模糊計算與模糊量累積控制方法之研究以模糊計算之系統動態學為例

學生：林國平　　　　　　　　　　　　　　　　　　指導教授：張炳騰 教授

東海大學工業工程與經營資訊研究所

摘要

在模糊理論中，模糊計算及其應用為眾多學者所探討，其中α截集法(α-cuts)是經常被利用來進行模糊運算的一種方式，然而根據 Zadeh 的延伸法則(Extension principle)所定義之運算三角形基準 (*t*-norm)，並未被深入探討於運算中且兩種類型之模糊計算於複雜系統中邊界擴張及合理性並未被探討。因此，本研究將著重於下列方向，探討模糊計算 *t*-norm 並擴展模糊計算應用性、模糊計算導入系統動態實際模型及解決模糊系統中模糊量不可控制累積問題。

首先提出延伸法則所定義出之模糊計算除法，在 *t*-norm 準則下加法、減法、乘法於近年此類型模糊計算已經應用在許多領域中，本研究證明推導 *t*-norm 準則下之模糊計算除法，此計算除法包含 Yager's *t*-norm 準則及 the weakest *t*-norm 準則，這些方法將會擴展模糊計算之應用性。

其次本研究將模糊計算導入系統動態學實際模型，傳統系統動態學已經廣泛的被應用於許多領域上，於傳統系統動態可以觀察出某些變數屬於模糊因子，因此有些系統動態之變數或參數可以擴展成模糊變數，此模糊計算之系統動態評估可以提供決策者在不確定環境下系統行為之資訊，於本研究將以α截集、Yager's *t*-norm 及 the weakest *t*-norm 等模糊計算方式導入系統動態傳染病模式，觀察不同模糊計算對於系統動態模式之變化，模糊計算以 the weakest *t*-norm 準則下可獲得模式變數最小模糊間距，而α截集獲得模式變數最大模糊間距，而 Yager's *t*-norm 可控制參數介於兩者計算之間，但於系統動態模式中可發現，變數模糊量隨著時間累積變成不可控制，因此最後本研究提出累積系統在每個時間區間結束時可以將其變數解模糊，這解模糊值代表系統可能預期值，再將此解模糊值模糊化帶入下一個時間點，此方法可以避免模糊量隨著時間累積而造成無法控制的情形，同時本研究以顧客-生產者-勞工模式為實例觀察結合模糊計算之系統動態模式及利用解模糊技巧控制模糊量累積且測試最大(α 截集)及最小間距(the weakest *t*-norm)計算準則，並測試觀察三角模糊數不同大小之模糊間距以及非對稱之三角模糊數於系統模式，可得到模糊計算之系統動態皆可得到穩定之結果，並可依據決策環境下不確定性之程度選擇模糊計算方式及不同之三角模糊數。

關鍵字：模糊計算、α截集法、Yager's *t*-norm 準則、the weakest *t*-norm 準則、系統動態

A Study of Fuzzy Arithmetic and Fuzziness Accumulation Controlling Method for System Dynamics with Fuzzy Arithmetic

Student: **Kuo-Ping Lin** Advisor: **Ping-Teng Chang**

Department of Industrial Engineering and Enterprise Information
Tunghai University

ABSTRACT

In fuzzy theory the fuzzy arithmetic has been widely studied. Among them, α-cut arithmetic is a popular method in fuzzy arithmetic. However, based on the Zadeh's extension principle, the *t*-norm operators have not widely been investigated. Moreover, in fuzzy system the problem, which the fuzzy accumulations become uncontrollable, has not been investigated. Therefore, this research thoroughly investigates the division of fuzzy arithmetic and system dynamics with fuzzy arithmetic, and solving uncontrollable accumulations in fuzzy system.

First of all this research provides division of fuzzy arithmetic based on extension principle. In recent years, the operations of fuzzy arithmetic have been developed and applied to many fields in addition, subtraction, and multiplication based on α-cut or *t*-norm. This research shows the division of fuzzy arithmetic based on Yager's *t*-norm and the weakest *t*-norm. These operations of division will extend application of fuzzy arithmetic.

In the second place this research applies fuzzy arithmetic to system dynamics analysis. Traditional crisp system dynamics can be observed that some variables/parameters may belong to the uncertain factors. It is necessary to extend the system dynamics to treating the vague variables/parameters too. The evaluation of fuzzy system dynamics may provide the decision maker information regarding the system's behavior uncertainties. In this research the epidemics model is examined with the fuzzy system dynamics in three types of fuzzy arithmetic, α-cut fuzzy arithmetic, T_p Yager's *t*-norm, and T_ω weakest *t*-norm operator. In this model we can observe that the α-cut fuzzy arithmetic variables get larger fuzzy spreads in epidemics model, and the weakest *t*-norm operator variables get smaller fuzzy spreads in epidemics model. Based on the Yager's *t*-norm operator variable of model can get intermediate fuzzy spreads with tuning parameter p. However, we can find that accumulations become uncontrollable with dynamic time in this model. Finally, this research uses defuzzification method to solve uncontrollable accumulations in fuzzy system. The fuzzy variables of the system at the end of each interval can be defuzzified to obtain the representative value similar to the expected values or *interval-end defuzzification* is performed. The representative values of the variables may be supplied to the next time interval with fuzzy inputs again. The purpose of defuzzification is obvious that it avoids the fuzziness from continually accumulating in the model and by time possibly becoming very uncontrollable. Moreover, in this research, the customer-producer-employment model is also examined with the fuzzy system dynamics in two types of fuzzy arithmetic, α-cut fuzzy arithmetic and the T_ω weakest *t*-norm operator, and this model uses defuzzification method to control fuzzy accumulations. Symmetrical and non-symmetrical triangular-fuzzy-number, varied amount of fuzzy inputs' fuzziness, and length of the system time delay are examined with useful results provided.

Keywords: Fuzzy Arithmetic, α-Cut Arithmetic, Yager's *t*-Norm, the Weakest *t*-Norm, System Dynamics

誌 謝

這本論文能順利付梓，首先得歸功於我的論文指導教授，張炳騰博士。在論文研究期間，老師不斷地給予我指引、教導、鼓勵與支援。同時，老師精準的洞察力、淵博的學識、與認真執著、要求盡善盡美的態度，在在都深刻影響著我。我覺得非常幸運能有這麼好的機會跟著他進行學術研究。

同時，我也非常感謝校內論文口試委員，包括陳潭教授、洪堯勳教授，以及校外論文口試委員，包括吳文言教授、白炳豐教授及陳琨太教授。他們的各項指正與寶貴意見使這本論文增色不少。

本研究室博班學長(弟)們的幫助，尤其是與宏武學長、國禎學長、志昇、龍廷及舜麟一起奮鬥的日子，既刻苦又銘心，在此之間的研究討論更是讓我受益匪淺。另外還有 IKS 的碩班學弟妹(已畢業或在校)替我打點諸多事宜，在此一併致上我最誠致的謝意。

尤其要感謝我的家人，爸、媽、大姐、二姐以及陪伴我的好友薇如，感謝你們的支持，沒有你們的支持我無法順利完成博士學位。

林國平　謹誌於

東海大學工業工程與經營資訊學系

民國九十六年六月

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisors, Professor Ping-Teng Chang, for his enlightenment, instructions, guidance, and support during my Ph.D. studies. His great insight, extensive knowledge, and earnest attitude tremendously inspired me. I feel very lucky to have such a good chance to carry out researches with him. I also thank other members of my dissertation committee, Professors Tam Chan, Jau-Shin Hon, Wen-Yen Wu, Ping-Feng Pai and Kuen-Tai Chen, for their invaluable suggestions and comments on my dissertation.

I greatly appreciate my research partners (Hung-Wu, Kuo-Chen, Chih-Sheng, Lung-Ting, and Shun-Lin) at Tunghai University. I have benefited from fruitful discussions with them. I owe them a great deal for what they have done for me. I also acknowledge the favor of colleagues in laboratory of Intelligent Knowledge Systems. They helped me prepare my dissertation defense.

Most of all, I am deeply grateful to my father, my mother, my sisters and my dearest friend (Wei-Ju). They devoted themselves to support me with their persistent passion, caring, and encouragement.

CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

INTRODUCTION

This chapter introduces the research background, motivation and purposes. Additionally, the organization of this dissertation is presented.

1.1 Research Background and Motivation

The concept of fuzzy sets introduced by Zadeh (1965) has led to the definition of fuzzy logic, fuzzy numbers, and the various implementations. Numerous studies on the field of fuzzy sets has been done during the past three decades. Many new theoretical, algorithmic, and computational contributions of fuzzy theory have been used to solve various problems in management science and engineering. Fuzzy theory has very widespread applications in control, approximate reasoning, design, and planning problems (Mizumoto and Tanaka, 1979; Zimmermann, 2001). Due to the fact that solving problems with fuzzy sets involves several fuzzy variables/parameters, it is necessary to have fuzzy arithmetic operations. As a result, the fuzzy arithmetic operations are crucial.

Previously, based on Zadeh's extension principle, fuzzy arithmetic has been investigated either in approximate or exact manners (see Wood et al., 1992; Chang, 2005 for reviews), including Zadeh's original sup-min operations (see Dubois and Prade, 1981; Giachetti and Young, 1997(a), (b) for reviews), other *t*-norm operations (see Hong, 2001 (a), (b) for a review, Kosheleva et al., 1997; Wagenknecht et al., 2001), the T_ω weakest *t*-norm (Hong and Do, 1997; Kolesárová, 1995; Mesiar, 1997), and other operations based on the inverse of membership functions (e.g., see Liou and Wang, 1992), among others. Particularly, Zadeh's original sup-min operations can be performed equivalently with the α-cuts of the fuzzy parameters and interval arithmetic (e.g., see Mizumoto and Tanaka, 1976) and thus is termed as the "*α-cut* (*fuzzy*) *arithmetic*".

However, based on Zadeh's extension principle, fuzzy arithmetic is developed mainly on additional and multiplication. Moreover, for extending application, this research develops system dynamics with fuzzy arithmetic. System dynamics, introduced by Forrester (1961), indicates that some systems may be modeled by level and rate variables. In effect, in system dynamics, variables and/or parameters may belong to uncertain factors. Thus, a system dynamics for treating the uncertain factors is also needed. Except for the probability theory, Zadeh's fuzzy set theory (Zadeh, 1965) has been considered useful for treating the uncertainties. It deals with complex systems where the interactions of variables appear too complex to be specified precisely. Therefore, in this research, the applications of fuzzy arithmetic are proposed for the system dynamics.

1.2 Research Purposes

According to the background and motivation, our research focuses on the fuzzy arithmetic problems, including fuzzy arithmetic operations, and system dynamics with fuzzy arithmetic. Furthermore, this research addresses the issues as follows.

1. The division of fuzzy arithmetic: the fuzzy arithmetic has been developed and applied to many fields in addition and multiplication based on *t*-norm. This research, based on Yager's *t*-norm and the weakest *t*-norm, shows the division of fuzzy arithmetic, which extends the application of fuzzy arithmetic.
2. System dynamics analysis based on the application of fuzzy arithmetic: This research presents a system dynamics analysis based on the application of fuzzy arithmetic. Traditional crisp system dynamics can be observed that some variables/parameters may belong to the uncertain factors. It is necessary to extend the system dynamics to treat the vague variables/parameters as well. The evaluation of fuzzy system dynamics may provide the decision maker information regarding the system's behavior uncertainties. In this

paper, the customer-producer-employment model is examined with the fuzzy system dynamics in two types of fuzzy arithmetic, the α-cut fuzzy arithmetic and the T_{ω} weakest t-norm operator.

3. Solving uncontrollable accumulations in fuzzy system: The fuzzy variable of the system at the end of each interval can be defuzzified to obtain the representative value similar to the expected values or interval-end defuzzification. The representative values of the variables may be supplied to the next interval with fuzzy inputs and parameters again. The purpose of defuzzification is to avoid the fuzziness from continually accumulating in the model and by time becoming very uncontrollable.

1.3 Organization of the Dissertation

The structure of this dissertation is showed in Figure 1.1. In Chapter 1, the research background, motivation and purposes are presented. Chapter 2 introduces the concepts of fuzzy sets, fuzzy number, α–cuts fuzzy arithmetic operations, the weakest t-norm operations, and Yager's t-norm operations. Furthermore, this research reviews the literatures of fuzzy arithmetic and system dynamics. Chapter 3 discusses fuzzy arithmetic (weakest t-norm operations, Yager's t-norm and the fuzziness accumulation controlling method) from several observations. Chapter 4 presents the system dynamic with fuzzy arithmetic. Moreover, numerical experiments perform comparison analysis from these algorithms are provided. In Chapter 5 the customer-producer-employment model is examined and analyzed. Finally, conclusions and future researches are in Chapter 6.

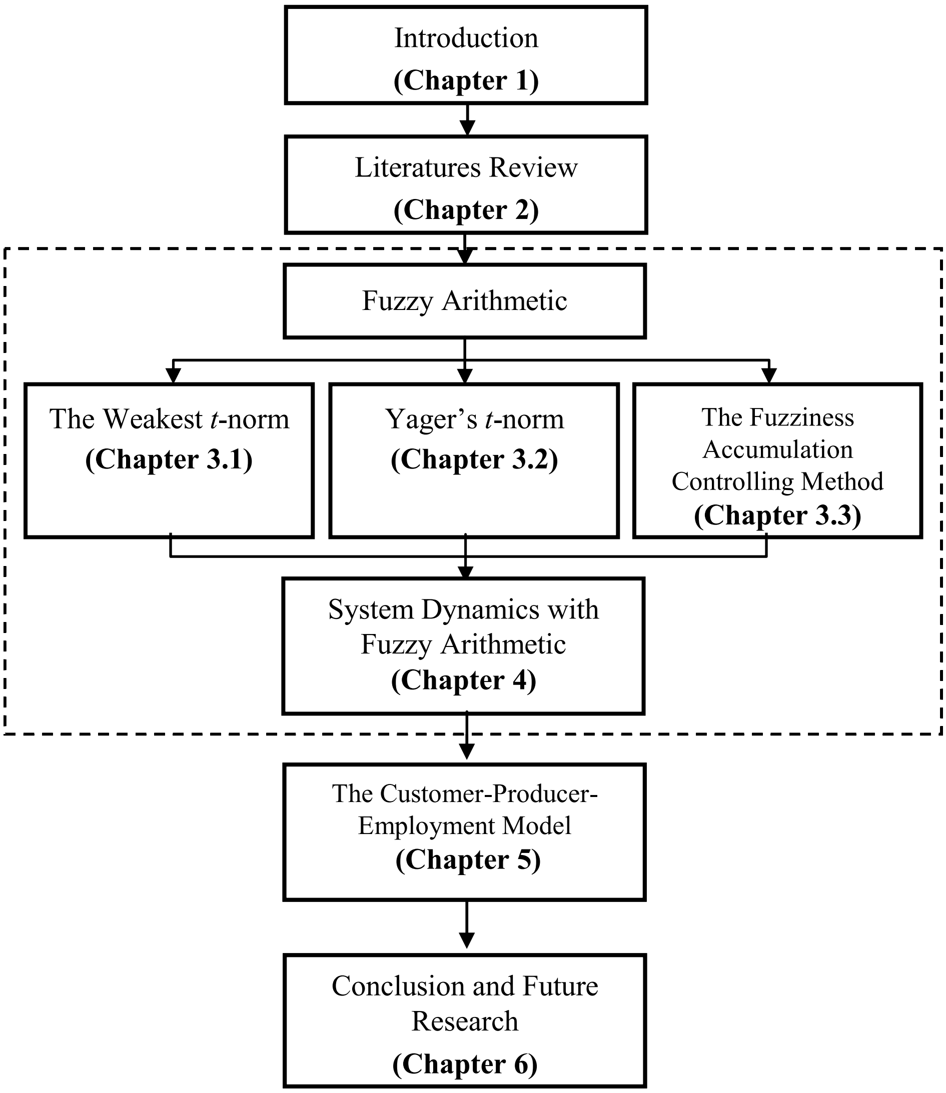


Figure 1.1 The research process and organization of the dissertation.

Chapter 2

LITERATURES REVIEW

This chapter contains with four sections. In Section 2.1, the fuzzy sets and fuzzy numbers are reviewed. Section 2.2 reviews fuzzy arithmetic operations (α-cut, the weakest t-norm, and Yager's t-norm). In addition, Section 2.3 reviews system dynamics and related problems.

2.1 Fuzzy Sets and Fuzzy Numbers

The theory of fuzzy sets was first introduced by Zadeh in 1965. A fuzzy set is defined by a membership function. The membership function maps elements (crisp inputs) in the universe of discourse to degrees of membership within a certain interval, which normally is [0, 1]. Then, the degree of membership specifies the extent to which a given element belongs to a set. Thus, for a fuzzy set A, it can be defined as follows.

Definition 2.1 (*Fuzzy set*)**.** Let U be the universe of discourse, A is a fuzzy subset in U if for all $x \in U$, there is a number $\mu_A(x) \in [0, 1]$ assigned to represent the membership of x to A, and $\mu_A(x)$ is called the membership function of A. It can be expressed by

$$A = \{(x, \mu_A(x)) |\, x \in U, \mu_A(x) \in [0, 1]\} \tag{2.1}$$

$\mu_A(x)$ is called the membership function or grade of membership (or degree of compatibility or truth) of x in A that maps the universal set U to the membership space E, which is a crisp set [0,1]. When E contains only two points 0 and 1, A is non-fuzzy and $\mu_A(x)$ is identical to a characteristic function of a non-fuzzy set. If the value lies within the interval [0,1], the element has a certain degree of membership (it belongs partially to the fuzzy set).

A fuzzy number is a fuzzy subset on R of numerical numbers, which possesses two properties of convexity and normality (Zadeh, 1965; Kaufman and Gupta, 1991). Special cases of fuzzy numbers include crisp real number and intervals of real numbers. The important definitions of fuzzy number and then the fuzzy arithmetic may be introduced as follows.

Definition 2.2 (*The convexity of fuzzy set*)**.** A fuzzy subset A of the universe of discourse U is convex, if

$$\mu_A[\lambda x_1+(1-\lambda)x_2] \geq \text{Min}[\mu_A(x_1), \mu_A(x_2)], x_1, x_2 \in U, \lambda \in [0, 1]. \quad (2.2)$$

Alternatively, a fuzzy subset is convex if all α-level sets (see Section 3.3) are convex.

Definition 2.3 (*The normality of fuzzy set*)**.** A fuzzy subset in the universe of discourse U is called a normal fuzzy set, if

$$\underset{x}{\text{Sup}}\ \mu_A(x) = 1. \quad (2.3)$$

A nonempty fuzzy subset A can always be normalized by $\mu_A(x) \Big/ \underset{x}{\sup}\ \mu_A(x)$.

Definition 2.4 (*Fuzzy number*)**.** A fuzzy number is a fuzzy subset in the universe of discourse R that is both convex and normal.

Therefore, A fuzzy number (FN, a fuzzy set) can be represented in the L-R type representation (Dubois and Prade, 1980) as A_i = $(x_i^L, x_i^M, x_i^U)_{L\text{-}R}$, where [x_i^L, x_i^U] defines the *support* (as *Supp*(A_i)), x_i^L and x_i^U the *lower* and *upper bounds*, x_i^M *mode*, and L and R the *left* and *right reference* (or *shape*) *functions* of A_i, respectively. A membership function of a FN A_i defines the belongingness of elements $x_i \in X_i$ to $A_i \in X_i$, denoted as $\mu_{A_i}(x_i)$, and is defined by L and R with $\mu_{A_i}(x_i^L) = L(x_i^L) = 0$, $\mu_{A_i}(x_i^R) = R(x_i^R) = 0$, $\mu_{A_i}(x_i^M) = L(x_i^M) = R(x_i^M) = 1$, and $\mu_{A_i}(x_i)$ $\in$ [0, 1]. Meanwhile, the α-cut of a FN A_i can be denoted as A_i^α = $[x_{i\alpha}^L, x_{i\alpha}^R]$ and $x_{i\alpha}^L$ and $x_{i\alpha}^R$ respectively denote the *left* and *right endpoints* of A_i^α. A_i^α is defined as an ordinary subset $\{x_i \mid \mu_{A_i}(x_i) \geq \alpha,\ \alpha \in [0, 1]\}$. For example, a triangular fuzzy number (TFN) may be denoted specially as $(x_i^L, x_i^M, x_i^U)_T$ with $L(x_i) = (x_i - x_i^L)/(x_i^M - x_i^L)$ for $x_i^L \le x_i \le x_i^M$, $R(x_i) = (x_i^U - x_i)/(x_i^U - x_i^M)$ for $x_i^M \le x_i \le x_i^U$, $L(x_i) = R(x_i) = 0$ otherwise, $x_{i\alpha}^L = L^{-1}(\alpha) = (x_i^M - x_i^L)\alpha + x_i^L$ and $x_{i\alpha}^R = R^{-1}(\alpha) = x_i^U - (x_i^U - x_i^M)\alpha$, where $L^{-1}(\alpha)$ and $R^{-1}(\alpha)$ are respectively denoting the inverses of L and R and $\alpha \in (0, 1]$.

2.2 Fuzzy Arithmetic Operations

The fuzzy arithmetic, following the Zadeh's extension principle (Zadeh, 1965) in the fuzzy set theory, was first investigated by Dubois and Prade (1980), Nahmias (1978), and Mizumoto and Tanaka (1976), amongst others. Among them, based on the Zadeh's extension principle, fuzzy arithmetic has been investigated in the approximate and exact manners (see Chang (2005), Chang and Chang (2005), and Chang and Hung (2006) for reviews). They include Zadeh's original sup-min operator (see Dubois and Prade (1987) and Giachetti and Young (1997a, b) for reviews), other *t*-norm operators (see Hong (2001a, b) for a review, Wagenknecht *et al.* (2001), the T_ω weakest *t*-norm operator (Hong and Do, 1997; Kolesárová, 1995; Mesiar, 1997)), and others based on the inverse of the membership functions (see Ma *et al.* (1999)). Among these operators, the α-cut fuzzy arithmetic are used most often and the T_ω weakest *t*-norm operator examined here.

For fuzzy sets or fuzzy numbers on the real line $\Re$, the following definitions and the fuzzy arithmetic may be introduced.

Fuzzy numbers—Let $\tilde{A}$ be a fuzzy set or a fuzzy number (FN) on $\Re$ and can be written as (a_1, a_2, a_3), where a_2 thus denotes the *mode* and a_1 and a_3 denote the *left and right bounds*, respectively, of $\tilde{A}$, with the *membership function* $\tilde{A}(x)$ defining the grade of membership of element $x \in \Re$ to $\tilde{A}$:

$$\tilde{A}(x)=\begin{cases}0, & x<a_1,\\ L((x-a_1)/(a_2-a_1)), & a_1\le x\le a_2,\\ R((a_3-x)/(a_3-a_2)), & a_2\le x\le a_3,\\ 0, & x>a_3,\end{cases} \tag{2.4}$$

where L and R respectively denote the *left and right shape functions* of $\tilde{A}(x)$. In particular, the fuzzy numbers with triangular membership functions or called triangular fuzzy numbers (TFNs) can be shown as Figure 2.1 and

$$\tilde{A}(x)=\begin{cases}0, & x<a_1,\\ (x-a_1)/(a_2-a_1), & a_1\le x\le a_2,\\ (a_3-x)/(a_3-a_2), & a_2\le x\le a_3,\\ 0, & x>a_3.\end{cases} \tag{2.5}$$

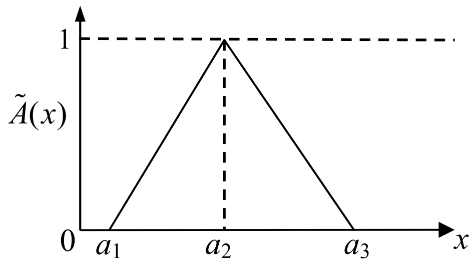


Figure 2.1 Triangular fuzzy number (a_1, a_2, a_3).

For an interval of confidence or α-cut at level $\alpha \in (0, 1)$, an ordinary subset A_α of $\tilde{A}$ can be defined:

$$A_\alpha=\left\{x\,\middle|\,\tilde{A}(x)\ge\alpha\right\}, \tag{2.6}$$

or given for a TFN

$$A_\alpha=[a_1^{(\alpha)},a_2^{(\alpha)}]=[a_1+\alpha(a_2-a_1),a_3-\alpha(a_3-a_2)], \tag{2.7}$$

where $a_1^{(\alpha)}$ and $a_2^{(\alpha)}$ respectively denote the *lower* and *upper bounds* of A_α.

1) *The α-cut* (*fuzzy*) *arithmetic*—In the fuzzy arithmetic approaches, Zadeh's sup-min operator can be stated as

$$(\tilde{A}\circ\tilde{B})(z)=\sup_{x\circ y}\min(\tilde{A}(x),\tilde{B}(y)), \tag{2.8}$$

where $\circ$ denotes any arithmetic operation. Eq. (2.8) shows that it can be performed in the equivalent manner by using the α-cuts of fuzzy numbers and the interval arithmetic (e.g., see Mizumoto and Tanaka (1976)). The resulting fuzzy arithmetic can be called the *α-cut* (*fuzzy*) *arithmetic*. Subsequently, the development of the α-cut arithmetic has been investigated by a number of researchers and can be seen with reviews in Chang and Hung (2006), Chang *et al.* (2006), and Chang and Hung (2006).

With the α-cut arithmetic, the addition/subtraction, multiplication/division, and others may be performed at each α on the intervals of confidence by the interval arithmetic (e.g., also see Kaufmann and Gupta (1988)). The following introduces the necessitated fuzzy arithmetic for our application.

Addition—Following the interval operation let $\tilde{A}$ and $\tilde{B}$ be two FNs and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$, $\alpha \in (0, 1]$. $\forall \tilde{A}, \tilde{B} \in \Re$, we can write

$$A_\alpha + B_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] + [b_1^{(\alpha)}, b_2^{(\alpha)}] = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}],\ \forall \alpha \in (0, 1]. \tag{2.9}$$

Subtraction—It can be defined as: $\forall\ \tilde{A}, \tilde{B}$ and $\alpha \in (0, 1]$,

$$A_\alpha - B_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] - [b_1^{(\alpha)}, b_2^{(\alpha)}] = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]. \tag{2.10}$$

Multiplication—It can be defined as: $\forall\ \tilde{A}, \tilde{B}$ and $\alpha \in (0, 1]$,

$$\begin{aligned} A_\alpha \times B_\alpha &= [a_1^{(\alpha)}, a_2^{(\alpha)}] \times [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\min(a_1^{(\alpha)} b_1^{(\alpha)}, a_1^{(\alpha)} b_2^{(\alpha)}, a_2^{(\alpha)} b_1^{(\alpha)}, a_2^{(\alpha)} b_2^{(\alpha)}), \\ &\max(a_1^{(\alpha)} b_1^{(\alpha)}, a_1^{(\alpha)} b_2^{(\alpha)}, a_2^{(\alpha)} b_1^{(\alpha)}, a_2^{(\alpha)} b_2^{(\alpha)})]. \end{aligned} \tag{2.11}$$

Division—It can be defined as: $\forall\ \tilde{A}, \tilde{B}$ and $\alpha \in (0, 1]$,

$$\begin{aligned} A_\alpha \div B_\alpha &= [a_1^{(\alpha)}, a_2^{(\alpha)}] \div [b_1^{(\alpha)}, b_2^{(\alpha)}] = [\min(a_1^{(\alpha)} / b_1^{(\alpha)}, a_1^{(\alpha)} / b_2^{(\alpha)}, a_2^{(\alpha)} / b_1^{(\alpha)}, a_2^{(\alpha)} / b_2^{(\alpha)}), \\ &\max(a_1^{(\alpha)} / b_1^{(\alpha)}, a_1^{(\alpha)} / b_2^{(\alpha)}, a_2^{(\alpha)} / b_1^{(\alpha)}, a_2^{(\alpha)} / b_2^{(\alpha)})].\ for\ b_1^{(\alpha)}, b_1^{(\alpha)} > 0, \alpha \in (0,1]. \end{aligned} \tag{2.12}$$

Thus, by Eqs. (2.9)-(2.12), the α-cut arithmetic if repeatedly performed in an equation, will accumulate the fuzziness of all fuzzy numbers involved. This property will be examined in the later Chapter 4 for the effect on the system dynamics as well when performed for each time interval.

