Abstract

In this thesis, we are interested in studying the stability of the unique positive equilibrium point of a nonselective harvesting of two competing fish species in the presence of toxicity with time delays. Firstly, we state the formulation of the model. Secondly, we drive different sufficient conditions for local and global stability of the positive equilibrium point of the system, respectively. Finally, we illustrate our results by some examples.

Keywords: Nonselective harvesting, Intraspecific Competition, Interspecific Competition,Time delay, Lyapunov functional.

Contents

1 Introduction

In recent years, the application of theories of functional differential equations in mathematical ecology has developed rapidly. Various mathematical models have been proposed in the study of population dynamics, ecology and epidemic [8]. Some of them are described as autonomous delay differential equations. Many people are doing research on the dynamics of population with delays, which is useful for the control of the population of mankind, animals and the environment. Clark [1] studied the problem of combined harvesting of two independent fish species governed by the logistic law of growth. Based on the work of Clark [1], Chaud-huri [2] proposed a model to study the combined harvesting of two competing fish species. In addition, Mesterton-Gibbons [7] extended the work of Clark [1] and found criteria for the survival of less productive species as a function of the system parameters and initial stocks.

In [5], they discuss nonselective harvesting of two competing fish species, each of which obeys the logistic law of growth. As pointed out above, it is assumed that the two fish species compete with each other for using a common source of food and each species releases a substance toxic to the other species as a biological measure of deterring the competitor from sharing the food resource. They develop a bioeconomic model of harvesting both the competing fish species which can demonstrate the toxin producing interspecific reaction as stated above. This is the first bioeconomic model of this kind. The local and the global stabilities of the dynamical system for the model are examined and the existence of a bionomic equilibrium is investigated. If we want to consider that it is a factor that the fish grows up from the seedling to multiply the ability of future generation through one periodic time, how is the dynamic behavior of this model? Since time delays occur so often in nature, a number of models in ecology can be formulated as systems of differential equations with time delays. One of the most important problems for this type of system is to analyze the effect of time delays on the stability of the system.

Because this assumption corresponds to the fact that the fish species cannot give birth to fishes when the species are infants, fishes have to mature for a duration of time. In order to make this model correspond with the factual factor, we assume that the system model obey the logistic law of growth with time delays.

The main purpose of this thesis is to establish local and global stability of the unique positive equilibrium of the system with two different time delays. In section 2, we introduce some definitions and theorems. In section 3, we give sufficient conditions for the unique positive equilibrium point of the system. In section 4, we analyze uniform persistence of the system. In section 5, we discuss the local and global stability by constructing respective differential Lyapunov functionals. Finally, in section 6, we illustrate our results by some examples.

2 Preliminaries

For ordinary differential equations, we have definitions and theorems of stability theory and we view the solution of initial value problem as maps in Euclidean space. In order to establish a similar view for the solution of delay differential equations, we need some definitions.

We denote $\mathcal{C} \equiv C([-\tau, 0], R^n)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n with the topology of uniform convergence. That is, for $\phi \in \mathcal{C}$, the norm of ϕ is defined as $\|\phi\| = \sup_{\theta \in [-\tau,0]} |\phi(\theta)|$, where $|\cdot|$ is a norm in R^n. We define $x_t \in \mathcal{C}$ as $x_t(\theta) = x(t+\theta),\ \theta \in [-\tau, 0]$. Assume that Ω is a subset of $\mathcal{C}$ and $f : \Omega \to R^n$ is a given function, then we consider the following general nonlinear autonomous system of delay differential equation

$$\dot{x}(t) = f(x_t) \tag{2.1}$$

Definition 2.1 *[6] Let $R^2_+ = \{x \in R^2 | x_i \geq 0\ ,\ i = 1, 2\}$. The notation $x > 0$ denotes $x \in \mathrm{Int}R^2_+$. The system (2.1) is said to be unifromly persistent if there exists a compact region $D \subseteq \mathrm{Int}R^2_+$ such that every positive solution $x(t)$ of the system (2.1) with the initial conditions eventually enters and remains in the region D.*

Definition 2.2 *[6] We say that $\phi \in B(0, \delta)$ if $\phi \in \mathcal{C}$ and $\|\phi\| \leq \delta$, where $\|\phi\| = \sup_{\theta \in [-\tau,0]} |\phi(\theta)|$.*

(i) The solution $x = 0$ of the system (2.1) is said to be stable if, for any $\sigma \in R$, $\epsilon > 0$, there is a $\delta = \delta(\epsilon, \sigma)$ such that $\phi \in B(0, \delta)$ implies $x_t(\sigma, \phi) \in B(0, \epsilon)$ for $t \geq \sigma$. Otherwise, we say that $x = 0$ is unstable.

(ii) *The solution* $x = 0$ *of the system (2.1) is said to be asymptotically stable if it is stable and there is a* $b_0 = b(\sigma) > 0$ *such that* $\phi \in B(0, b_0)$ *implies* $x(\sigma, \phi)(t) \to 0$ *as* $t \to \infty$.

(iii) *The solution* $x = 0$ *of the system (2.1) is said to be uniformly stable if the number* δ *in the definition of stable is independent of* σ.

(iv) *The solution* $x = 0$ *of the system (2.1) is said to be uniformly asymptotically stable if it is uniformly stable and there is a* $b_0 > 0$ *such that, for every* $\eta > 0$, *there is a* $t_0(\eta)$ *such that* $\phi \in B(0, b_0)$ *implies* $x_t(\sigma, \phi) \in B(0, \eta)$ *for* $t \geq \sigma + t_0(\eta)$, *for every* $\sigma \in R$.

Theorem 2.1 *[6]* *Assume that* $u(\cdot)$ *and* $w(\cdot)$ *are nonnegative continuous,* $u(0) = w(0) = 0$, $\lim_{s \to +\infty} u(s) = +\infty$, *and that* $V : \mathcal{C} \to R$ *is continuous and satisfies*

$$V(\phi) \geq u(|\phi(0)|)$$

and

$$\dot{V}(\phi) \leq -w(|\phi(0)|).$$

Then the solution $x = 0$ *of the system (2.1) is uniformly stable, and every solutions is bounded. If in addition,* $w(s) > 0$ *for* $s > 0$, *then* $x = 0$ *is globally asymptotically stable.*

Lemma 2.1 *[4]* ***(Barbălat's Lemma)*** *Let* f *be a nonnegative function defined on* $[0, \infty)$ *such that* f *is integrable on* $[0, \infty)$ *and uniformly continuous on* $[0, \infty)$. *Then*

$$\lim_{t \to \infty} f(t) = 0.$$

3 Formulation and Equilibrium Points of the Model

The combined harvesting of competition system with time delays is of the form

$$\begin{aligned} \dot{x}_1(t) &= r_1 x_1(t)[1 - \frac{x_1(t-\tau_1)}{k_1}] - \alpha_1 x_1(t) x_2(t) - \beta_1 x_2(t) x_1^2(t) - q_1 e x_1(t) \\ \dot{x}_2(t) &= r_2 x_2(t)[1 - \frac{x_2(t-\tau_2)}{k_2}] - \alpha_2 x_1(t) x_2(t) - \beta_2 x_1(t) x_2^2(t) - q_2 e x_2(t) \end{aligned} \tag{3.1}$$

with the initial conditions

$$\begin{aligned} & x_i(\theta) = \phi_i(\theta) > 0 \quad , \quad \theta \in [-\tau, 0] \quad , \quad \phi_i \in C\left([-\tau, 0], R_+\right) \\ & \tau = \max\{\tau_1, \tau_2\} \quad , \quad i = 1, 2 \end{aligned} \tag{3.2}$$

where $x_i(t)$ $(i = 1, 2)$denote the population densities of the two competing species at any time t. r_i, α_i, β_i, k_i, q_i and e $(i = 1, 2)$ are all positive constants. Here r_1 and r_2 denote the natural growth rates. k_1 and k_2 are the environmental carrying capacity of the two species. The delay τ_1 and τ_2 are constants representing a fixed period of time. After a fixed period of time, competing species growth will affect population density. α_1 and α_2 are the co-efficients of interspecific competition between the species. β_1 and β_2 are co-efficients of toxicity.

Since $d/dx_1(\beta_1 x_1^2) = 2\beta_1 x_1 > 0$ and $d^2/dx_1^2(\beta_1 x_1^2) = 2\beta_1 > 0$, there is an accelerated growth in the rate of production of the toxic substance as the density of the competing species increase. e denote the combined harvesting effort. q_1 and q_2 are the catchability coefficients of the species. The catch-rate function $q_1 e x_1$ and $q_2 e x_2$ are based on the catch-per–unit-effort (CPUE) hypothesis.[1]

All we want to discuss is biological population, so we just consider the first quadrant in the $x_1 - x_2$ plane.

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1(1 - \frac{x_1(t)}{k_1}) - \alpha_1 x_2(t) - \beta_1 x_1(t) x_2(t) - q_1 e] \\ \dot{x}_2(t) &= x_2(t)[r_2(1 - \frac{x_2(t)}{k_2}) - \alpha_2 x_1(t) - \beta_2 x_1(t) x_2(t) - q_2 e] \end{aligned}$$

Clearly, $\bar{E} \equiv (0,0)$ is an equilibrium point of the system (3.1). And all possible equilibrium points of the system (3.1) are

$$\tilde{E} \equiv (\tilde{x}_1, 0) \quad , \quad \tilde{x}_1 = \frac{k_1}{r_1}(r_1 - q_1 e)$$
$$\hat{E} \equiv (0, \hat{x}_2) \quad , \quad \hat{x}_2 = \frac{k_2}{r_2}(r_2 - q_2 e)$$

and $E^* \equiv (x_1^*, x_2^*)$ where x_1^* and x_2^* satisfy

$$\begin{aligned} r_1(1 - \frac{x_1^*}{k_1}) - \alpha_1 x_2^* - \beta_1 x_1^* x_2^* - q_1 e = 0 \\ r_2(1 - \frac{x_2^*}{k_2}) - \alpha_2 x_1^* - \beta_2 x_1^* x_2^* - q_2 e = 0 \end{aligned} \tag{3.3}$$

The ratio r_1/q_1 of the biotic potential r_1 to the catchability coefficient q_1 is known as the biotechnical productivity (BTP) of the species.[1] It is easy to see that the equilibrium point $\tilde{E}$ exists if

$$\frac{k_1}{r_1}(r_1 - q_1 e) > 0 \Rightarrow e < \frac{r_1}{q_1}$$

i.e. the harvesting effort is less than the biotechinical productivity (BTP_{x_1}). Similarly, $\hat{E}$ exists if

$$\frac{k_2}{r_2}(r_2 - q_2 e) > 0 \Rightarrow e < \frac{r_2}{q_2}$$

i.e. the harvesting effort is less than the biotechinical productivity (BTP_{x_2}).

