 3-wing. Recall that we have supposed that $d(u_1) = 2$, and thus exactly one of v, u_2 has degree two and the other has degree three. In case that $d(v) = 2$, making use of the triangle vu_1u_2 in a minimum clique partition, say $\mathcal{Q}$, will make $S_{\mathcal{Q}}(e_1)$ be contained in $S_{\mathcal{Q}}(e_2)$. Thus in this case the representation derived from the minimum clique partition of $L(G)$ making no use of the triangle vu_1u_2, i.e., the canonical one, is the unique approach to form an minimum antichain representation of $L(G)$. In case that $d(u_2) = 2$, whether or not we make use of the triangle vu_1u_2 in a minimum clique partition, say

$\mathcal{Q}$, of $L(G)$, there can't be inclusion relation between $S_{\mathcal{Q}}(e_1), S_{\mathcal{Q}}(e_2)$. But if we make use of the triangle vu_1u_2, then $S_{\mathcal{Q}}(u_1u_2)$ will be contained in both $S_{\mathcal{Q}}(e_1)$ and $S_{\mathcal{Q}}(e_2)$. Thus in this case we have the same conclusions as the former one.

Now what remained is the case that u_1 is not adjacent to u_2. For this case, we can without loss of generality assume that $d(u_1) = 1$ while leave $d(u_2)$ unappointed. See figure 8, where for the sake of generality we suppose that $u_1, ..., u_m$ are vertices in G with degree one and $u_{m+1}, ..., u_t$ with degree at least two. Immediately after we look for a minimum antichain representation of $L(G)$ in which the complete subgraph K_m with vertex set $vu_1, ..., vu_m$ is represented using exactly m elements. (Note that it is impossible to represent it by $m-1$ elements.) Assuming that we can, then this K_m's representation can only correspond to two types of clique partitions, say the corresponding clique partition being $\mathcal{Q}$, that is, near-pencil or projective plane. (When $m = 1$, we can represent K_m by m elements with respect to antichain. But in this case we can't make u_1 be adjacent to $u_{m+1}, ..., u_t$ by the single element in the representation set of u_1 so that the representation set of u_1 wouldn't be contained in the representation sets of $u_{m+1}, ..., u_t$, unless $t = 1$, that is, $G = K_2$.) Now to make $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$, we can't use more than one element in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ for securing the representation sets of any two vertices from overlapping on more than one element, neither can we use one (, unless G itself is a star, that is, $t = m$).

Thus we should yield by one step looking for a minimum antichain representation of $L(G)$ in which the complete subgraph K_m with vertex set $\{vu_1, ..., vu_m\}$ is represented by exactly $m+1$ elements. Assuming such a minimum antichain representation, then by tlheorem 3.2 and 4.1 this K_m's representation can only correspond to five types of clique partitions, say the corresponding clique partition being $\mathcal{Q}$, that is, near-pencil together with one trivial clique attached on it, projective plane together with one trivial clique attached on it, one K_m together with m trivial cliques attached on it, one as in figure 5, or projective plane with one vertex deleted.

For the first case, to make $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$, we can't use more than one element in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$ different from the monopolized one for securing the representation sets of any two vertices from overlapping on more than one element. Since we have only one monopolized element in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$, thus we must try to use one non-monopolized element in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$ to make $m-1$ vertices of the K_m be adjacent to $vu_{m+1}, ..., vu_t$ (, unless G is a star, i.e., $t = m$), and then use the monopolized element on the vertex of the K_m other than the aforementioned $m-1$ vertices to make this vertex be adjacent to $vu_{m+1}, ..., vu_t$. Clearly we have only one approach to do so, that is, at first take the element in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$ that correspond to the clique K_{m-1} in $\mathcal{Q}$ to make all vertices on this K_{m-1} be adjacent to $vu_{m+1}, ..., vu_t$, and then use the monopolized element on the vertex not on this K_{m-1} to make this vertex be adjacent to $vu_{m+1}, ..., vu_t$. But when $t > m+1$, using this method will make $|S(u_{m+1}) \cap S(u_{m+2})| \geq 2$. Thus, provided that G is not a star, this method can be carried out only if $t = m+1$.

As for the second case, i.e. projective plane together with one trivial clique attached on it, similarly we must try to use one non-monopolized element in $\bigcup_{i=1}^m S_{\mathcal{Q}}(vu_i)$ to make $m-1$ vertices of the K_m be adjacent to $vu_{m+1}, ..., vu_t$ (, unless G is a star, i.e., $t = m$). But we know that in a projective plane of order k each clique contain $k+1$ vertices, whereas there are k^2+k+1 vertices in total where $k \geq 2$, and thus each clique in a projective plane has

$$(k^2 + k + 1) - (k + 1) = k^2 \geq 4$$

vertices not on it. Thus in this case we have failed.