2) *The* T_ω (*weakest t-norm fuzzy*) *arithmetic*—The Zadeh's extension principle, (2.8), if it is generalized by using a general norm T to replace the original 'min', can be written as

$$(\widetilde{A} \circ \widetilde{B})(z) = \sup_{x \circ y} T(\widetilde{A}(x), \widetilde{B}(y)), , \tag{2.13}$$

where the binary T norm on interval [0, 1] is said to be a triangular norm (or simply called t-norm) iff it is associative, commutative, and monotonous in [0, 1] and $T(x, 1) = x$ for every $x \in [0, 1]$. Moreover, each t-norm can be shown that satisfies the following equation.

$$T_\omega(a_2, b_2) \le T(a_2, b_2) \le T_M(a_2, b_2) = \min(a_2, b_2), \tag{2.14}$$

where

$$T_\omega(a_2, b_2) = \begin{cases} a_2, & if\ b_2 = 1, \\ b_2, & if\ a_2 = 1, \\ 0, & otherwise, \end{cases} \tag{2.15}$$

which is the weakest t-norm. The importance of the t-norms, e.g., $\min(a_2, b_2)$, $a_2 \cdot b_2$, $\max(0, a_2 + b_2 - 1)$, $T_\omega(a_2, b_2)$, has been shown in Ling (1965), Hong (2001a), Garmendia *et al.* (2003), and Whalen (2003) and also references therein. It is well known that the fuzzy addition/subtraction by T_M and T_ω preserves the original shape of the fuzzy numbers. In the multiplication/division, the T_M may not preserve the shape of the original FNs. However, for given shapes, the T_ω preserves the original FN shape in the multiplication/division (Heshmaty and Kandel, 1985). The T_ω weakest t-norm based on its concept also results in taking only the largest fuzziness resultant and encountered in the operation. For these reasons, the T_ω arithmetic is also considered in this research

3) *The* T_p (*Yager's t-norm*) *arithmetic*— In many articles and textbooks related to the fuzzy set theory and its applications, the classical form of the extension principle is used with the minimum norm $T_{\min}$ (see Zimmermann (1991)). Alternatively, some authors used the parameterized Yager's t-norm T_p which has the form as following:

$$T_p(\mu_{\widetilde{A}_1}(x_1),\ldots,\mu_{\widetilde{A}_n}(x_n))$$
$$= \max\left\{0;1-\left(\sum_{i=1}^{n}(1-\mu_{\widetilde{A}_i}(x_i)^p)\right)^{1/p}\right\}. \tag{2.16}$$

This form of the extension principle allows the derivation of closed forms for the extended arithmetic operations of addition and multiplication only for special forms of fuzzy sets. For that reason the convexity of fuzzy sets and the term "fuzzy interval" are introduced in Definition 2.2.

Using Yager's *t*-norm T_p in the extension principle Keresztfalvi proves "fast computation formulas" for the extended addition and multiplication of two triangular fuzzy numbers, which are generalized to the usage of two trapezoid fuzzy intervals in Rommelfanger (1994). Fullér and Keresztfalvi(1992) compared the addition of Yager's *t*-norm with *t*-norm, and Hong (1995) provided an easier method to prove the addition of *t*-norm. Hauke (1999) developed Yager's *t*-norm for aggregation of fuzzy interval and used Yager's *t*-norm operations to information model.

2.3 System Dynamics

System dynamics, introduced by Forrester (1961), indicates that some systems may be modeled by level and rate variables. Forrester stated the system dynamics as "the study of information feedback characteristics of the industrial activities to show how the organization structure, amplification (in policies), and time delay intervals influence an enterprise." The study was to respond to the recognition that management sciences do not providing insight or understanding into the strategic problems of complex systems. Forrester espoused and settled instead for influence diagrams or called signed directed graphs of relationships between variables, but in effect that the influence diagrams could be drawn at several levels of details in more aggregated forms and became the causal loop diagrams. With the graphical simplification of the influence diagrams, formal techniques could be developed for formulations (Forrester, 1968; Forrester, 1969; Forrester, 1971; Forrester, 1973; Forrester, 1975; Alfeld and Graham, 1976; Lyneis, 1980; Coylea, 1997). Figure 2.2 shows an example of a causal loop diagram that depicts the deviation enhancing behavior (Reid and Koljonen, 1999). The causal loop diagram depicts the generic manufacturing environment. In particular, pressure for the efficient use of resources requires that materials are released ahead of schedule which, in turn, increases management difficulties. Increasing management difficulties lead to a reduction of the time that management has available to deal with other issues such as effective inventory control and production planning. The latter then forces planning changes at a higher rate, which, in turn, increases managerial difficulties. A vicious circle of increasing management difficulties ensues.

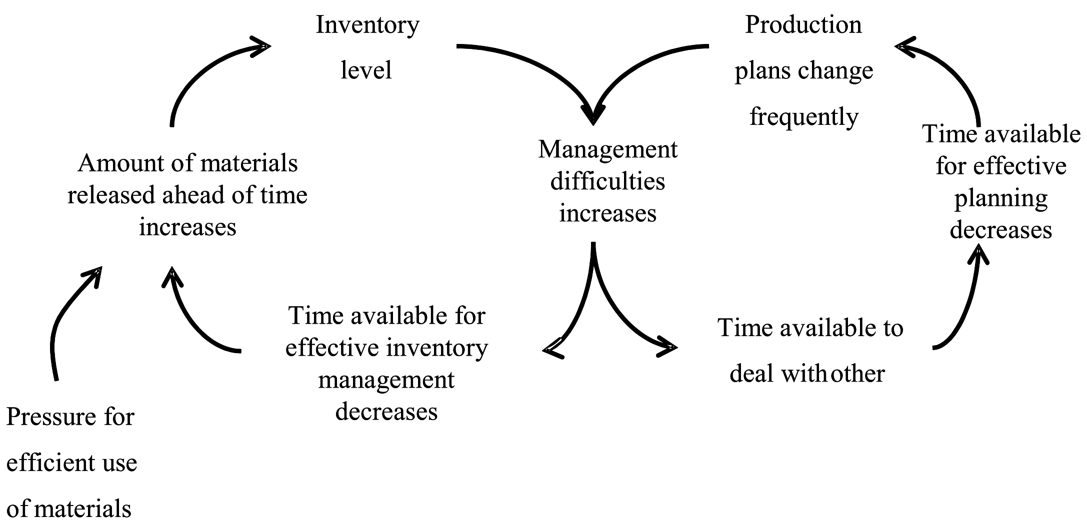


Figure 2.2 A causal loop diagram depicts the generic manufacturing environment.

System dynamics uses systems thinking as a conceptual tool for gaining insight into the structures that create the dynamic behavior often found in complex systems. System Dynamics is a model-and simulation-based method to study complex, dynamic systems in general. System dynamics proceeds on the assumption that a system can be represented as a collection of the following elements (Forrester, 1968):

(1) Things external to the system being modeled, such as sources for inputs to the system and sinks for outputs of the system.

(2) Levels, which represent things being accumulated within or used up by the system.

(3) Flows, which indicate where the items being accumulated come from and where those things being used up go.

(4) Auxiliaries, which are the fundamental variables within the system.

(5) Constants, which can be considered variables which remain constant during the course of a simulation.

(6) Links, which indicated how the basic elements of a system are interconnected.

With these basic elements, system dynamics can be used to explicate complex, dynamic systems such as those that exist in many instructional design and production settings. Using simulation techniques known as "management flight simulators," system dynamics can recreate the factors, features, and constraints of complex systems. These simulators have direct applications for the instructional design community in two contexts:

(1) As research tools —to study why people, facing complex, dynamic design, development, and production problems, act counterproductively due to their misperceptions of the structure underlying the manifest features of the problem.

(2) As training simulators — to improve the way people understand and create strategies to cope with complex design problems.

Essentially, the aim is to employ system dynamics to improve understanding of relationships between the structure and the behavior (dynamics) of problem-related systems comprised of many interacting instructional design variables. Given the assumption that the instructional design environment shares key features of complex systems, the research believe there are four kinds of structural problems that are particularly challenging (Sterman, 1989; Spector, 1995).

First, there is the origin of the dynamic behavior itself – the relationship between levels and flows. The level of a system determines its state at any point in time and the rates of the flows, accumulated by the levels, constitute the dynamics of the system. Although simple in principle, people often find it difficult to distinguish between real levels and flows and to identify the behavioral consequences of flows acting on levels.

Second, in real systems, not only do flow-rates effect levels, but through the causal relationships that constitute the structure of the system, levels feed back to influence flow rates. Real dynamic systems are consequently characterized by circular causality. Their structures contain feedback loops that transmit the dynamic behavior of one attribute to the next until the circle is closed and the signal, in a modified form, is fed back to its origin. Such loops have a tendency to counteract or reinforce each other in the stabilization or destabilization of the system. In the management of feedback systems, peoples' actions are typically being amplified or counteracted, depending on which feedback structures are dominating the system at a particular time.

Third, there are delays or lags in real systems. Delays distribute the effects of changes in variables throughout a system over time and often cause information to reach its destination in an untimely fashion. Delays and lags tend to cause people to discover and give priority to short-run gains and to ignore and postpone actions against future losses. Delayed reactions typically cause people to over and undershoot in their correcting (compensating) actions, and thus create systems exhibiting oscillatory behavior.

Finally, there is non-linearity, which implies that system attributes influence each other in a non-proportional way, and that they interact so that their partial effects cannot easily be distinguished. Such interactions may cause shifts in the structural dominance of a system over time. That is, substructures that have dominated a system's behavior for some time may, suddenly, or gradually, lose their significance, while other substructures gain influence. This typically causes a dramatic modification of the system's dynamic behavior. Notice that this happens endogenously, as a result of the behavior of the system itself, and that no structural change need take place, only a shift in relative dominance between substructures (Spector, 1995).

In essence, as shown in Figure 2.3, a system's pattern of behavior primarily results from the interaction of three core factors (Reid and Koljonen, 1999):

(1) The structure of the system, which is often expressed in the form of causal loop diagrams and/or stock-and-flow diagrams.

(2) The frequency and duration of time delays in feedback loops.

(3) The extent to which information flows and work are amplified through the system's feedback structure. The behavior of a system can often be described through interrelationships resulting from this set of three core factors.

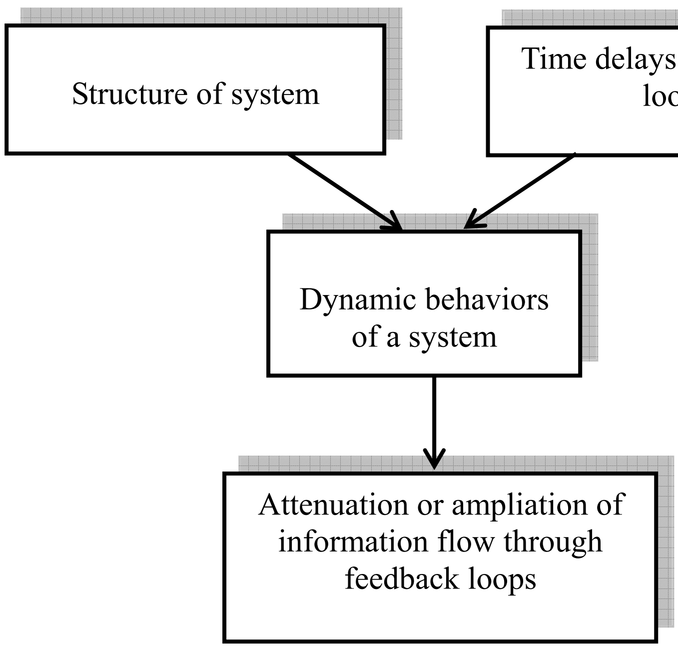


Figure 2.3 Dynamic interactions with a system.

The system dynamics has been developed and applied to a wide range of problems such as the analysis of public policies (Homer and Clair, 1991), biological and medical modeling (Hansen and Bie, 1987), dynamic decision-making (Sterman, 1989), supply chain management (Akkermans *et al.*, 1999), amongst others. A survey of the applications (Scholl, 1995) showed that the business policies have been the most active area in the system dynamics' application. Also, Kim and Kim (1997) applied a system dynamics to a mixed-strategy game between the police and driver. Richmonda (1997) applied the system dynamics to a design and alignment of strategies and business process with stated objectives. Maier (1998) discussed the new product diffusion models in the innovation management with a system dynamics perspective. Lyneis (2000) applied a system dynamics to the market forecasting and structural analysis. Grizzle and Pettijohn (2002) proposes a model for evaluating budget reforms which combines insights from budgeting, policy implementation, and system dynamics literatures. Stave (2002) applied a system dynamics to improving the public participation in environmental decisions. Disney and Towill (2003) applied system dynamics to simulate vendor managed inventory (VMI) supply chain model. Lai *et al.* (2003) applied a system dynamics to the just-in-time logistics concerning the total system perspective with an integrated framework of JIT and Kanban model and used the system dynamics tool as the modeling and simulation tool. Mingers (2003) provides a characterization of system dynamics using M–B framework. The M–B framework provides a basis for relating methodology and method to problem content and problem-solving activity with the purpose of alerting analysts to the appropriateness of different methodologies in different contexts. This analysis suggests that approximately half of the 'cells' reflecting stages in the problem-solving process are moderately or well addressed by system dynamics, leaving others less well addressed or not addressed at all. This suggests significant areas where system dynamics could be complemented by another methodology. Gui *et al.* (2005) applied system dynamics to analyze the area logistics system and establishes a system dynamics model for the area logistics system based on the characteristics of the area logistics system and system dynamics. Sehlke and Jacobson (2005) applied system dynamics to integrate surface water and ground water data for simulating the interactions between these sources within a given basin. In addition, this research also found that system dynamics modeling is useful for integrating complex hydrologic data with other information to produce a decision support system. Mabin *et al.* (2006) used the theory of constraints thinking processes to complement system dynamics' causal loop diagrams in developing fundamental solutions.

From observing previous researches, these characteristics of complex systems tend to mask relationships between cause and effect, and thus obscure current problems, while hiding the means of solving them. Successful solutions are often counterintuitive and hard to identify. In addition, there are often uncertainties associated with systems, and humans tend to state their perceptions, policy preferences, and attitudes vaguely. Hence, in system dynamics in reality, variables and/or parameters may belong to uncertain factors. A system dynamics for treating the uncertain factors is also necessitated. Fuzzy set and logic introduced by Zadeh (1965) represents another tool useful for treating the uncertainties in addition to the probability theory. Fuzzy set theory has been viewed particularly useful for dealing with complex systems where the interactions of variables can be too complex to be specified precisely. Levary (1990) has applied the fuzzy logic to a system dynamics. This research considered that in lack of the empirical verification of the variable relationships, some levels, delays and relations in the system dynamics might be treated as fuzzy variables. In his work, linguistic values and fuzzy if-then rules were adopted to state the conditions of the variables. Almost all the following researches have followed this idea in the fuzzy sets and applied the fuzzy logic to the system dynamics (Polat and Bozdag, 2002; Karavezyris *et al.*, 2002).

Chapter 3

THE FUZZY ARITHMETIC OPERATIONS AND FUZZINESS ACCUMULATION CONTROLLING METHOD

This chapter contains four sections. Section 3.1 describes the weakest *t*-norm operations and shows the division of weakest *t*-norm. Section 3.2 describes the Yager's *t*-norm operations and also shows the division of Yager's *t*-norm. Finally, Section 3.3 presents the fuzziness accumulation controlling method.

3.1 The Weakest *t*-norm Operations

In the Zadeh's extension principle (Zadeh, 1965), if generalized by using a general norm T that replaces the original 'min', four basic arithmetic operations of *t*–norm can be written as

(1) Addition :

$$(\tilde{A}+\tilde{B})(z)=\sup_{x+y=z} T(\tilde{A}(x),\tilde{B}(y)). \tag{3.1}$$

(2) Subtraction:

$$(\tilde{A}-\tilde{B})(z)=\sup_{x-y=z} T(\tilde{A}(x),\tilde{B}(y)). \tag{3.2}$$

(3) Multiplication:

$$(\tilde{A}\times\tilde{B})(z)=\sup_{x\times y=z} T(\tilde{A}(x),\tilde{B}(y)). \tag{3.3}$$

(4) Division :

$$(\tilde{A}/\tilde{B})(z)=\sup_{x/y=z} T(\tilde{A}(x),\tilde{B}(y)). \tag{3.4}$$

where the binary T norm on the interval [0, 1] is said to be a triangular norm (or called *t*-norm) iff it is associative, commutative, and monotonous in [0, 1] and $T(x, 1) = x$ for every $x \in [0, 1]$. Moreover, each *t*-norm may be shown that satisfies the following inequalities.

$$T_{\omega}(a_2,b_2) \le T(a_2,b_2) \le T_M(a_2,b_2) = \min(a_2,b_2), \tag{3.5}$$

where

$$T_{\omega}(a_2,b_2) = \begin{cases} a_2, & \text{if } b_2 = 1, \\ b_2, & \text{if } a_2 = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.6}$$

T_{ω} is the weakest *t*-norm. The importance of *t*-norms, e.g., $\min(a_2, b_2)$, $a_2 \cdot b_2$, $\max(0, a_2 + b_2 - 1)$, $T_{\omega}(a_2, b_2)$, has been shown in (Ling 1965, Hong 2001(a), Garmendia et al. 2003, and Whalen 2003), and also the references therein. It is well known that the addition/subtraction of fuzzy numbers by T_M and T_{ω} preserves the original shape of the fuzzy numbers. With the T_M in the multiplication/division the shapes of the original FNs may not be preserved. However, for given shapes, in the multiplication the T_{ω} preserves the original FNs' shape (Hong, 2001b). Based on the concept the T_{ω} weakest *t*-norm results in taking only the largest fuzziness encountered among the fuzzy numbers.

Theorem 3.1. (Kolesárová 1995, Mesiar 1997)

(a) *Let T be an arbitrary t-norm weaker than or equal to the Lukasiewicz t-norm T_L, $T(x, y) \le T_L(x,y)=\max(0, x+y-1)$, $x,y \in [0, 1]$. Then the addition $\oplus$ based on T coincides on linear fuzzy intervals with the addition $\oplus_{T_{\omega}}$ based on the weakest t-norm T_W, i.e.*

$$(a_1,\ a_2,\ a_3) \oplus_T (b_1,\ b_2,\ b_3) = (a_2 + b_2 - \max(a_2 - a_1,\ b_2 - b_1),\ a_2 + b_2,\\ a_2 + b_2 + \max\ (a_3 - a_2,\ b_3 - b_2)). \tag{3.7}$$

(b) *Let T be a continuous Archimedean t-norm with strictly convex additive generator f. Then the addition $\oplus$ based on T preserves the linearity of fuzzy intervals if and only if the t-norm T is a member of Yager's family of nilpotent t-norm* (Yager, 1980) *ith parameter $p \in (1, \infty)$, $T = T_p^Y$ and $f(x)=(1-x)^p$. Then*

$$(a_1, a_2,\ a_3) \oplus_{\mathrm{T}} (b_1, b_2, b_3) = (\ a_1 + a_2 - ((a_2 - a_1)^q +\\ (b_2 - b_1)^q)^{1/q}\ , a_1 + a_2,\ a_1 + a_2 + ((a_3 - a_2)^q +\ (b_3 - b_2)^q)^{1/q}) \tag{3.8}$$

where $(1/p+1/q)=1$, i.e. $q=p/(p-1)$.

It is also natural to preserving the shape of fuzzy numbers during subtraction, multiplication.

According to (Kolesárová 1995, Mesiar 1997, Hong and Do 1997), the addition, subtraction and multiplication can be defined as follow:

Let $T = T_\omega$ be the weakest *t*-norm, and let $\tilde{A} = (a_1, a_2, a_3)$, $\tilde{B} = (b_1, b_2, b_3)$ be two fuzzy numbers.

(1) *Addition of* T_ω: (see Kolesárová 1995, Mesiar 1997)

$$\begin{aligned}\tilde{A}+\tilde{B} = {} & (a_1,\ a_2,\ a_3)+(b_1,\ b_2,\ b_3)=(a_2+b_2-\max(a_2\\ & -a_1,\ b_2-b_1),\ a_2+b_2,\ a_2+b_2+\max(a_3-a_2,\ b_3-b_2)).\end{aligned} \tag{3.9}$$

(2) *Subtraction of* T_ω: (see Kolesárová 1995, Mesiar 1997)

$$\begin{aligned}\tilde{A}-\tilde{B} & = (a_1,\ a_2,\ a_3)-(b_1,\ b_2,\ b_3)\\ & = (a_2-b_2-\max(a_2-a_1,\ b_3-b_2),\ a_2-b_2,\ a_2-b_2+\max(b_2-b_1,\ a_3-a_2)).\end{aligned} \tag{3.10}$$

(3) *Multiplication of* T_ω: (see Hong and Do 1997)

Case I: For $a_2 > 0$, $b_2 > 0$,

$$\begin{aligned}\tilde{A}\times\tilde{B} = {} & (a_2b_2-\max((a_2-a_1)b_2,\ (a_3-a_2)a_2)),\ a_2b_2,\\ & a_2b_2+\max((a_3-a_2)b_2,\ (b_3-b_2)a_2))).\end{aligned} \tag{3.11}$$

Case II: For $a_2 < 0$, $b_2 < 0$,

$$\begin{aligned}\tilde{A}\times\tilde{B} = {} & (a_2b_2-\max((a_3-a_2)b_2,\ (b_3-b_2)a_2),\\ & a_2b_2,\ a_2b_2+\max((a_2-a_1)b_2,\ (b_2-b_1)a_2))).\end{aligned} \tag{3.12}$$

Case III: For $a_2 = 0$, $b_2 > 0$,

$$\tilde{A}\times\tilde{B} = (-(a_2 - a_1)b_2,\ 0,\ (a_3-a_2)b_2). \tag{3.13}$$

Case IV: For $a_2 = 0$, $b_2 < 0$,

$$\tilde{A}\times\tilde{B} = ((a_3-a_2)b_2,\ 0, -(a_2-a_1)b_2). \tag{3.14}$$

Case V: For $a_2 = 0$, $b_2 = 0$,

$$\tilde{A}\times\tilde{B} = (0, 0, 0). \tag{3.15}$$

Case VI: For $a_2 < 0$, $b_2 > 0$ and $L = R$,

$$\widetilde{A}\times\widetilde{B}=(a_2b_2-\max((a_2-a_1)b_2,-(b_3-b_2)a_2),\ a_2b_2,\ a_2b_2+\max((a_3-a_2)b_2,\ -(b_2-b_1)a_2)). \tag{3.16}$$

(4) *Division of* T_ω:

Recent year the operations of weakest *t*-norm have been developed and applied to many fields in addition, subtraction, and multiplication. Therefore, the research extends the division of T_ω-based as follows:

Case I: Let $a_2 > 0$, $b_2 > 0$ and $b_1 > 0$, then for $z \le a_2/b_2$, $z/b_2 \le a_2$, and $z/a_2 \le 1/b_2$,

$$(\widetilde{A}/\widetilde{B})(z)=\sup_{x/y=z} T_\omega(\widetilde{A}(x),\widetilde{B}(y))$$
$$=\max\left\{A(z\times b_2),B(a_2/z)\right\}$$
$$=\max\left\{L\left(\frac{a_2-z\times b_2}{a_2-a_1}\right),R\left(\frac{1/b_2-z/a_2}{1/b_2-1/b_3}\right)\right\}$$
$$=\max\left\{L\left(\frac{a_2/b_2-z}{(a_2-a_1)/b_2}\right),R\left(\frac{a_2/b_2-z}{a_2(1/b_2-1/b_3)}\right)\right\}$$
$$=L\left\{(a_2/b_2-z)/\max((a_2-a_1)/b_2,a_2(1/b_2-1/b_3))\right\}$$

for $z \ge a_2/b_2-\max((a_2-a_1)/b_2,a_2(1/b_2-1/b_3))$; it is 0 otherwise. Similarly, for $z > a_2/b_2$, $z/b_2 > a_2$ and $z/a_2 > 1/b_2$, we can get the result as follows:

$$(\widetilde{A}/\widetilde{B})(z)=\sup_{x/y=z} T_\omega(\widetilde{A}(x),\widetilde{B}(y))$$
$$=\max\left\{A(z\times b_2),B(a_2/z)\right\}$$
$$=\max\left\{R\left(\frac{z\times b_2-a_2}{a_3-a_2}\right),L\left(\frac{z/a_2-1/b_2}{1/b_1-1/b_2}\right)\right\}$$
$$=\max\left\{R\left(\frac{z-a_2/b_2}{(a_3-a_2)/b_2}\right),L\left(\frac{z-a_2/b_2}{a_2(1/b_1-1/b_2)}\right)\right\}$$
$$=R\left\{(z-a_2/b_2)/\max((a_3-a_2)/b_2,a_2(1/b_1-1/b_2))\right\}$$

for $z < a_2/b_2+\max((a_3-a_2)/b_2,a_2(1/b_1-1/b_2))$; it is 0 otherwise.

Therefore, we can obtained that

$$\widetilde{A}/\widetilde{B}=(a_2/b_2-\max((a_2-a_1)/b_2,a_2(1/b_2-1/b_3)),\ a_2/b_2,a_2/b_2+\max((a_3-a_2)/b_2,a_2(1/b_1-1/b_2))). \tag{3.17}$$

In a similar manner, the other cases can be derived.

Case II: For $a_2 < 0$, $b_2 < 0$ and $b_3 < 0$,

$$\tilde{A}/\tilde{B}=(a_2/b_2-\max((a_2-a_3)/b_2,a_2(1/b_2-1/b_1)),\ a_2/b_2,\ a_2/b_2+\max((a_1-a_2)/b_2,a_2(1/b_3-1/b_2))) \quad (3.18)$$

Case III: For $a_2 = 0$, $b_2 > 0$ and $b_1 > 0$,

$$\tilde{A}/\tilde{B}=(-(a_2-a_1)/b_2,0,(a_3-a_2)/b_2) \quad (3.19)$$

Case IV: For $a_2 = 0$, $b_2 < 0$ and $b_3 < 0$,

$$\tilde{A}/\tilde{B}=(-(a_2-a_3)/b_2,0,(a_1-a_2)/b_2) \quad (3.20)$$

Case V: For $a_2 > 0$, $b_2 < 0$ and $b_3 < 0$,

$$\tilde{A}/\tilde{B}=(a_2/b_2-\max((a_2-a_3)/b_2,a_2(1/b_2-1/b_3)),\ a_2/b_2,a_2/b_2+\max((a_1-a_2)/b_2,a_2(1/b_1-1/b_2))) \quad (3.21)$$

Case VI: For $a_2 < 0$, $b_2 > 0$ and $b_1 > 0$,

$$\tilde{A}/\tilde{B}=(a_2/b_2-\max((a_2-a_1)/b_2,a_2(1/b_2-1/b_1)),a_2/b_2,a_2/b_2+\max((a_3-a_2)/b_2,a_2(1/b_3-1/b_2))). \quad (3.22)$$

3.2 The Yager's *t*-norm Operations

The parameterized Yager's *t*-norms T_p which has the form $T_p(\mu_{\tilde{A}_1}(x_1),\ldots,\mu_{\tilde{A}_n}(x_n))=\max\left\{0;\ 1-\left(\sum_{i=1}^{n}(1-\mu_{\tilde{A}_i}(x_i))^p\right)^{1/p}\right\}.$

This form of the extension principle allows the derivation of closed forms for the extended arithmetic operations of addition, subtraction, multiplication only for special forms of fuzzy sets.

Using Yager's *t*-norms in the extension principle Keresztfalvi proves "fast computation formulas" for the extended addition and multiplication of two triangular (Keresztfalvi, 1993).

We can get the exact formula as following:

(1) Addition of Yager's *t*-norms:

$$\tilde{A}+\tilde{B}=(a_1,a_2,a_3)+(b_1,b_2,b_3)=(a_2+b_2-((a_2-a_1)^q+(b_2-b_1)^q)^{1/q},\ a_2+b_2,\ a_2+b_2+((a_3-a_2)^q+(b_3-b_2)^q)^{1/q}). \quad (3.23)$$

where $1/p+1/q=1$ for the extended addition. The extreme cases for the parameter p in Yager's t-norms yield the same result as the classical min-operator and the so-called bounded difference (Rommelfanger, 1994). The bounded difference considers the minimum bounded. Hence, this research uses the weakest *t*-norm to regard as the bounded difference and approaches the min-operator by the bounded difference.

- q=1 resp. $p\to\infty$ (min-operator)

$$\tilde{A}+\tilde{B}= (a_1,\ a_2,\ a_3)+(b_1,\ b_2,\ b_3)=(a_2+b_2-((a_2 -a_1)\ +(b_2-b_1)),\ a_2+b_2,\ a_2+b_2+((a_3-a_2)+(b_3-b_2))). \tag{3.24}$$

- p=1 resp. $q\to\infty$ (bounded difference)

$$\tilde{A}+\tilde{B}= (a_1,\ a_2,\ a_3)+(b_1,\ b_2,\ b_3)=(a_2+b_2-\max(a_2 -a_1,\ b_2-b_1),\ a_2+b_2,\ a_2+b_2+\max(a_3-a_2,\ b_3-b_2)). \tag{3.25}$$

(2) Subtraction of Yager's *t*-norms:

$$\begin{aligned}\tilde{A}-\tilde{B}&=(a_1,\ a_2,\ a_3)-(b_1,\ b_2,\ b_3)\\ &=(a_2-b_2-((a_2-a_1)^q+(b_3-b_2)^q)^{1/q},\ a_2-b_2,\ a_2-b_2+((b_2-b_1)^q+(a_3-a_2)^q)^{1/q}).\end{aligned} \tag{3.26}$$

where $1/p+1/q=1$ for the extended subtraction. As addition manner, the extreme cases of subtraction can be derived.