Remark 3.1

Let x_1^* and x_2^* satisfy the equation (3.3), and A_i, B_i, C_i satisfy $A_i {x_i^*}^2 + B_i x_i^* + C_i = 0$, $(i = 1, 2)$, where

$$A_1 = \frac{-k_2\beta_2}{k_1} \frac{(r_1\beta_2 - k_1\alpha_2\beta_1)}{(r_2\beta_1 - k_2\alpha_1\beta_2)}$$
$$B_1 = \frac{\beta_2}{(r_2\beta_1 - k_2\alpha_1\beta_2)}[\frac{-r_1 r_2}{k_1} + k_2(\alpha_1\alpha_2 - r_2\beta_1 + r_1\beta_2 + \beta_1 q_2 e - \beta_2 q_1 e)]$$
$$C_1 = \frac{\beta_2}{(r_2\beta_1 - k_2\alpha_1\beta_2)}(r_1 r_2 - r_2 k_2 \alpha_1 - r_2 q_1 e + k_2 \alpha_1 q_2 e)$$

$$A_2 = \frac{-k_1\beta_1}{k_2}\frac{(r_2\beta_1 - k_2\alpha_1\beta_2)}{(r_1\beta_2 - k_1\alpha_2\beta_1)}$$
$$B_2 = \frac{\beta_1}{(r_1\beta_2 - k_1\alpha_2\beta_1)}[\frac{-r_1r_2}{k_2} + k_1(\alpha_1\alpha_2 - r_1\beta_2 + r_2\beta_1 + \beta_2q_1e - \beta_1q_2e)]$$
$$C_2 = \frac{\beta_1}{(r_1\beta_2 - k_1\alpha_2\beta_1)}(r_1r_2 - r_1k_1\alpha_2 - r_1q_2e + k_1\alpha_2q_1e)$$

If the follows holds

$$A_1C_1 < 0 \quad and \quad A_2C_2 < 0$$

Then E^* is the unique positive equilibrium point of the system (3.1).

4 Uniform Persistence

The system (3.1) has an unique positive equilibrium point if Remark 3.1 holds. In the following, we always assume that such a positive equilibrium exists and denote it by $E^*(x_1^*, x_2^*)$. The following lemmas are elementary concerned with the qualitative nature of solutions of the system (3.1).

Lemma 4.1 *All solutions of the system (3.1) with the initial conditions (3.2) are positive for all* $t \geq 0$

Proof: It is true because

$$\begin{aligned} x_1(t) &= x_1(0)\exp\left\{\int_0^t [r_1(1-\frac{x_1(s-\tau_1)}{k_1})-\alpha_1 x_2(s)-\beta_1 x_1(s)x_2(s)-q_1 e]ds\right\} \\ x_2(t) &= x_2(0)\exp\left\{\int_0^t [r_2(1-\frac{x_2(s-\tau_2)}{k_2})-\alpha_2 x_1(s)-\beta_2 x_1(s)x_2(s)-q_2 e]ds\right\} \end{aligned} \tag{4.1}$$

and $x_i(0) > 0$ $(i = 1, 2)$. Therefore, we obtain that all solutions $(x_1(t), x_2(t))$ of the system (3.1) with the initial conditions (3.2) are positive. ∎

Lemma 4.2 *Let* $(x_1(t), x_2(t))$ *denote the solution of the system(3.1) with the initial conditions (3.2), then*

$$0 < x_i(t) \leq M_i \text{ , for } i = 1, 2 \tag{4.2}$$

eventually for all large t*, where*

$$M_1 = k_1 e^{r_1 \tau_1} \tag{4.3}$$

$$M_2 = k_2 e^{r_2 \tau_2} \tag{4.4}$$

Proof: By Lemma 4.1, we know that the solutions of the system (3.1) with initial conditions (3.2) are positive, and hence, by first equation of the system (3.1),

$$\begin{aligned}\dot{x}_1(t) &= r_1x_1(t)[1-\frac{x_1(t-\tau_1)}{k_1}]-\alpha_1x_1(t)x_2(t)-\beta_1x_1^2(t)x_2(t)-q_1ex_1(t)\\ &\leq r_1x_1(t)[1-\frac{x_1(t-\tau_1)}{k_1}] \qquad (4.5)\end{aligned}$$

Taking $M_1^* = k_1\,(1+b_1)$, where $0 < b_1 < e^{r_1\tau_1}-1$. Due to the variation of $x_1(t)$ with respect to m_1^*, we discuss the following two cases.

Case 1: Suppose $x_1(t)$ is not oscillatory about M_1^*. That is, there exists a $T_0 > 0$ such that either

$$x_1(t) \leq M_1^* \qquad \text{for} \;\; t > T_0 \qquad (4.6)$$

or

$$x_1(t) > M_1^* \qquad \text{for} \;\; t > T_0 \qquad (4.7)$$

(i)If (4.6) holds, then for $t > T_0$

$$x_1(t) \leq M_1^* = k_1(1+b_1) < k_1e^{r_1\tau_1} = M_1$$

That is, (4.2) holds for i=1.

(ii)Suppose (4.7) holds, equation (4.5) implies that, for $t > T_0+\tau_1$

$$\begin{aligned}\dot{x}_1(t) &\leq r_1x_1(t)[1-\frac{x_1\left(t-\tau_1\right)}{k_1}]\\ &< r_1x_1(t)[1-\frac{M_1^*}{k_1}]\\ &= -r_1x_1(t)b_1\end{aligned}$$

It follows that

$$\begin{aligned}\int_{T_0+\tau_1}^{t}\frac{\dot{x}_1(s)}{x_1(s)}ds &< \int_{T_0+\tau_1}^{t}-b_1r_1ds\\ &= -b_1r_1(t-T_0-\tau_1).\end{aligned}$$

Then $0 < x_1(t) < x_1(T_0+\tau_1)e^{-b_1 r_1(t-T_0-\tau_1)} \to 0$ as $t \to \infty$. By the Squeeze Theorem, $\lim_{t\to\infty} x_1(t) = 0$. It contradics to (4.7). Therefore, there must exist a $T_1 > T_0$ such that $x_1(T_1) \le M_1^*$. If $x_1(t) \le M_1^*$ for all $t \ge T_1$, then (4.2) follows. If not, then there must exists a $T_2 > T_1$ such that T_2 be the first time which $x_1(T_2) > M_1^*$. Therefore, there exists a $T_3 > T_2$ such that $x_1(T_3) \le M_1^*$ by above discussion. By above, we know that $x_1(T_1) \le M_1^*$, $x_1(T_2) > M_1^*$, and $x_1(T_3) \le M_1^*$ where $T_1 < T_2 < T_3$. Then, by the Intermediate Value Theorem, there exists T_4 and T_5 such that

$$x_1(T_4) = M_1^* \quad , \quad T_1 \le T_4 < T_2$$
$$x_1(T_5) = M_1^* \quad , \quad T_2 < T_5 \le T_3$$

and $x_1(t) > M_1^*$ for $T_4 < t < T_5$. Hence there is a $T_6 \in (T_4, T_5)$ such that $x_1(T_6)$ is local maximum, and it follows from (4.5) that

$$\begin{aligned} 0 &= \dot{x_1}(T_6) \\ &= r_1 x_1(T_6)[1 - \frac{x_1(T_6-\tau_1)}{k_1}] - \alpha_1 x_1(T_6)x_2(T_6) - \beta_1 x_1^2(T_6)x_2(T_6) - q_1 e x_1(t) \\ &\le r_1 x_1(T_6)[1 - \frac{x_1(T_6-\tau_1)}{k_1}] \end{aligned} \tag{4.8}$$

and this implies

$$x_1(T_6 - \tau_1) \le k_1. \tag{4.9}$$

Integrating both sides of (4.5) on the interval $[T_6 - \tau_1, T_6]$, we can have

$$\begin{aligned} \int_{T_6-\tau_1}^{T_6} \frac{\dot{x_1}(s)}{x_1(s)} ds &\le \int_{T_6-\tau_1}^{T_6} r_1[1 - \frac{x_1(s-\tau_1)}{k_1}]ds \\ &\le \int_{T_6-\tau_1}^{T_6} r_1 ds \\ &= r_1\tau_1 \end{aligned} \tag{4.10}$$

By (4.9) and (4.10) imply

$$x_1(T_6) \le x_1(T_6-\tau_1)\exp(r_1\tau_1) \le k_1 \exp(r_1\tau_1) = M_1$$

Thus

$$x_1(t) \le M_1, \quad t \in [T_1, T_5].$$

Since any local maximum is less than or equal to M_1, thus there exist

$$x_1(t) \le M_1 \qquad \text{for } t \ge T_6. \tag{4.11}$$

That is, (4.2) holds.

Case 2: Suppose $x_1(t)$ is oscillatory about M_1^*, for this case, the proof is similarly to above one. And we can conclude that there exists a $\hat{T} \ge T_6$, such that $x_1(t) \le M_1$, for all $t \ge \hat{T}$.

By above conclude, we can conclude $x_2(t) \le M_2$, $M_2 = k_2 e^{r_2 \tau_2}$, for all $t \ge \hat{T}$.Thus

$$0 < x_i(t) \le M_i, \quad i = 1, 2 \qquad \text{for } \; t \ge T.$$

This completes the proof.

The following result shows that the system (3.1) is uniformly persistent.

Theorem 4.1 *Suppose that the system (3.1) satisfy the following:*

$$\begin{aligned} r_1 - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e &> 0 \\ r_2 - \alpha_2 M_1 - \beta_2 M_1 M_2 - q_2 e &> 0 \end{aligned} \tag{4.12}$$

in which M_i $(i = 1, 2)$ is defined by (4.3) and (4.4). Then the system (3.1) is uniformly persistent.

That is, there exist m_1, m_2 and $T^* > 0$ such that $m_1 \le x_1 \le M_1$ and $m_2 \le x_2 \le M_2$ for $t \ge T^*$, where

$$m_1 = \frac{k_1}{2r_1}(r_1 - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e)\exp\left\{[r_1(1 - \frac{M_1}{k_1}) - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e]\tau_1\right\}$$

$$m_2 = \frac{k_2}{2r_2}(r_2 - \alpha_2 M_1 - \beta_2 M_1 M_2 - q_2 e)\exp\left\{[r_2(1 - \frac{M_2}{k_2}) - \alpha_2 M_1 - \beta_2 M_1 M_2 - q_2 e]\tau_2\right\}$$

Proof: By Lemma4.2, equation (3.1) follows that for $t \geq T + \tau_1$

$$\begin{aligned} \dot{x_1}(t) &= x_1(t)[r_1(1 - \frac{x_1(t-\tau_1)}{k_1}) - \alpha_1 x_2(t) - \beta_1 x_1(t) x_2(t) - q_1 e] \\ &\geq x_1(t)[r_1(1 - \frac{M_1}{k_1}) - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e] \end{aligned} \tag{4.13}$$

$$\frac{\dot{x_1}(t)}{x_1(t)} \geq r_1(1 - \frac{M_1}{k_1}) - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e \tag{4.14}$$

Integrating both sides of (4.14) on $[t - \tau_1, t]$, where $t \geq T + \tau_1$, then we have

$$x_1(t) \geq x_1(t-\tau_1) \exp\left\{[r_1(1 - \frac{M_1}{k_1}) - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e]\tau_1\right\}$$

That is

$$x_1(t-\tau_1) \leq x_1(t) \exp\left\{-[r_1(1 - \frac{M_1}{k_1}) - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e]\tau_1\right\} \tag{4.15}$$

From (4.13) and (4.15), for $t \geq T + \tau_1$

$$\dot{x_1}(t) \geq x_1(t)\{r_1[1 - \frac{x_1(t)}{k_1 \exp\{[r_1(1 - \frac{M_1}{k_1}) - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e]\tau_1\}}]$$

$$-\alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e\}$$

$$\geq (r_1 - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e) x_1(t)\{1 -$$

$$\frac{x_1(t)}{\frac{k_1}{r_1}(r_1 - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e) \exp\{[r_1(1 - \frac{M_1}{k_1}) - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e]\tau_1\}}\}$$

$$x_1(t) \leq \frac{k_1}{r_1}(r_1 - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e) \exp\{[r_1(1 - \frac{M_1}{k_1}) - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e]\tau_1\}$$

It follows that

$$\begin{aligned} \liminf_{t \to \infty} x_1(t) &\geq \frac{k_1}{r_1}(r_1 - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e) \exp\{[r_1(1 - \frac{M_1}{k_1}) - \alpha_1 M_2 - \beta_1 M_1 M_2 - q_1 e]\tau_1\} \\ &\equiv \bar{m}_1 \end{aligned}$$

and $\bar{m}_1 > 0$, by (4.12). So for large t, $x_1(t) > \bar{m}_1/2 \equiv m_1 > 0$.