For the third case, i.e., one clique K_m together with m trivial cliques attached on it, for the sake not to make two representation sets overlap on more than one element, we can only use the element in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ corresponding to the clique K_m in $\mathcal{Q}$ or all monopolized elements in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ to make $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$. But when $t > m + 1$ and there is one vertex, say u_{m+1}, in $\{u_{m+1}, ..., u_t\}$ which is not adjacent to any other vertex in $\{u_{m+1}, ..., u_t\}$, then for the sake that we should make vu_{m+1} be adjacent to $vu_{m+2}, ..., vu_t$, we can only use the element in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ corresponding to this K_m for vu_{m+1} to be adjacent to $vu_1, ..., vu_m$. (If we use the monopolized elements corresponding to all trivial cliques in $\mathcal{Q}$ for vu_{m+1} to be adjacent to $vu_1, ..., vu_m$, then since there is no triangle in G which contains v and u_{m+1} by our supposition before, so in any clique partition of $L(G)$ we can only cover the edge $\{vu_{m+1}, vu_{m+2}\}$ by a clique induced by some star in G centered at v. Thus we can use neither the element in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ corresponding to K_m nor all monopolized elements in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ for vu_{m+2}, or otherwise either we can't make vu_{m+2} be adjacent to vu_{m+1} or we will make the representation sets of vu_{m+1}, vu_{m+2} overlap on more than one element.) Thus in this case, when $t > m + 1$ we have only one method to make a vertex belonging to $vu_{m+1}, ..., vu_t$ but not adjacent to any member of it be adjacent to $vu_1, ..., vu_m$ using elements in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$, while when $t = m + 1$ we have two methods to make vu_{m+1} be adjacent to $vu_1, ..., vu_m$ using elements in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$.

As for the forth case, i.e. one as in figure 5, for securing the representation sets of any two vertices from overlapping on more than one element, we need one pair of vertex-disjoint cliques in the clique partition as in figure 5, and the unique two vertex-disjoint pairs of cliques, refer to figure 5, are $\{Q_3, Q_4\}$ and $\{Q_5, Q_6\}$. If we use Q_3, Q_4 to make $vu_{m+1}, ..., vu_t$ be adjacent to v_2, v_3, v_4, v_5, then to make v_1 be adjacent to $vu_{m+1}, ..., vu_t$ we can use neither 1 nor 2 for the sake of two representation sets overlapping on more than one element. Similarly for the use of Q_5, Q_6. Thus in this case we have failed.

For the fifth case, i.e. projective plane, say of order $k \geq 2$, with one vertex, say x, deleted, we know that a clique in this clique partition has at most $k+1$ vertices, whereas there are k^2+k vertices in total. Thus in this case each clique has at least

$$(k^2 + k) - (k + 1) = k^2 - 1 \geq 3$$

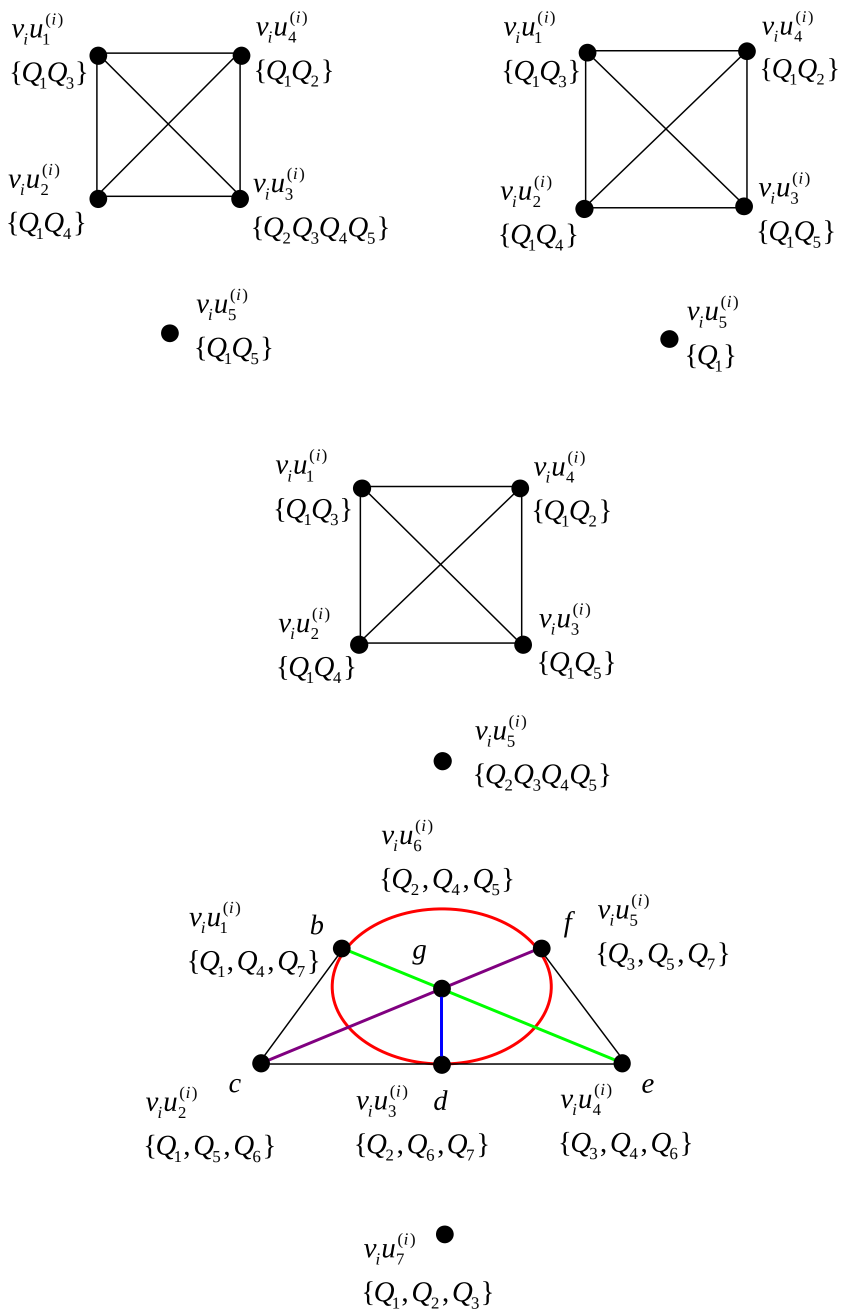
vertices not on it. Thus in order that $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$, we need more than one element from $\mathcal{F}(\mathcal{Q})$. Recall that a projective plane with k^2+k+1 points for some $k \geq 2$ has point and line regularity $k+1$. Thus deleting one vertex from a projective plane of order $k \geq 2$ leaves a clique partition consisting of $k+1$ cliques of cardinality k and k^2 cliques of cardinality $k+1$. Besides, recall that any two cliques in a projective plane intersect on a common vertex. Thus we couldn't adopt two elements in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ which correspond to two cliques in $\mathcal{Q}$ of cardinality $k+1$ to make $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$, or otherwise the representation set (turned out after we take $\mathcal{F}(\mathcal{Q})$) of the vertex on which the two cliques intersect and the representation sets of $vu_{m+1}, ..., vu_t$ would overlap on more than one element (unless $t = m$, that is, G is a star). Nor could we adopt two elements in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ corresponding to two cliques in $\mathcal{Q}$ respectively of cardinality $k, k+1$, for the same reason. Now the only permissible choice is the adoption of elements in $\bigcup_{i=1}^{m} S_{\mathcal{Q}}(vu_i)$ corresponding to the $k+1$ cliques in $\mathcal{Q}$ of cardinality k. The vertex, say x, on which these $k+1$ cliques would intersect but for the deletion of x from the primitive projective plane of order k, having been deleted, these $k+1$ cliques are pairwisely vertex-disjoint. (Recall the property of one linear space that any two lines intersect on at most one point.) There are altogether $k(k+1) = k^2 + k$ vertices in these $k+1$ cliques, tantamount to the sum total of vertices in $\mathcal{Q}$. Thus we could utilize the $k+1$ elements corresponding to these $k+1$ cliques in order that $vu_{m+1}, ..., vu_t$ be adjacent to $vu_1, ..., vu_m$. Note that this method can be carried out only when $t = m+1$ for securing two vertices from having their representation sets overlapping on more than one element.