- q=1 resp. $p\to\infty$ (min-operator)

$$\begin{aligned}\tilde{A}-\tilde{B}&=(a_1,\ a_2,\ a_3)-(b_1,\ b_2,\ b_3)\\ &=(a_2-b_2-((a_2-a_1)+(b_3-b_2)),\ a_2-b_2,\ a_2-b_2+((b_2-b_1)+(a_3-a_2))).\end{aligned} \tag{3.27}$$

- p=1 resp. $q\to\infty$ (bounded difference)

$$\begin{aligned}\tilde{A}-\tilde{B}&=(a_1,\ a_2,\ a_3)-(b_1,\ b_2,\ b_3)\\ &=(a_2-b_2-\max(a_2-a_1,\ b_3-b_2),\ a_2-b_2,\ a_2-b_2+\max(b_2-b_1,\ a_3-a_2)).\end{aligned} \tag{3.28}$$

(3) Multiplication of Yager's *t*-norms:

Tradition multiplication of Yager's *t*-norms holds for small bounded differences compared mode values (Dubois and Prade, 1980). However, different case can not express exactly in tradition multiplication of Yager's *t*-norms, and the weakest *t*-norm is the minimum bounded which has been proven by (Ling 1965; Hong 2001; Garmendia 2003). This research uses the weakest *t*-norm to consider the bounded difference in extreme cases. From the bounded, difference the research can exactly derive min-operator of multiplication with different cases as follows.

Case I: For $a_2 > 0$, $b_2 > 0$,

$$\widetilde{A}\times\widetilde{B}=(a_2b_2-(((a_2-a_1)b_2)^q+((a_3-a_2)a_2)^q))^{1/q},\ a_2b_2,\ a_2b_2+(((a_3-a_2)b_2)^q+((b_3-b_2)a_2))^q)^{1/q}). \tag{3.29}$$

where $1/p+1/q=1$ for the multiplication. As previous manner, the extreme cases of multiplication can be derived in this case I.

- $q=1$ resp. $p\to\infty$ (min-operator)

$$\widetilde{A}\times\widetilde{B}=(a_2b_2-((a_2-a_1)b_2+(a_3-a_2)a_2),\ a_2b_2,\ a_2b_2+((a_3-a_2)b_2+(b_3-b_2)a_2)). \tag{3.30}$$

- $p=1$ resp. $q\to\infty$ (bounded difference)

$$\widetilde{A}\times\widetilde{B}=(a_2b_2-\max((a_2-a_1)b_2,\ (a_3-a_2)a_2)),\ a_2b_2,\ a_2b_2+\max((a_3-a_2)b_2,\ (b_3-b_2)a_2))). \tag{3.31}$$

Case II: For $a_2 < 0$, $b_2 < 0$,

$$\widetilde{A}\times\widetilde{B}=(a_2b_2-(((a_3-a_2)b_2)^q+((b_3-b_2)a_2)^q)^{1/q},\ a_2b_2,\ a_2b_2+(((a_2-a_1)b_2)^q+((b_2-b_1)a_2))^q)^{1/q}). \tag{3.32}$$

where $1/p+1/q=1$ for the multiplication. As previous manner, the extreme cases of multiplication can be derived in this case II.

- $q=1$ resp. $p\to\infty$ (min-operator)

$$\widetilde{A}\times\widetilde{B}=(a_2b_2-((a_3-a_2)b_2+(b_3-b_2)a_2),\ a_2b_2,\ a_2b_2+((a_2-a_1)b_2+(b_2-b_1)a_2)). \tag{3.33}$$

- $p=1$ resp. $q\to\infty$ (bounded difference)

$$\widetilde{A}\times\widetilde{B}=(a_2b_2-\max((a_3-a_2)b_2,\ (b_3-b_2)a_2),\ a_2b_2,\ a_2b_2+\max((a_2-a_1)b_2,\ (b_2-b_1)a_2))). \tag{3.34}$$

Case III: For $a_2 = 0$, $b_2 > 0$, due to $a_2 = 0$, the multiplication of Yager's *t*-norms is always the same with different cases and extreme cases as follows.

$$\widetilde{A}\times\widetilde{B}=(-(a_2-a_1)b_2,\ 0,\ (a_3-a_2)b_2). \tag{3.35}$$

Case IV: For $a_2 = 0$, $b_2 < 0$, due to $a_2 = 0$, the multiplication of Yager's *t*-norms is always the same with different cases and extreme cases as follows.

$$\tilde{A}\times\tilde{B} = ((a_3 - a_2)b_2,\ 0, -(a_2 - a_1)b_2). \tag{3.36}$$

Case V: For $a_2 = 0$, $b_2 = 0$, due to $a_2 = 0$ and $b_2 = 0$, the multiplication of Yager's *t*-norms is always the same with different cases and extreme cases as follows.

$$\tilde{A}\times\tilde{B} = (0, 0, 0). \tag{3.37}$$

Case VI: For $a_2 < 0$, $b_2 > 0$ and $L = R$,

$$\begin{aligned}\tilde{A}\times\tilde{B} = (&a_2 b_2 - (((a_2 - a_1)b_2)^q + (-(b_3 - b_2)a_2)^q)^{1/q},\ a_2 b_2,\\ &a_2 b_2 + (((a_3 - a_2)b_2)^q + (-(b_2 - b_1)a_2)^q)^{1/q}).\end{aligned} \tag{3.38}$$

where $1/p+1/q=1$ for the multiplication. As previous manner, the extreme cases of multiplication can be derived in this case VI.

- $q=1$ resp. $p\to\infty$ (min-operator)

$$\tilde{A}\times\tilde{B} = (a_2 b_2 - ((a_2 - a_1)b_2 - (b_3 - b_2)a_2),\ a_2 b_2,\ a_2 b_2 + ((a_3 - a_2)b_2 - (b_2 - b_1)a_2)). \tag{3.39}$$

- $p=1$ resp. $q\to\infty$ (bounded difference)

$$\begin{aligned}\tilde{A}\times\tilde{B} = (&a_2 b_2 - \max((a_2 - a_1)b_2, -(b_3 - b_2)a_2),\ a_2 b_2,\\ &a_2 b_2 + \max((a_3 - a_2)b_2,\ -(b_2 - b_1)a_2)).\end{aligned} \tag{3.40}$$

(4) Division of Yager's *t*-norms:

The division of Yager's *t*-norms is similar as multiplication and can be showed as follows:

Case I: Let $a_2 > 0$, $b_2 > 0$ and $b_1 > 0$,

Theorem 3.2. If the division is extended by Yager's t-norm T_P ($p\geq 1$) *then the division of two positive triangular fuzzy numbers* $(a_1, a_2, a_3)/(b_1, b_2, b_3)$ *can be calculated by the following approximation formula*:

$$\begin{aligned}\tilde{A}/\tilde{B} \simeq (&a_2/b_2 - (((a_2 - a_1)/b_2)^q + (a_2(1/b_2 - 1/b_3))^q)^{1/q},\ a_2/b_2,\\ &a_2/b_2 + (((a_3 - a_2)/b_2)^q + (a_2(1/b_1 - 1/b_2))^q)^{1/q}).\end{aligned} \tag{3.41}$$

with $1/p+1/q=1$, *provided that the spreads of A and B are small compared with their mean* *values*, *i.e.* $((a_2 - a_1), (a_3 - a_2) \ll a_2$ *and* $((b_2 - b_1), (b_3 - b_2) \ll b_2$.

Proof: A general form of Nguyen's formula on

$$[A/B]_\lambda = a_2/b_2 + \bigcup_{\|(1-\xi,1-\eta)\|_p \le 1-\lambda} \begin{matrix} ([-((a_2-a_1)/b_2)(1-\xi) - a_2(1/b_2 - 1/b_3)(1-\eta), \\ ((a_3-a_2)/b_2(1-\xi) + a_2(1/b_1 - 1/b_2)(1-\eta)] + \\ [(a_2-a_1)(1-\xi)(1/b_2-1/b_3)(1-\eta), \\ (a_3-a_2)(1-\xi)(1/b_1-1/b_2)(1-\eta))]) \end{matrix} \tag{3.42}$$

Based on assumption $((a_2 - a_1), (a_3 - a_2) \ll a_2$ *and* $((b_2 - b_1), (b_3 - b_2) \ll b_2$, we can neglect the last serial of Eq. (3.42). Hence, we can get the following equation:

$$\begin{aligned} [A/B]_\lambda &\simeq a_2/b_2 + \bigcup_{\|(1-\xi,1-\eta)\|_p \le 1-\lambda} \begin{matrix} [-((a_2-a_1)/b_2)(1-\xi) - a_2(1/b_2-1/b_3)(1-\eta), \\ ((a_3-a_2)/b_2(1-\xi) + a_2(1/b_1-1/b_2)(1-\eta)] \end{matrix} \\ &= a_2/b_2 + (1-\lambda)\cdot[-\|((a_2-a_1)/b_2, a_2(1/b_2-1/b_3))\|_q, \\ &\|((a_3-a_2)/b_2, a_2(1/b_1-1/b_2))\|_q] \end{aligned} \tag{3.43}$$

This is a λ-level set of TFNs. That is

$$\begin{aligned} \tilde{A}/\tilde{B} &\simeq (a_2/b_2 - \|((a_2-a_1)/b_2), a_2(1/b_2-1/b_3)\|_q,\ a_2/b_2, \\ &a_2/b_2 + \|((a_3-a_2)/b_2), a_2(1/b_1-1/b_2)\| \\ &\simeq (a_2/b_2 - (((a_2-a_1)/b_2)^q + (a_2(1/b_2-1/b_3))^q)^{1/q},\ a_2/b_2, \\ &a_2/b_2 + (((a_3-a_2)/b_2)^q + (a_2(1/b_1-1/b_2))^q)^{1/q}) \end{aligned} \tag{3.44}$$

Hence the general formula can be proved, and extreme cases can be shown as following:

- q=1 resp. $p \to \infty$ (min-operator)

$$\begin{aligned} \tilde{A}/\tilde{B} &= (a_2/b_2 - ((a_2-a_1)/b_2 + a_2(1/b_2-1/b_3)),\ a_2/b_2, \\ &a_2/b_2 + ((a_3-a_2)/b_2 + a_2(1/b_1-1/b_2))). \end{aligned} \tag{3.45}$$

- p=1 resp. $q \to \infty$ (bounded difference)

$$\begin{aligned} \tilde{A}/\tilde{B} &= (a_2/b_2 - \max((a_2-a_1)/b_2, a_2(1/b_2-1/b_3)),\ a_2/b_2, \\ &a_2/b_2 + \max((a_3-a_2)/b_2, a_2(1/b_1-1/b_2))). \end{aligned} \tag{3.46}$$

In a similar manner, the other cases can be derived.

Case II: For $a_2 < 0$, $b_2 < 0$ and $b_3 < 0$,

$$\tilde{A}/\tilde{B} = (a_2/b_2 - (((a_2-a_3)/b_2)^q + (a_2(1/b_2 - 1/b_1))^q)^{1/q},$$
$$a_2/b_2,\ a_2/b_2 + (((a_1-a_2)/b_2)^q + (a_2(1/b_3 - 1/b_2))^q)^{1/q}). \quad (3.47)$$

- q=1 resp. $p \to \infty$ (min-operator)

$$\tilde{A}/\tilde{B} = (a_2/b_2 - ((a_2-a_3)/b_2 + a_2(1/b_2 - 1/b_1)),$$
$$a_2/b_2,\ a_2/b_2 + ((a_1-a_2)/b_2 + a_2(1/b_3 - 1/b_2))). \quad (3.48)$$

- p=1 resp. $q \to \infty$ (bounded difference)

$$\tilde{A}/\tilde{B} = (a_2/b_2 - \max((a_2-a_3)/b_2, a_2(1/b_2 - 1/b_1)),$$
$$a_2/b_2,\ a_2/b_2 + \max((a_1-a_2)/b_2, a_2(1/b_3 - 1/b_2))) \quad (3.49)$$

Case III: For $a_2 = 0$, $b_2 > 0$ and $b_1 > 0$, due to $a_2 = 0$, the division of Yager's t-norms is always the same with different cases and extreme cases as follows.

$$\tilde{A}/\tilde{B} = (-(a_2-a_1)/b_2, 0, (a_3-a_2)/b_2) \quad (3.50)$$

Case IV: For $a_2 = 0$, $b_2 < 0$ and $b_3 < 0$, due to $a_2 = 0$, the division of Yager's t-norms is always the same with different cases and extreme cases as follows.

$$\tilde{A}/\tilde{B} = (-(a_2-a_3)/b_2, 0, (a_1-a_2)/b_2) \quad (3.51)$$

Case V: For $a_2 > 0$, $b_2 < 0$ and $b_3 < 0$,

$$\tilde{A}/\tilde{B} = (a_2/b_2 - (((a_2-a_3)/b_2)^q + (a_2(1/b_2 - 1/b_3))^q)^{1/q},$$
$$a_2/b_2, a_2/b_2 + (((a_1-a_2)/b_2)^q + (a_2(1/b_1 - 1/b_2))^q)^{1/q}). \quad (3.52)$$

- q=1 resp. $p \to \infty$ (min-operator)

$$\tilde{A}/\tilde{B} = (a_2/b_2 - ((a_2-a_3)/b_2 + a_2(1/b_2 - 1/b_3)),$$
$$a_2/b_2, a_2/b_2 + ((a_1-a_2)/b_2 + a_2(1/b_1 - 1/b_2))). \quad (3.53)$$

- p=1 resp. $q \to \infty$ (bounded difference)

$$\tilde{A}/\tilde{B} = (a_2/b_2 - \max((a_2-a_3)/b_2, a_2(1/b_2 - 1/b_3)),$$
$$a_2/b_2, a_2/b_2 + \max((a_1-a_2)/b_2, a_2(1/b_1 - 1/b_2))) \quad (3.54)$$

Case VI: For $a_2 < 0$, $b_2 > 0$ and $b_1 > 0$,

$$\tilde{A}/\tilde{B} = (a_2/b_2 - (((a_2-a_1)/b_2)^q + (a_2(1/b_2 - 1/b_1))^q)^{1/q}$$
$$, a_2/b_2, a_2/b_2 + (((a_3-a_2)/b_2)^q + (a_2(1/b_3 - 1/b_2))^q)^{1/q}). \tag{3.55}$$

- q=1 resp. $p\to\infty$ (min-operator)

$$\tilde{A}/\tilde{B} = (a_2/b_2 - ((a_2-a_1)/b_2 + a_2(1/b_2 - 1/b_1))$$
$$, a_2/b_2, a_2/b_2 + ((a_3-a_2)/b_2 + a_2(1/b_3 - 1/b_2))). \tag{3.56}$$

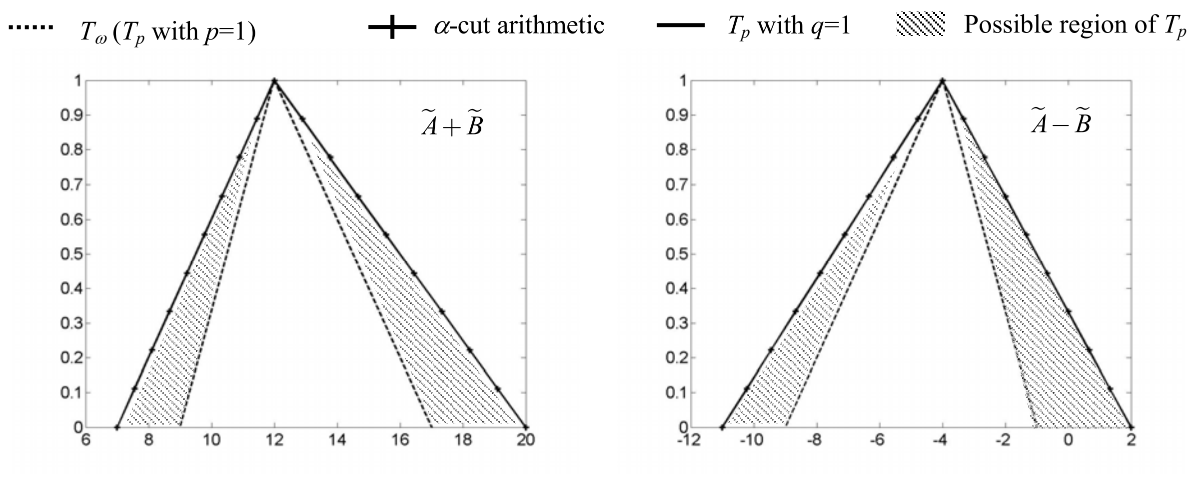
- p=1 resp. $q\to\infty$ (bounded difference)

$$\tilde{A}/\tilde{B} = (a_2/b_2 - \max((a_2-a_1)/b_2, a_2(1/b_2 - 1/b_1))$$
$$, a_2/b_2, a_2/b_2 + \max((a_3-a_2)/b_2, a_2(1/b_3 - 1/b_2))). \tag{3.57}$$

For example, let $\tilde{A}$ = (2, 4, 7), $\tilde{B}$ = (5, 8, 13) be two triangular fuzzy numbers. Table 3.1 can obtain the results of α-cut, T_ω, and T_p operations, and the illustration also shows in Figure 3.1. In Figure 3.1 we can observe that the T_ω or T_p with p=1 obtains smaller fuzzy spread in the addition, subtraction, multiplication and division. The α-cut arithmetic has larger fuzzy spread than others except the addition and subtraction which is equal to T_p with q=1.

Table 3.1 The results of α-cut, T_ω, and T_p operations.

	α-cut arithmetic	T_p with q=1	T_ω or T_p with p=1
Addition	(7, 12, 20)	(7, 12, 20)	(9, 12, 17)
Subtraction	(-11, -4, 2)	(-11, -4, 2)	(-9, -4, -1)
Multiplication	(10, 32, 91)	(4, 32, 76)	(16, 32, 56)
Division	(0.154, 0.5, 1.4)	(0.058, 0.5, 1.175)	(0.25 0.5 0.875)



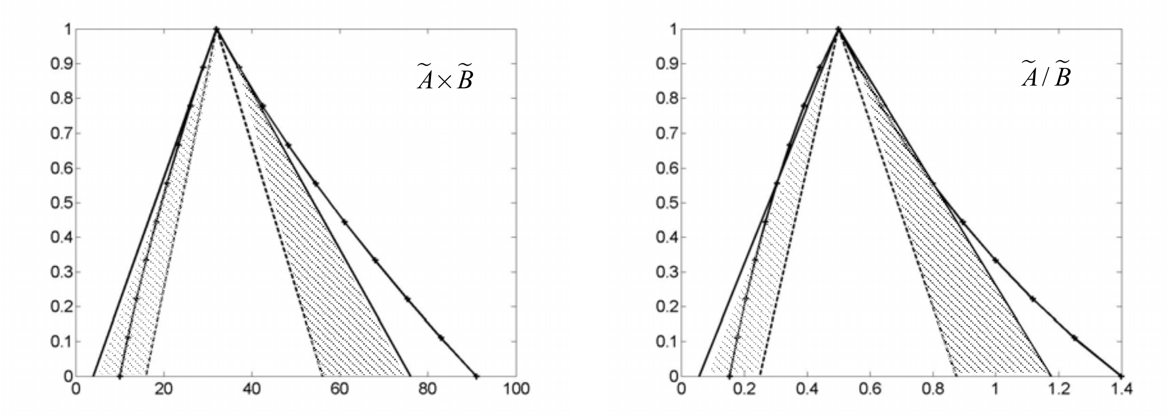


Figure 3.1 Fuzzy numbers operation.

3.3 The Fuzziness Accumulation Controlling Method

From principle of system dynamics we can observe that system dynamics mainly accumulates value with dynamic time by level variables. The level of a system determines its state at any point in time and the rates of the flows, accumulated by the levels, constitute the dynamics of the system. The accumulated phenomenon may become uncontrollable with dynamic time in fuzzy spread. However, previous research has not mentioned any related solution of problem. Hence, this research uses an *interval-end defuzzification* method to control fuzziness accumulation. The fuzziness except fuzzy inputs and fuzzy parameters of the system at the end of each interval can be defuzzified to obtain the representative value similar to the expected values or *interval-end defuzzification* is performed. The representative values of the variables may be supplied to the next interval with fuzzy inputs and parameters again (see Figure 3.2).

A *defuzzification* of fuzzy numbers can be defined as a mapping from a fuzzy number to a best representative crisp value. It is similar to the concept of mean values of random variables. The following three criteria should be considered in choosing a defuzzification scheme (Kaufman and Gupta, 1991):

(1) Plausibility—the crisp point should represent fuzzy number from an intuitive point of view; for example, it may lie approximately in the middle of the support of fuzzy number

or has a high degree of membership in a fuzzy number.

(2) Computational simplicity—this criterion is particularly important for fuzzy operator in real time.

(3)Continuity—a small change in a fuzzy number should not result in a large change in crisp point.

The defuzzification may be utilized with such techniques as, e.g., the *center of area* (COA) or $COA(x)$ or others (e.g., those and the ranking indices, e.g., see Leekwijck and Kerre (1999)). The COA can be defined as

$$COA(x) = \frac{\int_{\Re} \tilde{A}(x)xdx}{\int_{\Re} \tilde{A}(x)dx}. \tag{3.58}$$

The *center of area* defuzzifier is the most commonly used defuzzifier in fuzzy systems. It is computationally simple and intuitively plausible. Hence, this research adopts *center of area* defuzzifier to control fuzziness accumulation.

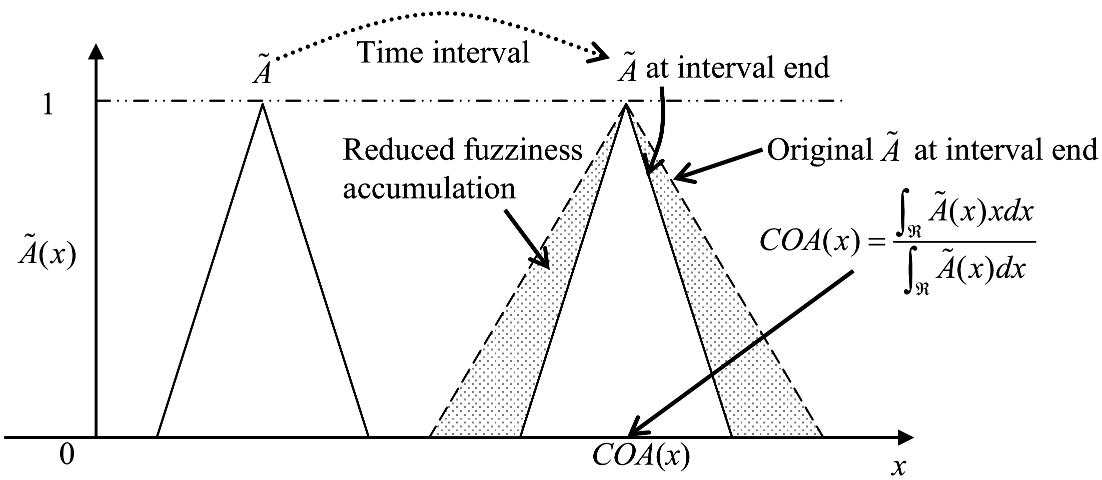


Figure 3.2 An illustrative example of the *interval-end defuzzification* method.

Moreover, the *interval-end defuzzification* method must assume the fuzziness in intervals independent of each other or *interval independence of fuzziness* (IIF). The purpose of this IIF is obvious that it avoids the fuzziness continually to accumulate in the model and that by time may become very uncontrollable. The rationale of this measure is obviously that because the operations of a system can be done and checked period by period, at the end of each interval, crisp values may be obtained. The fuzziness of a system in interactions of variables may be influenced by fuzzy inputs and/or fuzzy parameters in the interval instead of that being carried over by time and causing uncontrollable accumulations. At each interval, the results can still be examined with the fuzziness to consider the fuzziness involved in the interval of a system.

Chapter 4

SYSTEM DYNAMICS WITH FUZZY ARITHMETIC

This chapter discusses system dynamics with fuzzy arithmetic and shows an example. Section 4.1 depicts the idea of system dynamics with fuzzy arithmetic. Section 4.2 examines a simple example "epidemics mode".

4.1 The Concept of System Dynamics with Fuzzy Arithmetic

In system dynamics, the level and rate equations can be representing the node activities or events in a system represented in the causal loop diagram. The dynamics of the system is characterized by the level equations reflecting the change and interactions of the variables over time intervals and also affected by other variables/parameters of the system. Figure 4.1 depicts the fundamental relationship between a level and a rate equation. When the linear independent type relationship between the input and output rates of an interval is assumed, a typical level equation can be defined by

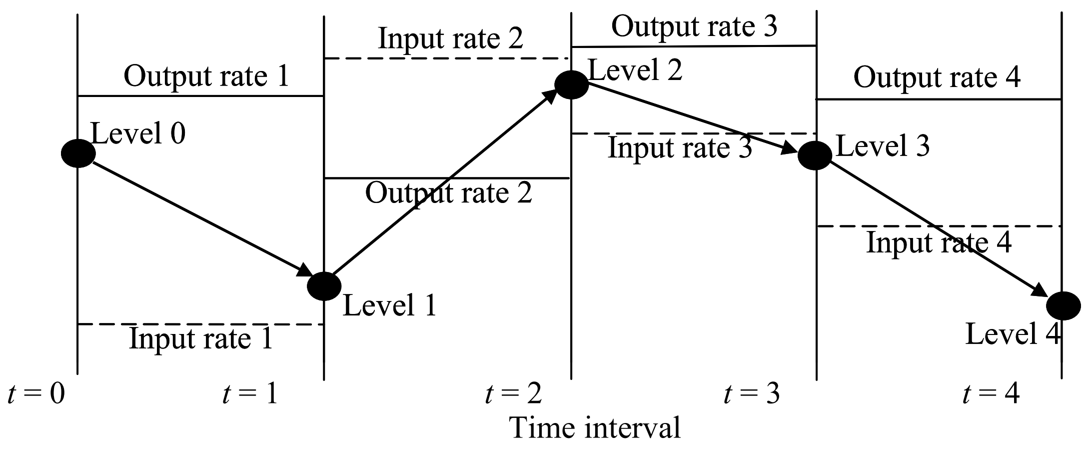


Figure 4.1 The fundamental relationship between level and rate variables.

$$L_{t+1}(x) = L_t(x) + \int_t^{t+1} R(x',t)dt = L_t(x) + DT \times \left(RI_{t+1}(x') - RO_{t+1}(x') \right), \quad (4.1)$$

where $L_{t+1}(x)$ denotes the level of variable x at time t+1, $L_t(x)$ at time t, $RI_{t+1}(x')$ and

$RO_{t+1}(x')$ are constant linear independent input and output rates of x', respectively, to and from x from time t to t+1, DT is the time interval, and $R(x',t)$ is the general rate function and can be given by

$$R(x',t) = f\left(L_t(x), C(x)\right), \tag{4.2}$$

which denotes the rate equation of x' and combines the level of the variable x at time t and a system constant $C(x)$. From (4.1) and (4.2) a level at time t is influenced by itself at the previous interval and rate variable(s) at the same interval. It can be complex since the relation may exist with more level and/or rate variables interacting frequently and affected by other parameters or variables even in the linear independent type system.

Thus, with the interactive level and rate variables of a system, the overall effects of inputs resulting in the system behavior or interaction of variables in a long run are the important characteristic of system dynamics. With the fuzzy variable and fuzzy arithmetic, a fuzzy system dynamics shown in as Figure 4.1 shows the fundamental illustration. The application of the fuzzy arithmetic may result in a system dynamics even more complex in the linear independent type of relationships, since the interaction of the variables inhere and fuzzy arithmetic has to be performed in the interactive variables. Yet, the fuzzy system dynamics can provide more information regarding the system behavior's uncertainties of a system because of the following reason. The overall effects on the model behavior can be that due to the variable interactions, all variables or behaviors of the system can be rendered fuzzy and affected by the fuzzy inputs and/or parameters of the system. These results may be provided to the decision-makers for realizing the system's fuzziness (uncertainty or risk). This approach is very different from that has been adopted the fuzzy logic. In the fuzzy system dynamics, the variables and/or parameters as in Eqs. (4.1) and (4.2) may be characterized by the fuzzy variables and/or parameters as $\widetilde{L_{t+1}}(x)$, $\widetilde{L_t}(x)$, $\widetilde{RI_t}(x')$, $\widetilde{RO_t}(x')$, $\widetilde{R}(x',t)$, $\widetilde{C}(x)$ and the corresponding fuzzy arithmetic could apply.