Similar to above one, we can conclude

$$\lim_{t\to\infty}\inf x_2(t) \geq \frac{k_2}{r_2}(r_2-\alpha_2M_1-\beta_2M_1M_2-q_2e)\exp\{[r_2(1-\frac{M_2}{k_2})-\alpha_2M_1-\beta_2M_1M_2-q_2e]\tau_2\}$$
$$\equiv \bar{m}_2$$

and $\bar{m}_2 > 0$, by (4.12). So for large t, $x_2(t) > \bar{m}_2/2 \equiv m_2 > 0$.

Let

$$D = \{(x_1(t), x_2(t))|m_1 \leq x_1(t) \leq M_1, m_2 \leq x_2(t) \leq M_2\}$$

For $t > T+\tau$ where $\tau = \max\{\tau_1, \tau_2\}$, and t is large enough. Then D is bounded compact region in $\mathbb{R}^2_+$ that positive distance from coordinate hyperplancds.

Hence we obtain that exists a $T^* > 0$ such that if $t \geq T^*$, then every positive solution of system (3.1) with the initial condition (3.2) eventually enters and remains in the regin D, that is, system (3.1) is uniformly persistent.

5 Stability

In this chapter, we discuss the local stability and the global stability of the equilibrium point E^* of the system (3.1).

5.1 Local Stability

To investigate the local stability of the equilibrium point E^*, we linearize the system (3.1). Let

$$y_1(t) = x_1(t) - x_1^*$$

$$y_2(t) = x_2(t) - x_2^*$$

be the perturbed variables. After removing nonlinear terms, we obtain the linear variational system by using equilibria conditions as

$$\begin{aligned}
\dot{y_1}(t) &= [r_1(1 - \frac{x_1^*}{k_1}) - \alpha_1 x_2^*(t) - 2\beta_1 x_1^* x_2^* - q_1 e] y_1(t) \\
&\quad + [-\alpha_1 x_1^* - \beta_1 x_1^{*2}] y_2(t) - \frac{r_1 x_1^*}{k_1} y_1(t - \tau_1) \\
&= -\beta_1 x_1^* x_2^* y_1(t) + [-\alpha_1 x_1^* - \beta_1 x_1^{*2}] y_2(t) - \frac{r_1 x_1^*}{k_1} y_1(t - \tau_1) \\
\dot{y_2}(t) &= [r_2(1 - \frac{x_2^*}{k_2}) - \alpha_2 x_1^*(t) - 2\beta_2 x_1^* x_2^* - q_2 e] y_2(t) \\
&\quad + [-\alpha_2 x_2^* - \beta_2 x_2^{*2}] y_1(t) - \frac{r_2 x_2^*}{k_2} y_2(t - \tau_2) \\
&= [-\alpha_2 x_2^* - \beta_2 x_2^{*2}] y_1(t) - \beta_2 x_1^* x_2^* y_2(t) - \frac{r_2 x_2^*}{k_2} y_2(t - \tau_2)
\end{aligned} \tag{5.1}$$

It is noticed that the asymptotical stability of E^* of the system (3.1) is determined by the asymptotical stability of the zero solution of the system (5.1).[3]

Theorem 5.1 *Let $E^* = (x_1^*, x_2^*)$ be the unique equilibrium point of the system (3.1) and the delays τ_1 and τ_2 satisfy*

$$a_1 - a_2\tau_1 - a_3\tau_2 > 0 \tag{5.2}$$

and

$$b_1 - b_2\tau_1 - b_3\tau_2 > 0 \tag{5.3}$$

where

$$\begin{aligned}
a_1 &= 2\beta_1 x_1^* x_2^* + \frac{2r_1 x_1^*}{k_1} - \alpha_1 x_1^* - \beta_1 x_1^{*2} - \alpha_2 x_2^* - \beta_2 x_2^{*2} \\
a_2 &= \frac{r_1 x_1^*}{k_1}(2\beta_1 x_1^* x_2^* + \frac{2r_1 x_1^*}{k_1} + \alpha_1 x_1^* + \beta_1 x_1^{*2}) \\
a_3 &= \frac{r_2 x_2^*}{k_2}(\beta_2 x_2^{*2} + \alpha_2 x_2^*) \\
b_1 &= 2\beta_2 x_1^* x_2^* + \frac{2r_2 x_2^*}{k_2} - \alpha_1 x_1^* - \beta_1 x_1^{*2} - \alpha_2 x_2^* - \beta_2 x_2^{*2} \\
b_2 &= \frac{r_1 x_1^*}{k_1}(\beta_1 x_1^{*2} + \alpha_1 x_1^*) \\
b_3 &= \frac{r_2 x_2^*}{k_2}(2\beta_2 x_1^* x_2^* + \frac{2r_2 x_2^*}{k_2} + \alpha_2 x_2^* + \beta_2 x_2^{*2})
\end{aligned}$$

then the unique equilibrium point E^* of the system (3.1) is local asymptotically stable.

Proof: The equation (5.1) can be written as

$$\begin{aligned}
\frac{d}{dt}[y_1(t) - \frac{r_1 x_1^*}{k_1}\int_{t-\tau_1}^{t} y_1(s)ds] &= [-\beta_1 x_1^* x_2^* - \frac{r_1 x_1^*}{k_1}]y_1(t) + [-\alpha_1 x_1^* - \beta_1 x_1^{*2}]y_2(t) \\
\frac{d}{dt}[y_2(t) - \frac{r_2 x_2^*}{k_2}\int_{t-\tau_2}^{t} y_2(s)ds] &= [-\alpha_2 x_2^* - \beta_2 x_2^{*2}]y_1(t) + [-\beta_2 x_1^* x_2^* - \frac{r_2 x_2^*}{k_2}]y_2(t)
\end{aligned} \tag{5.4}$$

Let

$$W_1(y(t)) = [y_1(t) - \frac{r_1 x_1^*}{k_1}\int_{t-\tau_1}^{t} y_1(s)ds]^2 + [y_2(t) - \frac{r_2 x_2^*}{k_2}\int_{t-\tau_2}^{t} y_2(s)ds]^2 \tag{5.5}$$

then

$$\begin{aligned}
\frac{dW_1(y(t))}{dt} &= 2[y_1(t) - \frac{r_1 x_1^*}{k_1}\int_{t-\tau_1}^{t} y_1(s)ds][(-\beta_1 x_1^* x_2^* - \frac{r_1 x_1^*}{k_1})y_1(t) + (-\alpha_1 x_1^* - \beta_1 x_1^{*2})y_2(t)] \\
&\quad + 2[y_2(t) - \frac{r_2 x_2^*}{k_2}\int_{t-\tau_2}^{t} y_2(s)ds][(-\alpha_2 x_2^* - \beta_2 x_2^{*2})y_1(t) + (-\beta_2 x_1^* x_2^* - \frac{r_2 x_2^*}{k_2})y_2(t)] \\
&= -2(\beta_1 x_1^* x_2^* + \frac{r_1 x_1^*}{k_1})y_1^2(t) - 2(\beta_2 x_1^* x_2^* + \frac{r_2 x_2^*}{k_2})y_2^2(t)
\end{aligned}$$

$$+2(-\alpha_1 x_1^* - \beta_1 x_1^{*2}) y_1(t) y_2(t) + 2(-\alpha_2 x_2^* - \beta_2 x_2^{*2}) y_1(t) y_2(t)$$

$$+\frac{2r_1 x_1^*}{k_1}(\beta_1 x_1^* x_2^* + \frac{r_1 x_1^*}{k_1}) \int_{t-\tau_1}^{t} y_1(t) y_1(s) ds$$

$$+\frac{2r_1 x_1^*}{k_1}(\alpha_1 x_1^* + \beta_1 x_1^{*2}) \int_{t-\tau_1}^{t} y_2(t) y_1(s) ds$$

$$+\frac{2r_2 x_2^*}{k_2}(\alpha_2 x_2^* + \beta_2 x_2^{*2}) \int_{t-\tau_2}^{t} y_1(t) y_2(s) ds$$

$$+\frac{2r_2 x_2^*}{k_2}(\beta_2 x_1^* x_2^* + \frac{r_2 x_2^*}{k_2}) \int_{t-\tau_2}^{t} y_2(t) y_2(s) ds$$

$$\leq [-2(\beta_1 x_1^* x_2^* + \frac{r_1 x_1^*}{k_1}) + \alpha_1 x_1^* + \beta_1 x_1^{*2} + \alpha_2 x_2^* + \beta_2 x_2^{*2}] y_1^2(t)$$

$$+[-2(\beta_2 x_1^* x_2^* + \frac{r_2 x_2^*}{k_2}) + \alpha_1 x_1^* + \beta_1 x_1^{*2} + \alpha_2 x_2^* + \beta_2 x_2^{*2}] y_2^2(t)$$

$$+\frac{r_1 x_1^*}{k_1}(\beta_1 x_1^* x_2^* + \frac{r_1 x_1^*}{k_1}) \int_{t-\tau_1}^{t} [y_1^2(t) + y_1^2(s)] ds$$

$$+\frac{r_1 x_1^*}{k_1}(\alpha_1 x_1^* + \beta_1 x_1^{*2}) \int_{t-\tau_1}^{t} [y_2^2(t) + y_1^2(s)] ds$$

$$+\frac{r_2 x_2^*}{k_2}(\alpha_2 x_2^* + \beta_2 x_2^{*2}) \int_{t-\tau_2}^{t} [y_1^2(t) + y_2^2(s)] ds$$

$$+\frac{r_2 x_2^*}{k_2}(\beta_2 x_1^* x_2^* + \frac{r_2 x_2^*}{k_2}) \int_{t-\tau_2}^{t} [y_2^2(t) + y_2^2(s)] ds$$