Now we have obtained the following lemma.

LEMMA 6.3. *Let G be a connected simple graph, and $G \neq K_3, K_4, 3K_2 \vee K_1$, or $W_t, t \geq 2$. In addition, we suppose that G is not a star. And let $u_1^{(i)}, ..., u_{m_i}^{(i)}$ be all vertices in G of degree one and adjacent to v_i with $d(v_i) > 1$, while $u_{m_i+1}^{(i)}, ..., u_{t_i}^{(i)}$ be all vertices in G of degree more than one and adjacent to v_i. We suppose that there are k vertices with its degree more than one in G in total which are adjacent to some vertex of degree one, i.e., $1 \leq i \leq k$.*

Then for any $1 \leq i \leq k$ so that $t_i = m_i + 1$, we have exactly four distinct minimum antichain representations of $L(G)$ respectively corresponding to four distinct methods for representing the clique of $L(G)$ with vertex set $\{v_iu_1^{(i)}, ..., v_iu_{t_i}^{(i)}\}$. Figure 10 illustrates these four distinct methods where for illustration we suppose that $m_i = 4$ in the upper three graphs and that $v_iu_1^{(i)}, ..., v_iu_6^{(i)}$ form a projective plane of order 2 with one vertex deleted in the lowermost graph. (Note that the method corresponding to the lowermost graph of figure 10 rely on the existence of projective plane with t_i vertices.)

On the other hand, for any $1 \leq i \leq k$ so that $t_i > m_i + 1$, all minimum antichain representations of $L(G)$ have the same method for representing the clique of $L(G)$ with vertex set $\{v_iu_1^{(i)}, ..., v_iu_{m_i}^{(i)}\}$, and for any vertex in


$v_iu_1^{(i)}$
$\{Q_1Q_3\}$
$v_iu_4^{(i)}$
$\{Q_1Q_2\}$
$v_iu_2^{(i)}$
$\{Q_1Q_4\}$
$v_iu_3^{(i)}$
$\{Q_2Q_3Q_4Q_5\}$
$v_iu_5^{(i)}$
$\{Q_1Q_5\}$
$v_iu_1^{(i)}$
$\{Q_1Q_3\}$
$v_iu_4^{(i)}$
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$v_iu_2^{(i)}$
$\{Q_1Q_4\}$
$v_iu_3^{(i)}$
$\{Q_1Q_5\}$
$v_iu_5^{(i)}$
$\{Q_1\}$
$v_iu_1^{(i)}$
$\{Q_1Q_3\}$
$v_iu_4^{(i)}$
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$\{Q_1Q_4\}$
$v_iu_3^{(i)}$
$\{Q_1Q_5\}$
$v_iu_5^{(i)}$
$\{Q_2Q_3Q_4Q_5\}$
$v_iu_6^{(i)}$
$\{Q_2, Q_4, Q_5\}$
$v_iu_1^{(i)}$
$\{Q_1, Q_4, Q_7\}$
b
g
f
$v_iu_5^{(i)}$
$\{Q_3, Q_5, Q_7\}$
c
d
e
$v_iu_2^{(i)}$
$\{Q_1, Q_5, Q_6\}$
$v_iu_3^{(i)}$
$\{Q_2, Q_6, Q_7\}$
$v_iu_4^{(i)}$
$\{Q_3, Q_4, Q_6\}$
$v_iu_7^{(i)}$
$\{Q_1, Q_2, Q_3\}$

Figure 10:

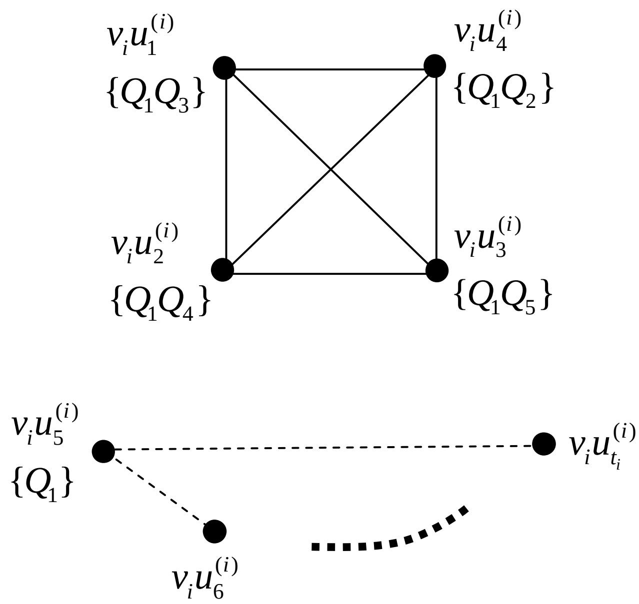

Figure 11:

$\{v_iu_{m_i+1}^{(i)}, ..., v_iu_{t_i}^{(i)}\}$ which is not adjacent to any other member of it, all minimum antichain representations of $L(G)$ also have the same method to make this vertex be adjacent to $v_iu_1^{(i)}, ..., v_iu_{m_i}^{(i)}$ using elements in $\bigcup_{j=1}^{m_i} S(v_iu_j^{(i)})$. Figure 11 illustrate this unique method, where for illustration we suppose that $m_i = 4$ and $v_iu_{m_i+1}^{(i)}$ is a such vertex, which is not adjacent to any other member of $\{v_iu_{m_i+1}^{(i)}, ..., v_iu_{t_i}^{(i)}\}$.

On the other hand, for any vertex in $\{v_iu_{m_i+1}^{(i)}, ..., v_iu_{t_i}^{(i)}\}$ which is adjacent to some other member of it, a minimum antichain representation of $L(G)$ would make this vertex be adjacent to $v_iu_1^{(i)}, ..., v_iu_{m_i}^{(i)}$ using elements in $\bigcup_{j=1}^{m_i} S(v_iu_j^{(i)})$ by one of the two methods as described in figure 12, where for illustration we suppose that $m_i = 4$ and $v_iu_{m_i+1}^{(i)}$ is a such vertex, which is adjacent to $v_iu_{m_i+2}^{(i)}$.

(We should note that once a vertex in $\{vu_{m_i+1}^{(i)}, ..., vu_{t_i}^{(i)}\}$ adopts the representation method as the right in figure 12, then all other vertices in $\{vu_{m_i+1}^{(i)}, ..., vu_{t_i}^{(i)}\}$

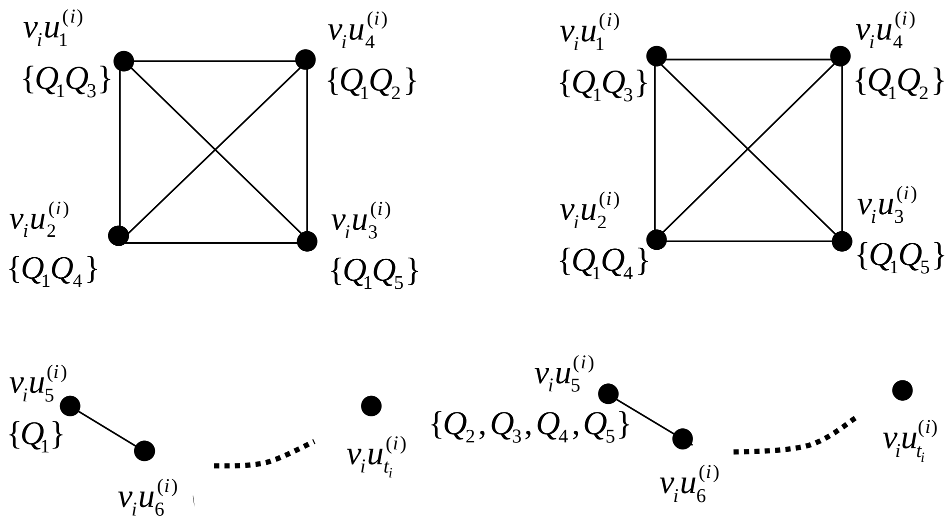

Figure 12:

must all adopt the representation method as the left in figure 12, or otherwise there will be two vertices in $\{vu^{(i)}_{m_i+1}, ..., vu^{(i)}_{t_i}\}$ *overlapping on more than one element on their representation sets.)*

Due to the above lemma, what is still vague is mere the case that $d(u_1) = 1$ and there is some triangle on v in G, see figure 13 where for illustration we suppose that $u_1, ..., u_m$ are the all vertices in G adjacent to v and with degree one, u_{m+1} is a vertex adjacent to v and with degree at least two so that there is no triangle in G containing the edge vu_{m+1}, and v, u_{m+2}, u_{m+3} form a triangle in G.

By lemma 6.3, if only we can prove that using the method as the left in figure 12 is always not worst than the one as the right in figure 12 in sense of the intent to minimize a representation of $L(G)$, where G is connected, $\neq K_3, K_4, 3K_2 \vee K_1,$ or $W_t, t \geq 2$ and is not a star, and characterize all situations under which the two methods in figure 12 is equally fine, then we can determine the antichain intersection number of any line graph and whether or not any line graph is uniquely intersectable with respect to antichain.