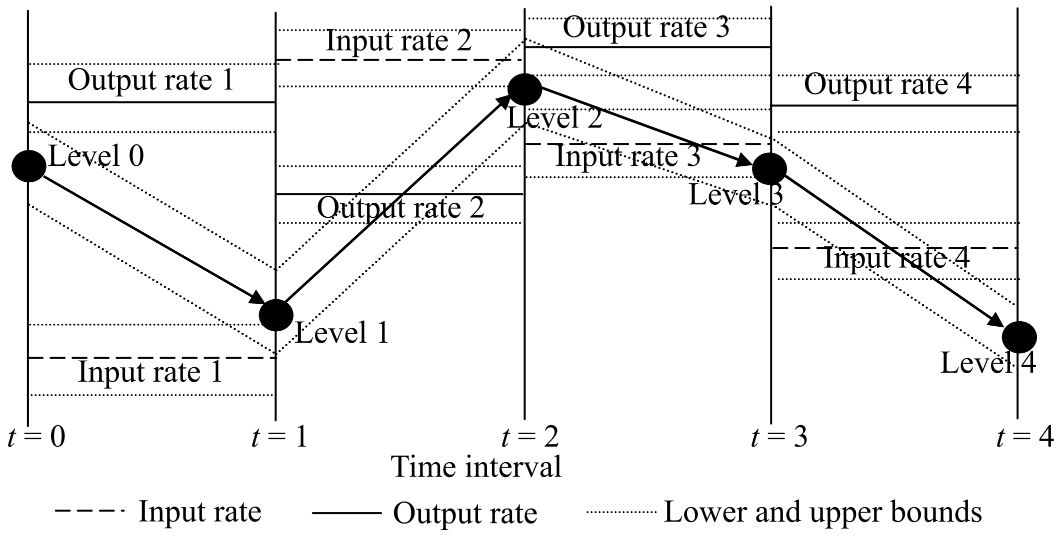


Figure 4.2 An illustrative example of the fuzzy rate and level relationship.

In the following sections, the epidemics people will be examined in detail for this purpose. The fuzzy system dynamics can be examined in the fuzzy-arithmetic system operational equations.

Moreover, for the definition of *nonnegativity* of a fuzzy value (number), the following definition $Noneg(\tilde{A})$ may be utilized.

$$Noneg(\tilde{A}) = \{x \in \tilde{A} \mid \forall x \geq 0 \ and \ \tilde{A}(x) > 0\}. \tag{4.3}$$

4.2 An Example: Epidemics Model

This research examines epidemics people model of system dynamics with fuzzy arithmetic. Epidemics people model is a typical second older model with *S-shaped* growth (The epidemics model is explained in more detail in (Duhon and Glick, 1994). The *S-shaped* growth has representative of the significance of the system dynamic in real world. The initial system state grows quickly. When system arrives one turning point, the system state will grow slowly. Finally, the system state arrives objective (steady state). Many cases can be described by the system state of *S-shaped* growth (for instance, gossip spreads, life cycle… etc.).

In Figure 4.3 an overview of the model structure is depicted. The source of sick people is catching illness from healthy people, and recovery people leave group of sick people to group of healthy people. The relation circulates between healthy and sick people continuously. However, the epidemics people model has some uncertain factors and vague information. Generally, the number of the epidemic people is not known clearly because some sick people may hide or leave while the epidemics were spread. Therefore, this research uses fuzzy numbers to substitute for crisp values of healthy and sick peoples, and the others variables use crisp values. We use the T_ω-based addition, subtract, multiplication and division for fuzzy numbers to epidemics people model. Due to this model, the following notations can be used.

Variables	
L_HP:	Healthy people (Men/Month), level variable
RRP:	Recovery People (Men/Month), rate variable
RCI:	Catching illness people (Men/Month), rate variable
L_SP:	Sick people (Men/Month), level variable
Auxiliary variables	
PS:	Probability of contact with sick people
Parameters	
DT:	Time interval or system delay
t:	*t*-th time interval, signifying its end or for the entire interval
PI:	Chance of population interactions (10 is used)
DI:	Duration of illness,(month) (0.5 is used)
PCI:	Probability of Catching Illness (0.5 is used)

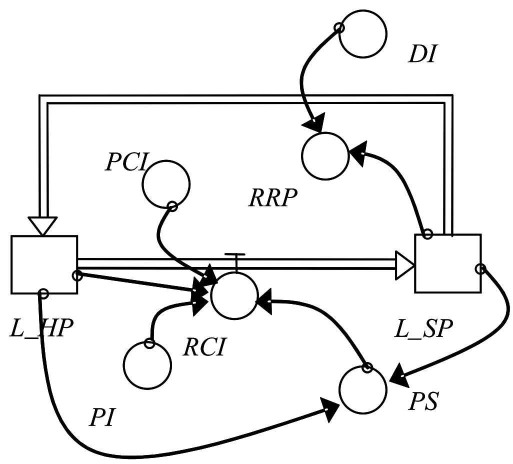


Figure 4.3 An overview of the epidemics people model.

The following steps depict the operational equations of the model.

Step 1. Define the initial inputs $\widetilde{L\_HP}_{t=0}$ and $\widetilde{L\_SP}_{t=0}$. A fuzzy variable will be also written as, e.g., $\widetilde{RRP}_{t=0}=(rrp_{1,t=0}, rrp_{2,t=0}, rrp_{3,t=0})$.

Step 2.Define $\widetilde{PS}_t$, $\widetilde{RCI}_t$ and $\widetilde{RRP}_t$ according to the basic operation,

$$\widetilde{PS}_t = \frac{\widetilde{L\_HP}_{t-1}}{\widetilde{L\_SP}_{t-1} + \widetilde{L\_HP}_{t-1}} \tag{4.4}$$

$$\widetilde{RCI}_t = \widetilde{L\_HP}_{t-1} \times \widetilde{PS}_t \times \widetilde{PI} \times \widetilde{PCI} \tag{4.5}$$

$$\widetilde{RRP}_t = \frac{\widetilde{L\_SP}_{t-1}}{\widetilde{DI}} \tag{4.6}$$

Step 3. Define $\widetilde{L\_SP}_t$ and its operational equation according to the basic operations as

$$\widetilde{L\_SP}_t = \widetilde{L\_SP}_{t-1} + \int_{t-1}^{t} (\widetilde{RCI}_t - \widetilde{RRP}_t)dt \tag{4.7}$$

Step 4. Define $\widetilde{L\_HP}_t$ and its operational equation according to the basic operations as

$$\widetilde{L\_HP}_t = \widetilde{L\_HP}_{t-1} + \int_{t-1}^{t} (\widetilde{RRP}_t - \widetilde{RCI}_t)dt \tag{4.8}$$

This research has be coded the above model in MATLAB 6.5. With the model in the fuzzy arithmetic operational equations, Table 4.1 shows the input (including the parameters) and initial values for the two cases, crisp and symmetrical triangular fuzzy numbers. Although for the fuzzy case most of the data are crisp, as *L_HP* and *L_SP* are fuzzy and affect the entire model, we shall see that the entire model or all variables will be rendered fuzzy as well. In the following, we discuss the results obtained from the crisp, Yager's *t*-norm, and T_ω weakest *t*-norm fuzzy arithmetic.

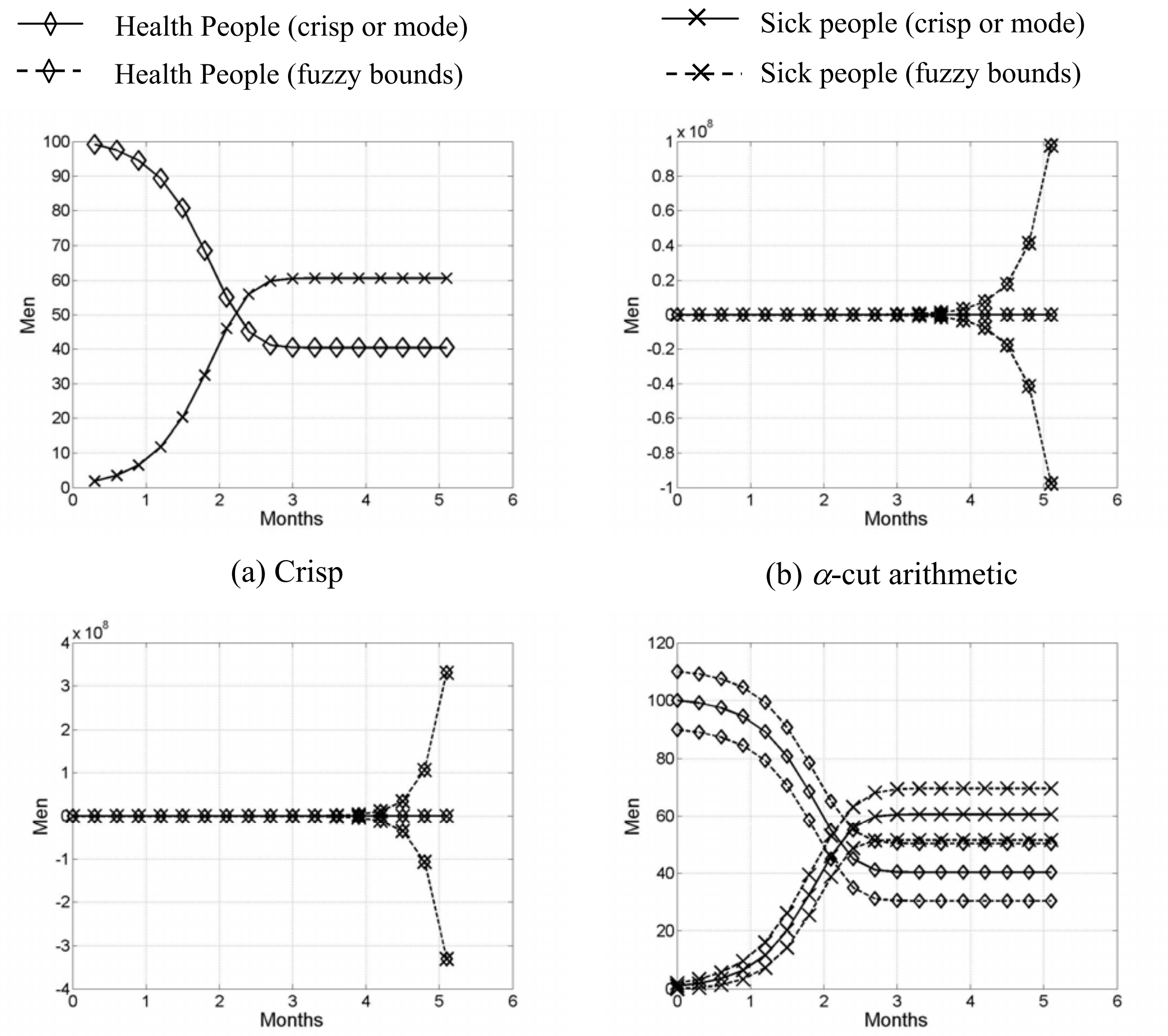
Table 4.1 Fuzzy input data in epidemics people model.

Variables(t=0)	*L_HP*	*L_SP*	*PI**	*DI**	*PCI**
Crisp	100	1	10	0.5	0.5
Fuzzy numbers	(90, 100, 110)	(0, 1, 2)	(10, 10, 10)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)

* represents that the variable is constant value

The interval DT in the system dynamics represents the system time delay in the operational equations and also controls the frequency of the system variable updates. From Figure 4.4 to 4.6 and Table 4.2, we can obtains various results with DT increasing in crisp arithmetic, α-cut arithmetic, T_p with q=1 and T_ω (T_p with p=1). In Figure 4.4 (a) the *L_HP* and *L_SP* respectively stabilize from at 40 and 60 after three months with DT=0.3. This indicates that the number of sick people gradually increases to 60 from the first to the third months, and the number of healthy people gradually decreases to 40 from the first to the third months. In Figure 4.4 (b), the bounds of *L_HP* and *L_SP* can not stabilize due to the accumulating phenomenon of fuzziness of the α-cut arithmetic. In Figure 4.4 (c), the result of T_p with q=1 is similar α-cut arithmetic. The fuzzy arithmetic also has accumulating phenomenon of fuzziness by time. In Figure 4.4 (d) the left and right bounds of *L_HP* respectively stabilize at 30 and 50 after three months with DT=0.3. This indicates that the number of healthy people ranges between 30 and 50 when the model arrives a stable state. On the other hand, the left and right bounds of *L_SP* respectively stabilize at 50 and 70 after three months with DT=0.3. We can observe that the fuzzy intervals of *L_SP* are the same with *L_HP* after the model stabilizes, and the fuzzy intervals of *L_SP* are smaller than *L_HP* three months ago. This means that the fuzzy intervals of *L_SP* would growth gradually. The fuzzy result can effectively observe fuzzy growth trend of *L_SP*.

In Figure 4.5 (a) the *L_HP* and *L_SP* respectively stabilize at 40 and 60 after four months with DT=0.5. However, the two curve are not monotonic functions with DT=0.5 because the numbers of observation decrease with DT=0.5. Hence, the model may increase time for arriving stable states. In Figure 4.5 (b)-(c) we can observe that the fuzzy accumulation can not be controlled. Moreover we can find that the α-cut arithmetic has larger fuzzy accumulations and T_ω (T_p with p=1) has smaller accumulations.



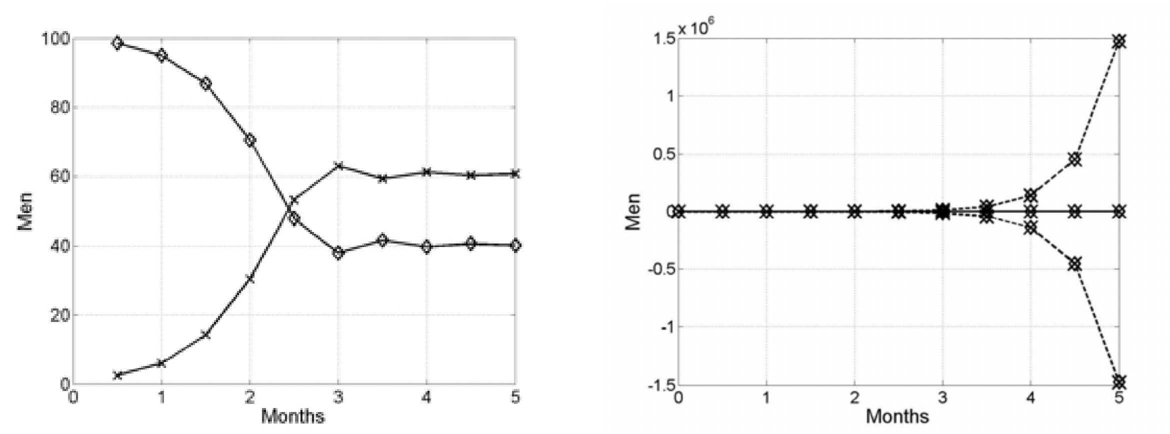
(a) Crisp

(b) α-cut arithmetic

(c) T_p with q=1

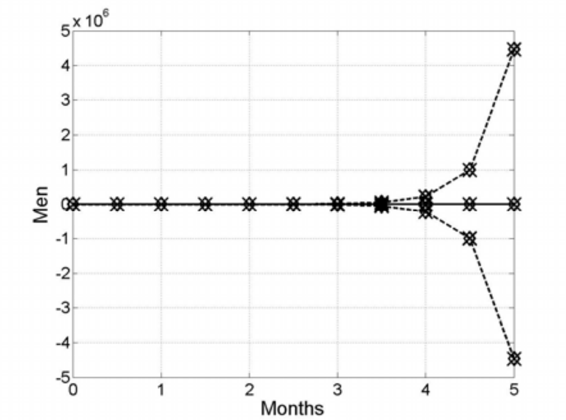
(d) T_ω (T_p with p=1)

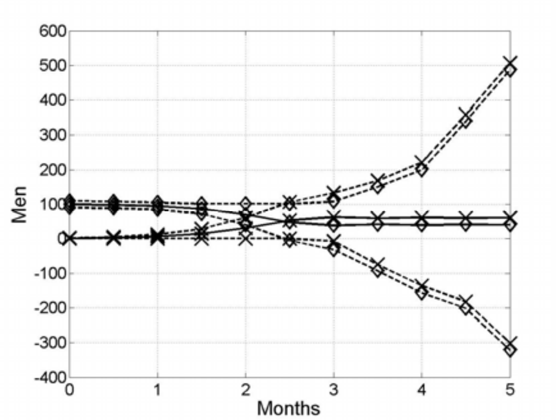
Figure 4.4 The simulate results of various arithmetic with DT=0.3.



(a) Crisp

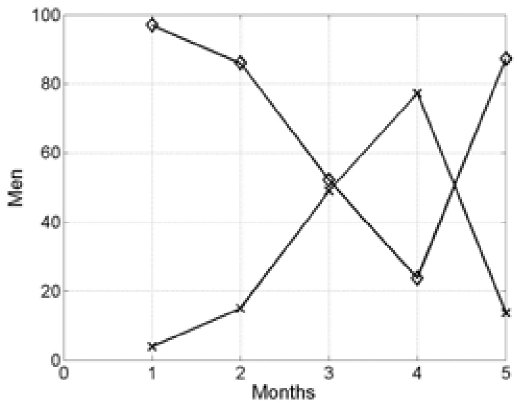
(b) α-cut arithmetic



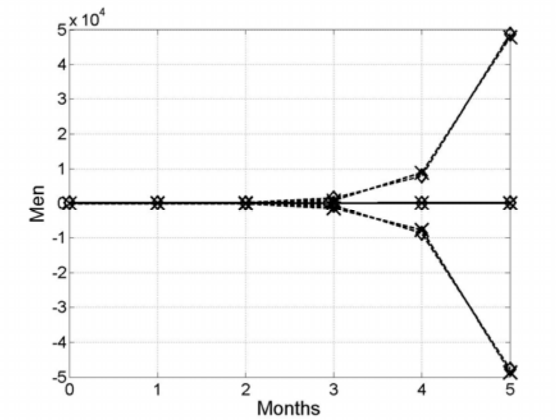


(c) T_p with q=1 (d) T_ω (T_p with p=1)

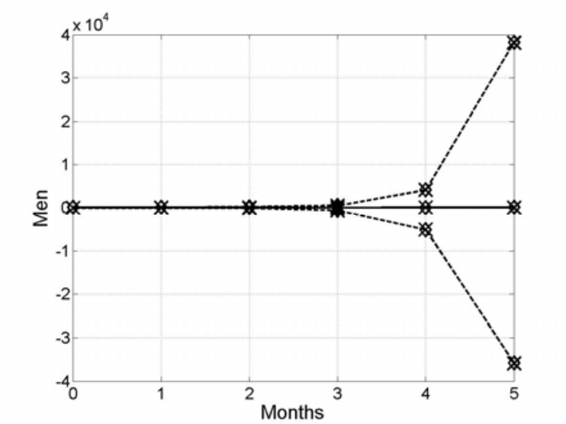
Figure4.5 The simulate results of various arithmetic with *DT*=0.5.

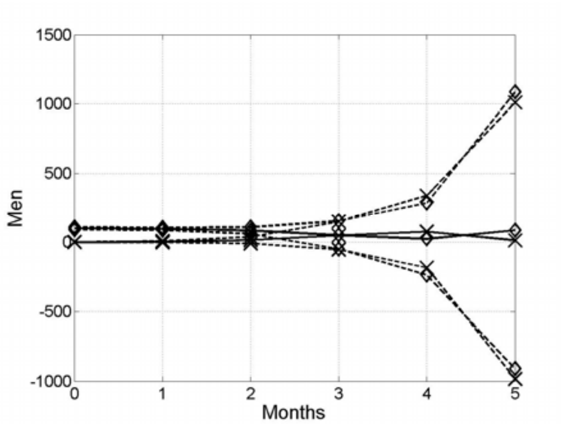


(a) Crisp



(b) α-cut arithmetic





(c) T_p with q=1 (d) T_ω (T_p with p=1)

Figure4.6 The simulate results of various arithmetic with *DT*=1.0.

In Figure 4.6 (a) since the *DT* is too large, the model can not arrive at stable states. It means that the trend of the model can not be easily controlled when the numbers of observation are few. In Figure 4.6 (b)-(c), fuzzy arithmetic also can not arrive at the stable states, and the fuzzy accumulations can not be controlled.

Although with DT=0.5, 1.0 the fuzzy arithmetic is not suitable for epidemics people model, with DT=0.3 the fuzzy arithmetic can be evidenced to result in the correct and suitable result in Figure 4.4(d). The fuzzy arithmetic can get more information than crisp arithmetic, and these observational results will help decision-makers to make correct policy for dealing or controlling epidemic.

Table 4.2 Results and comparison of the crisp, α-cut, T_ω arithmetic and Yager's t-norm with the symmetrical TFN input for the model.

			t=1	t=2	t=3	t=4	t=5
L_HP	Crisp	DT=0.3	89.24	55	40.49	40.4	40.4
		DT=0.5	94.96	70.51	38.06	39.79	40.25
		DT=1.0	97.05	85.97	52.07	23.806	87.22
	α-cut arithmetic	DT=0.3	(-729.4, 89.24, 1571)	(-18814.8, 55, 19028.3)	(-246413.3, 40.49, 246614.8)	(-3200056.6, 40.4, 3200258.7)	(-41532075.1, 40.4, 41532277.1)
		DT=0.5	(54.2, 94.96, 127)	(-1041, 70.51, 1289.2)	(-12988.73, 38.06, 13277.24)	(-139422.17, 39.79, 139739.71)	(-1475135.5, 40.25, 1475432)
		DT=1.0	(77.8, 97.05, 114)	(-40.1, 85.97, 173.4)	(-697.55, 52.06, 1544.35)	(-8761.46, 23.81, 7869.01)	(-47756.2, 87.22, 48769)
	T_p with q=1	DT=0.3	(-38.2, 89.24, 253.8)	(-4004.9, 55, 4070.9)	(-120472.40, 40.49, 119764.46)	(-3581201.8, 40.4, 3575397.9)	(-106621360.2, 40.4, 106580604.2)
		DT=0.5	(62.1, 94.96, 128.5)	(-494.6, 70.51, 596.9)	(-11176.2, 38.06, 10617.2)	(-220036.29, 39.79, 221083.7)	(-4466716.6, 40.25, 4465767.1)
		DT=1.0	(79.12, 97.05, 115.1)	(-0.5, 85.97, 175.8)	(-630.2, 52.06, 540.9)	(-5120.63, 23.81, 4143.95)	(-35826.3, 87.22, 38231.1)
	T_ω or T_p with p=1	DT=0.3	(79.24, 89.24, 99.24)	(45, 55, 65)	(30.49, 40.49, 50.49)	(30.4, 40.4, 50.4)	(30.4, 40.4, 50.4)
		DT=0.5	(85, 94.96, 105)	(40, 70.51, 101)	(-32, 38.06, 108.2)	(-157.55, 39.79, 200.06)	(-322.1, 40.25, 486.4)
		DT=1.0	(87.1, 97.05, 107.1)	(62.2, 85.97, 109.8)	(-49.2, 52.07, 153.3)	(-237.1, 23.806, 284.71)	(-909.8, 87.22, 1084.3)
L_SP	Crisp	DT=0.3	11.76	46	60.51	60.6	60.6
		DT=0.5	6.04	30.49	62.94	61.21	60.75
		DT=1.0	3.95	15.03	48.94	77.19	13.78
	α-cut arithmetic	DT=0.3	(-1461, 11.76, 821.4)	(-18918, 46, 18907)	(-246504.77, 60.51, 246505.27)	(-3200148.7, 60.6, 3200148.6)	(-41532167.1, 60.6, 41532167.1)
		DT=0.5	(-16.95, 6.04, 37.85)	(-1179.2, 30.49, 1133)	(-13167, 62.943, 13081)	(-139629.7, 61.21, 139514.2)	(-1475322, 60.75, 1475227.5)
		DT=1.0	(-4, 3.95, 14.22)	(-63.35, 15.03, 132.1)	(-1434.35, 48.94, 789.55)	(-7759, 77.19, 8853.5)	(-48659, 13.78, 47848.2)
	T_p with q=1	DT=0.3	(-106.7, 11.8, 167.3)	(-4004.9, 46, 4052.8)	(-120443.4, 60.51, 119775.5)	(-3581172.6, 60.6, 3575409.1)	(-106621331, 60.6, 106580615.4)
		DT=0.5	(-17.88, 6.04 , 30.5)	(-525.6, 30.49, 547.9)	(-11142.3, 62.94, 10633.1)	(-220005.9, 61.21, 221096.2)	(-4466687, 60.75, 4465778.6)
		DT=1.0	(-4.98, 3.95, 13.00)	(-62.4, 15.03, 95.8)	(-624.4, 48.94, 528.8)	(-5058.24, 77.19, 4188.34)	(-35890.7, 13.78, 38148.6)
	T_ω or T_p with p=1	DT=0.3	(7.32, 11.76, 16.2)	(38.8, 46, 53.2)	(51.6, 60.5, 69.4)	(51.6, 60.6, 69.6)	(51.6, 60.6, 69.6)
		DT=0.5	(0, 6.04, 12.07)	(0, 30.49, 60.98)	(-7.2, 62.9, 133)	(-136.12, 61.21, 221.48)	(-301.6, 60.7, 506.9)
		DT=1.0	(-1, 3.95, 8.9)	(-8.76, 15.03, 38.81)	(-52.3, 48.9, 150.2)	(-183.71, 77.19, 338.1)	(-983.3, 13.8, 1010.8)

Chapter 5

THE CUSTOMER-PRODUCER-EMPLOYMENT MODEL

The customer, producer, employment model was originally considered by Forrester (1961) and later also by other researchers in the crisp manner. This model is especially important as it observes the employment-level variation of a company resulting from the interactions of purchase orders of customers, inventory level, production, and employment practice of this company. The model utilizes the usual practice that the customer orders are first filled by the inventory, followed by that manufacturing may be ordered to make insufficient orders' quantities, and inventory thus must be maintained above a minimum level. Due to the practice, the company must manufacture for both insufficient customer orders' quantity and inventory. Therefore, the men power for both manufacturing must be estimated. This model provides the variations predicted and to be observed in the inventory, production, and employment level of this company. In industries, production quantities and required men power are the important issues. Due to this model, the following notations may be used.

Variables

L_OrtF:	Customer orders' quantity to fill (units), level variable
R_OrR:	Order quantity receiving rate (units/week), rate variable (system input)
R_OrFI:	Customer orders' quantity filled by inventory (units/week), rate variable
R_OrRM:	Rate of orders' quantity requisitioning manufacturing (units/week), rate variable
L_OrB:	Customer order backlog (units), level variable
R_OrM:	Orders' manufactured quantity (units/week), rate variable
L_IRB:	Inventory replenishment backlog (units), level variable
R_RIRB:	Reduction of inventory replenishment backlog (units/week), rate variable
L_AI:	Expected inventory level (units), level variable
R_MI:	Manufacturing rate for inventory (units/week), rate variable
R_SI:	Shipment from inventory (units/week), rate variable

Auxiliary variables

DWOrRM:	Desired labor (men) for the *R_OrRM*
TMP:	Total man power (men)
WOrM:	Labor for orders' manufactured quantity (men), i.e., min{*DWOrRM*, *TMP*} can be assigned
WI:	Labor for inventory replenishment manufacturing (men)

WC: Labor change rate (men)

Parameters

DT: Time interval or system delay (initially a week is used)

t: *t*-th time interval, signifying its end or the entire interval

WPC: Labor productivity (units/man-week)

MIL: Minimum inventory level

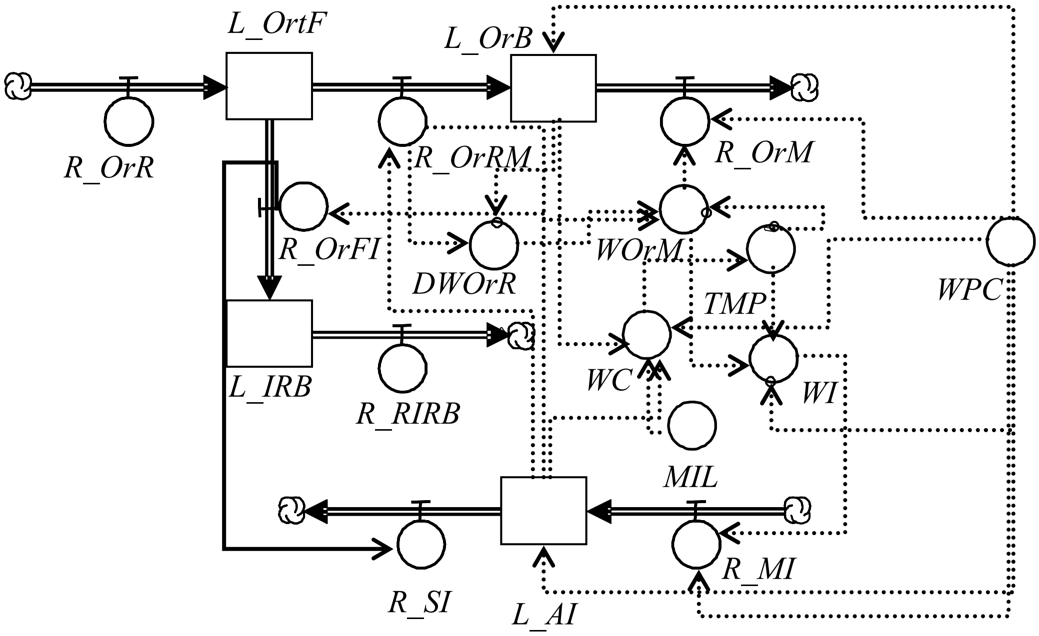


Figure 5.1 An overview of system structure.