$$= [-2(\beta_1 x_1^* x_2^* + \frac{r_1 x_1^*}{k_1}) + \alpha_1 x_1^* + \beta_1 x_1^{*2} + \alpha_2 x_2^* + \beta_2 x_2^{*2}$$

$$+\frac{r_1 x_1^*}{k_1}(\beta_1 x_1^* x_2^* + \frac{r_1 x_1^*}{k_1})\tau_1 + \frac{r_2 x_2^*}{k_2}(\alpha_2 x_2^* + \beta_2 x_2^{*2})\tau_2] y_1^2(t)$$

$$+[-2(\beta_2 x_1^* x_2^* + \frac{r_2 x_2^*}{k_2}) + \alpha_1 x_1^* + \beta_1 x_1^{*2} + \alpha_2 x_2^* + \beta_2 x_2^{*2}$$

$$+\frac{r_1x_1^*}{k_1}(\alpha_1x_1^*+\beta_1x_1^{*2})\tau_1+\frac{r_2x_2^*}{k_2}(\beta_2x_1^*x_2^*+\frac{r_2x_2^*}{k_2})\tau_2]y_2^2(t)$$

$$+\frac{r_1x_1^*}{k_1}(\beta_1x_1^*x_2^*+\frac{r_1x_1^*}{k_1}+\alpha_1x_1^*+\beta_1x_1^{*2})\int_{t-\tau_1}^t y_1^2(s)ds$$

$$+\frac{r_2x_2^*}{k_2}(\alpha_2x_2^*+\beta_2x_2^{*2}+\beta_2x_1^*x_2^*+\frac{r_2x_2^*}{k_2})\int_{t-\tau_2}^t y_2^2(s)ds \tag{5.6}$$

Now, we let

$$\begin{aligned} W_2(y(t)) &= \frac{r_1x_1^*}{k_1}(\beta_1x_1^*x_2^*+\frac{r_1x_1^*}{k_1}+\alpha_1x_1^*+\beta_1x_1^{*2})\int_{t-\tau_1}^t\int_s^t y_1^2(\rho)d\rho ds \\ &\quad +\frac{r_2x_2^*}{k_2}(\alpha_2x_2^*+\beta_2x_2^{*2}+\beta_2x_1^*x_2^*+\frac{r_2x_2^*}{k_2})\int_{t-\tau_2}^t\int_s^t y_2^2(\rho)d\rho ds \end{aligned} \tag{5.7}$$

then

$$\begin{aligned} \frac{dW_2(y(t))}{dt} &= \frac{r_1x_1^*}{k_1}(\beta_1x_1^*x_2^*+\frac{r_1x_1^*}{k_1}+\alpha_1x_1^*+\beta_1x_1^{*2})[\tau_1y_1^2(t)-\int_{t-\tau_1}^t y_1^2(s)ds] \\ &\quad +\frac{r_2x_2^*}{k_2}(\alpha_2x_2^*+\beta_2x_2^{*2}+\beta_2x_1^*x_2^*+\frac{r_2x_2^*}{k_2})[\tau_2y_2^2(t)-\int_{t-\tau_2}^t y_2^2(s)ds] \end{aligned} \tag{5.8}$$

Now, we define a Lyapunov functional $W(y(t))$ as

$$W(y(t)) = W_1(y(t))+W_2(y(t)) \tag{5.9}$$

then we have (5.10) from (5.6) and (5.8) that

$$\begin{aligned} \frac{dW(y(t))}{dt} &= \frac{dW_1(y(t))}{dt}+\frac{dW_2(y(t))}{dt} \\ &\le [-2(\beta_1x_1^*x_2^*+\frac{r_1x_1^*}{k_1})+\alpha_1x_1^*+\beta_1x_1^{*2}+\alpha_2x_2^*+\beta_2x_2^{*2} \\ &\quad +\frac{r_1x_1^*}{k_1}(\beta_1x_1^*x_2^*+\frac{r_1x_1^*}{k_1})\tau_1+\frac{r_2x_2^*}{k_2}(\alpha_2x_2^*+\beta_2x_2^{*2})\tau_2]y_1^2(t) \\ &\quad +[-2(\beta_2x_1^*x_2^*+\frac{r_2x_2^*}{k_2})+\alpha_1x_1^*+\beta_1x_1^{*2}+\alpha_2x_2^*+\beta_2x_2^{*2} \\ &\quad +\frac{r_1x_1^*}{k_1}(\alpha_1x_1^*+\beta_1x_1^{*2})\tau_1+\frac{r_2x_2^*}{k_2}(\beta_2x_1^*x_2^*+\frac{r_2x_2^*}{k_2})\tau_2]y_2^2(t) \\ &\quad +[\frac{r_1x_1^*}{k_1}(\beta_1x_1^*x_2^*+\frac{r_1x_1^*}{k_1}+\alpha_1x_1^*+\beta_1x_1^{*2})]\tau_1y_1^2(t) \end{aligned}$$

$$
\begin{aligned}
& +[\frac{r_2x_2^*}{k_2}(\alpha_2x_2^* + \beta_2x_2^{*2} + \beta_2x_1^*x_2^* + \frac{r_2x_2^*}{k_2})]\tau_2y_2^2(t) \\
= \; & [-2(\beta_1x_1^*x_2^* + \frac{r_1x_1^*}{k_1}) + \alpha_1x_1^* + \beta_1x_1^{*2} + \alpha_2x_2^* + \beta_2x_2^{*2} \\
& +\frac{r_1x_1^*}{k_1}(2\beta_1x_1^*x_2^* + \frac{2r_1x_1^*}{k_1} + \alpha_1x_1^* + \beta_1x_1^{*2})\tau_1 \\
& +\frac{r_2x_2^*}{k_2}(\alpha_2x_2^* + \beta_2x_2^{*2})\tau_2]y_1^2(t) \\
& +[-2(\beta_2x_1^*x_2^* + \frac{r_2x_2^*}{k_2}) + \alpha_1x_1^* + \beta_1x_1^{*2} + \alpha_2x_2^* + \beta_2x_2^{*2} \\
& +\frac{r_1x_1^*}{k_1}(\alpha_1x_1^* + \beta_1x_1^{*2})\tau_1 \\
& +\frac{r_2x_2^*}{k_2}(2\beta_2x_1^*x_2^* + \frac{2r_2x_2^*}{k_2} + \alpha_2x_2^* + \beta_2x_2^{*2})\tau_2]y_2^2(t) \\
= \; & -\{[2(\beta_1x_1^*x_2^* + \frac{r_1x_1^*}{k_1}) - \alpha_1x_1^* - \beta_1x_1^{*2} - \alpha_2x_2^* - \beta_2x_2^{*2}] \\
& -[\frac{r_1x_1^*}{k_1}(2\beta_1x_1^*x_2^* + \frac{2r_1x_1^*}{k_1} + \alpha_1x_1^* + \beta_1x_1^{*2})]\tau_1 \\
& -[\frac{r_2x_2^*}{k_2}(\alpha_2x_2^* + \beta_2x_2^{*2})]\tau_2\}y_1^2(t) \\
& -\{[2(\beta_2x_1^*x_2^* + \frac{r_2x_2^*}{k_2}) - \alpha_1x_1^* - \beta_1x_1^{*2} - \alpha_2x_2^* - \beta_2x_2^{*2}] \\
& -[\frac{r_1x_1^*}{k_1}(\alpha_1x_1^* + \beta_1x_1^{*2})]\tau_1 \\
& -[\frac{r_2x_2^*}{k_2}(2\beta_2x_1^*x_2^* + \frac{2r_2x_2^*}{k_2} + \alpha_2x_2^* + \beta_2x_2^{*2})]\tau_2\}y_2^2(t) \\
\equiv \; & -\eta_1y_1^2(t) - \eta_2y_2^2(t) \qquad (5.10)
\end{aligned}
$$

Clearly, (5.2) and (5.3) implies that $\eta_1 > 0$ and $\eta_2 > 0$. Denote $\eta = \min\{\eta_1, \eta_2\}$ then (5.10) leads to

$$
W(t) + \eta \int_T^t [y_1^2(s) + y_2^2(s)]ds \leq W(T) \text{ for } \; t \geq T \qquad (5.11)
$$

and which implies $y_1^2(t) + y_2^2(t) \in L_1[T, \infty)$. We can see from (5.1) and boundness of $y(t)$ that $y_1^2(t)+y_2^2(t)$ is uniform continuous and then, using Bărbalat's Lemma[4], we can conclude that $\lim_{t\to\infty}[y_1^2(t)+y_2^2(t)] = 0$. Therefore, the zero solution of (5.1) is asymptotically stable and this completes the proof.

5.2 Global Stability

In this section, we drive sufficient conditions which guarantee that the positive equilibrium point E^* of the system (3.1) is globally asymptotically stable. Our method in the proof of the global asymptotic stability of the positive equilibrium E^* of the system (3.1) is to construct a suitable Lyapunov functional.

Theorem 5.2 *Let $E^* = (x_1^*, x_2^*)$ be the unique equilibrium point of the system (3.1) and the delays τ_1 and τ_2 satisfy*

$$\mu_1 - \mu_2\tau_1 - \mu_3\tau_2 > 0 \tag{5.12}$$

and

$$\omega_1 - \omega_2\tau_1 - \omega_3\tau_2 > 0 \tag{5.13}$$

where $M_i, m_i (i = 1, 2)$ defined by lemma 4.2 and Theorem 4.1, and

$$\begin{aligned}
\mu_1 &= \frac{r_1 x_1^*}{k_1} + \beta_1 x_1^* m_2 - \frac{1}{2}(\alpha_1 x_2^* + \alpha_2 x_1^* + \beta_1 x_1^* x_2^* + \beta_2 x_1^* x_2^*)\\
\mu_2 &= \frac{r_1\beta_1 x_2^* x_1^{*2}}{2k_1} + \frac{r_1\beta_1 x_1^* x_2^* M_1}{k_1} + \frac{r_1\alpha_1 x_2^* M_1}{2k_1} + \frac{r_1^2 x_1^* M_1}{k_1^2} + \frac{r_1\beta_1 x_1^*(M_1 - x_1^*)M_2}{k_1}\\
\mu_3 &= \frac{r_2\alpha_2 x_1^* M_2}{2k_2} + \frac{r_2\beta_2 x_1^* x_2^*(2M_2 - x_2^*)}{2k_2}\\
\omega_1 &= \frac{r_2 x_2^*}{k_2} + \beta_2 x_2^* m_1 - \frac{1}{2}(\alpha_1 x_2^* + \alpha_2 x_1^* + \beta_1 x_1^* x_2^* + \beta_2 x_1^* x_2^*)\\
\omega_2 &= \frac{r_1\alpha_1 x_2^* M_1}{2k_1} + \frac{r_1\beta_1 x_1^* x_2^*(2M_1 - x_1^*)}{2k_1}\\
\omega_3 &= \frac{r_2\beta_2 x_1^* x_2^{*2}}{2k_2} + \frac{r_2\beta_2 x_1^* x_2^* M_2}{k_2} + \frac{r_2\alpha_2 x_1^* M_2}{2k_2} + \frac{r_2^2 x_2^* M_2}{k_2^2} + \frac{r_2\beta_2 x_2^*(M_2 - x_2^*)M_1}{k_2}
\end{aligned}$$

then the unique equilibrium point E^* of the system (3.1) is global asymptotically stable.