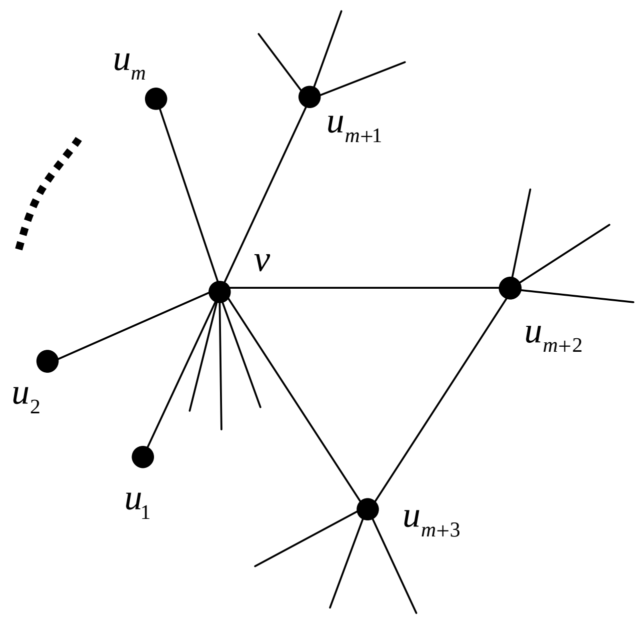

Figure 13:

We examine the method as the left in figure 12. In this method, refer to figure 13, we use one element to make the vertices $vu_1, ..., vu_t$, where we say that $d(v) = t$, be adjacent to each other, and use m monopolized elements respectively in the representation sets of $vu_1, ..., vu_m$. Thus in the whole $L(G)$, we use $|V_2(G)| + \sum_{i=1}^{k} m_i$ elements, where $V_2(G)$ denote the set of vertices of degree at least two in G and we let $v_i, 1 \leq i \leq k$ be all vertices of degree more than one in G which is adjacent to some vertex of degree one and for $1 \leq i \leq k$, $u_1^{(i)}, ..., u_{m_i}^{(i)}$ be all vertices in G of degree one and adjacent to v_i.

Immediately after we examine the method as the right in figure 12. In this method, refer to figure 13, we use m elements to make vu_{m+2} be adjacent to $vu_1, ..., vu_m$, respectively; and use one more element to make all u_i with $1 \leq i \leq t$ and $i \neq m + 2$ be adjacent to each other. Note that now we have made use of $m + 1$ elements, that is exactly equal to the number of elements

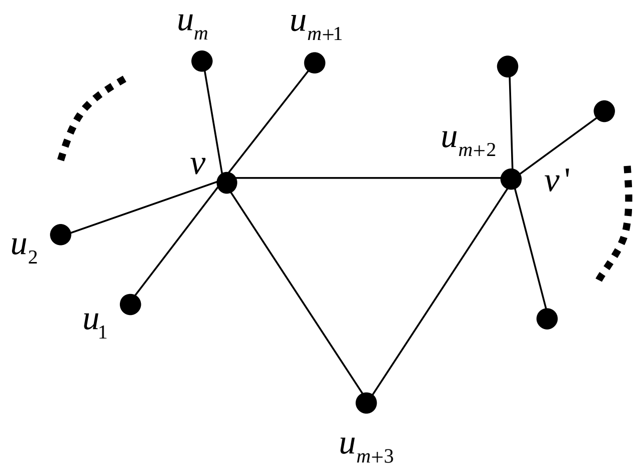

Figure 14:

we should have used for the v-star if we had adopted the left method in figure 12. But now we should use still another element to make vu_{m+2} be adjacent to vu_{m+1} (, unless u_{m+1} is adjacent to u_{m+2} and thus we can shake off the responsibility to make vu_{m+2} be adjacent to vu_{m+1} to the triangle $\{vu_{m+1}, vu_{m+2}, u_{m+1}u_{m+2}\}$ just like how we will deal with the responsibility to make vu_{m+2} be adjacent to vu_{m+3}). But even if u_{m+1} is adjacent to u_{m+2}, where to dispose of the triangle $\{vu_{m+1}, vu_{m+2}, u_{m+1}u_{m+2}\}$? If only $d(u_{m+1}) = 2$, we can shake off this triangle to the star $\{vu_{m+1}, u_{m+1}u_{m+2}\}$. Thus to attain a minimum antichain representation we should have either that u_{m+1} is adjacent to u_{m+2} and $d(u_{m+1}) = d(u_{m+3}) = 2$, or that $d(u_{m+1}) = 1$ and $d(u_{m+3}) = 2$.

For the latter case, see figure 14, where note that by symmetry we also have all neighbors of u_{m+2} being of degree one.

In figure 14, if we use the triangle $\{vu_{m+2}, vu_{m+3}, u_{m+2}u_{m+3}\}$, i.e., use one element to make the three vertices $vu_{m+2}, vu_{m+3}, u_{m+2}u_{m+3}$ be adjacent to each other, and either use one more element to make all vu_i with $1 \leq i \leq m+3$ and $i \neq m+2$ be adjacent to each other and $m+1$ more elements to

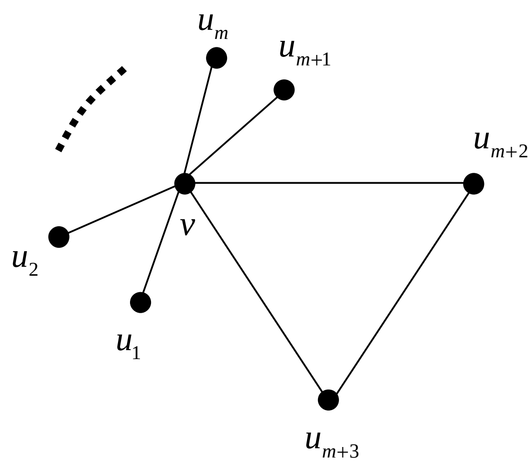

Figure 15:

respectively make vu_{m+2} be adjacent to $vu_1, ..., vu_{m+1}$ or exchange the roles of u_{m+2} and u_{m+3} and do the same as before, and then similarly for the v'-star, then we can obtain four more distinct minimum antichain representations different from "the canonical one".