In Figure 5.1, an overview of the system's structure is depicted. In the following, we will discuss the basic operations (in Figure 5.1) and followed by that of the fuzzy arithmetic denoted equations for the model. Since the linear independent relationship is assumed, the rate variables are assumed constant but fuzzy during an interval of time.

5.1 Basic Operations

In this model, the situation is triggered by the customer order-receiving rate R_OrR_t in each period. In Figure 5.1 (the overview of the model structure), as mentioned earlier the system fills the customer orders first by the inventory. If there is an insufficient inventory to completely fill the customer orders, then requests for manufacturing the insufficient orders' quantities R_OrRM_t happen. Therefore, R_OrRM_t occurs in this model (see Figure 5.1) due to the customer orders' quantity to fill, L_OrtF_{t-1}, from the last period plus the order-receiving rate, R_OrR_t, at the current period and the inventory replenishment backlog, L_IRB_{t-1}, from the last period (i.e., the total quantity to fill) and minus the inventory L_AI_{t-1} from the last period. If this calculated R_OrRM_t is negative, it signifies that the inventory is sufficient and R_OrRM_t is reset to zero. Conversely, if it is positive, it then goes to or increase the customer order backlog L_OrB_t. If L_OrB_t cannot be accomplished at the period, the system will increase the men power in the next period.

Therefore, since the customer orders are first filled by the inventory, if the calculated R_OrRM_t is positive, it means that the inventory is insufficient for the orders. In this case, the customer orders' quantity filled by the inventory, R_OrFI_t, can be estimated as follows. It is the quantity that needs to be filled by inventory and manufacturing together (i.e., the order-receiving rate R_OrR_t at the current period plus the orders' quantity to fill L_OrtF_{t-1} from the last period) minus the rate of orders' quantity requisitioning manufacturing, R_OrRM_t, at the current period (Figure 5.1). On the other hand, if the calculated R_OrRM_t before taking the nonnegativity is negative, it means that the inventory is sufficient at the current period. Therefore, in this case, the R_OrFI_t is simply the order-receiving rate R_OrR_t plus the orders' quantity to fill L_OrtF_{t-1} and the inventory replenishment backlog L_IRB_{t-1} from the last period. Thus, also *R_OrFI* is the quantity that will be shipped form the inventory and determines the shipment from the inventory *R_SI*.

The variable, customer orders' quantity to fill *L_OrtF*, is determined by the rate variables, *R_OrR*, *R_OrFI*, and *R_OrRM*, as in Figure4. The level, customer order backlog *L_OrB*, is determined by the rate variables, orders' manufactured quantity *R_OrM* and rate of orders' quantity requisitioning manufacturing *R_OrRM*, when an inventory is insufficient. Moreover, the rate variable, orders' manufactured quantity *R_OrM*, is estimated from the labor for orders' manufactured quantity *WOrM* and *WOrM* equals min{*DWOrRM*, *TMP*}, which can be assigned. And *DWOrRM*, i.e., the desired labor for *R_OrRM*, occurs due to the current *R_OrRM* plus the *L_OrB* from the last period. The level, inventory replenishment backlog *L_IRB*, can be estimated from the rates, *R_OrFI* and reduction of inventory replenishment backlog *R_RIRB*. *L_IRB* should be kept as close to zero as possible, which means that the inventory is maintained sufficiently. The *R_RIRB* is determined by the customer orders' quantity filled by inventory R_OrFI_t at the current period plus the inventory replenishment backlog L_IRB_{t-1} and L_OrB_{t-1} from the last period (see Figure 5.1). This also means that the customer order backlog L_OrB_{t-1} from the last period will also be accomplished by the inventory manufactured at the current period first.

Moreover, the rate variable *R_MI* is estimated from the *WI* and *WPC*, and the *WI* is estimated from the *TMP* minus *WOrM*, which signifies that the manufacturing for fulfilling the *R_OrRM* has been given a higher priority over the manufacturing for replenishing the inventories.

The *L_AI* is determined from its previous level and the current *R_MI* and *R_SI*. In addition, the customer order backlog L_OrB_t plus the inventory replenishment backlog *L_IRB* (i.e., the total backlog) stands for the quantity needed to be manufactured above the production capacity with the current *TMP* and is influenced by the *TMP* and *WPC*. Additional hiring (i.e., a positive *WC*) may occur. In this case, in addition to this *WC*, unfilled desired labor for the current *R_OrRM* should be added too (i.e., *DWOrRM* − *TMP*). Conversely, if the *L_AI* is more than the minimum inventory level *MIL* required, layoff may occur (or a negative *WC*). However, in order to avoid the total men power from becoming too small in a period, the *TMP* may be constrained. This can be done, e.g., by constraining the *TMP* to be above some lower limit or the initial *TMP*.

For these variables, except the *L_OrtF* and *WC*, the nonnegativity requirements naturally hold. For the *WC* the negative meaning is obvious. For *L_OrtF*, the negative means that the excessive capacities including inventory and manufacturing occur. It also means that the order fulfillment is being completely met by inventory, and the manufacturing is performed for the inventory but not for the orders. The purpose of the negative *L_OrtF* is avoiding assigning any labor for the *R_OrRM* when the inventory is enough and balances the model in controlling the total men power by avoiding unnecessary increases.

In the next section, the systematical consecutive steps of the system can be formulated for the process of the model and fuzzy-arithmetic denoted equations can be introduced.

5.2 Fuzzification and Defuzzification of the System Dynamics Model

In the following, we will use the fuzzy variables to estimate the rate and level variables. In this case, it is complex due to that the fuzzy interactive equations have to be carried out. In the following, especially the level variable $\widetilde{L\_OrB}$ will be shown in more details by the equation expanded from the α-cut arithmetic and the T_ω weakest *t*-norm operator. The detailed fuzzy arithmetic of the other variables can be performed in a similar manner. The following depicts the steps and the operational equations of the above model described in Section 4.1.

Step 1. Define the initial $\widetilde{L\_OrB}_{t=0}$, $\widetilde{L\_IRB}_{t=0}$, and $\widetilde{L\_AI}_{t=0}$. A fuzzy variable will also be written as, e.g., $\widetilde{R\_OrR}_t = (r\_orr_{1,t}, r\_orr_{2,t}, r\_orr_{3,t})$.

Step 2. Define $\widetilde{L\_OrtF}_{t=0}$ and its operational equation according to the above basic operations as

$$\widetilde{L\_OrtF}_t = \widetilde{L\_OrtF}_{t-1} + \int_{t-1}^{t} (\widetilde{R\_OrR}_t - (\widetilde{R\_OrFI}_t + \widetilde{R\_OrRM}_t))dt, \tag{5.1}$$

where $\widetilde{R\_OrRM}_t$ and $\widetilde{R\_OrFI}_t$ can be estimated as follows.

(1) For $\widetilde{R\_OrRM}_t$ according to the basic operations,

$$\widetilde{R\_OrRM}_t = Noneg(\widetilde{R\_OrR}_t + \frac{\widetilde{L\_OrtF}_{t\text{-}1} + \widetilde{L\_IRB}_{t\text{-}1} - \widetilde{L\_AI}_{t\text{-}1}}{DT}), \quad (5.2)$$

where *Neneg*() signifies taking the nonnegative elements of the results in order to satisfy the nonnegative requirement.

(2) For the $\widetilde{R\_OrFI}_t$, if $Mod(\widetilde{R\_OrR}_t \times DT + \widetilde{L\_OrtF}_{t\text{-}1} + \widetilde{L\_IRB}_{t\text{-}1} - \widetilde{L\_AI}_{t\text{-}1}) \leq 0$, then

$$\widetilde{R\_OrFI}_t = Noneg(\widetilde{R\_OrR}_t + \frac{\widetilde{L\_OrtF}_{t\text{-}1} + \widetilde{L\_IRB}_{t-1}}{DT}); \quad (5.3)$$

otherwise,

$$\widetilde{R\_OrFI}_t = Noneg(\widetilde{R\_OrR}_t + \frac{\widetilde{L\_OrtF}_{t\text{-}1}}{DT} - \widetilde{R\_OrRM}_t), \quad (5.4)$$

where *Mod*() denotes the operation of taking only the mode of the result and thus the highest possibility of the premise for the Eqs. (5.3) and (5.4). It has been observed that the mode, i.e., the most possible element, is more appropriate than the defuzzification in this step in this model for the following reasons:

(a) In this step, the purpose is to judge whether inventories at the current period are sufficient or not and it results in two different operations to be selected. If a defuzzification method is used in the premise, misjudgments may result. In this case, either Eq. (5.3) or (5.4) may be taken alone and continually due to the fuzziness before defuzzification. $\widetilde{R\_OrFI}_t$ may be found continually to increase or decrease and cause the system instability even in the mode of the variable. If however the mode (most possible element) comparison is used, a decisive discretion decision can be made. The incorrectness as above can be prevented and Eqs. (5.3) and (5.4) may alternate. Even the system fuzziness can reach the steady state naturally.

(b) In addition, the $\widetilde{R\_OrFI_t}$ is to be obtained as the right-hand-side of Eq. (5.3) or (5.4) but not the premise.

Step 3. Determine the requisition rate manufactured.

(1) Define the $\widetilde{DWOrRM_t}$ according to the basic operations.

$$\widetilde{DWOrRM_t} = Noneg(\frac{\widetilde{R\_OrRM_t} \times DT + \widetilde{L\_OrB_{t-1}}}{\widetilde{WPC_t} \times DT}). \tag{5.5}$$

(2) Define $\widetilde{WOrM_t}$ by the basic operations. As mentioned earlier, since *WOrM* = min{*DWOrRM*, *TMP*},

$$\widetilde{WOrM_t} = \min(\widetilde{DWOrRM_t}, \widetilde{TMP_{t-1}}) = \begin{cases} \widetilde{DWOrRM_t}, & if\ COA(\widetilde{DWOrRM_t}) \le COA(\widetilde{TMP_{t-1}}), \\ \widetilde{TMP_{t-1}}, & otherwise. \end{cases} \tag{5.6}$$

Since $\widetilde{WOrM_t}$ is equal to either $\widetilde{DWOrRM_t}$ or $\widetilde{TMP_t}$, the defuzzification method is used here to rank the $\widetilde{DWOrRM_t}$ and $\widetilde{TMP_{t-1}}$. This operation is different from that for the premise of Eqs. (5.3) and (5.4).

(3) Finally, calculate the $\widetilde{R\_OrM_t}$ by the equation

$$\widetilde{R\_OrM_t} = \widetilde{WPC_t} \times \widetilde{WOrM_t}. \tag{5.7}$$

Step 4. Determine the order backlog. Get $\widetilde{R\_OrRM_t}$ (Eq. (5.2)). Calculate the $\widetilde{L\_OrB_t}$ by the equation according to the basic operations,

$$\widetilde{L\_OrB_t} = Noneg(\widetilde{L\_OrB_{t-1}} + \int_{t-1}^{t} (\widetilde{R\_OrRM_t} - \widetilde{R\_OrM_t})dt) \tag{5.8}$$

The following illustrates $\widetilde{L\_OrB_t}$ by using the α-cut fuzzy arithmetic for the demonstration. Using the α-cuts, $\widetilde{L\_OrB}$ can be written as before taking the nonnegative requirement,

$$L\_OrB'_{\alpha,t} = \left[l\_orb'^{(\alpha)}_{1,t}, l\_orb'^{(\alpha)}_{2,t} \right], \forall \alpha \in (0,1], \tag{5.9}$$

where $l\_orb_{1,t}'^{(\alpha)}$ and $l\_orb_{2,t}'^{(\alpha)}$ can be determined by the α-cut arithmetic as

$$l\_orb_{1,t}'^{(\alpha)} = l\_orb_{1,t-1}^{(\alpha)} + (r\_orrm_{1,t}^{(\alpha)} - r\_orm_{2,t}^{(\alpha)}) \times DT, \tag{5.10a}$$

$$l\_orb_{2,t}'^{(\alpha)} = l\_orb_{2,t-1}^{(\alpha)} + (r\_orrm_{2,t}^{(\alpha)} - r\_orm_{1,t}^{(\alpha)}) \times DT. \tag{5.10b}$$

On the other hand, if the T_{ω} arithmetic is used,

$$\begin{aligned}\widetilde{L\_OrB}'_t = \Big(& l\_orb_{2,t-1} + (r\_orrm_{2,t} - r\_orm_{2,t}) \times DT - \max\{l\_orb_{2,t-1} - l\_orb_{1,t-1}, \\ & (r\_orrm_{2,t} - r\_orrm_{1,t}) \times DT, (r\_orm_{3,t} - r\_orm_{2,t}) \times DT\}, \\ & l\_orb_{2,t-1} + (r\_orrm_{2,t} - r\_orm_{2,t}) \times DT, \\ & l\_orb_{2,t-1} + (r\_orrm_{2,t} - r\_orm_{2,t}) \times DT + \max\{l\_orb_{3,t-1} - l\_orb_{2,t-1}, \\ & (r\_orrm_{3,t} - r\_orrm_{2,t}) \times DT, (r\_orm_{2,t} - r\_orm_{1,t}) \times DT\}\Big).\end{aligned} \tag{5.11}$$

Step 5. Determine the inventory replenishment backlog.

(1) Determine the $\widetilde{R\_RIRB}_t$ as $(\widetilde{L\_OrB}_{t-1} + \widetilde{L\_IRB}_{t-1}) / DT + \widetilde{R\_OrFI}_t$.

(2) Define the $\widetilde{L\_IRB}_t$ by the equation

$$\widetilde{L\_IRB}_t = Noneg(\widetilde{L\_IRB}_{t-1} + \int_{t-1}^{t} (\widetilde{R\_OrFI}_t - \widetilde{R\_RIRB}_t) dt). \tag{5.12}$$

Step 6. Determine the inventory level.

(1) Determine the $\widetilde{WI}_t$ as

$$\widetilde{WI}_t = \widetilde{TMP}_{t-1} - \widetilde{WOrM}_t. \tag{5.13}$$

(2) Determine the $\widetilde{R\_MI}_t$ as

$$\widetilde{R\_MI}_t = \widetilde{WI}_t \times \widetilde{WPC}_t. \tag{5.14}$$

(3) $\widetilde{R\_SI}_t = \widetilde{R\_OrFI}_t$.

(4) Define $\widetilde{L\_AI}_t$ by the following equation.

$$\widetilde{L\_AI}_t = Noneg(\widetilde{L\_AI}_{t-1} + \int_{t-1}^{t} (\widetilde{R\_MI}_t - \widetilde{R\_SI}_t) dt). \tag{5.15}$$

Step 7. Determine the labor change rate. If $\widetilde{L\_AI}_t$ is more than $\widetilde{MIL}$ (that is, a similar operation to that of the premise for Eqs. (5.3) and (5.4) applies here too), $\widetilde{WC}$ can be estimated as

$$\widetilde{WC}_t = \frac{\widetilde{MIL} - \widetilde{L\_AI}_t}{\widetilde{WPC}_t \times DT}, \tag{5.16}$$

which signifies a decrease in the men power of the factory in the next period; otherwise, again by the basic operations,

$$\widetilde{WC}_t = \frac{\widetilde{L\_OrB}_t + \widetilde{L\_IRB}_t}{\widetilde{WPC}_t \times DT}, \tag{5.17}$$

which signifies an increase in the men power of the factory in the next period.

Step 8. The total manpower $\widetilde{TMP}_t$, customer order backlog $\widetilde{L\_OrB}_{t+1}$, and un-replenished inventory will be adjusted in the next interval with the labor change rate $\widetilde{WC}_t$. If $\widetilde{L\_AI}_t$ is more than $\widetilde{MIL}$,

$$\widetilde{TMP}_t = \widetilde{TMP}_{t-1} + \widetilde{WC}_t. \tag{5.18}$$

Otherwise,

$$\widetilde{TMP}_t = \widetilde{TMP}_{t-1} + \widetilde{WC}_t + (\widetilde{DWOrRM}_t - \widetilde{TMP}_{t-1}) = \widetilde{WC}_t + \widetilde{DWOrRM}_t. \tag{5.19}$$

Moreover, in order to avoid the $\widetilde{TMP}_t$ being planned too low in an interval $\widetilde{TMP}_t$ may be constrained. In this case, the initial $\widetilde{TMP}_{t=0}$ will be used as the lower limit. With the α-cut arithmetic, the $TMP_{\alpha,t} = \left[tmp_{1,t}^{(\alpha)}, tmp_{2,t}^{(\alpha)}\right]$ $\forall \alpha \in (0, 1]$ can be constrained by

$$tmp_{j,t}^{(\alpha)} = \begin{cases} tmp_{j,t}^{(\alpha)}, & if\ tmp_{j,t}^{(\alpha)} \geq tmp_{j,t=0}^{(\alpha)}, \\ tmp_{j,t=0}^{(\alpha)}, & otherwise, \end{cases} \quad \text{for } j = 1, 2. \tag{5.20}$$

With the T_ω fuzzy arithmetic, the $\widetilde{TMP}_t = (tmp_{1,t}, tmp_{2,t}, tmp_{3,t})$ may be constrained by

$$tmp_{k,t} = \begin{cases} tmp_{k,t}, & if\ tmp_{k,t} \geq tmp_{k,t=0}, \\ tmp_{k,t=0}, & otherwise, \end{cases} \quad \text{for } k = 1, 2, 3. \tag{5.21}$$

In addition, the following measure or assumption can be used in the model. That is, the fuzziness in intervals independent of each other or *interval independence of fuzziness* (IIF) may be used. In this IIF, the fuzziness except fuzzy inputs and fuzzy parameters of the system at the end of each interval can be defuzzified to obtain the representative value similar to the expected values or *interval-end defuzzification* is performed. The representative values of the variables may be supplied to the next interval with fuzzy inputs ($\widetilde{R\_OrR}_t$ in this model) and parameters again.

In the next section, we shall show numerically the effects of the interactive (fuzzy) variables of the above model resulting from the system's fuzzy arithmetic operational equations.

5.3 Numerical Analysis

This research has coded the above model in MATLAB 6.5 and included the defuzzification of fuzzy numbers. With the model in the fuzzy arithmetic operational equations, Table 5.1 shows the input (including the parameters) and initial values for the two cases, crisp and symmetrical TFNs. Although for the fuzzy case most of the data are crisp, as *WPC* and *R_OrR* are fuzzy and affect the entire model, we shall see that the entire model or all variables will be rendered fuzzy as well. In the following, we shall discuss the results obtained from the crisp, the α-cut and T_ω weakest *t*-norm fuzzy arithmetic.

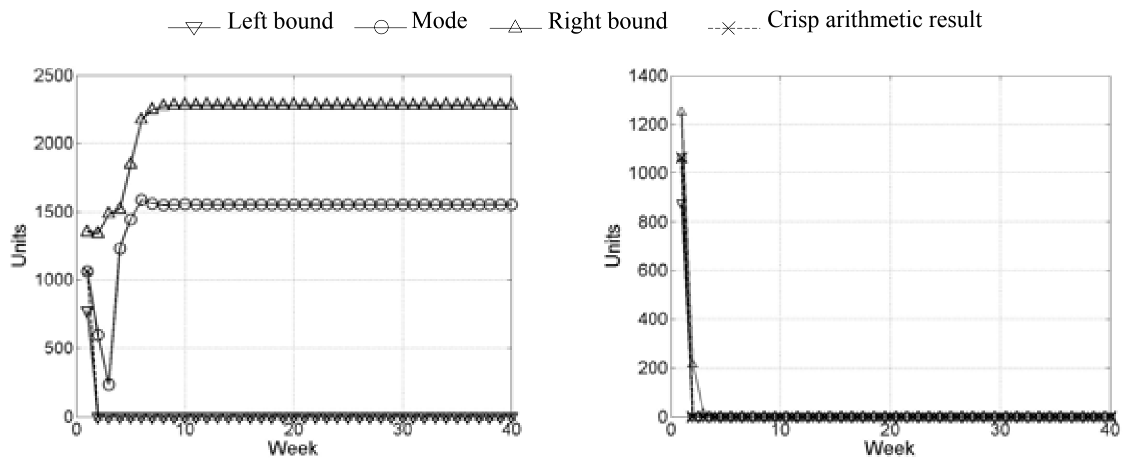
As shown in Figures 5.2-5.7, the results of $\widetilde{L\_OrB}$, $\widetilde{L\_AI}$, $\widetilde{WOrM}$, $\widetilde{TMP}$, $\widetilde{WI}$ and $\widetilde{R\_OrM}$ with the symmetrical-TFN input are plotted, as they are the important variables of this model. These results and that of the other variables are provided in Table 5.2 too, where for the space limitation only those of the 15 intervals are listed. For these variables, the left bound, mode, and right bound are shown in these figures with the time intervals. In addition, the crisp arithmetic values are provided in both parts ((a) and (b)) of each figure for comparisons.

Table 5.1 Initial and input data (the crisp and symmetrical TFN cases).

	WPC (constant)	*L_OrtF* (*t* = 0)	*L_OrB* (*t* = 0)	*L_IRB* (*t* = 0)	*L_AI* (*t* = 0)
Crisp	2.5	1000	1200	2800	4000
Symmetrical	(2, 2.5, 3)	(1000, 1000, 1000)	(1200, 1200, 1200)	(2800, 2800, 2800)	(4000, 4000, 4000)
	R_OrR (constant)	*MIL* (constant)	*TMP* (*t* = 0)		
Crisp	1000	4000	375		
Symmetrical	(900, 1000, 1100)	(4000, 4000, 4000)	(375, 375, 375)		

In Figure 5.2 and Table 5.2, the customer order backlog $\widetilde{L\_OrB}$ is affected by itself at the previous period and the $\widetilde{R\_OrM}$ and $\widetilde{R\_OrRM}$ at the current period interactively. It first shows a fuzzily decreasing phenomenon, then increases, and finally stabilizes with the α-cut arithmetic and fuzzily reduces and finally stabilizes with the T_ω arithmetic. The modes by the T_ω arithmetic are the same as the results of the crisp arithmetic. The fuzziness also reduces and finally stabilizes at the crisp zero with the T_ω arithmetic, due to the nonnegativity requirement and that when fuzziness occurs in the negative it is truncated. With the α-cut arithmetic, the modes of $\widetilde{L\_OrB}$ show a higher value than that of the crisp arithmetic, due to the nonnegativity requirement and that the α-cut arithmetic accumulates fuzziness of all fuzzy numbers involved in an operation. Still the α-cut arithmetic finds the fuzzy stable-state results (from t = 5, Figure 5.2 and Table 5.2). The left bound of $\widetilde{L\_OrB}$ from t = 2 is the same as the crisp arithmetic results. With the fuzzy arithmetic, $\widetilde{L\_OrB}$ may be seen fuzzily stabilizing. But still there is the uncertainty or fuzziness involved by these results. By these results, we also found that the α-cut arithmetic due to the accumulating phenomenon of involved fuzziness has a wider or fuzzier result than that by the T_ω arithmetic.

In Figure 5.3 and Table 5.2, the level of inventory $\widetilde{L\_AI}$ is affected by itself at the previous period and the $\widetilde{R\_MI}$ and $\widetilde{R\_SI}$ interactively. In addition, as there is a minimum inventory to maintain, it is also affected by other variables, $\widetilde{WC}$, $\widetilde{L\_OrB}$ and $\widetilde{WPC}$. Thus, we observe that with the T_ω arithmetic, the $\widetilde{L\_AI}$ reaches the cyclically steady pattern (Figure 5.3 (b)). The fuzziness also gradually reduces and stabilizes from $t = 5$ with the cyclic steady pattern, due to that the T_ω arithmetic takes only the largest fuzziness encountered in the operations. With the α-cut arithmetic (Figure 5.3 and Table 5.2), the stable pattern appears to be very different from that of the crisp arithmetic and the T_ω arithmetic. The fuzziness also stabilizes from $t = 6$. But a much wider fuzziness result also results with the α-cut arithmetic. In the case, the $\widetilde{L\_AI}$ may be viewed cyclic stabilized with the T_ω arithmetic and stabilized with the α-cut arithmetic fuzzily. Still there is an uncertainty or fuzziness and it may mean that excessive or below-*MIL* inventories may exist.

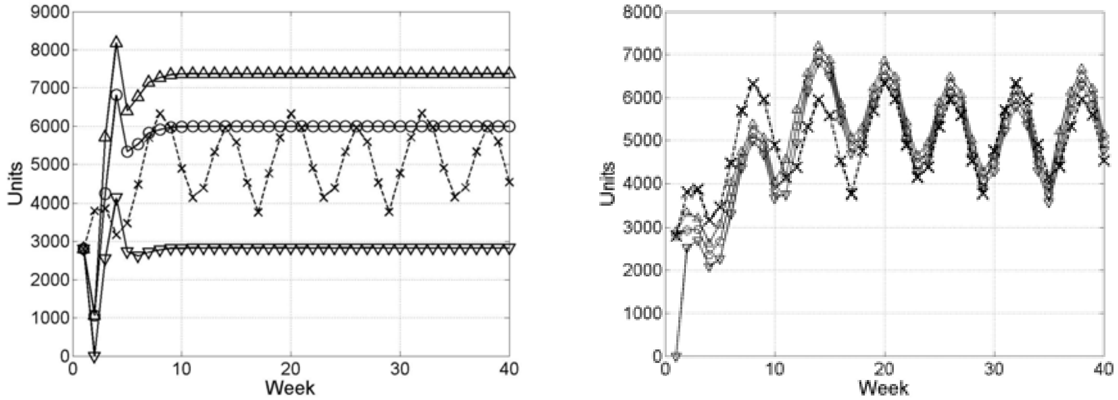


(a) α-cut and crisp arithmetic (b) T_ω and crisp arithmetic

Figure 5.2 Customer order backlog $\widetilde{L\_OrB}$ over the time intervals.

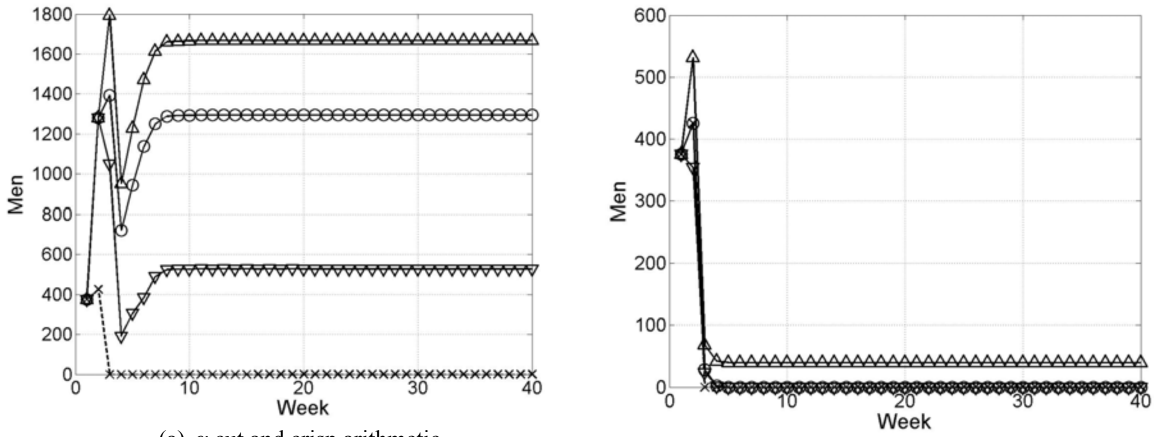
Moreover, in Figure 5.4 and Table 5.2, the labor for orders' manufactured quantity $\widetilde{WOrM}$ is interactively affected by $\widetilde{TMP}$ and $\widetilde{DWOrRM}$ and exhibits a stable condition with the T_ω arithmetic (Figure 5.4 (b)) with the zero stable mode and left bound due to the nonnegativity requirement, which are the same as the crisp arithmetic results. With the α-cut arithmetic, the mode and left and right bounds are stabilized at a much higher range between

524 and 1668 from $t = 8$ (Figure 5.4 (a) and Table 5.2) and much fuzzier. In this case, with the T_ω arithmetic, the $\widetilde{L\_OrB}$ is fuzzily zero at $t = 2$ but $\widetilde{WOrM}$ is nonzero fuzzily at the same period (Table 5.2). This is because at the end of $t = 1$ there exists a fuzzily nonzero $\widetilde{L\_OrB}$ and that requires $\widetilde{WOrM}$ to produce in $t = 2$ still fuzzily.