Proof:

Define

$$z(t) = (z_1(t), z_2(t))$$

by

$$z_i(t) = \frac{x_i(t) - x_i^*}{x_i^*} \qquad (i = 1, 2).$$

$$\begin{aligned}
\frac{dz_1(t)}{dt} &= \frac{1}{x_1^*}\frac{dx_1(t)}{dt} \\
&= (1 + z_1(t))[r_1 - \frac{r_1 x_1^*}{k_1} - \frac{r_1 x_1^* z_1(t-\tau_1)}{k_1} - \alpha_1 x_2^* - \alpha_1 x_2^* z_2(t) \\
&\quad -\beta_1 x_1^* x_2^* - \beta_1 x_1^* x_2^* z_1(t) - \beta_1 x_1^* x_2^* z_2(t) - \beta_1 x_1^* x_2^* z_1(t) z_2(t) - q_1 e] \\
&= (1 + z_1(t))[-\beta_1 x_1^* x_2^* z_1(t) - (\alpha_1 x_2^* + \beta_1 x_1^* x_2^*) z_2(t) \\
&\quad -\frac{r_1 x_1^* z_1(t-\tau_1)}{k_1} - \beta_1 x_1^* x_2^* z_1(t) z_2(t)] \qquad (5.14)
\end{aligned}$$

$$\begin{aligned}
\frac{dz_2(t)}{dt} &= \frac{1}{x_2^*}\frac{dx_2(t)}{dt} \\
&= (1 + z_2(t))[r_2 - \frac{r_2 x_2^*(1 + z_2(t-\tau_2))}{k_2} - \alpha_2 x_1^* - \alpha_2 x_1^* z_1(t) \\
&\quad -\beta_2 x_1^*(1 + z_1(t)) x_2^*(1 + z_2(t)) - q_2 e] \\
&= (1 + z_2(t))[-(\alpha_2 x_1^* + \beta_2 x_1^* x_2^*) z_1(t) - \beta_2 x_1^* x_2^* z_2(t) \\
&\quad -\frac{r_2 x_2^* z_2(t-\tau_2)}{k_2} - \beta_2 x_1^* x_2^* z_1(t) z_2(t)] \qquad (5.15)
\end{aligned}$$

Let

$$V_1(z(t)) = \{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\} \qquad (5.16)$$

then we have (5.17) from (5.14) and (5.15) that

$$\begin{aligned}
\dot{V}_1(z(t)) &= \frac{\dot{z}_1(t) z_1(t)}{1 + z_1(t)} + \frac{\dot{z}_2(t) z_2(t)}{1 + z_2(t)} \\
&= -\beta_1 x_1^* x_2^* z_1^2(t) - (\alpha_1 x_2^* + \beta_1 x_1^* x_2^*) z_1(t) z_2(t)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{r_1 x_1^*}{k_1} z_1(t) z_1(t-\tau_1) - \beta_1 x_1^* x_2^* z_1^2(t) z_2(t) \\
& -\beta_2 x_1^* x_2^* z_2^2(t) - (\alpha_2 x_1^* + \beta_2 x_1^* x_2^*) z_1(t) z_2(t) \\
& -\frac{r_2 x_2^*}{k_2} z_2(t) z_2(t-\tau_2) - \beta_2 x_1^* x_2^* z_1(t) z_2^2(t) \\
\leq\ & -\beta_1 x_1^* x_2^* z_1^2(t) - \beta_2 x_1^* x_2^* z_2^2(t) - \beta_1 x_1^* x_2(t) z_1^2(t) - \beta_2 x_2^* x_1(t) z_2^2(t) \\
& +\frac{1}{2}(\alpha_1 x_2^* + \alpha_2 x_1^* + \beta_1 x_1^* x_2^* + \beta_2 x_1^* x_2^*)[z_1^2(t) + z_2^2(t)] \\
& +\frac{r_1 x_1^*}{k_1} \int_{t-\tau_1}^{t} z_1(t) \dot{z}_1(s) ds + \frac{r_2 x_2^*}{k_2} \int_{t-\tau_2}^{t} z_2(t) \dot{z}_2(s) ds \\
& -\frac{r_1 x_1^*}{k_1} z_1^2(t) - \frac{r_2 x_2^*}{k_2} z_2^2(t) + \beta_1 x_1^* x_2^* z_1^2(t) + \beta_2 x_1^* x_2^* z_2^2(t) \qquad (5.17)
\end{aligned}
$$

By Theorem 4.1, there exists a $T^* > 0$ such that $m_1 \leq x_1^*[1 + z_1(t)] \leq M_1$, and $m_2 \leq x_2^*[1 + z_2(t)] \leq M_2$ for $t > T^*$. Then (5.17) implies that

$$
\begin{aligned}
\dot{V}_1(z(t)) \leq & [-\beta_1 x_1^* x_2^* + \frac{1}{2}(\alpha_1 x_2^* + \alpha_2 x_1^* + \beta_1 x_1^* x_2^* + \beta_2 x_1^* x_2^*) - \frac{r_1 x_1^*}{k_1} - \beta_1 x_1^* m_2 + \beta_1 x_1^* x_2^*] z_1^2(t) \\
& + [-\beta_2 x_1^* x_2^* + \frac{1}{2}(\alpha_1 x_2^* + \alpha_2 x_1^* + \beta_1 x_1^* x_2^* + \beta_2 x_1^* x_2^*) - \frac{r_2 x_2^*}{k_2} - \beta_2 x_2^* m_1 + \beta_2 x_1^* x_2^*] z_2^2(t) \\
& + \frac{r_1 x_1^*}{k_1} \int_{t-\tau_1}^{t} z_1(t) \dot{z}_1(s) ds + \frac{r_2 x_2^*}{k_2} \int_{t-\tau_2}^{t} z_2(t) \dot{z}_2(s) ds \\
= & [\frac{1}{2}(\alpha_1 x_2^* + \alpha_2 x_1^* + \beta_1 x_1^* x_2^* + \beta_2 x_1^* x_2^*) - \frac{r_1 x_1^*}{k_1} - \beta_1 x_1^* m_2] z_1^2(t) \\
& + [\frac{1}{2}(\alpha_1 x_2^* + \alpha_2 x_1^* + \beta_1 x_1^* x_2^* + \beta_2 x_1^* x_2^*) - \frac{r_2 x_2^*}{k_2} - \beta_2 x_2^* m_1] z_2^2(t) \\
& - \frac{r_1 \beta_1 x_2^* x_1^{*2}}{k_1} \int_{t-\tau_1}^{t} z_1(t) z_1(s) ds - (\alpha_1 x_2^* + \beta_1 x_1^* x_2^*) \frac{r_1 x_1^*}{k_1} \int_{t-\tau_1}^{t} z_1(t) z_2(s) ds \\
& - (\frac{r_1 x_1^*}{k_1})^2 \int_{t-\tau_1}^{t} z_1(t) z_1(s-\tau_1) ds - \frac{r_1 \beta_1 x_2^* x_1^{*2}}{k_1} \int_{t-\tau_1}^{t} z_1(t) z_1(s) z_2(s) ds
\end{aligned}
$$

$$-\frac{r_1\beta_1 x_2^* x_1^{*2}}{k_1}\int_{t-\tau_1}^t z_1(t)z_1^2(s)ds-(\alpha_1 x_2^*+\beta_1 x_1^* x_2^*)\frac{r_1 x_1^*}{k_1}\int_{t-\tau_1}^t z_1(t)z_1(s)z_2(s)ds$$

$$-(\frac{r_1 x_1^*}{k_1})^2\int_{t-\tau_1}^t z_1(t)z_1(s)z_1(s-\tau_1)ds-\frac{r_1\beta_1 x_2^* x_1^{*2}}{k_1}\int_{t-\tau_1}^t z_1(t)z_1^2(s)z_2(s)ds$$

$$-\frac{r_2\beta_2 x_1^* x_2^{*2}}{k_2}\int_{t-\tau_2}^t z_2(t)z_2(s)ds-(\alpha_2 x_1^*+\beta_2 x_1^* x_2^*)\frac{r_2 x_2^*}{k_2}\int_{t-\tau_2}^t z_2(t)z_1(s)ds$$

$$-(\frac{r_2 x_2^*}{k_2})^2\int_{t-\tau_2}^t z_2(t)z_2(s-\tau_2)ds-\frac{r_2\beta_2 x_1^* x_2^{*2}}{k_2}\int_{t-\tau_2}^t z_2(t)z_1(s)z_2(s)ds$$

$$-\frac{r_2\beta_2 x_1^* x_2^{*2}}{k_2}\int_{t-\tau_2}^t z_2(t)z_2^2(s)ds-(\alpha_2 x_1^*+\beta_2 x_1^* x_2^*)\frac{r_2 x_2^*}{k_2}\int_{t-\tau_2}^t z_2(t)z_1(s)z_2(s)ds$$

$$-(\frac{r_2 x_2^*}{k_2})^2\int_{t-\tau_2}^t z_2(t)z_2(s)z_2(s-\tau_2)ds-\frac{r_2\beta_2 x_1^* x_2^{*2}}{k_2}\int_{t-\tau_2}^t z_2(t)z_1(s)z_2^2(s)ds$$

$$\le[-\frac{r_1 x_1^*}{k_1}-\beta_1 x_1^* m_2+\frac{1}{2}(\alpha_1 x_2^*+\alpha_2 x_1^*+\beta_1 x_1^* x_2^*+\beta_2 x_1^* x_2^*)]z_1^2(t)$$

$$+[-\frac{r_2 x_2^*}{k_2}-\beta_2 x_2^* m_1+\frac{1}{2}(\alpha_1 x_2^*+\alpha_2 x_1^*+\beta_1 x_1^* x_2^*+\beta_2 x_1^* x_2^*)]z_2^2(t)$$

$$+\frac{r_1\beta_1 x_2^* x_1^{*2}}{2k_1}[\tau_1 z_1^2(t)+\int_{t-\tau_1}^t z_1^2(s)ds]+\frac{1}{2}(\frac{r_1 x_1^*}{k_1})^2[\tau_1 z_1^2(t)+\int_{t-\tau_1}^t z_1^2(s-\tau_1)ds]$$

$$+(\alpha_1 x_2^*+\beta_1 x_1^* x_2^*)\frac{r_1 x_1^*}{2k_1}[\tau_1 z_1^2(t)+\int_{t-\tau_1}^t z_2^2(s)ds]$$

$$-\frac{r_1\beta_1 x_1^* x_2^*[x_1(t)-x_1^*]}{k_1}\int_{t-\tau_1}^t z_1(s)z_2(s)ds-\frac{r_1\beta_1 x_1^* x_2^*[x_1(t)-x_1^*]}{k_1}\int_{t-\tau_1}^t z_1^2(s)ds$$