When $d(u_{m+1}) = 1$ and $d(u_{m+2}) = d(u_{m+3}) = 2$, see figure 15.

In figure 15, if we use the triangle $\{vu_{m+2}, vu_{m+3}, u_{m+2}u_{m+3}\}$, and use one more element to make all vu_i with $1 \leq i \leq m+3, i \neq m+2$ be adjacent to each other, and use $m+1$ more elements to make vu_{m+2} be adjacent to $vu_1, ..., vu_{m+1}$, respectively, and then attach one monopolized element to the representation set of $u_{m+2}u_{m+3}$ then we obtain one more minimum antichain representation other than "the canonical one".

As for the case that u_{m+1} is adjacent to u_{m+2} and $d(u_{m+1}) = d(u_{m+3}) = 2$, see figure 16.

In figure 16, if we use the $t - m$ triangles

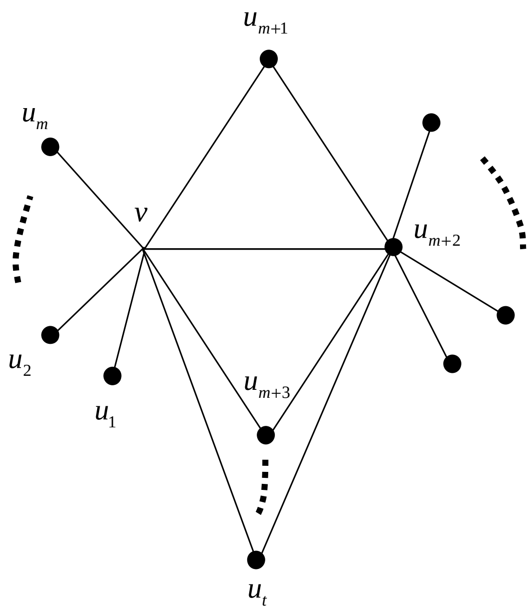

Figure 16:

$$\{vu_{m+1}, vu_{m+2}, u_{m+1}u_{m+2}\}, \{vu_{m+2}, vu_{m+3}, u_{m+2}u_{m+3}\},$$
$$..., \{vu_{m+2}, vu_t, u_{m+2}u_t\},$$

and use one more element to make all vu_i with $1 \leq i \leq t, i \neq m+2$ be adjacent to each other, and use m more elements to make vu_{m+2} be adjacent to $vu_1, ..., vu_m$, respectively, and then do the same for the u_{m+2}-star, we will obtain one more minimum antichain representation other than "the canonical one".

There is another case remained, see figure 17.

In figure 17, if we use the $t-m$ triangles

$$\{vu_{m+1}, vu_{m+2}, u_{m+1}u_{m+2}\}, \{vu_{m+2}, vu_{m+3}, u_{m+2}u_{m+3}\},$$
$$..., \{vu_{m+2}, vu_t, u_{m+2}u_t\},$$

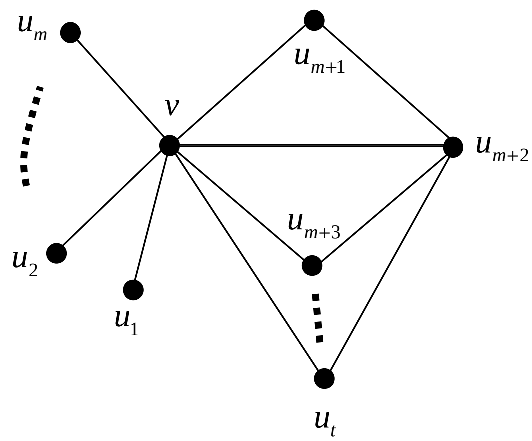

Figure 17:

and use one more element to make all vu_i with $1 \le i \le t, i \ne m+2$ be adjacent to each other, and then use one more element to make all $u_{m+2}u_i$ with $m+1 \le i \le t$ and $i \ne m+2$ be adjacent to each other, we will obtain one more minimum antichain representation other than "the canonical one".

Summarizing the above, we have the following theorem.

THEOREM 6.4. *Let G be a connected simple graph, and $G \ne K_3, K_4, 3K_2 \vee K_1$, or $W_t, t \ge 2$. In addition, we suppose that G is not a star, and is not a graph as in figure 14, 15, 16, 17. Let $V_2(G)$ denote the set of vertices in G with degree at least two, and let w_3 denote the number of 3-wing in G. And let $u_1^{(i)}, ..., u_{m_i}^{(i)}$ be all vertices in G of degree one and adjacent to v_i with $d(v_i) > 1$. We suppose that there are k vertices with its degree more than one in G in total which are adjacent to some vertex of degree one, i.e., $1 \le i \le k$. And we suppose that there are altogether k' such numbers i in $\{1, ..., k\}$ so that $t_i = m_i + 1$, and that among the k' numbers there are k'' such numbers i so that there exists projective plane with t_i vertices. Then when regarding $L(G)$ as a multigraph, $\omega_{\text{ai}}(L(G)) = |V_2(G)| + \sum_{i=1}^{k} m_i$ and there are exactly $3^{k'-k''}4^{k''}$ distinct minimum antichain representations of $L(G)$.*