(a) α-cut and crisp arithmetic (b) T_ω and crisp arithmetic

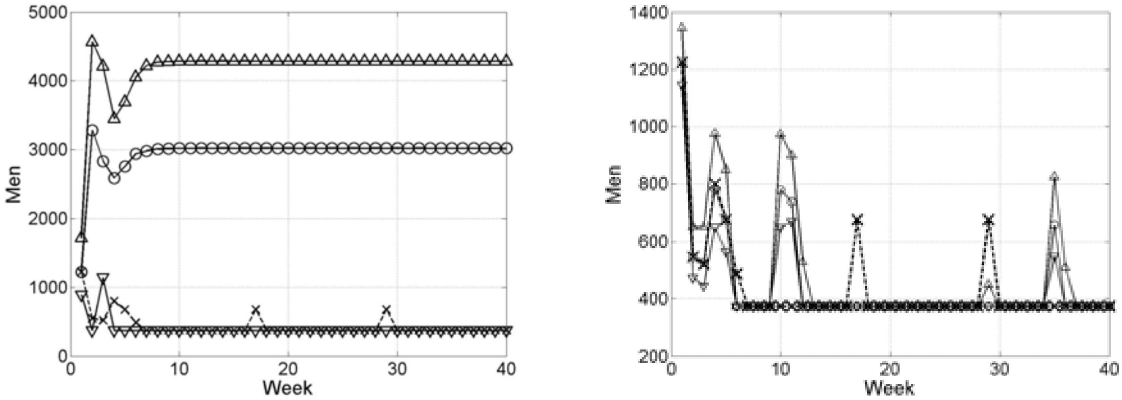
Figure 5.3 Expected inventory level $\widetilde{L\_AI}$ over the time intervals.



(a) α-cut and crisp arithmetic (b) T_ω and crisp arithmetic

Figure 5.4 Labor for orders' manufactured quantity $\widetilde{WOrM}$ over the time intervals.

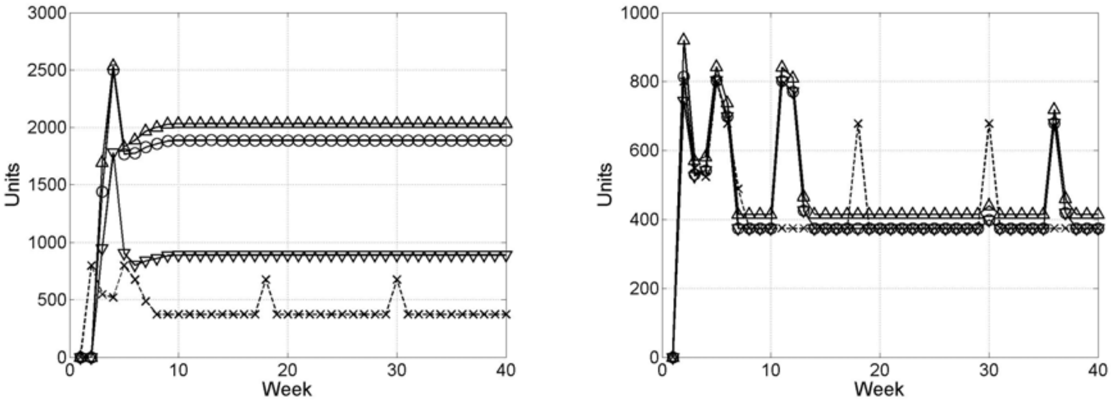
In Figure 5.5 and Table 5.2, the total men power $\widetilde{TMP}$ is affected by the requirements of $\widetilde{WI}$ and $\widetilde{WOrM}$ and other variables interactively from the previous period. As seen, with the T_ω arithmetic, the $\widetilde{TMP}$ reaches the stable and also cyclic mixed pattern (Figure 5.5 (b)). The T_ω arithmetic obtains the increases in $\widetilde{TMP}$ at $t = 10$, 11, 29, and 34. This is because the inventories at these periods could be below *MIL* and more manpower is required. A similar situation also shows with the crisp arithmetic. With the α-cut arithmetic, a straight fuzzy stable condition is reached (Figure 5.5 (a) and Table 5.2). The $\widetilde{TMP}$ increases fuzzily quickly at t = 1 (Figure 5.5 (a) and Table 5.2) to fulfill the customer orders, then gradually adjusts, and reaches the stable condition in the mode, left and right bounds.



(a) α-cut and crisp arithmetic (b) T_ω and crisp arithmetic

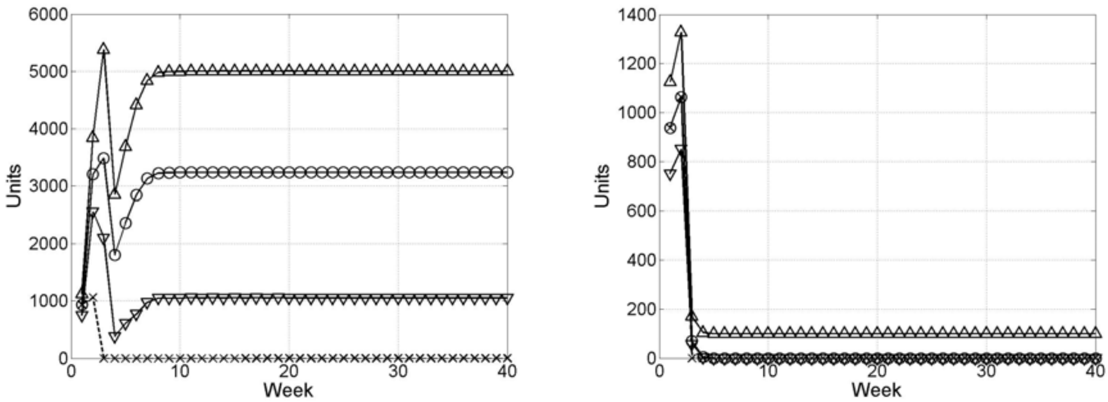
Figure 5.5 Total men power $\widetilde{TMP}$ over the time intervals.

In Figure 5.6 and Table 5.2, the labor for inventory replenishment manufacturing $\widetilde{WI}$ is affected by $\widetilde{TMP}$ and $\widetilde{WOrM}$ interactively. It is used to adjust the manufacturing for fulfilling the *MIL*. As seen, $\widetilde{WI}$ has a similar pattern to that of $\widetilde{TMP}$ with the two types of fuzzy arithmetic but lags one period behind the $\widetilde{TMP}$. In Figure 5.7 and Table 5.2, the orders' manufactured quantity $\widetilde{R\_OrM}$ is affected directly by the $\widetilde{WOrM}$. The variation is very similar to that of $\widetilde{WOrM}$ in the two types of fuzzy arithmetic (Figure 5.4 and Table 5.2) and the fuzziness is also bigger than that of $\widetilde{WOrM}$. Also, we observe that $\widetilde{R\_OrM}$ fuzzily increases from t = 1 to 2 and becomes stabilized at fuzzy zero from t = 5 with the T_ω arithmetic. This is because the $\widetilde{L\_OrB}$ will be generally fulfilled at these periods. With the α-cut arithmetic, due to the accumulating phenomenon of the involved fuzziness and the nonnegativity requirement of the other variables, the stabilization has been obtained at a higher level.



(a) α-cut and crisp arithmetic (b) T_ω and crisp arithmetic

Figure 5.6 Labor for inventory replenishment manufacturing $\widetilde{WI}$ over the time intervals.



(a) α-cut and crisp arithmetic (b) T_ω and crisp arithmetic

Figure 5.7 Orders' manufactured quantity $\widetilde{R\_OrM}$ over the time intervals.

Moreover, from Table 5.2 the left bound of $\widetilde{R\_OrFI}$ reaches zero from t = 2 and even some right bounds are zero (at t = 3-5) with the α-cut arithmetic. This means that the inventory at these intervals might be depleted (or is depleted at t = 3-5) and order fulfillment might be turned over to the production still fuzzily. With the T_ω arithmetic, a more stable planning is reached and only at t = 7, 13 the left bound of $\widetilde{R\_OrFI}$ is zero. Both the left bound and mode of the $\widetilde{R\_OrRM}$ reach zero with the T_ω arithmetic from t = 2, which means that there will be generally no $\widetilde{L\_OrB}$ at these intervals but still fuzzily. With the α-cut arithmetic, there is only the left bound of $\widetilde{R\_OrRM}$ reaching zero at t = 2, 4, 5. Moreover, due to the mutual influence relationships of the variables $\widetilde{R\_OrM}$, $\widetilde{DWOrRM}$, $\widetilde{R\_OrRM}$ and $\widetilde{L\_OrB}$, $\widetilde{R\_OrM}$ and $\widetilde{DWOrRM}$ have behaved similarly to $\widetilde{R\_OrRM}$, again since there is generally no $\widetilde{L\_OrB}$. As mentioned earlier, in order to fulfill the customer orders, $\widetilde{TMP}$ may increase fuzzily and quickly and adjust gradually. Consequently, the $\widetilde{L\_OrB}$ and $\widetilde{WC}$ may reduce fuzzily and quickly and adjust gradually, in which the left bound of $\widetilde{L\_OrB}$ reaches the zeros from t = 2 with both types of fuzzy arithmetic, and $\widetilde{WC}$ reaches the fuzzy negative from t = 3 with the α-cut arithmetic and from t = 7 with the T_ω arithmetic. Moreover, from the inventory, $\widetilde{L\_AI}$ is first used to suffice the customer orders. When $\widetilde{L\_AI}$ is sufficient, the $\widetilde{TMP}$ simply manufactures for inventory and may be decreased. From t = 7 (Figure 5.5 (b) and Table 5.2), we observe that $\widetilde{TMP}$ is cyclically stabilized with the T_ω arithmetic. This is because starting from t = 6 (Figure 5.3 (b) and Table 2) $\widetilde{L\_AI}$ could be possibly sufficient and above *MIL* and thus the manpower is only adjusting for maintaining the *MIL*. The $\widetilde{TMP}$ is straight stabilized with the α-cut arithmetic. From Table 5.2, the manufacturing rate for inventory $\widetilde{R\_MI}$ is affected by the $\widetilde{WI}$ and thus the pattern is similar to that of $\widetilde{WI}$. The $\widetilde{L\_OrtF}$ reaches the fuzzy zero or fuzzy negative from t = 1 with both fuzzy arithmetic. This means that the inventory and manufacturing capacity may be planned generally sufficient together in this example. Nevertheless, we notice that the system's fuzziness exists in the interaction of these variables, caused by the fuzzy input and parameter. Fuzzy outcomes planned result. The crisp system dynamics analysis certainly cannot provide this information.

Table 5.2 Results and comparison of the crisp, α-cut, and T_ω arithmetic with the symmetrical TFN input for the model.

Variable	Method	For t = 1, …, 5, then t = 6, …, 10, and t = 11, …, 15				
L_OrB	Crisp	1062.5	0	0	0	0
		0	0	0	0	0
		0	0	0	0	0
	α-cut arithmetic	(775, 1062.5, 1350)	(0, 598.6, 1338.9)	(0, 234.59, 1485.1)	(0, 1228.9, 1520)	(0, 1442.7, 1848.2)
		(0, 1588.4, 2173.8)	(0, 1561, 2249.8)	(0, 1548.7, 2273.5)	(0, 1552.7, 2278.4)	(0, 1556, 2281.8)
		(0, 1552.3, 2280.4)	(0, 1555.2, 2282.5)	(0, 1552.7, 2280.7)	(0, 1555.2, 2282.4)	(0, 1552.7, 2280.8)
	T_ω arithmetic	(875, 1062.5, 1250)	(0, 0, 212.5)	(0, 0, 14.1)	(0, 0, 0.87419)	(0, 0, 0)
		(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
		(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
L_AI	Crisp	2800	3800	3862.5	3168.6	3474.7
		4474.7	5693.9	6325.3	5956.6	4894.1
		4137.8	4381.4	5318.9	5950.3	5581.6
	α-cut arithmetic	(2800, 2800, 2800)	(0, 1062.5, 1062.5)	(2542.1, 4247.3, 5718)	(4126, 6818.6, 8182.3)	(2726.2, 5336.2, 6403.5)
		(2616.7, 5528, 6765.4)	(2720.2, 5823.5, 7150.6)	(2777.5, 5907.4, 7269.7)	(2821.4, 5976.8, 7352.8)	(2823.4, 5990.6, 7368.8)
		(2830.3, 5994, 7373.8)	(2825.3, 5992.6, 7372)	(2831.4, 5996.1, 7376.3)	(2825.8, 5993.3, 7372.8)	(2831.2, 5996, 7376.2)
	T_ω arithmetic	(0, 2800, 2900)	(2526.6, 2933.3, 3339.9)	(2690, 2954.8, 3219.5)	(2087.6, 2357.9, 2628.2)	(2265.6, 2666.9, 3068.2)
		(3313.06, 3662, 4010.94)	(4360, 4547.5, 4735)	(4995.1, 5182.6, 5370.1)	(4682.1, 4869.6, 5057.1)	(3671.6, 3859.1, 4046.6)
		(3752, 4152.5, 4552.9)	(4943.73, 5328.9, 5714)	(6140.1, 6353.1, 6566.1)	(6811.8, 6999.3, 7186.8)	(6498.8, 6686.3, 6873.8)
WOrM	Crisp	375	425	0	0	0
		0	0	0	0	0
		0	0	0	0	0
	α-cut arithmetic	(375, 375, 375)	(1280.6, 1280.6, 1280.6)	(1050.5, 1394.5, 1793.1)	(191.1, 720.89, 951.11)	(305.44, 943.62, 1229.5)
		(385.38, 1137.8, 1472.3)	(488.75, 1250.9, 1613.6)	(524.75, 1289.2, 1661.5)	(525.65, 1291.9, 1664.8)	(525.85, 1293.4, 1666.8)
		(528.1, 1294.7, 1668.3)	(527.22, 1294.8, 1668.4)	(528.02, 1294.7, 1668.4)	(527.28, 1294.8, 1668.5)	(528.04, 1294.7, 1668.4)
	T_ω arithmetic	(375, 375, 375)	(354.167, 425, 531.25)	(23.4999, 28.2, 68.2)	1.457, 1.7484, 41.7484)	(0, 0, 40)
		(0, 0, 40)	(0, 0, 40)	(0, 0, 40)	(0, 0, 40)	(0, 0, 40)
		(0, 0, 40)	(0, 0, 40)	(0, 0, 40)	(0, 0, 40)	(0, 0, 40)
WI	Crisp	0	800	547.45	522.45	800
		677.55	487.65	375	375	375
		375	375	375	375	375
	α-cut arithmetic	(0, 0, 0)	(0, 0, 0)	(948.16, 1440.6, 1690.7)	(1776.3, 2498.1, 2536.4)	(904.97, 1768, 1829.1)
		(802.58, 1772.4, 1889.5)	(840.63, 1827.8, 1965.5)	(863.08, 1854.9, 1999.8)	(887.07, 1881.1, 2026.2)	(889.67, 1885.4, 2030.6)
		(891.3, 1885.9, 2031.5)	(890.26, 1886, 2031.5)	(892.02, 1886.8, 2032.4)	(890.46, 1886.2, 2031.7)	(891.95, 1886.7, 2032.3)
	T_ω arithmetic	(0, 0, 0)	(742.497, 813.33, 919.58)	(524.7699, 529.47, 569.47)	(540.2586, 540.55, 580.55)	(802.6, 802.6, 842.6)
		(697.87, 697.87, 737.87)	(375, 375, 415)	(375, 375, 415)	(375, 375, 415)	(375, 375, 415)
		(800.85, 800.85, 840.85)	(770.35, 770.35, 810.35)	(426.02, 426.02, 466.02)	(375, 375, 415)	(375, 375, 415)
TMP	Crisp	1225	547.45	522.45	800	677.55
		487.65	375	375	375	375
		375	375	375	375	375
	α-cut arithmetic	(891.67, 1225, 1725)	(375, 3279.4, 4569.4)	(1139.7, 2830, 4212.7)	(375, 2583.1, 3445.5)	(375, 2754.2, 3695.5)
		(375, 2934.4, 4053.4)	(375, 2979.7, 4219)	(375, 3008.2, 4272.6)	(375, 3014, 4280.4)	(375, 3018.2, 4285.7)
		(375, 3016.5, 4284.7)	(375, 3018.9, 4287.3)	(375, 3016.8, 4285.1)	(375, 3018.9, 4287.3)	(375, 3016.9, 4285.2)
	T_ω arithmetic	(1145, 1225, 1345)	(475.027, 545.7, 652.11)	(445.034, 528.4, 653.45)	(651.09, 780.96, 975.76)	(565.84, 679, 848.76)
		(375, 375, 375)	(375, 375, 375)	(375, 375, 375)	(375, 375, 375)	(649.34, 779.2, 974.01)
		(671.188, 739.8, 900.01)	(375, 375, 529.07)	(375, 375, 375)	(375, 375, 375)	(375, 375, 375)
R_MI	Crisp	0	2000	1368.6	1306.1	2000
		1693.9	1219.1	937.5	937.5	937.5
		937.5	937.5	937.5	937.5	937.5
	α-cut arithmetic	(0, 0, 0)	(0, 0, 0)	(1896.3, 3601.5, 5072.1)	(3552.7, 6245.3, 7609.1)	(1809.9, 4419.9, 5487.2)
		(1605.2, 4431, 5668.5)	(1681.3, 4569.4, 5896.5)	(1726.2, 4637.2, 5999.5)	(1774.1, 4702.8, 6078.7)	(1779.3, 4713.6, 6091.8)
		(1782.6, 4714.7, 6094.5)	(1780.5, 4715.1, 6094.5)	(1784, 4716.9, 6097.1)	(1780.9, 4715.5, 6095)	(1783.9, 4716.8, 6097)
	T_ω arithmetic	(0, 0, 0)	(2033.3, 406.67, 406.67)	(1323.7, 264.73, 264.73)	(1351.4, 270.27, 270.27)	(2006.5, 401.3, 401.3)
		(1744.7, 348.94, 348.94)	(937.5, 187.5, 187.5)	(937.5, 187.5, 187.5)	(937.5, 187.5, 187.5)	(937.5, 187.5, 187.5)
		(2002.1, 400.43, 400.43)	(1925.9, 385.17, 385.17)	(1065.1, 213.01, 213.01)	(937.5, 187.5, 187.5)	(937.5, 187.5, 187.5)
R_OrM	Crisp	937.5	1062.5	0	0	0
		0	0	0	0	0
		0	0	0	0	0
	α-cut arithmetic	(750, 937.5, 1125)	(2561.1, 3201.4, 3841.7)	(2101, 3486.1, 5379.2)	(382.19, 1802.2, 2853.3)	(610.87, 2359.1, 3688.6)
		(770.77, 2844.6, 4416.9)	(977.5, 3127.3, 4840.9)	(1049.5, 3223, 4984.5)	(1051.3, 3229.6, 4994.5)	(1051.7, 3233.5, 5000.3)
		(1056.2, 3236.6, 5004.9)	(1054.4, 3236.9, 5005.3)	(1056, 3236.7, 5005.1)	(1054.6, 3237, 5005.5)	(1056.1, 3236.9, 5005.3)
	T_ω arithmetic	(750, 937.5, 1125)	(850, 1062.5, 1328.12)	(56.4, 70.501, 170.5)	(3.45, 4.37, 104.37)	(0, 0, 100)
		(0, 0, 100)	(0, 0, 100)	(0, 0, 100)	(0, 0, 100)	(0, 0, 100)
		(0, 0, 100)	(0, 0, 100)	(0, 0, 100)	(0, 0, 100)	(0, 0, 100)
L_OrtF	Crisp	0	0	-306.12	-1306.1	-2000
		-1693.9	-693.88	0	-306.12	-1306.1
		-2000	-1693.9	-693.88	0	-306.12
	α-cut arithmetic	(-200, 0, 200)	(-3037.5, 1000, 1100)	(-2354, -1919.6, -1719)	(-2426, -997.79, -897)	(-2083.7, -440.94, -341)
		(-1988.4, -114.62, 85.37)	(-1961, 15.071, 215.07)	(-1948.7, 18.95, 218.95)	(-1952.7, 26.738, 226.74)	(-1956, 32.982, 232.98)
		(-1952.3, 31.557, 231.56)	(-1955.2, 32.777, 232.78)	(-1952.7, 31.885, 231.88)	(-1955.2, 32.922, 232.92)	(-1955.2, 32.922, 232.92)

	T_ω arithmetic	(-100, 0, 100) (-1797, -1697.5, -1597) (-2059, -1959.1, -1859)	(-100, 0, 100) (-849, -749.5, -649) (-1808, -1708.7, -1608)	(-402, -302.15, -202) (-151.98, -51.982, 48.02) (-849.5, -749.5, -649.5)	(-1350, -1250.5, -1150) (-402.5, -302.48, -202.5) (-140.85, -40.855, 59.14)	(-2048, -1948, -1848) (-1350, -1250.5, -1150) (-391.3, -291.35, -191.3)
R_OrFI	Crisp	1200 693.88 1693.9	1000 0 693.88	1306.1 306.12 0	2000 1306.1 306.12	1693.9 2000 1306.1
	α-cut arithmetic	(1200, 1200, 1200) (0, 0, 85.376) (0, 31.557, 231.56)	(0, 1000, 1100) (0, 15.071, 215.07) (0, 32.777, 232.78)	(0, 0, 0) (0, 18.95, 218.95) (0, 31.885, 231.88)	(0, 0, 0) (0, 26.738, 226.74) (0, 32.922, 232.92)	(0, 0, 0) (0, 32.982, 232.98) (0, 31.906, 231.91)
	T_ω arithmetic	(1100, 1200, 1300) (649.5, 749.5, 849.5) (1608.7, 1708.7, 1808.7)	(900, 1000, 1100) (0, 51.981, 151.981) (649.5, 749.5, 849.5)	(1202.2, 1302.2, 1402.2) (202.48, 302.48, 402.48) (0, 40.852, 140.852)	(1848.3, 1948.3, 2048.3) (1150.5, 1250.5, 1350.5) (191.35, 291.35, 391.35)	(1597.5, 1697.5, 1797.5) (1848, 1948, 2048) (1150.5, 1250.5, 1350.5)
R_OrRM	Crisp	800 0 0	0 0 0	0 0 0	0 0 0	0 0 0
	α-cut arithmetic	(700, 800, 900) (59.218, 1747.7, 1847.7) (305.07, 1957.4, 2057.4)	(0, 2737.5, 2837.5) (212.2, 1873.2, 1973.2) (304.11, 1959.3, 2059.3)	(2505.7, 2840.3, 2940.3) (304.03, 1952.7, 2052.7) (304.86, 1957.5, 2057.5)	(0, 1228.9, 1328.9) (302.91, 1955.6, 2055.6) (304.06, 1959.2, 2059.2)	(0, 1442.7, 1542.7) (300.55, 1956.5, 2056.5) (304.96, 1957.7, 2057.7)
	T_ω arithmetic	(700, 800, 900) (0, 0, 100) (0, 0, 100)	(0, 0, 100) (0, 0, 100) (0, 0, 100)	(0, 0, 100) (0, 0, 100) (0, 0, 100)	(0, 0, 100) (0, 0, 100) (0, 0, 100)	(0, 0, 100) (0, 0, 100) (0, 0, 100)
DWOrRM	Crisp	800 0 0	425 0 0	0 0 0	0 0 0	0 0 0
	α-cut arithmetic	(633.33, 800, 1050) (385.38, 1137.8, 1472.3) (528.1, 1294.7, 1668.3)	(354.17, 1520, 1950) (488.75, 1250.9, 1613.6) (527.22, 1294.8, 1668.4)	(1050.5, 1394.5, 1793.1) (524.75, 1289.2, 1661.5) (528.02, 1294.7, 1668.4)	(191.1, 720.89, 951.11) (525.65, 1291.9, 1664.8) (527.28, 1294.8, 1668.5)	(305.44, 943.62, 1229.5) (525.85, 1293.4, 1666.8) (528.04, 1294.7, 1668.4)
	T_ω arithmetic	(720, 800, 920) (0, 0, 40) (0, 0, 40)	(354.167, 425, 531.25) (0, 0, 40) (0, 0, 40)	(23.4999, 28.2, 68.2) (0, 0, 40) (0, 0, 40)	(1.457, 1.7484, 41.7484) (0, 0, 40) (0, 0, 40)	(0, 0, 40) (0, 0, 40) (0, 0, 40)
L_IRB	Crisp	0 693.88 1693.9	306.12 0 693.88	1306.1 306.12 0	2000 1306.1 306.12	1693.9 2000 1306.1
	α-cut arithmetic	(0, 0, 0) (0, 1646.2, 2070.9) (0, 1582.7, 2151.6)	(0, 3800, 3900) (0, 1592.7, 2161.2) (0, 1586.8, 2155.2)	(440.13, 1222.3, 1689.3) (0, 1585.1, 2156.9) (0, 1582.9, 2151.7)	(0, 1556.3, 1699) (0, 1581.8, 2150.3) (0, 1586.7, 2155.1)	(0, 1683.5, 1951.5) (0, 1587.5, 2155.6) (0, 1583.1, 2151.8)
	T_ω arithmetic	(0, 0, 100) (649.5, 749.5, 849.5) (1608.7, 1708.7, 1808.7)	(202.15, 302.15, 402.15) (0, 51.981, 103.962) (649.5, 749.5, 849.5)	(1150.5, 1250.5, 1350.5) (202.48, 302.48, 402.48) (0, 40.852, 81.704)	(1848, 1948, 2048) (1150.5, 1250.5, 1350.5) (191.35, 291.35, 391.35)	(1597.5, 1697.5, 1797.5) (1848, 1948, 2081.39) (1150.5, 1250.5, 1350.5)
R_RIRB	Crisp	4000 1693.9 2000	693.88 693.88 1693.9	306.12 0 693.88	1306.1 306.12 0	2000 1306.1 306
	α-cut arithmetic	(4000, 4000, 4000) (814.44, 3036.2, 3236.8) (879.92, 3005.5, 3237.6)	(0, 0, 0) (853.9, 3019.2, 3246.7) (877.6, 3005.6, 3237.4)	(1110.7, 1810.9, 2125.3) (862.04, 2990.5, 3224.2) (880.22, 3006, 3238)	(1101, 3089.7, 3136.9) (876.44, 3003.2, 3234.8) (877.81, 3005.7, 3237.5)	(848.48, 3078.2, 3184.5) (877.34, 3006.9, 3238.4) (880.12, 3005.9, 3237.9)
	T_ω arithmetic	(0, 4000, 4000) (1697.5, 1697.5, 1794.8) (1959.1, 1959.1, 2056.9)	(637.074, 697.85, 789) (749.5, 749.5, 829.447) (1708.7, 1708.7, 1797.4)	(350.67, 353.81, 380.5) (51.982, 51.982, 57.53) (749.5, 749.5, 819.872)	(1250.1, 1250.8, 1343.3) (302.48, 302.48, 334.7) (40.855, 40.855, 45.21)	(1948, 1948, 2045.085) (1250.5, 1250.5, 1383.7) (291.35, 291.35, 322.42)
R_SI	Crisp	1200 693.88 1693.9	1000 0 693.88	1306.1 306.12 0	2000 1306.1 306.12	1693.9 2000 1306.1
	α-cut arithmetic	(1200, 1200, 1200) (0, 0, 85.376) (0, 31.557, 231.56)	(0, 1000, 1100) (0, 15.071, 215.07) (0, 32.777, 232.78)	(0, 0, 0) (0, 18.95, 218.95) (0, 31.885, 231.88)	(0, 0, 0) (0, 26.738, 226.74) (0, 32.922, 232.92)	(0, 0, 0) (0, 32.982, 232.98) (0, 31.906, 231.91)
	T_ω arithmetic	(1100, 1200, 1300) (649.5, 749.5, 849.5) (1608.7, 1708.7, 1808.7)	(900, 0, 1100) (0, 51.981, 151.981) (649.5, 749.5, 849.5)	(1202.2, 1302.2, 1402.2) (202.48, 302.48, 402.48) (0, 40.852, 140.852)	(1848.3, 1948.3, 2048.3) (1150.5, 1250.5, 1350.5) (191.35, 291.35, 391.35)	(1597.5, 1697.5, 1797.5) (1848, 1948, 2048) (1150.5, 1250.5, 1350.5)
WC	Crisp	425 -189.9 -55.1	122.45 -677.5 -152.55	522.45 -930.1 -527.55	800 -782.65 -780.1	677.55 -357.65 -632.65
	α-cut arithmetic	(258.33, 425, 675) (-1382.7, 24.188, 691.64) (-1686.9, -164.04, 584.7)	(0, 1759.4, 2619.4) (-1575.3, -98.947, 639.8) (-1686, -161.86, 587.36)	(-858.98, -5.1012, 728.9) (-1634.8, -135.9, 611.27) (-1688.2, -164.62, 584.3)	(-2091.2, -635.88, -41.9) (-1676.4, -158.95, 589.3) (-1686.4, -162.1, 587.12)	(-1201.7, 42.602, 636.88) (-1684.4, -160.63, 588.3) (-1688.1, -164.54, 584.4)
	T_ω arithmetic	(0, 425, 531.25) (249.833, 299.8, 374.75) (-129.7, -61, 99.1)	(80.86, 120.86, 205.86) (-273.78, -219.02, -144) (-664, -531.56, -377)	(416.8, 500.2, 625.25) (-591.3, -473.03, -394.2) (-1176, -941.24, -784)	(649.34, 779.21, 974.01) (-434.8, -347.83, -272.8) (-1499, -1199.7, -999)	(565.84, 679.01, 848.76) (649.34, 779.21, 974.01) (-1343, -1074.5, -895)

In the following sections, we shall examine the above model with the varied fuzziness of input and varied lengths of the system time interval for the further examinations of the two types of fuzzy arithmetic in the fuzzy system dynamics evaluations.