$$-(\alpha_1 x_2^*+\beta_1 x_1^* x_2^*)\frac{r_1[x_1(t)-x_1^*]}{k_1}\int_{t-\tau_1}^t z_1(s)z_2(s)ds$$

$$-\frac{r_1^2 x_1^*[x_1(t)-x_1^*]}{k_1^2}\int_{t-\tau_1}^t z_1(s)z_1(s-\tau_1)ds$$

$$-\frac{r_1\beta_1 x_1^*[x_1(t)-x_1^*]}{k_1}\int_{t-\tau_1}^t z_1^2(s)[x_2(s)-x_2^*]ds$$

$$
\begin{aligned}
&+\frac{r_2\beta_2 x_1^* x_2^{*2}}{2k_2}[\tau_2 z_2^2(t)+\int_{t-\tau_2}^{t} z_2^2(s)ds]+\frac{1}{2}(\frac{r_2x_2^*}{k_2})^2[\tau_2 z_2^2(t)+\int_{t-\tau_2}^{t} z_2^2(s-\tau_2)ds]\\
&+(\alpha_2x_1^*+\beta_2x_1^*x_2^*)\frac{r_2x_2^*}{2k_2}[\tau_2 z_2^2(t)+\int_{t-\tau_2}^{t} z_1^2(s)ds]\\
&-\frac{r_2\beta_2x_1^*x_2^*[x_2(t)-x_2^*]}{k_2}\int_{t-\tau_2}^{t} z_1(s)z_2(s)ds-\frac{r_2\beta_2x_1^*x_2^*[x_2(t)-x_2^*]}{k_2}\int_{t-\tau_2}^{t} z_2^2(s)ds\\
&-(\alpha_2x_1^*+\beta_2x_1^*x_2^*)\frac{r_2[x_2(t)-x_2^*]}{k_2}\int_{t-\tau_2}^{t} z_1(s)z_2(s)ds\\
&-\frac{r_2^2x_2^*[x_2(t)-x_2^*]}{k_2^2}\int_{t-\tau_2}^{t} z_2(s)z_2(s-\tau_2)ds\\
&-\frac{r_2\beta_2x_2^*[x_2(t)-x_2^*]}{k_2}\int_{t-\tau_2}^{t} z_2^2(s)[x_1(s)-x_1^*]ds\\
\leq\ &\{[-\frac{r_1x_1^*}{k_1}-\beta_1x_1^*m_2+\frac{1}{2}(\alpha_1x_2^*+\alpha_2x_1^*+\beta_1x_1^*x_2^*+\beta_2x_1^*x_2^*)]\\
&+[\frac{r_1\beta_1x_2^*x_1^{*2}}{2k_1}+(\alpha_1x_2^*+\beta_1x_1^*x_2^*)\frac{r_1x_1^*}{2k_1}+\frac{1}{2}(\frac{r_1x_1^*}{k_1})^2]\tau_1\}z_1^2(t)\\
&+\{[-\frac{r_2x_2^*}{k_2}-\beta_2x_2^*m_1+\frac{1}{2}(\alpha_1x_2^*+\alpha_2x_1^*+\beta_1x_1^*x_2^*+\beta_2x_1^*x_2^*)]\\
&+[\frac{r_2\beta_2x_1^*x_2^{*2}}{2k_2}+(\alpha_2x_1^*+\beta_2x_1^*x_2^*)\frac{r_2x_2^*}{2k_2}+\frac{1}{2}(\frac{r_2x_2^*}{k_2})^2]\tau_2\}z_2^2(t)\\
&+\{\frac{r_1\beta_1x_2^*x_1^{*2}}{2k_1}+\frac{r_1\beta_1x_1^*x_2^*(M_1-x_1^*)}{2k_1}+\frac{r_1^2x_1^*(M_1-x_1^*)}{2k_1^2}\\
&+\frac{r_1\beta_1x_1^*x_2^*(M_1-x_1^*)}{k_1}+(\alpha_1x_2^*+\beta_1x_1^*x_2^*)\frac{r_1(M_1-x_1^*)}{2k_1}\\
&+\frac{r_1\beta_1x_1^*(M_1-x_1^*)(M_2-x_2^*)}{k_1}\}\int_{t-\tau_1}^{t} z_1^2(s)ds\\
&+\{(\alpha_2x_1^*+\beta_2x_1^*x_2^*)\frac{r_2x_2^*}{2k_2}+\frac{r_2\beta_2x_1^*x_2^*(M_2-x_2^*)}{2k_2}
\end{aligned}
$$

$$+(\alpha_2 x_1^* + \beta_2 x_1^* x_2^*)\frac{r_2(M_2 - x_2^*)}{2k_2}\}\int_{t-\tau_2}^{t} z_1^2(s)ds$$

$$+\{\frac{r_2\beta_2 x_1^* x_2^{*2}}{2k_2} + \frac{r_2\beta_2 x_1^* x_2^*(M_2 - x_2^*)}{2k_2} + \frac{r_2^2 x_2^*(M_2 - x_2^*)}{2k_2^2}$$

$$+\frac{r_2\beta_2 x_1^* x_2^*(M_2 - x_2^*)}{k_2} + (\alpha_2 x_1^* + \beta_2 x_1^* x_2^*)\frac{r_2(M_2 - x_2^*)}{2k_2}$$

$$+\frac{r_2\beta_2 x_2^*(M_2 - x_2^*)(M_1 - x_1^*)}{k_2}\}\int_{t-\tau_2}^{t} z_2^2(s)ds$$

$$+\{(\alpha_1 x_2^* + \beta_1 x_1^* x_2^*)\frac{r_1 x_1^*}{2k_1} + \frac{r_1\beta_1 x_1^* x_2^*(M_1 - x_1^*)}{2k_1}$$

$$+(\alpha_1 x_2^* + \beta_1 x_1^* x_2^*)\frac{r_1(M_1 - x_1^*)}{2k_1}\}\int_{t-\tau_1}^{t} z_2^2(s)ds$$

$$+\{\frac{r_1^2 x_1^*(M_1 - x_1^*)}{2k_1^2} + \frac{r_1^2 x_1^{*2}}{2k_1^2}\}\int_{t-\tau_1}^{t} z_1^2(s-\tau_1)ds$$

$$+\{\frac{r_2^2 x_2^*(M_2 - x_2^*)}{2k_2^2} + \frac{r_2^2 x_2^{*2}}{2k_2^2}\}\int_{t-\tau_2}^{t} z_2^2(s-\tau_2)ds \tag{5.18}$$

Now, we let

$$V_2(z(t)) = \{\frac{r_1\beta_1 x_2^* x_1^{*2}}{2k_1} + \frac{r_1\beta_1 x_1^* x_2^*(M_1 - x_1^*)}{k_1} + \frac{r_1\alpha_1 x_2^*(M_1 - x_1^*)}{2k_1}$$

$$+\frac{r_1^2 x_1^*(M_1 - x_1^*)}{2k_1^2} + \frac{r_1\beta_1 x_1^*(M_1 - x_1^*)M_2}{k_1}\}\int_{t-\tau_1}^{t}\int_{s}^{t} z_1^2(\rho)d\rho ds$$

$$+\{\frac{r_2\beta_2 x_1^* x_2^{*2}}{2k_2} + \frac{r_2\beta_2 x_1^* x_2^*(M_2 - x_2^*)}{k_2} + \frac{r_2\alpha_2 x_1^*(M_2 - x_2^*)}{2k_2}$$

$$+\frac{r_2^2 x_2^*(M_2 - x_2^*)}{2k_2^2} + \frac{r_2\beta_2 x_2^*(M_2 - x_2^*)M_1}{k_2}\}\int_{t-\tau_2}^{t}\int_{s}^{t} z_2^2(\rho)d\rho ds$$

$$+\{\frac{r_2\alpha_2 x_1^* M_2}{2k_2} + \frac{r_2\beta_2 x_1^* x_2^*(2M_2 - x_2^*)}{2k_2}\}\int_{t-\tau_2}^{t}\int_{s}^{t} z_1^2(\rho)d\rho ds$$

$$+\{\frac{r_1\alpha_1 x_2^* M_1}{2k_1} + \frac{r_1\beta_1 x_1^* x_2^*(2M_1 - x_1^*)}{2k_1}\}\int_{t-\tau_1}^{t}\int_{s}^{t} z_2^2(\rho)d\rho ds$$

$$+\{\frac{r_1^2x_1^*M_1}{2k_1^2}\}\int_{t-\tau_1}^{t}\int_{s}^{t} z_1^2(\rho-\tau_1)d\rho ds$$

$$+\{\frac{r_2^2x_2^*M_2}{2k_2^2}\}\int_{t-\tau_2}^{t}\int_{s}^{t} z_2^2(\rho-\tau_2)d\rho ds \tag{5.19}$$

then

$$\begin{aligned}
\dot{V}_2(z(t)) \quad = \quad & \{\frac{r_1\beta_1x_2^*x_1^{*2}}{2k_1}+\frac{r_1\beta_1x_1^*x_2^*(M_1-x_1^*)}{k_1}+\frac{r_1\alpha_1x_2^*(M_1-x_1^*)}{2k_1} \\
&+\frac{r_1^2x_1^*(M_1-x_1^*)}{2k_1^2}+\frac{r_1\beta_1x_1^*(M_1-x_1^*)M_2}{k_1}\}[\tau_1z_1^2(t)-\int_{t-\tau_1}^{t} z_1^2(s)ds] \\
&+\{\frac{r_2\beta_2x_1^*x_2^{*2}}{2k_2}+\frac{r_2\beta_2x_1^*x_2^*(M_2-x_2^*)}{k_2}+\frac{r_2\alpha_2x_1^*(M_2-x_2^*)}{2k_2} \\
&+\frac{r_2^2x_2^*(M_2-x_2^*)}{2k_2^2}+\frac{r_2\beta_2x_2^*(M_2-x_2^*)M_1}{k_2}\}[\tau_2z_2^2(t)-\int_{t-\tau_2}^{t} z_2^2(s)ds] \\
&+\{\frac{r_2\alpha_2x_1^*M_2}{2k_2}+\frac{r_2\beta_2x_1^*x_2^*(2M_2-x_2^*)}{2k_2}\}[\tau_2z_1^2(t)-\int_{t-\tau_2}^{t} z_1^2(s)ds] \\
&+\{\frac{r_1\alpha_1x_2^*M_1}{2k_1}+\frac{r_1\beta_1x_1^*x_2^*(2M_1-x_1^*)}{2k_1}\}[\tau_1z_2^2(s)-\int_{t-\tau_1}^{t} z_2^2(s)ds] \\
&+\{\frac{r_1^2x_1^*M_1}{2k_1^2}\}[\tau_1z_1^2(t-\tau_1)-\int_{t-\tau_1}^{t} z_1^2(s-\tau_1)ds] \\
&+\{\frac{r_2^2x_2^*M_2}{2k_2^2}\}[\tau_2z_2^2(t-\tau_2)-\int_{t-\tau_2}^{t} z_2^2(t-\tau_2)ds]
\end{aligned} \tag{5.20}$$