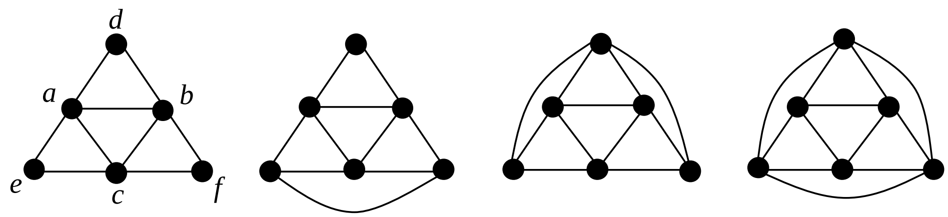

Figure 18:

7 Clique-Helly graph, maximal clique irreducible graph, and strongly chordal graph.

A *clique-Helly* (CH) graph is a graph G whose maximal clique obey the so-called *Helly-property*: For any set of pairwisely intersecting maximal cliques, the total intersection of these maximal cliques is nonempty. G is *hereditary clique-Helly* (HCH) if every induced subgraph of G is clique-Helly. The smallest graph that is not CH is the *Hajós graph*, see the first graph in figure 18.

It is well-known that CH graphs are easy to be constructed — for any graph G, the graph $G \vee K_1$ (formed by adding a vertex v that is adjacent to all of $V(G)$) is CH as every clique contain v. So every graph is an induced subgraph of a CH graph with just one more vertex. In particular, the $k-$ partite complete graph $K_{3,\ldots,3}$ with all parts of size 3 has $3k$ vertices but 3^k cliques, and is not CH when $k \geq 3$; on the other hand $K_{3,\ldots,3} \vee K_1$ is CH with $3k+1$ vertices and 3^k cliques. We have just seen that there are both CH graphs and non-CH graphs with exponentially many cliques. In Prisner [14], Prisner regarded this observation as the reason why there had been no polynomial recognition algorithm known for CH graphs up to now, and thus in that paper he considered the HCH graphs proving the following theorem, where ocular graphs are the four graphs in figure 18.

THEOREM 7.1. *A graph is HCH if and only if it contains no ocular graph as induced subgraph.*

A *hypergraph* consists of a collection of vertices and a collection of edges; if the vertex set is V, then the edges are subsets of V. Any hypergraph on n vertices and m edges yields an n by m $(0,1)$-matrix $A = [a_{ij}]$, where $a_{ij} = 1$ if and only if vertex j is in edge i. The matrix A is a (edge-vertex) incidence matrix of the hypergraph. We avoid dealing with row and column permutations of the matrix here.

Let a *chordless chain* of length k be a chain $x_1E_1x_2E_2...x_kE_kx_{k+1}$ of distinct vertices x_i and distinct edges E_j with

$$E_i \cap \{x_1, x_2, ..., x_{k+1}\} = \{x_i, x_{i+1}\} \text{ for } i = 1, 2, ..., k.$$

A *chordless cycle* of length k is defined to be a chordless chain of length k except that $x_1 = x_{k+1}$.

A $(0,1)$-matrix is *totally balanced* if it doesn't contain an incidence matrix of any graph cycle, of length at least 3, as a submatrix, where *graph cycle* means cycle in graph rather than cycle in hypergraph.

In Anstee and Farber [2], Anstee and Farber suggested the following remark.

REMARK 7.2. *The incidence matrix of a hypergraph is totally balanced if and only if the hypergraph contains no chordless cycle of size greater than 2.*

The clique matrix of a graph G on the vertices $v_1, ..., v_n$ with maximal cliques $C_1, ..., C_k$ is the k by n matrix $C(G)$ whose (i, j) entry is 1 if v_j is in C_i and is 0 otherwise.

In Farber [9], Farber proved the following theorem.

THEOREM 7.3. *The graph G is strongly chordal if and only if $C(G)$ is totally balanced.*

A *partial hypergraph* of a hypergraph $H = (V, \{S_1, ..., S_t\})$ is any hypergraph we can obtain by deleting hyperedges and vertices, that is, any hyperedge $P = (W, \{W \cap S_j : j \in J\})$, where $W \subseteq V$ and $J \subseteq \{1, ..., t\}$. The *underlying graph* $U(H)$ of the hypergraph has the same vertex set as H, and two distinct vertices are adjacent in $U(H)$ if they lie in some common hyperedge. A hypergraph H is *conformal*, if the set of its hyperedges are exactly the set of maximal cliques of $U(H)$.

In Prisner [14], Prisner proved the following theorem.

THEOREM 7.4. *Let Θ denote the class of all conformal hypergraphs without graph cycle of length 3 as partial hypergraph. Then the underlying graphs of the members of Θ are exactly the HCH graphs.*

Note that it is straightforward to realize the fact that for any conformal hypergraph H, the edge-vertex incidence matrix of H are exactly the clique matrix of $U(H)$. And note that the incidence matrix of a partial hypergraph $P = (W, \{W \cap S_j : j \in J\})$ of H is exactly the submatrix of the incidence matrix of H with certain rows and columns corresponding to W, $\{S_j : j \in J\}$. Hence the underlying graphs of the members of the class Θ of hypergraphs in the above theorem have their clique matrices containing no incidence matrix of one graph cycle of length 3 as a submatrix. But theorem 7.4 state that the underlying graphs of the members of the class Θ are exactly the HCH graphs, and therefore we know that all HCH graphs have their clique matrices containing no incidence matrix of one graph cycle of length 3 as a submatrix.