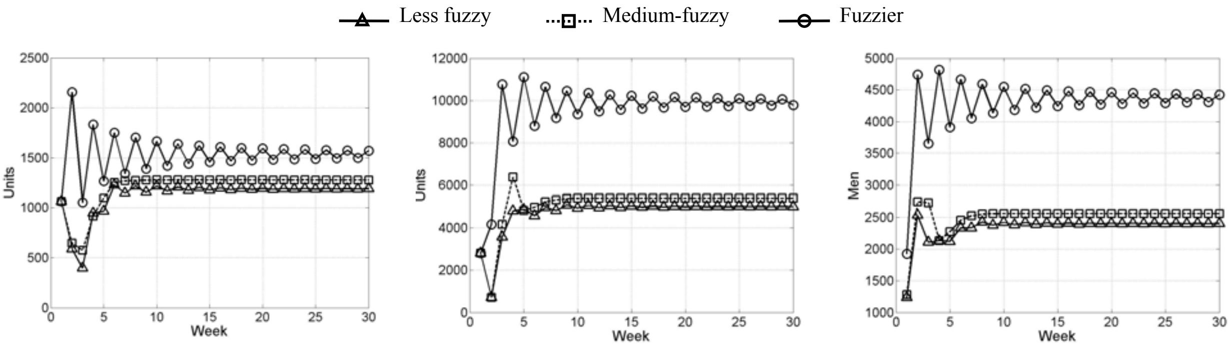
5.3.1 Comparisons in Varied Fuzziness and Skewed Membership Functions of Input Data

In this subsection, we further examine the above model of fuzzy system dynamics in varied fuzziness and skewed membership functions of the input. For the discussions, the results will be discussed after the defuzzification by using Eq. (3.58) to provide the decision-maker the representative value.

1) *Varied fuzziness*—In this test, the symmetrical input data as shown in Table 5.1 (here also termed the 'medium-fuzzy' case for the distinguishing purpose) are made both fuzzier and less fuzzy in the tests. For the fuzzier case, the mode minus the left bound and the right bound minus the mode of $\widetilde{R\_OrR}$ are tripled as $\widetilde{R\_OrR}$ = (800, 1000, 1200) and $\widetilde{WPC}$ as $\widetilde{WPC}$ = (1, 2.5, 4), as they influence the entire model. For the less fuzzy case, likewise, but the fuzziness is cut to a half as $\widetilde{R\_OrR}$ = (950, 1000, 1050) and $\widetilde{WPC}$ = (2.25, 2.5, 2.75). The results with the fuzzy arithmetic are again computed and as shown in Figs. 5.8 and 5.9 after defuzzification.

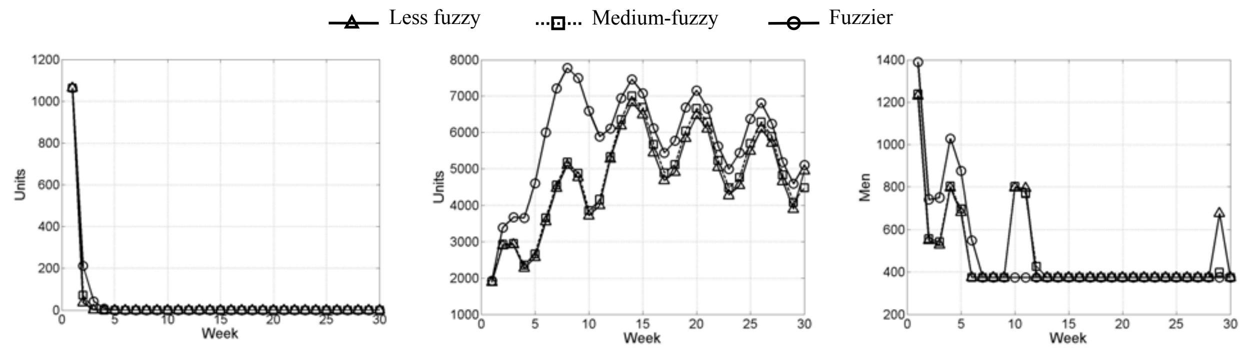
In Figure 5.8, with the α-cut arithmetic, thus we observe that the *L_OrB*, *L_AI*, and *TMP* after the defuzzification in all three cases of fuzziness show the gradual stabilization too with the additional results. It indicates that the fuzzier the input, the higher the level and oscillations of the cyclic pattern of these variables. In addition, due to the space limitation, without the data given herein, it can be pointed out that because of the nonnegativity of these variables, the membership functions of these variables have been rendered non-symmetrical even with the symmetrical input case as Table 5.2.

In Figure 5.9, with the T_ω arithmetic, the results of *L_OrB* in all three cases of input fuzziness still stabilize at the same level. And it is unaffected by the different amounts of input fuzziness. Mathematically, this is due to the nonnegativity requirement and non-symmetricity of membership functions resultant even with the symmetrical input case. The pattern of the *L_AI* also remains in the cyclic steady pattern as before in the three cases of fuzziness. However, the fuzzier the input, the higher slightly the cyclic level of the variable. Of course, this is due to the influence of the different amount of fuzziness included. Also, a similar situation occurs in the *TMP*, except that it has a stable or stable and cyclic mixed condition.



(a) *L_OrB* (b) *L_AI* (c) *TMP*

Figure 5.8 The comparison of α-cut arithmetic results for fuzzier, medium-fuzzy, and less fuzzy input data after defuzzification.



(a) *L_OrB* (b) *L_AI* (c) *TMP*

Figure 5.9 The comparison of T_ω arithmetic results for fuzzier, medium-fuzzy, and less fuzzy input data after defuzzification.

Similar to the earlier observation, the levels of these variables with the α-cut arithmetic are higher than that with the T_ω arithmetic, due to the accumulating phenomenon of fuzziness of the α-cut arithmetic and the non-symmetricity of the membership functions. Moreover, generally closer results are found with the T_ω arithmetic in the three cases of input fuzziness examined than that in the α-cut arithmetic.

2) *Skewed fuzziness*—For the skewed fuzziness, first we use the slant-right membership for the above three cases of input fuzziness. For the fuzzier case, $\widetilde{R\_OrR}$ = (800, 1000, 1100) and $\widetilde{WPC}$ = (1.75, 2.5, 2.625). For the medium-fuzzy case, $\widetilde{R\_OrR}$ = (850, 1000, 1100) and $\widetilde{WPC}$ = (2.25, 2.5, 2.625). For the less fuzzy case, $\widetilde{R\_OrR}$ = (875, 1000, 1100) and $\widetilde{WPC}$ = (2.365, 2.5, 2.625). In addition, we also use the slant-left membership function, and for the fuzzier case, $\widetilde{R\_OrR}$ = (900, 1000, 1200) and $\widetilde{WPC}$ = (2.365, 2.5, 3.25). For the medium-fuzzy case, $\widetilde{R\_OrR}$ = (900, 1000, 1150) and $\widetilde{WPC}$ = (2.365, 2.5, 2.75), and for the less fuzzy case, $\widetilde{R\_OrR}$ = (900, 1000, 1125) and $\widetilde{WPC}$ = (2.365, 2.5, 2.625). The results by the two types of fuzzy arithmetic are again computed and shown in Figs. 5.10 and 5.11 after defuzzification.

In Figure 5.10, with the α-cut arithmetic, three important observations may be made:

(a) No matter whether the membership functions of the inputs are slant-right or slant-left, the levels of the cyclic patterns of the variables are still influenced by the amount of fuzziness.

(b) The levels are generally slightly lower with the slant-left membership-function than that with the slant-right membership-function, and more importantly, both are generally lower than that in the symmetrical membership-function input (compared with Figure 5.8).

(c) Moreover, in the case both non-symmetrical cases give closer results in the three cases of input fuzziness than that in the symmetrical input case (compared to Figure 5.8).

The reasons for these happenings of (b) and (c) are because the nonsymmetrical inputs, whether they are slant-right or slant-left, in this case may have the chance to decrease the fuzziness in the positive side on the real line and/or to increase the fuzziness in the negative side to even more negative. Thus, when the nonnegativity requirements hold, they may give lower steady levels and closer results for the three cases of fuzziness. Still this may occur in this case only; other results in different cases might be obtained. Another point maybe worthy of mention is that in the slant-left membership function input, the fuzziness may influence the levels of the *TMP* reversely and differently from the other two variables'; that is, the lowest level of *TMP* occurs in the fuzzier case, next lowest level in the medium-fuzzy case, and the highest level in the less fuzzy case.

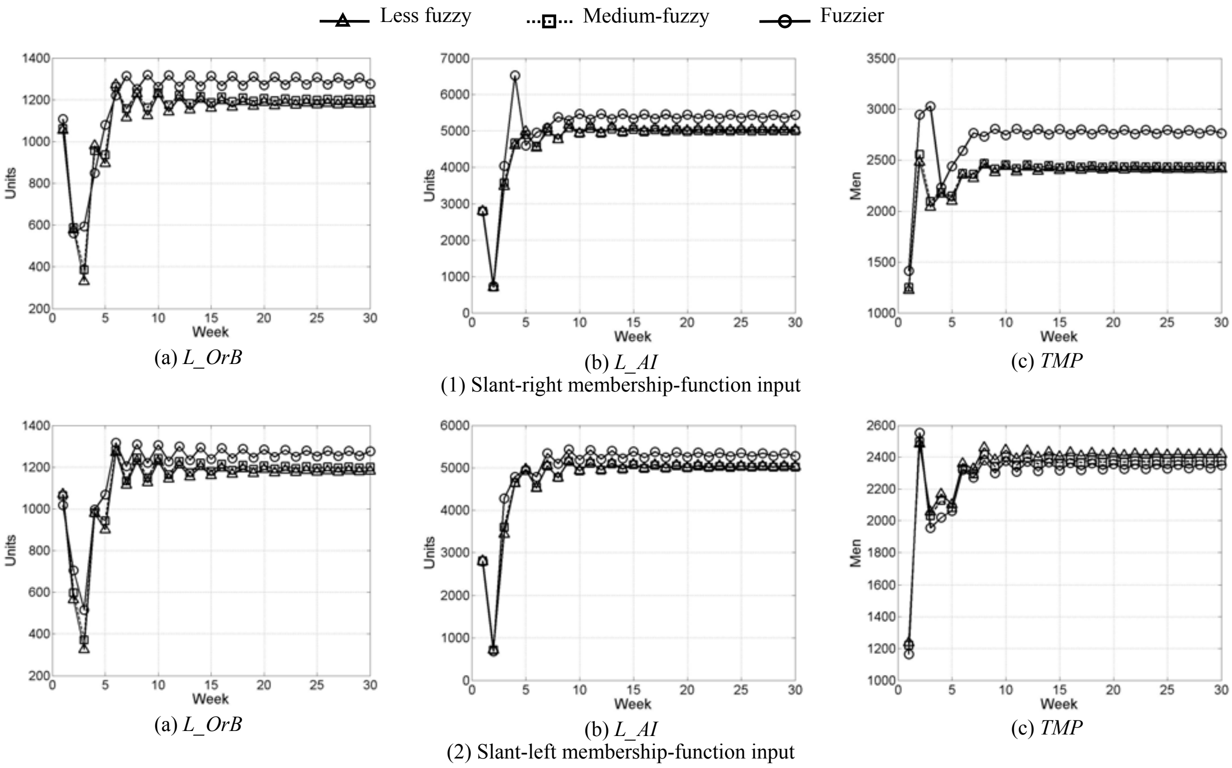


Figure 5.10 The results of α-cut arithmetic with skewed-membership-function and varied-fuzziness input data after defuzzification.

In Figure 5.11, in the T_ω fuzzy arithmetic, the last two observations (b) and (c) above on the α-cut arithmetic are generally not occurring. The only exception is for the first 11 periods of *L_AI* and first six periods of *TMP*, where the non-symmetrical input data have realized closer results in the three cases of input fuzziness (Figure 5.11 (1) and (2), (b) and (c), compared with Figure 5.9). This is because the T_ω fuzzy arithmetic takes only the largest fuzziness encountered in the operation. Another point, which may be mentioned, is that the nonsymmetrical membership function input may change the level size-sequence of the *L_AI* in the three cases of input fuzziness compared with the symmetrical case. A slightly similar situation also occurs in the *TMP* in this case.

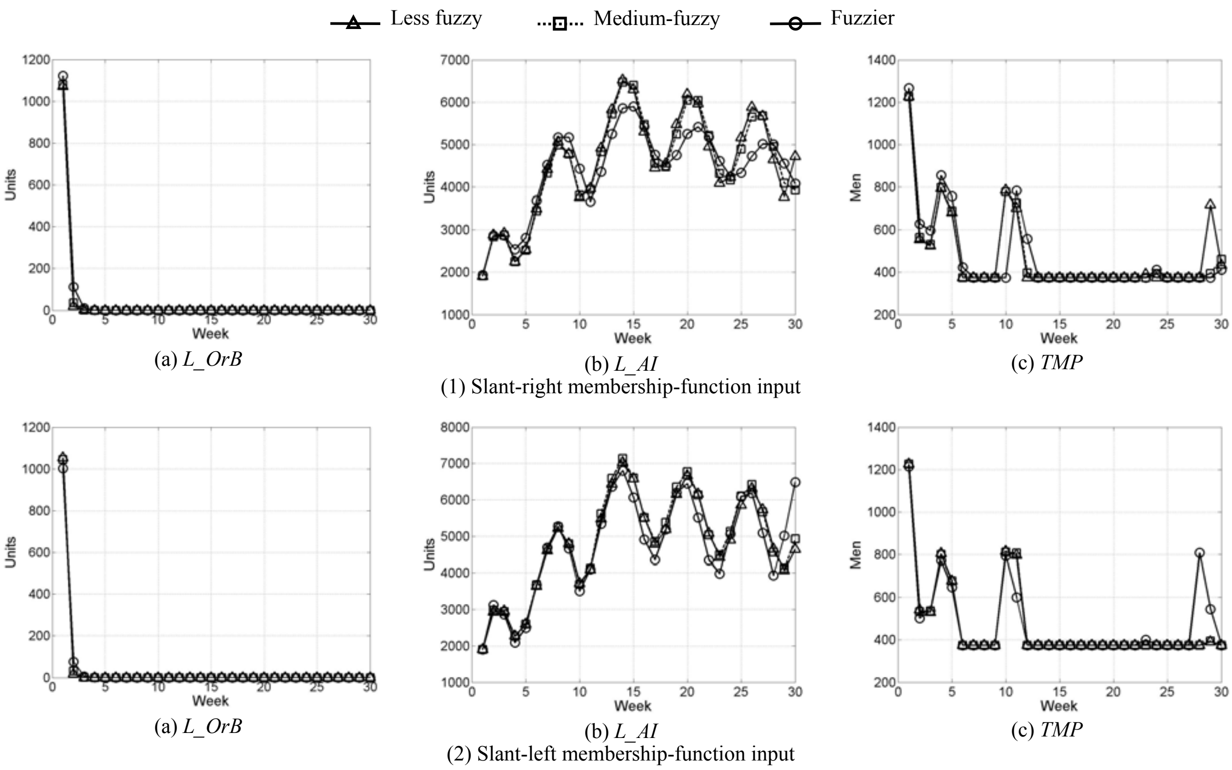


Figure 5.11 The results of T_ω fuzzy arithmetic with skewed-membership-function and varied-fuzziness input data after defuzzification.

In this case, the T_ω arithmetic provides more stable results (or conversely less sensitive results) to the amount of fuzziness and non-symmetricity of the input data.

From these results, tests 1) and 2) along with that of Figs. 5.8-5.11, the consideration of the influence of the system fuzziness should be taken in the system dynamics. The crisp analysis has lack of this consideration and different results maybe result.

5.3.2 A Sensitivity Test for the System Time Delay and a Further Comparison with the Crisp Model

The interval *DT* in the system dynamics represents the system time delay in the operational equations and also controls the frequency of the system variable updates. In this subsection, the results of a sensitivity test for the effects of *DT* (= 0.5, 1, 1.5) with the symmetrical input data and the two type of fuzzy arithmetic are provided for the examination (Figs. 5.10-5.11). By these results, the fuzzy arithmetic results as their crisp counterpart are sensitive to the system time delay *DT*.

From Figure 5.12, with the α-cut arithmetic, it shows that different *DT*s result in different stable statuses for all these variables. Among these variables, *DT* = 1.5 results in cyclic steady patterns, while the other *DT*s result in straight stable conditions for these variables. *DT* = 1.5 also results in the highest level of *L_OrB* and of *L_AI* but lowest level of *TMP*. On the other hand, the smallest *DT* results in the lowest level of *L_OrB* and of *L_AI* but the highest level of *TMP*. If the lowest *TMP* is desired, *DT* = 1.5 may be chosen. However, if a straight stable *TMP* is desired, *DT* = 1.0 may be elected, in which the levels of *L_OrB* and *L_AI* are also moderate. However, the costs of the *L_OrB*, *L_AI* and *TMP* may thus play an important role in determining the *DT*.

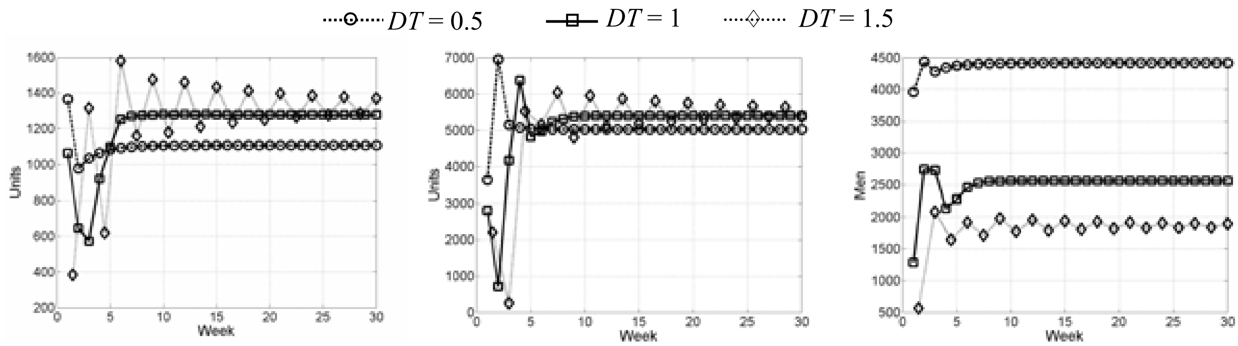
Figure 5.13 shows that the T_ω arithmetic results are also sensitive to the system time delay *DT*. The *DT* may only affect the *L_OrB* in the first five intervals (with *DT* = 1.5) and affect the *TMP* for the first 12 weeks and for the long run, the *DT* may have only little influence on these variables. For the *L_AI*, it has been shown that the longer the *DT*, the less the number of cycles and the higher the oscillation and higher the level of the cyclic pattern exhibit. In addition, a smaller *DT* cyclically stabilizes the *L_AI* quicker at first fewer intervals. In the T_ω arithmetic and in this case, *DT* = 0.5 gives a better performance for the system than the others. Furthermore, although *DT* = 0.5 gives the best performance of the system in view of the *L_AI*, the *TMP* may require a big number at the initial periods and thus this may again influence the final decision of the decision-maker.

Further Figure 5.14 shows the result of the crisp arithmetic evaluation for the model. In addition to the comparisons already made by Figs. 5.2-5.7, the following comparison may be made additionally.

In Figure 5.12 (the α-cut arithmetic) and Figure 5.13, the significant difference appears in both stabilizing patterns and/or levels of these variables. It indicates that the fuzziness or uncertainty shall not be omitted in the system dynamics evaluations. Between Figs. 5.13 (the T_ω arithmetic) and 5.14, although the T_ω arithmetic result patterns after defuzzification may appear to be somewhat similar to that of the crisp arithmetic evaluation results, still differences in stabilizing levels are detected in the *L_AI*. Apparently due to the influence of the fuzziness, at $DT = 0.5$ and 1.5, yet lower general levels are revealed, while $DT = 1.5$ also translates into the larger oscillation than that with the crisp evaluation. At $DT = 1.0$, although the general level of the *L_AI* with the T_ω arithmetic (after defuzzification) may appear to somewhat equal the crisp evaluation, but that from the T_ω arithmetic after consideration of the fuzziness decreases somewhat from a higher general level (starting from $t = 12$ approximately in Figure 5.13 (b)) and that of the crisp evaluation appears constant. The reason for this difference may be that while considering the fuzziness, the *R_MI* must be conditioned positive. Therefore, by the T_ω arithmetic the *L_AI* becomes slant right in this case. Thus, also with $DT = 1$, *L_AI* starts with a higher (general) level. Also, with the $DT = 1$, *TMP* shows a pattern (after defuzzification) that is different from the crisp evaluation to a degree (see Figs. 5.13 and 5.14, (c)).

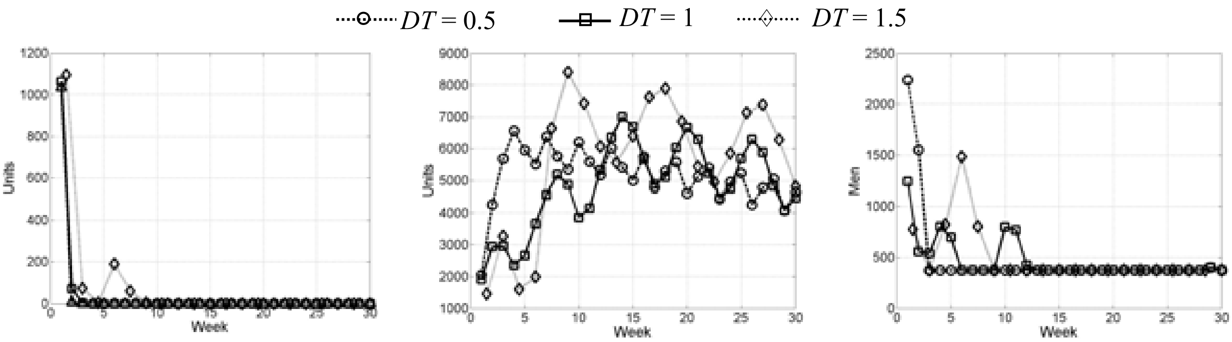
These observations again indicate the difference between the consideration of the fuzziness (uncertainties) in the system dynamics and the crisp evaluations.

In addition, a difference is detected between Figure 5.14 with $DT = 1$ and Figure 5.10. The amount of fuzziness influences the system behaviors (level and/or pattern).



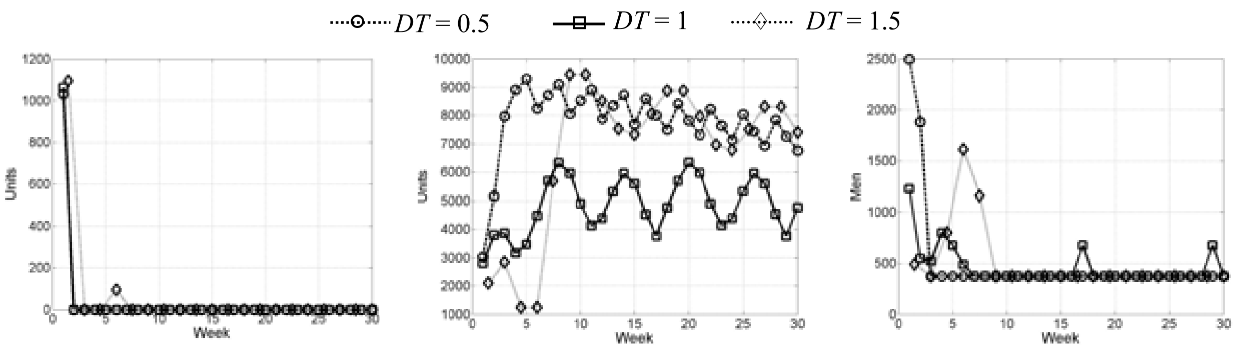
(a) *L_OrB* (b) *L_AI* (c) *TMP*

Figure 5.12 The results of α-cut arithmetic on $DT = 0.5, 1, 1.5$ after defuzzification.



(a) *L_OrB* (b) *L_AI* (c) *TMP*

Figure 5.13 The results of T_ω fuzzy arithmetic on $DT = 0.5, 1, 1.5$ after defuzzification.



(a) *L_OrB* (b) *L_AI* (c) *TMP*

Figure 5.14 The crisp arithmetic results with $DT = 0.5, 1, 1.5$.

Chapter 6

Conclusion and Future Research

Fuzzy arithmetic has been widely applied to many fields. In this research we extend (the) fuzzy operator on fuzzy numbers which include α-cut arithmetic, weakest *t*-norms and Yager's *t*-norms. In the meanwhile, this research develops a system dynamics analysis based on the applications of fuzzy arithmetic. The system dynamics has been proven an effective tool in analyzing complex systems' dynamics. Thus, this paper has proposed a fuzzy system dynamics evaluation based on further the applications of fuzzy arithmetic for the uncertain variables/parameters. The proposed fuzzy system dynamics has been successfully applied to a customer-producer-employment model with useful results obtained. In addition, this proposed fuzzy system has been tested dynamics with the varied fuzziness, symmetry and non-symmetry of fuzzy input, and interval of system time delay also with useful results obtained. In particular, the following observations can be made.

(1) Both types of fuzzy arithmetic provided the steady-state analysis of the model for the system variables.

(2) In the α-cut arithmetic, the fuzziness of the variables stabilized but which was also estimated fuzzier than that with the T_{ω} fuzzy arithmetic, due to the accumulating phenomenon of fuzziness of the α-cut arithmetic. In addition, the fuzzier the input, the higher the level and/or oscillation of stable or cyclic steady pattern of the variables. In addition, the defuzzified general levels of these variables in the nonsymmetrical input cases in this case were generally lower than that of the symmetrical case, due to that the nonnegativity was required with these variables and the accumulating phenomenon of fuzziness of the α-cut arithmetic. The nonsymmetrical input cases thus also gave closer results in the three cases of input fuzziness examined in this model than the symmetrical case.

(3) The T_ω arithmetic provided a smaller fuzziness and defuzzified general levels of variables due to the concept of taking only the maximal fuzziness encountered and resultant in the operation. In the case, also the T_ω arithmetic provided more stable results or conversely less sensitive ones to the varying amounts of input fuzziness and the nonsymmetricity (membership function) of inputs.

Therefore, it can be concluded that due to the inclusion of fuzziness (uncertainties), the system dynamics (variable interaction) may change the behaviors (variables') either in the stable or cyclic steady pattern or level of these variables. The fuzzy system dynamics may provide more information for the system dynamics regarding uncertainties if the system is consisting of variable interaction and fuzzy inputs and/or fuzzy parameters.

Further research may investigate a number of issues remaining. The system fuzziness may be carried over time intervals. However, how to effectively control the system fuzziness from being uncontrollably accumulating may demand more effort. Particularly, a different type of fuzzy arithmetic or controlling mechanism for the fuzziness of the fuzzy arithmetic may be demanded. The other types of system dynamics, e.g., the linear dependent, nonlinear independent and nonlinear dependent, may need to be investigated. In particular, a linear dependent type of system dynamics can be investigated that there could exist a certain type of constraints in a system (e.g., the total amount constraint of input and output resources) on rate variables to and from a level variable. In this case, all variables may be modeled and constrained too due to the interaction of the variables. Thus, with the interaction of the variables, a different type of fuzzy system dynamics may be investigated with fuzzy constraints in the linear dependent type relationships among or between the rate variables. Also, other types of system dynamics or applications may deserve the investigations too, e.g., those investigated in Coylea (1997) and applications reviewed in this research.