and we have (5.21) from (5.18) and (5.20) that for $t \geq \hat{T}$

$$\begin{aligned}
\dot{V}_1(z(t))+\dot{V}_2(z(t)) \leq & \{[-\frac{r_1x_1^*}{k_1}-\beta_1x_1^*m_2+\frac{1}{2}(\alpha_1x_2^*+\alpha_2x_1^*+\beta_1x_1^*x_2^*+\beta_2x_1^*x_2^*)] \\
&+[\frac{r_1\beta_1x_2^*x_1^{*2}}{2k_1}+(\alpha_1x_2^*+\beta_1x_1^*x_2^*)\frac{r_1x_1^*}{2k_1}+\frac{1}{2}(\frac{r_1x_1^*}{k_1})^2]\tau_1\}z_1^2(t) \\
&+\{[-\frac{r_2x_2^*}{k_2}-\beta_2x_2^*m_1+\frac{1}{2}(\alpha_1x_2^*+\alpha_2x_1^*+\beta_1x_1^*x_2^*+\beta_2x_1^*x_2^*)] \\
&+[\frac{r_2\beta_2x_1^*x_2^{*2}}{2k_2}+(\alpha_2x_1^*+\beta_2x_1^*x_2^*)\frac{r_2x_2^*}{2k_2}+\frac{1}{2}(\frac{r_2x_2^*}{k_2})^2]\tau_2\}z_2^2(t)
\end{aligned}$$

$$+\{[\frac{r_1\beta_1x_2^*x_1^{*2}}{2k_1}+\frac{r_1\beta_1x_1^*x_2^*(M_1-x_1^*)}{k_1}+\frac{r_1\alpha_1x_2^*(M_1-x_1^*)}{2k_1}$$

$$+\frac{r_1^2x_1^*(M_1-x_1^*)}{2k_1^2}+\frac{r_1\beta_1x_1^*(M_1-x_1^*)M_2}{k_1}]\tau_1\}z_1^2(t)$$

$$+\{[\frac{r_2\beta_2x_1^*x_2^{*2}}{2k_2}+\frac{r_2\beta_2x_1^*x_2^*(M_2-x_2^*)}{k_2}+\frac{r_2\alpha_2x_1^*(M_2-x_2^*)}{2k_2}$$

$$+\frac{r_2^2x_2^*(M_2-x_2^*)}{2k_2^2}+\frac{r_2\beta_2x_2^*(M_2-x_2^*)M_1}{k_2}]\tau_2\}z_2^2(t)$$

$$+\{[\frac{r_2\alpha_2x_1^*M_2}{2k_2}+\frac{r_2\beta_2x_1^*x_2^*(2M_2-x_2^*)}{2k_2}]\tau_2\}z_1^2(t)$$

$$+\{[\frac{r_1\alpha_1x_2^*M_1}{2k_1}+\frac{r_1\beta_1x_1^*x_2^*(2M_1-x_1^*)}{2k_1}]\tau_1\}z_2^2(s)$$

$$+\{[\frac{r_1^2x_1^*M_1}{2k_1^2}]\tau_1\}z_1^2(t-\tau_1)+\{[\frac{r_2^2x_2^*M_2}{2k_2^2}]\tau_2\}z_2^2(t-\tau_2) \qquad (5.21)$$

Let

$$V_3(z(t))=\{[\frac{r_1^2x_1^*M_1}{2k_1^2}]\tau_1\}\int_{t-\tau_1}^{t}z_1^2(s)ds+\{[\frac{r_2^2x_2^*M_2}{2k_2^2}]\tau_2\}\int_{t-\tau_2}^{t}z_2^2(s)ds \qquad (5.22)$$

then

$$\dot{V}_3(z(t))=\frac{r_1^2x_1^*M_1}{2k_1^2}[z_1^2(t)-z_1^2(t-\tau_1)]\tau_1+\frac{r_2^2x_2^*M_2}{2k_2^2}[z_2^2(t)-z_2^2(t-\tau_2)]\tau_2 \qquad (5.23)$$

Now define a Lyapunov functional $V(z(t))$ as

$$V(z(t))=V_1(z(t))+V_2(z(t))+V_3(z(t)) \qquad (5.24)$$

then we have (5.25) from (5.21) and (5.23) that for $t\geq\tilde{T}$

$$\dot{V}(z(t))=\dot{V}_1(z(t))+\dot{V}_2(z(t))+\dot{V}_3(z(t))$$

$$\leq\{[-\frac{r_1x_1^*}{k_1}-\beta_1x_1^*m_2+\frac{1}{2}(\alpha_1x_2^*+\alpha_2x_1^*+\beta_1x_1^*x_2^*+\beta_2x_1^*x_2^*)]$$

$$+[\frac{r_1\beta_1x_1^{*2}x_2^*}{2k_1}+(\alpha_1x_2^*+\beta_1x_1^*x_2^*)\frac{r_1x_1^*}{2k_1}+\frac{1}{2}(\frac{r_1x_1^*}{k_1})^2+\frac{r_1^2x_1^*M_1}{2k_1^2}$$

$$+\frac{r_1\beta_1 x_2^* x_1^{*2}}{2k_1}+\frac{r_1\beta_1 x_1^* x_2^*(M_1-x_1^*)}{k_1}+\frac{r_1\alpha_1 x_2^*(M_1-x_1^*)}{2k_1}$$

$$+\frac{r_1^2 x_1^*(M_1-x_1^*)}{2k_1^2}+\frac{r_1\beta_1 x_1^*(M_1-x_1^*)M_2}{k_1}]\tau_1$$

$$+[\frac{r_2\alpha_2 x_1^* M_2}{2k_2}+\frac{r_2\beta_2 x_1^* x_2^*(2M_2-x_2^*)}{2k_2}]\tau_2\}z_1^2(t)$$

$$+\{[-\frac{r_2 x_2^*}{k_2}-\beta_2 x_2^* m_1+\frac{1}{2}(\alpha_1 x_2^*+\alpha_2 x_1^*+\beta_1 x_1^* x_2^*+\beta_2 x_1^* x_2^*)]$$

$$+[\frac{r_2\beta_2 x_1^* x_2^{*2}}{2k_2}+(\alpha_2 x_1^*+\beta_2 x_1^* x_2^*)\frac{r_2 x_2^*}{2k_2}+\frac{1}{2}(\frac{r_2 x_2^*}{k_2})^2+\frac{r_2^2 x_2^* M_2}{2k_2^2}$$

$$+\frac{r_2\beta_2 x_1^* x_2^{*2}}{2k_2}+\frac{r_2\beta_2 x_1^* x_2^*(M_2-x_2^*)}{k_2}+\frac{r_2\alpha_2 x_1^*(M_2-x_2^*)}{2k_2}$$

$$+\frac{r_2^2 x_2^*(M_2-x_2^*)}{2k_2^2}+\frac{r_2\beta_2 x_2^*(M_2-x_2^*)M_1}{k_2}]\tau_2$$

$$+[\frac{r_1\alpha_1 x_2^* M_1}{2k_1}+\frac{r_1\beta_1 x_1^* x_2^*(2M_1-x_1^*)}{2k_1}]\tau_1\}z_2^2(t)$$

$$=-\{[\frac{r_1 x_1^*}{k_1}+\beta_1 x_1^* m_2-\frac{1}{2}(\alpha_1 x_2^*+\alpha_2 x_1^*+\beta_1 x_1^* x_2^*+\beta_2 x_1^* x_2^*)]$$

$$-[\frac{r_1\beta_1 x_2^* x_1^{*2}}{2k_1}+\frac{r_1\beta_1 x_1^* x_2^* M_1}{k_1}+\frac{r_1\alpha_1 x_2^* M_1}{2k_1}$$

$$+\frac{r_1^2 x_1^* M_1}{k_1^2}+\frac{r_1\beta_1 x_1^*(M_1-x_1^*)M_2}{k_1}]\tau_1$$

$$-[\frac{r_2\alpha_2 x_1^* M_2}{2k_2}+\frac{r_2\beta_2 x_1^* x_2^*(2M_2-x_2^*)}{2k_2}]\tau_2\}z_1^2(t)$$

$$-\{[\frac{r_2 x_2^*}{k_2}+\beta_2 x_2^* m_1-\frac{1}{2}(\alpha_1 x_2^*+\alpha_2 x_1^*+\beta_1 x_1^* x_2^*+\beta_2 x_1^* x_2^*)]$$

$$-[\frac{r_2\beta_2 x_1^* x_2^{*2}}{k_2}+\frac{r_2\beta_2 x_1^* x_2^* M_2}{k_2}+\frac{r_2\alpha_2 x_1^* M_2}{2k_2}$$

$$+\frac{r_2^2 x_2^* M_2}{k_2^2} + \frac{r_2\beta_2 x_2^*(M_2 - x_2^*)M_1}{k_2}]\tau_2$$

$$-[\frac{r_1\alpha_1 x_2^* M_1}{2k_1} + \frac{r_1\beta_1 x_1^* x_2^*(2M_1 - x_1^*)}{2k_1}]\tau_1\} z_2^2(t)$$

$$\equiv -\xi_1 z_1^2(t) - \xi_2 z_2^2(t) \tag{5.25}$$

Then it follows from (5.12) and (5.13) that $\xi_1 > 0$ and $\xi_2 > 0$. Let $\omega(s) = \xi s^2$, where $\xi = \min\{\xi_1, \xi_2\}$, then ω is nonnegative continuous on $[0, \infty)$, $\omega(0) = 0$, and $\omega(s) > 0$ for $s > 0$. Follows from (5.25) that for $t \geq \tilde{T}$

$$\dot{V}(z_t) \leq -\xi[z_1^2(t) + z_2^2(t)] = -\xi|z(t)|^2 = -\omega(|z(t)|) \tag{5.26}$$

Now, we want to find a function u such that $V(z_t) \geq u(|z(t)|)$. It follows from (5.16), (5.19), and (5.22) that

$$V(z_t) \geq \{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\} \tag{5.27}$$

By the Taylor Theorem, we have that

$$z_i(t) - \ln[1 + z_i(t)] = \frac{z_i^2(t)}{2[1 + \theta_i(t)]^2} \tag{5.28}$$

where $\theta_i(t) \in (0, z_i(t))$ or $(z_i(t), 0)$ for $i = 1, 2$.

In the following Case 1 $\sim$ Case 4, we discuss the different relation between $\theta_i(t)$ and $z_i(t)$.