THEOREM 7.5. *All HCH graphs have their clique matrices containing no incidence matrix of one graph cycle of length 3 as a submatrix.*

Now let G be strongly chordal, then by theorem 7.3 we know that $C(G)$ contain no incidence matrix of any graph cycle of length at least 3 as a submatrix. Now we construct a hypergraph H with its vertex set $V(H)$ being exactly $V(G)$ and with its edge set being exactly the set of all maximal cliques of G. Then H is conformal and have its incidence matrix identical with $C(G)$ which contain no incidence matrix of any graph cycle of length at least 3 as a submatrix. Thus $H \in \Theta$. Thus by theorem 7.4 $U(H) = G$ is HCH.

THEOREM 7.6. *Every strongly chordal graph is HCH.*

As an extension of theorem 7.1, Szwarcfiter [15] characterize the CH graphs using the terminology of extended triangle. Let G be a graph and T a triangle of it. The *extended triangle* of G, relative to T, is the subgraph of G induced by the vertices which form a triangle with at least one edge of T. Let H be a subgraph of G. A vertex $v \in V(H)$ is *universal* in H whenever v is adjacent to every other vertex of H. In terms of extended triangle, Szwarcfiter characterize CH graphs in the following theorem.

THEOREM 7.7. *G is CH graphs if and only if every extended triangle of G contains an universal vertex.*

This lead to a polynomial time algorithm for recognizing CH graphs.

A graph is called irreducible if each maximal clique of G contains an edge which is not contained in any other maximal clique of G. Otherwise G is called reducible. Wallis and Zhang [18] characterized irreducible graph in the following theorem.

THEOREM 7.8. *A graph G is reducible if and only if there exists a set of maximal cliques*

$$\mathcal{F} = \{M_1, ..., M_t\}$$

such that the set of vertices contained in at least two maximal cliques in $\mathcal{F}$ form a maximal clique different from those in $\mathcal{F}$.

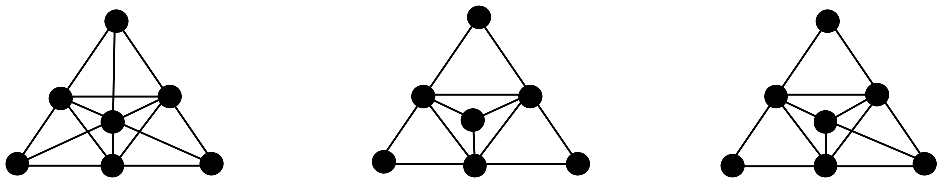

Figure 19:

In Prisner [14], Prisner characterized hereditary irreducible graph as well as HCH graph in the following theorem.

THEOREM 7.9. *A graph G is HCH if and only if G is hereditary irreducible.*

By the above theorem, we know that CH and irreducible are equivalent under "hereditariness". However, CH and irreducible graphs are incompatible. For example, the first graph in figure 19 is CH but not irreducible, while the second and third in figure 19 are irreducible but not CH.

In fact , in some measure, the first graph can represent the class of graphs which are CH but not HCH, and the second and third can represent the class of graphs which are irreducible but not hereditary irreducible.

Let G be CH but not HCH. Then by theorem 7.1, G containan induced ocular graph, say H, and by theorem 7.7 the extended triangle of the middle triangle, say T, of this ocular graph must have universal vertex. But none of the 6 vertices of this ocular graph can be universal vertex of this extended triangle. Thus there must be another vertex in G adjacent to the 6 vertices of this ocular graph as the first graph in figure 19.

Let G be irreducible but not hereditary irreducible. Then by theorem 7.1 and 7.9, G contain an induced ocular graph, say H. Now assume that the middle triangle $\{a, b, c\}$ of H is a maximal clique, refer to the first graph in figure 18, then three different maximal cliques containing the three triangles $\{a, c, e\}, \{a, b, d\}, \{b, c, f\}$, respectively together with the maximal clique

$\{a, b, c\}$ form a contradiction with theorem 7.8. Thus there must be another vertex, say v_1, in G adjacent to a, b, c, since $\{a, b, c\}$ is not a maximal clique. Now if v_1 is adjacent to at least two of the three vertices d, e, f, say d, e , and $\{v_1, a, b, c\}$ is a maximal clique, then the three maximal cliques $\{v_1, a, c, e\}, \{v_1, a, b, d\}, \{b, c, f\}$ together with the middle maximal clique $\{v_1, a, b, c\}$ again form a contradiction with theorem 7.8. Thus there exists another vertex, say v_2, in G adjacent to v_1, a, b, c, since $\{v_1, a, b, c\}$ is not a maximal clique. Now if v_2 is still adjacent to at least two of the three vertices d, e, f and $\{v_1, v_2, a, b, c\}$ is a maximal clique, then similarly we again arrive at a contradiction with theorem 7.8. Eventually we will find a vertex $v' \notin \{a, b, c, d, e, f\}$ adjacent to at most one of the three vertices d, e, f as the second or third graph in figure 19.

THEOREM 7.10. *Let G be CH but not HCH. Then G contain an induced ocular graph H together with another v adjacent to all vertices of H.*

On the other hand, let G be irreducible but not hereditary irreducible. Then G contain an induced ocular graph H together with another vertex v adjacent to all vertices of the middle triangle of H and adjacent to at most one of the other vertices of H.

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