APPENDIX A (The division of weakest *t*-norm)

Appendix A shows that the division of *L*-*R* fuzzy number. First of all, we must decide the left or right fuzzy number as compared bounded based on basic supposed $z \le a_2/b_2$. Then, we can get $z = \frac{x}{b_2} \le \frac{a_2}{b_2}$ while the basic supposed is $z = \frac{x}{b_2}$. Based on initial consider the case of a_2 and b_2, which is positive or negative, we can obtain $x \le a_2$ or $x \ge a_2$. If $x \le a_2$, then $\widetilde{A}_L$ can be obtained. Otherwise, $\widetilde{A}_R$ can be obtained. Similarly, we can obtain $z = \frac{a_2}{y} \le \frac{a_2}{b_2}$ to decide compared bounded of $\widetilde{B}$. Here this research find that the variable y must consider inverse when compared bounded, i.e. $R\left(\frac{1/b_2 - z/a_2}{1/b_2 - 1/b_3}\right)$ in Case I. The results of inverse made that the bounded can compare in condition of a_2/b_2-z. Now, we suppose L=R, then division can be derived as follows:

Case I: For $a_2 > 0$, $b_2 > 0$ and $b_1 > 0$,

1) For $z \le a_2/b_2$,

$z = \frac{x}{b_2} \le \frac{a_2}{b_2}$, since $x \le a_2$, therefore, $\widetilde{A}_L$ can be obtained,

$z = \frac{a_2}{y} \le \frac{a_2}{b_2}$, since $y \ge b_2$, therefore, $\widetilde{B}_R$ can be obtained.

$$
\begin{aligned}
(\widetilde{A}/\widetilde{B})(z) &= \sup_{x/y=z} T_w(\widetilde{A}(x), \widetilde{B}(y)) \\
&= \max\{A(z \times b_2), B(a_2/z)\} = \max\left\{L\left(\frac{a_2 - z \times b_2}{a_2 - a_1}\right), R\left(\frac{1/b_2 - z/a_2}{1/b_2 - 1/b_3}\right)\right\} \\
&= \max\left\{L\left(\frac{a_2/b_2 - z}{(a_2 - a_1)/b_2}\right), R\left(\frac{a_2/b_2 - z}{a_2(1/b_2 - 1/b_3)}\right)\right\} \\
&= L\{(a_2/b_2 - z)/\max((a_2 - a_1)/b_2, a_2(1/b_2 - 1/b_3))\}.
\end{aligned}
$$

For $z \ge a_2/b_2 - \max((a_2 - a_1)/b_2, a_2(1/b_2 - 1/b_3))$; it is 0 otherwise.

2) for $z > a_2/b_2$, ($\widetilde{A}_R$ and $\widetilde{B}_L$ can be obtained)

$$
\begin{aligned}
(\widetilde{A}/\widetilde{B})(z) &= \sup_{x/y=z} T_w(\widetilde{A}(x),\widetilde{B}(y)) \\
&= \max\left\{A(z\times b_2), B(a_2/z)\right\} = \max\left\{R\left(\frac{z\times b_2 - a_2}{a_3 - a_2}\right), L\left(\frac{z/a_2 - 1/b_2}{1/b_1 - 1/b_2}\right)\right\} \\
&= \max\left\{R\left(\frac{z - a_2/b_2}{(a_3 - a_2)/b_2}\right), L\left(\frac{z - a_2/b_2}{a_2(1/b_1 - 1/b_2)}\right)\right\} \\
&= R\left\{(z - a_2/b_2)/\max((a_3 - a_2)/b_2, a_2(1/b_1 - 1/b_2))\right\}
\end{aligned}
$$

for $z < a_2/b_2 + \max((a_3 - a_2)/b_2, a_2(1/b_1 - 1/b_2))$; it is 0 otherwise.

Therefore, due to above 1) and 2), we can obtain that

$$
\begin{aligned}
\widetilde{A}/\widetilde{B} = (&a_2/b_2 - \max((a_2 - a_1)/b_2, a_2(1/b_2 - 1/b_3)), \\
&a_2/b_2, a_2/b_2 + \max((a_3 - a_2)/b_2, a_2(1/b_1 - 1/b_2))).
\end{aligned}
\tag{A-1}
$$

Case II: For $a_2 < 0$, $b_2 < 0$ and $b_3 < 0$,

1) for $z \le a_2/b_2$, $\widetilde{A}_R$ and $\widetilde{B}_L$ can be obtained.⇒

$$
\begin{aligned}
(\widetilde{A}/\widetilde{B})(z) &= \sup_{x/y=z} T_w(\widetilde{A}(x),\widetilde{B}(y)) \\
&= \max\left\{A(z\times b_2), B(a_2/z)\right\} = \max\left\{R\left(\frac{z\times b_2 - a_2}{a_3 - a_2}\right), L\left(\frac{z/a_2 - 1/b_2}{1/b_1 - 1/b_2}\right)\right\} \\
&= \max\left\{R\left(\frac{a_2/b_2 - z}{(a_2 - a_3)/b_2}\right), L\left(\frac{a_2/b_2 - z}{a_2(1/b_2 - 1/b_1)}\right)\right\} \\
&= L\left\{(a_2/b_2 - z)/\max((a_2 - a_3)/b_2, a_2(1/b_2 - 1/b_1))\right\}
\end{aligned}
$$

for $z \ge a_2/b_2 - \max((a_2 - a_3)/b_2, a_2(1/b_2 - 1/b_1))$; it is 0 otherwise.

2) for $z > a_2/b_2$, $\widetilde{A}_L$ and $\tilde{B}_R$ can be obtained.

$$
\begin{aligned}
(\widetilde{A}/\widetilde{B})(z) &= \sup_{x/y=z} T_w(\widetilde{A}(x),\widetilde{B}(y)) \\
&= \max\left\{A(z\times b_2), B(a_2/z)\right\} = \max\left\{L\left(\frac{a_2 - z\times b_2}{a_2 - a_1}\right), R\left(\frac{1/b_2 - z/a_2}{1/b_2 - 1/b_3}\right)\right\} \\
&= \max\left\{L\left(\frac{z - a_2/b_2}{(a_1 - a_2)/b_2}\right), R\left(\frac{z - a_2/b_2}{a_2(1/b_3 - 1/b_2)}\right)\right\} \\
&= R\left\{(z - a_2/b_2)/\max((a_1 - a_2)/b_2, a_2(1/b_3 - 1/b_2))\right\}
\end{aligned}
$$

for $z < a_2 / b_2 + \max((a_1 - a_2)/b_2, a_2(1/b_3 - 1/b_2))$; it is 0 otherwise.

Therefore, due to above 1) and 2), we can obtain that

$$\tilde{A}/\tilde{B} = (a_2/b_2 - \max((a_2 - a_3)/b_2, a_2(1/b_2 - 1/b_1)), a_2/b_2, \\ a_2/b_2 + \max((a_1 - a_2)/b_2, a_2(1/b_3 - 1/b_2))) \tag{A-2}$$

Case III: For $a_2 = 0$, $b_2 > 0$ and $b_1 > 0$

1) for $z \le a_2/b_2$, $\tilde{A}_L$ and $\tilde{B}_R$ can be obtained.

$$\begin{aligned}(\tilde{A}/\tilde{B})(z) &= \sup_{x/y=z} T_w(\tilde{A}(x), \tilde{B}(y)) \\ &= \max\{A(z\times b_2), B(a_2/z)\} = \max\left\{L\left(\frac{a_2 - z\times b_2}{a_2 - a_1}\right), R\left(\frac{1/b_2 - z/a_2}{1/b_2 - 1/b_3}\right)\right\} \\ &= \max\left\{L\left(\frac{a_2/b_2 - z}{(a_2 - a_1)/b_2}\right), R\left(\frac{a_2/b_2 - z}{a_2(1/b_2 - 1/b_3)}\right)\right\} \\ &= L\{(a_2/b_2 - z)/((a_2 - a_1)/b_2)\}\end{aligned}$$

for $z \ge a_2/b_2 - (a_2 - a_1)/b_2$; it is 0 otherwise.

2) for $z > a_2/b_2$, $\tilde{A}_R$ and $\tilde{B}_L$ can be obtained.⇒

$$\begin{aligned}(\tilde{A}/\tilde{B})(z) &= \sup_{x/y=z} T_w(\tilde{A}(x), \tilde{B}(y)) \\ &= \max\{A(z\times b_2), B(a_2/z)\} = \max\left\{R\left(\frac{z\times b_2 - a_2}{a_3 - a_2}\right), L\left(\frac{z/a_2 - 1/b_2}{1/b_1 - 1/b_2}\right)\right\} \\ &= \max\left\{R\left(\frac{z - a_2/b_2}{(a_3 - a_2)/b_2}\right), L\left(\frac{z - a_2/b_2}{a_2(1/b_1 - 1/b_2)}\right)\right\} \\ &= R\{(z - a_2/b_2)/((a_3 - a_2)/b_2)\}\end{aligned}$$

for $z < a_2/b_2 + (a_3 - a_2)/b_2$; it is 0 otherwise.

Therefore, due to above 1) and 2), we can obtain that

$$\tilde{A}/\tilde{B} = (a_1/b_2, 0, a_3/b_2) \tag{A-3}$$

Case IV: For $a_2 = 0$, $b_2 < 0$ and $b_3 < 0$,

1) for $z \le a_2/b_2$, $\tilde{A}_R$ and $\tilde{B}_L$ can be obtained. ⇒

$$
\begin{aligned}
(\widetilde{A}/\widetilde{B})(z) &= \sup_{x/y=z} T_w(\widetilde{A}(x),\widetilde{B}(y)) \\
&= \max\left\{A(z\times b_2), B(a_2/z)\right\} = \max\left\{R\left(\frac{z\times b_2 - a_2}{a_3 - a_2}\right), L\left(\frac{z/a_2 - 1/b_2}{1/b_1 - 1/b_2}\right)\right\} \\
&= \max\left\{R\left(\frac{a_2/b_2 - z}{(a_2 - a_3)/b_2}\right), L\left(\frac{a_2/b_2 - z}{a_2(1/b_2 - 1/b_1)}\right)\right\} \\
&= L\left\{(a_2/b_2 - z)/((a_2 - a_3)/b_2)\right\}
\end{aligned}
$$

for $z \ge a_2/b_2 - (a_2 - a_3)/b_2$; it is 0 otherwise.

2) for $z > a_2/b_2$, $\Rightarrow \widetilde{A}_L$ and $\widetilde{B}_R$ can be obtained. $\Rightarrow$

$$
\begin{aligned}
(\widetilde{A}/\widetilde{B})(z) &= \sup_{x/y=z} T_w(\widetilde{A}(x),\widetilde{B}(y)) \\
&= \max\left\{A(z\times b_2), B(a_2/z)\right\} = \max\left\{L\left(\frac{a_2 - z\times b_2}{a_2 - a_1}\right), R\left(\frac{1/b_2 - z/a_2}{1/b_2 - 1/b_3}\right)\right\} \\
&= \max\left\{L\left(\frac{z - a_2/b_2}{(a_1 - a_2)/b_2}\right), R\left(\frac{z - a_2/b_2}{a_2(1/b_3 - 1/b_2)}\right)\right\} \\
&= R\left\{(z - a_2/b_2)/((a_1 - a_2)/b_2)\right\}
\end{aligned}
$$

for $z < a_2/b_2 + (a_1 - a_2)/b_2$; it is 0 otherwise.

Therefore, due to above 1) and 2), we can obtain that

$$
\widetilde{A}/\widetilde{B} = (a_3/b_2, 0, a_1/b_2) \qquad \text{(A-4)}
$$

Case V: For $a_2 > 0$, $b_2 < 0$ and $b_3 < 0$,

1) for $z \le a_2/b_2$, $\widetilde{A}_R$ and $\widetilde{B}_R$ can be obtained.

$$
\begin{aligned}
(\widetilde{A}/\widetilde{B})(z) &= \sup_{x/y=z} T_w(\widetilde{A}(x),\widetilde{B}(y)) \\
&= \max\left\{A(z\times b_2), B(a_2/z)\right\} = \max\left\{R\left(\frac{z\times b_2 - a_2}{a_3 - a_2}\right), R\left(\frac{1/b_2 - z/a_2}{1/b_2 - 1/b_3}\right)\right\} \\
&= \max\left\{R\left(\frac{a_2/b_2 - z}{(a_2 - a_3)/b_2}\right), R\left(\frac{a_2/b_2 - z}{a_2(1/b_2 - 1/b_3)}\right)\right\} \\
&= L\left\{(a_2/b_2 - z)/\max((a_2 - a_3)/b_2, a_2(1/b_2 - 1/b_3))\right\}
\end{aligned}
$$

for $z \ge a_2/b_2 - \max((a_2 - a_3)/b_2, a_2(1/b_2 - 1/b_3))$; it is 0 otherwise.

2) for $z > a_2/b_2, \Rightarrow \widetilde{A}_L$ and $\widetilde{B}_L$ can be obtained.

$$
\begin{aligned}
(\widetilde{A}/\widetilde{B})(z) &= \sup_{x/y=z} T_w(\widetilde{A}(x), \widetilde{B}(y)) \\
&= \max\{A(z\times b_2), B(a_2/z)\} = \max\left\{L\left(\frac{a_2 - z\times b_2}{a_2 - a_1}\right), L\left(\frac{1/b_2 - z/a_2}{1/b_1 - 1/b_2}\right)\right\} \\
&= \max\left\{L\left(\frac{z - a_2/b_2}{(a_1 - a_2)/b_2}\right), L\left(\frac{z - a_2/b_2}{a_2(1/b_1 - 1/b_2)}\right)\right\} \\
&= R\{(z - a_2/b_2)/\max((a_1 - a_2)/b_2, a_2(1/b_1 - 1/b_2))\}
\end{aligned}
$$

for $z < a_2/b_2 + \max((a_1 - a_2)/b_2, a_2(1/b_1 - 1/b_2))$; it is 0 otherwise.

Therefore, due to above 1) and 2), we can obtain that

$$
\begin{aligned}
\widetilde{A}/\widetilde{B} = (&a_2/b_2 - \max((a_2 - a_3)/b_2, a_2(1/b_2 - 1/b_3)), a_2/b_2, \\
&a_2/b_2 + \max((a_1 - a_2)/b_2, a_2(1/b_1 - 1/b_2)))
\end{aligned}
\tag{A-5}
$$

Case VI: For $a_2 < 0$, $b_2 > 0$ and $b_1 > 0$,

1) for $z \le a_2/b_2$, $\widetilde{A}_L$ and $\widetilde{B}_L$ can be obtained.

$$
\begin{aligned}
(\widetilde{A}/\widetilde{B})(z) &= \sup_{x/y=z} T_w(\widetilde{A}(x), \widetilde{B}(y)) \\
&= \max\{A(z\times b_2), B(a_2/z)\} = \max\left\{L\left(\frac{a_2 - z\times b_2}{a_2 - a_1}\right), L\left(\frac{z/a_2 - 1/b_2}{1/b_1 - 1/b_2}\right)\right\} \\
&= \max\left\{L\left(\frac{a_2/b_2 - z}{(a_2 - a_1)/b_2}\right), L\left(\frac{a_2/b_2 - z}{a_2(1/b_2 - 1/b_1)}\right)\right\} \\
&= L\{(a_2/b_2 - z)/\max((a_2 - a_1)/b_2, a_2(1/b_2 - 1/b_1))\}
\end{aligned}
$$

for $z \ge a_2/b_2 - \max((a_2 - a_1)/b_2, a_2(1/b_2 - 1/b_1))$; it is 0 otherwise.

2) for $z > a_2/b_2$, $\widetilde{A}_R$ and $\widetilde{B}_R$ can be obtained.

$$
\begin{aligned}
(\widetilde{A}/\widetilde{B})(z) &= \sup_{x/y=z} T_w(\widetilde{A}(x), \widetilde{B}(y)) \\
&= \max\{A(z\times b_2), B(a_2/z)\} = \max\left\{R\left(\frac{z\times b_2 - a_2}{a_3 - a_2}\right), R\left(\frac{1/b_2 - z/a_2}{1/b_2 - 1/b_3}\right)\right\} \\
&= \max\left\{R\left(\frac{z - a_2/b_2}{(a_3 - a_2)/b_2}\right), R\left(\frac{z - a_2/b_2}{a_2(1/b_3 - 1/b_2)}\right)\right\} \\
&= R\{(z - a_2/b_2)/\max((a_3 - a_2)/b_2, a_2(1/b_3 - 1/b_2))\}
\end{aligned}
$$

for $z < a_2/b_2 + \max((a_3 - a_2)/b_2, a_2(1/b_3 - 1/b_2))$; it is 0 otherwise.

Therefore, due to above 1) and 2), we can obtain that

$$\tilde{A}\big/\tilde{B} = (a_2/b_2 - \max((a_2 - a_1)/b_2, a_2(1/b_2 - 1/b_1)), a_2/b_2, \\ a_2/b_2 + \max((a_3 - a_2)/b_2, a_2(1/b_3 - 1/b_2))). \tag{A-6}$$

From different case proof, we see that T_ω-based division preserves the shape of LR-fuzzy numbers with bounded or unbounded supports. In this research we do not discuss that denominator is zero and some special cases.

REFERENCES

[1]Akkermans, H. A., Bogerd, P. and Vos, B., "Virtuous and vicious cycles on the road towards international supply chain management," *International Journal of Operations & Production Management*, vol. 19, pp. 565–581, 1999.

[2]Alfeld, L. W. and Graham, A. K. (Ed.), *Introduction to Urban Dynamics*, Pegasus Communication, 1976.

[3]Chang, P.-T., "Fuzzy Strategic Replacement Analysis," *European Journal of Operational Research*, Vol. 160, pp. 532–559, 2005.

[4]Chang, P.-T. and Chang, C.-H., "An elaborative unit cost structure-based fuzzy economic production quantity model," *Mathematical and Computer Modelling*, vol. 43, pp. 1337–1356, 2006.

[5]Chang, P.-T. and Hung, K.-C., "Applying the fuzzy-weighted-average approach to evaluate network security systems," *Computers and Mathematics with Applications*, vol. 49, pp. 1797–1814 , 2005.

[6]Chang, P.-T. and Hung, K.-C., "α-cut fuzzy arithmetic: Simplifying rules and a fuzzy function optimization with a decision variable," *IEEE Transactions on Fuzzy Systems*, vol. 14, pp. 496–510, 2006.

[7]Chang, P.-T., Hung, K.-C., Lin, K.-P., and Chang, C.-H., "A comparison of discrete algorithms for fuzzy weighted average," *IEEE Transactions on Fuzzy Systems*, vol. 14, pp. 663–675, 2006.

[8]Coylea, R. G., "System dynamics at Bradford University: A silver jubilee review," *System Dynamics Review*, vol. 13, pp. 311–324, 1997.

[9]Disney S. M. and Towill, D. R., "The effect of vendor managed inventory (VMI) dynamics on the Bullwhip Effect in supply chains," *International Journal of Production Economics*, vol. 85, pp. 199–215, 2003.

[10]Dubois, D. and Prade, H., *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, 1980.

[11]Dubois, D. and Prade, H. (Ed.), Fuzzy numbers: An overview, in *Analysis of Fuzzy Information*, CRC Press, 1987.

[12]Dubois, D. and Prade, H., "Additions of interactive fuzzy numbers", *IEEE Transactions*

on Automatic Control, vol. 26, pp. 926–936, 1981.

[13]Duhon, T. and Glick, M., *Generic Structures: S-Shaped growth*. MIT, 1994, D-4432

[14]Forrester, J. W., *Industrial Dynamic*, Productivity, 1961.

[15]Forrester, J. W. (Ed.), *Principle of Systems*, Pegasus Communication, 1968.

[16]Forrester, J. W. (Ed.), *Urban Dynamics*, Pegasus Communication, 1969.

[17]Forrester, J. W. (Ed.), *World Dynamics*, Pegasus Communication, 1971.

[18]Forrester, J. W. (Ed.), *Collected Papers of Jay W. Forrester*, Pegasus Communication, 1975.

[19]Forrester, N. B., *The Life Cycle of Economic Development*, Pegasus Communication, 1973.

[20]Fullér, R. and Keresztfalvi, T., "t-Norm-based addition of fuzzy intervals", *Fuzzy set and systems*, vol. 51, pp. 155–159, 1992.

[21]Garmendia, L., Yager, R. R., Trillas, E. and Salvador, A., "On *t*-norms based measures of specificity," *Fuzzy Sets and Systems*, vol. 133, pp. 237–248, 2003.

[22]Giachetti, R. E. and Young, R. E., "Analysis of the error in the standard approximation used for multiplication of triangular and trapezoidal fuzzy numbers and the development of a new approximation," *Fuzzy Sets and Systems*, vol. 91, pp. 1–13, 1997.

[23]Giachetti, R. E. and Young, R. E., "A parametric representation of fuzzy numbers and their arithmetic operators," *Fuzzy Sets and Systems*, vol. 91, pp. 185–202, 1997.

[24]Grizzle, G. A. and Pettijohn, C. D., "Implementing performance-based program budgeting: A system-dynamics perspective," *Public Administration Review*, vol. 62, pp. 51–62, 2002.

[25]Gui, S.-P., Zhu, Q. and Lu, L., "Area logistics system based on system dynamics model," *Tsinghua Science and Technology*, vol. 10, pp. 265–269, 2005.

[26]Hansen, J. E. and Bie, P., "Distribution of body fluids, plasma protein, and sodium in dogs: A system dynamics model," *System Dynamics Review*, vol. 3, pp. 116–135, 1987

[27]Hauke, W., "Using Yager's t-norms for aggregation of fuzzy intervals," *Fuzzy Sets and Systems*, vol. 101, pp. 59–65, 1999.

[28]Heshmaty, B. and Kandel, A., "Fuzzy linear regression and its applications to forecasting

in uncertain environment," *Fuzzy Sets and Systems*, vol. 15, pp. 159–191, 1985.

[29]Homer, J. B. and Clair, C. L. St., "A model useful in analyzing public policies, such as a needle cleaning campaign," *Inference*, vol. 21, pp. 26–29, 1991.

[30]Hong, D. H., "A note on t-norm-based addition of fuzzy intervals," *Fuzzy Sets and Systems*, vol. 75, pp. 73–76, 1995.

[31]Hong, D. H., "Some results on the addition of fuzzy intervals," *Fuzzy Sets and Systems*, vol. 122, pp. 349–352, 2001a.

[32]Hong, D. H., "Shape preserving multiplications of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 123, pp. 81–84, 2001b.

[33]Hong, D. H. and Do, H. Y., "Fuzzy system reliability analysis by the use of Tω (the weakest t-norm) on fuzzy number arithmetic operations," *Fuzzy Sets and Systems*, vol. 90, pp. 307–316, 1997.

[34]Karavezyris, V., Timpe, K. P., and Marzi, R., "Application of system dynamics and fuzzy logic to forecasting of municipal solid waste," *Mathematics and Computers in Simulation*, vol. 60, pp. 149−158, 2002.

[35]Kaufmann, A. and Gupta, M. M., *Fuzzy Mathematical Models in Engineering and Management Science*, North-Holland, Amsterdam, 1988.

[36]Kaufman, A. and Gupta, M. M., *Introduction to Fuzzy Arithmetic*, Van Nostrand Reinhold, New York, 1991.

[37]Keresztfalvi, T., "Operations on fuzzy numbers extended by Yager's family of t-norm," In: H. Bandemer (Ed.), *Modelling Uncertain Data, Mathematical Research*, vol. 88, pp. 163−167, 1993.

[38]Kim, D.-H. and Kim, D.-H., "A system dynamics model for a mixed-strategy game between police and driver," *System Dynamics Review*, vol. 13, pp. 33−52, 1997.

[39]Kolesárová, A., "Additive preserving the linearity of fuzzy intervals," *Tatra Mountains Mathematical Publications*, vol. 6, pp. 75−81, 1995.

[40]Kosheleva, O., Cabrera, S. D., Gibson, G. A. and Koshelev, M., "Fast implementations of fuzzy arithmetic operations using fast Fourier transform (FFT)," *Fuzzy Sets and Systems*, vol. 91, pp. 269–277, 1997.

[41]Lai, C.-L., Lee, W.-B. and Ip, W.-H., "A study of system dynamics in just-in-time

logistics," *Journal of Materials Processing Technology*, vol. 138, pp. 265−269, 2003.

[42]Leekwijck, W.-V. and Kerre, E.-E., "Defuzzification: Criteria and classification," *Fuzzy Sets and Systems*, vol. 108, pp. 159−178, 1999.

[43]Levary, R. R., "Systems dynamics with fuzzy logic," *International Journal of Systems Science,* vol. 21, pp. 1701−1707, 1990.

[44]Ling, C.-H., "Representation of associative functions," *Publicationes Mathematicae Debrecen*, vol. 12, 189–212, 1965.

[45]Liou, T.-S. and Wang, M.-J. J., "Fuzzy weighted average: An improved algorithm," *Fuzzy Sets and Systems*, vol. 49, pp. 307–315, 1992.

[46]Lyneis, J. M. (Ed.), *Corporate Planning and Policy Design: A System Dynamics Approach*, Pegasus Communication, 1980.

[47]Lyneis, J. M., "System dynamics for market forecasting and structural analysis," *System Dynamics Review*, vol. 16, pp. 3−25, 2000.

[48]Ma, M., Friedman, M., and Kandel, A., "A new fuzzy arithmetic," *Fuzzy Sets and Systems*, vol. 108, pp. 83–90, 1999.

[49]Mabin, V. J., Davies, J. and Cox, J. F., "Using the theory of constraints thinking processes to complement system dynamics' causal loop diagrams in developing fundamental solutions," *International Transactions in Operational Research*, vol. 13, pp. 33–57, 2006.

[50]Maier, F. H., "New product diffusion models in innovation management−A system dynamics perspective," *System Dynamics Review*, vol. 14, pp. 285−308, 1998.

[51]Mesiar, R., "Shape preserving additions of fuzzy intervals," *Fuzzy Sets and Systems*, vol. 86, pp. 73–78, 1997.

[52]Mingers, J., "A classification of the philosophical assumptions of management science methods," *Journal of the Operational Research Society*, vol. 54, pp. 559–570, 2003.

[53]Mizumoto, M. and Tanaka, K., "The four operations of arithmetic on fuzzy numbers," *Systems Computers Controls*, vol. 7, pp. 73–81, 1976.

[54]Nahmias, S., "Fuzzy variables," *Fuzzy Sets and Systems*, vol. 1, pp. 97–111, 1978.

[55]Polat, S. and Bozdag, C.E., "Comparison of fuzzy and crisp systems via system dynamics

simulation," *European Journal of Operational Research*, vol.138, pp. 178−190, 2002.

[56]Reid, R. A. and Koljonen, E. L., "Validating a manufacturing paradigm: A system dynamics modeling approach," in *Proceedings of the 1999 Winter Simulation Conference*, pp. 759−765, 1999.

[57]Richmonda, B., "The strategic forum: Aligning objectives, strategy and process," *System Dynamics Review*, vol. 13, pp. 131–148, 1997.

[58]Rommelfanger, H., *Fuzzy decision support-system-entscheiden bej unschäfe*, 2nd ed., Springer, Berlin, 1994.

[59]Scholl, G. A., "Benchmarking the system dynamics community: Research results," *System Dynamics Review*, vol. 11, pp. 139–155, 1995.

[60]Sehlke, G. and Jacobson, J., "System dynamics modeling of transboundary systems: The bear river basin model," *Ground Water*, vol. 43, pp. 722–730, 2005.

[61]Spector, J. M., "Using system dynamics to model courseware development: the project dynamics of complex problem solving," in *Proceedings of the 1995 ACM symposium on Applied computing SAC '95*, pp. 32−35, 1995.

[62]Stave, K. A., "Using system dynamics to improve public participation in environmental decisions," *System Dynamics Review*, vol. 18, pp. 139−167, 2002.

[63]Sterman, J. D., "Modeling managerial behavior: Misperception of feedback in a dynamic decision making experiment," *Management Science*, vol. 35, pp. 321−339, 1989.

[64]Wagenknecht, M., Hampel, R. and Schneider, V., "Computational aspects of fuzzy arithmetics based on Archimedean t-norms," *Fuzzy Sets and Systems*, vol. 123, pp. 49–62, 2001.

[65]Whalen, T., "Parameterized R-implications," *Fuzzy Sets and Systems*, vol. 134, pp. 231–281, 2003

[66]Wood, K. L., Otto, K. N., and Antonsson, E. K., "Engineering design calculations with fuzzy parameters," *Fuzzy Sets and Systems*, vol. 52, pp. 1–20, 1992.

[67]Zadeh, L. A., "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

[68]Zimmermann, H.J., *Fuzzy Set Theory—and its Applications*, Kluwer, Massachusertts, 2001.