Case 1: If $0 < \theta_i(t) < z_i(t)$ for $i = 1, 2$. ,then

$$\frac{z_i^2(t)}{[1 + z_i(t)]^2} < \frac{z_i^2(t)}{[1 + \theta_i(t)]^2} < z_i^2(t). \tag{5.29}$$

By Theorem 4.1, it follows that for $t \geq T^*$

$$m_i \leq x_i^*[1 + z_i(t)] = x_i(t) \leq M_i, \quad \text{for} \quad i = 1, 2. \tag{5.30}$$

Then (5.29) implies that

$$(\frac{x_1^*}{M_1})^2 z_1^2(t) \leq \frac{z_1^2(t)}{[1 + \theta_1(t)]^2} < z_1^2(t)$$

$$(\frac{x_2^*}{M_2})^2 z_2^2(t) \leq \frac{z_2^2(t)}{[1 + \theta_2(t)]^2} < z_2^2(t). \tag{5.31}$$

It follows that $(5.27), (5.28)$ and (5.31) that for $t \geq T^*$

$$
\begin{aligned}
V(z_t) &\geq \frac{z_1^2(t)}{2[1+\theta_1(t)]^2} + \frac{z_2^2(t)}{2[1+\theta_2(t)]^2} \\
&\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\
&\geq min\left\{\frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2, \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2\right\}\left[z_1^2(t)+z_2^2(t)\right] \\
&\equiv \widetilde{m}|z(t)|^2. \qquad (5.32)
\end{aligned}
$$

Case 2: If $-1 < z_i(t) < \theta_i(t) < 0$ for i=1,2 ,then

$$
z_i^2(t) < \frac{z_i^2(t)}{[1+\theta_i(t)]^2} < \frac{z_i^2(t)}{[1+z_i(t)]^2}. \qquad (5.33)
$$

By $(5.30), (5.33)$ implies that

$$
\begin{aligned}
z_1^2(t) &< \frac{z_1^2(t)}{[1+\theta_1(t)]^2} \leq \left(\frac{x_1^*}{m_1}\right)^2 z_1^2(t) \\
z_2^2(t) &< \frac{z_2^2(t)}{[1+\theta_2(t)]^2} \leq \left(\frac{x_2^*}{m_2}\right)^2 z_2^2(t). \qquad (5.34)
\end{aligned}
$$

It follows $(5.27), (5.28)$ and (5.34) that for $t \geq T^*$

$$
\begin{aligned}
V(z_t) &\geq \frac{z_1^2(t)}{2[1+\theta_1(t)]^2} + \frac{z_2^2(t)}{2[1+\theta_2(t)]^2} \\
&> \frac{1}{2}z_1^2(t) + \frac{1}{2}z_2^2(t) \\
&\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\
&\geq \widetilde{m}\left[z_1^2(t)+z_2^2(t)\right] \\
&= \widetilde{m}|z(t)|^2. \qquad (5.35)
\end{aligned}
$$

Case 3: If $0 < \theta_1(t) < z_1(t)$ and $-1 < z_2(t) < \theta_2(t) < 0$, then it follows (5.27), (5.28), (5.31) and (5.34) that for $t \geq T^*$

$$\begin{aligned} V(z_t) &\geq \frac{z_1^2(t)}{2[1+\theta_1(t)]^2} + \frac{z_2^2(t)}{2[1+\theta_2(t)]^2} \\ &> \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}z_2^2(t) \\ &\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq \widetilde{m}\left[z_1^2(t) + z_2^2(t)\right] \\ &= \widetilde{m}|z(t)|^2. \end{aligned} \tag{5.36}$$

Case 4: If $-1 < z_1(t) < \theta_1(t) < 0$ and $0 < \theta_2(t) < z_2(t)$, then it follows that (5.27),(5.28),(5.31) and (5.34) that for $t \geq T^*$

$$\begin{aligned} V(z_t) &\geq \frac{z_1^2(t)}{2[1+\theta_1(t)]^2} + \frac{z_2^2(t)}{2[1+\theta_2(t)]^2} \\ &> \frac{1}{2}z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq \widetilde{m}\left[z_1^2(t) + z_2^2(t)\right] \\ &= \widetilde{m}|z(t)|^2. \end{aligned} \tag{5.37}$$

Let $u(s) = \widetilde{m}s^2$, then u is nonnegative continuous on $[0, \infty)$, $u(0) = 0, u(s) > 0$ for $s > 0$, and $\lim_{s\to\infty} u(s) = +\infty$. So, by Case1 $\sim$ Case4, we have

$$V(z_t) \geq u(|z(t)|) \quad \text{for} \quad t \geq T^*. \tag{5.38}$$

Therefore, the unique equilibrium point E^* of the system (3.1) is globally asymptotically stable.

6 Examples & Conclusions

In this chaper, we want to illustrate our results by some examples.

Example 6.1 *Consider the following system:*

$$\begin{aligned} \dot{x}_1(t) &= 9.89x_1(t)\left[1 - \tfrac{x_1(t-0.001)}{360}\right] - 0.001x_1(t)x_2(t) \\ &\quad -0.00008x_2(t)x_1^2(t) - 0.04 \times 10x_1(t) \\ \dot{x}_2(t) &= 7.97x_2(t)\left[1 - \tfrac{x_2(t-0.002)}{300}\right] - 0.002x_1(t)x_2(t) \\ &\quad -0.00005x_1(t)x_2^2(t) - 0.01 \times 10x_2(t) \end{aligned} \tag{6.1}$$

Comparing the system (6.1) with the system (3.1), we get $r_1 = 9.89, r_2 = 7.97, k_1 = 360, k_2 = 300, q_1 = 0.04, q_2 = 0.01, \alpha_1 = 0.001, \alpha_2 = 0.002, \beta_1 = 0.00008, \beta_2 = 0.00005, e = 10$. So the system (5.1) has the unique positive equilibrium point $E^* \equiv (213.80, 199.76)$.

And

$$a_1 - a_2\tau_1 - a_3\tau_2 = 12.1621 > 0$$

$$b_1 - b_2\tau_1 - b_3\tau_2 = 8.4146 > 0$$

Then we conclude that the unique positive equilibrium point E^* of the system (6.1) is locally asymptotically stable by Theorem5.1. The trajectory of the system (6.1) is depicted in Figure 6.1.

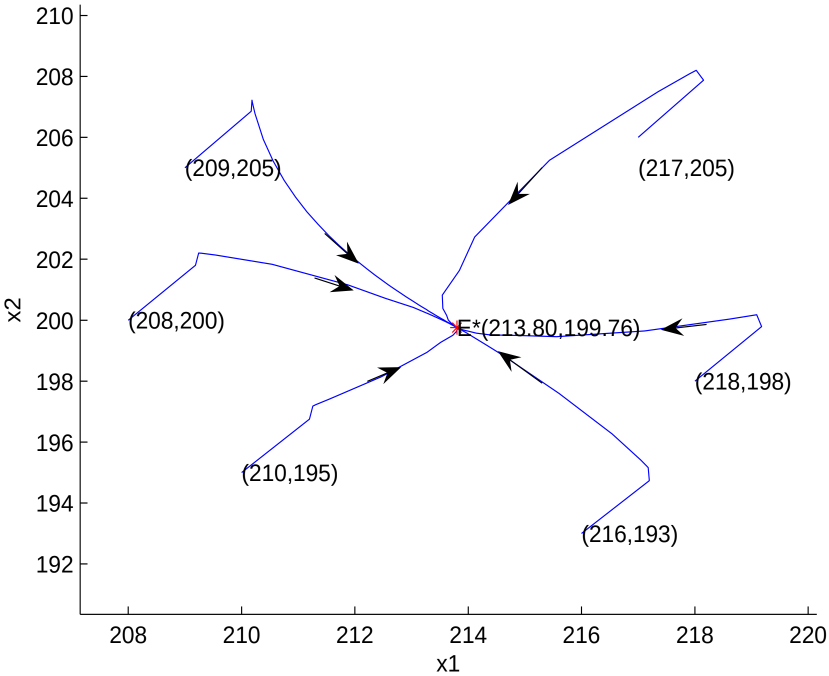

Figure 6.1: Phase portrait of the system (6.1).

Example 6.2 *Consider the following system:*

$$\begin{aligned}
\dot{x}_1(t) &= 9.89x_1(t)\left[1 - \tfrac{x_1(t-0.001)}{320}\right] - 0.001x_1(t)x_2(t) \\
&\quad -0.00008x_2(t)x_1^2(t) - 0.04 \times 10x_1(t) \\
\dot{x}_2(t) &= 7.97x_2(t)\left[1 - \tfrac{x_2(t-0.002)}{250}\right] - 0.002x_1(t)x_2(t) \\
&\quad -0.00005x_1(t)x_2^2(t) - 0.01 \times 10x_2(t)
\end{aligned} \tag{6.2}$$

Comparing the system (6.2) with the system (3.1), we get $r_1 = 9.89, r_2 = 7.97, k_1 = 320, k_2 = 250, q_1 = 0.04, q_2 = 0.01, \alpha_1 = 0.001, \alpha_2 = 0.002, \beta_1 = 0.00008, \beta_2 = 0.00005, e = 10$. So the system (6.2) has the unique positive equilibrium point $E^* \equiv (206.80, 176.61)$.

And

$$\mu_1 - \mu_2\tau_1 - \mu_3\tau_2 = 2.2667 > 0$$

$$\omega_1 - \omega_2\tau_1 - \omega_3\tau_2 = 1.2220 > 0$$

Then we conclude that the unique positive equilibrium point E^* of the system (6.2) is global asymptotically stable by Theorem 5.2. The trajectory of the system (6.2) is depicted in Figure 6.2.

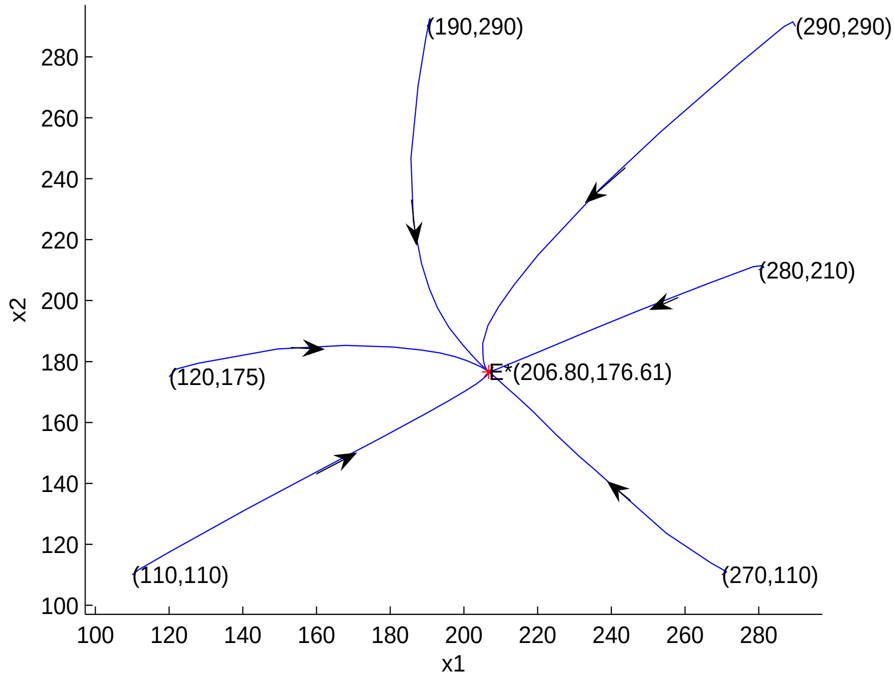

Figure 6.2: Phase portrait of the system (6.2) .

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