Abstract

In this thesis, we analyse a time-delay model to study the dynamics of fishery resource system. Firstly, we propose a criteria for the unique positive equilibrium point of the system. Secondly, we derive different sufficient conditions for local and global stability of the positive equilibrium point. Finally, we illustrate our results by some examples.

Keywords: Fishery resources; Delay; Stability; Lyapunov functional; Reserved zone

Contents

1 Introduction

Fishery resource modeling is concerned with the harvesting of fish. That is, how do we harvest the fish with the result that the fishery resource does not disappear. In recent years, many research workers propose and analyse a mathematical model to study the dynamics of a fishery resource system. In particular, Leung and Wang [7] presented a mathematical model for commerical fishing to study the phenomena of non-explosive fishing capital and non-extinctive fishery resources. Kitabatake [5] developed a dynamic model for fishery resources with predator-prey relationship based on data for Lake Kasumigaura in Japan. Clark [2] studied the problem of combined harvesting of two independent fish species governed by the logistic law of growth. Based on thee work of Clark [2], Chaud-huri [1] proposed a model to study the combined harvesting of two competeing fish species. Mesterton-Gibbons [8] extended the work of Clark [2] and found criteria for the survival of less produtive species as a function of the system parameters and initial stocks.

In 2003, B.Dubey, Peeyush Chandra, Prawal Sinha proposed a mathematical model of a fishery resource system in an aquatic enviroment that consists of reserved area. [3] showed that if fishery is exploited continuously in the unreserved area, fish population can be maintained at an appropriate equilibrium level. But is the real state like this? If we must consider that it has to be the factor that the fish grows up from the seedling to multiply the ability of future generation through one periodic time, how is the dynamic behavior of this model? In order to make this model correspond with the factual factor, so we assume the fishery resource system model with reserved area in which both species in reserved and unreserved obey the logistic law of growth with time delay. Because this assumption corresponds to the fact that the fish species cannot give birth to fishes when the species are infants, fishes have to mature for a duration of time. The main purpose of this thesis is to analyse the stability of the unique positive equilibrium of the system. In section 2, we introduce some useful definitions and theorems. In section 3, we give sufficient conditions for the unique positive equilibrium point of the system. In section 5, we analyse uniform persistence of the system. In section 4 and 6, we discuss the local and global stability by constructing respective differential Lyapunov functionals. Finally, in section 7, we illustrate our results by some examples.

2 Preliminaries

For ordinary differential equations, we have definitions and theorems of stability theory and we view the solution of initial value problem as maps in Euclidean space. In order to establish a similar view for the solution of delay differential equations, we need some definitions.

We denote $\mathcal{C} \equiv C([-\tau, 0], R^n)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n with the topology of uniform convergence. That is, for $\phi \in \mathcal{C}$, the norm of ϕ is defined as $\|\phi\| = \sup_{\theta \in [-\tau,0]} |\phi(\theta)|$, where $|\cdot|$ is a norm in R^n. We define $x_t \in \mathcal{C}$ as $x_t(\theta) = x(t+\theta),\ \theta \in [-\tau, 0]$. Assume that Ω is a subset of $\mathcal{C}$ and $f : \Omega \to R^n$ is a given function, then we consider the following general nonlinear autonomous system of delay differential equation

$$\dot{x}(t) = f(x_t) \tag{2.1}$$

Definition 2.1 *[6]* *Let $R_+^2 = \{x \in R^2 | x_i \geq 0\ ,\ i = 1, 2\}$. The notation $x > 0$ denotes $x \in \mathrm{Int}R_+^2$. The system (2.1) is said to be unifromly persistent if there exists a compact region $D \subseteq \mathrm{Int}R_+^2$ such that every positive solution $x(t)$ of the system (2.1) with the initial conditions eventually enters and remains in the region D.*

Definition 2.2 *[6]* *We say that $\phi \in B(0, \delta)$ if $\phi \in \mathcal{C}$ and $\|\phi\| \leq \delta$, where $\|\phi\| = \sup_{\theta \in [-\tau,0]} |\phi(\theta)|$.*

(i) The solution $x = 0$ of the system (2.1) is said to be stable if, for any $\sigma \in R$, $\epsilon > 0$, there is a $\delta = \delta(\epsilon, \sigma)$ such that $\phi \in B(0, \delta)$ implies $x_t(\sigma, \phi) \in B(0, \epsilon)$ for $t \geq \sigma$. Otherwise, we say that $x = 0$ is unstable.

(ii) The solution $x = 0$ of the system (2.1) is said to be asymptotically stable if it is stable and there is a $b_0 = b(\sigma) > 0$ such that $\phi \in B(0, b_0)$ implies $x(\sigma, \phi)(t) \to 0$ as $t \to \infty$.

(iii) The solution $x = 0$ of the system (2.1) is said to be uniformly stable if the number δ in the definition of stable is independent of σ.

(iv) The solution $x = 0$ of the system (2.1) is said to be uniformly asymptotically stable if it is uniformly stable and there is a $b_0 > 0$ such that, for every $\eta > 0$, there is a $t_0(\eta)$ such that $\phi \in B(0, b_0)$ implies $x_t(\sigma, \phi) \in B(0, \eta)$ for $t \geq \sigma + t_0(\eta)$, for every $\sigma \in R$.

Theorem 2.1 *[6] Assume that $u(\cdot)$ and $w(\cdot)$ are nonnegative continuous, $u(0) = w(0) = 0$, $\lim_{s \to +\infty} u(s) = +\infty$, and that $V : \mathcal{C} \to R$ is continuous and satisfies*

$$V(\phi) \geq u(|\phi(0)|)$$

and

$$\dot{V}(\phi) \leq -w(|\phi(0)|).$$

Then the solution $x = 0$ of the system (2.1) is uniformly stable, and every solutions is bounded. If in addition, $w(s) > 0$ for $s > 0$, then $x = 0$ is globally asymptotically stable.

Lemma 2.1 *[4]* **(Barbălat's Lemma)** *Let f be a nonnegative function defined on $[0, \infty)$ such that f is integrable on $[0, \infty)$ and uniformly continuous on $[0, \infty)$. Then*

$$\lim_{t \to \infty} f(t) = 0.$$

3 The Model

3.1 Formulation of the Model

An aquatic ecosystem consisting of reserved and unreserved areas with time delays is of the form

$$\begin{aligned} \dot{x_1}(t) &= r_1 x_1(t)\left[1 - \frac{x_1(t-\tau_1)}{k_1}\right] - \sigma_1 x_1(t) + \sigma_2 x_2(t) - qe x_1(t) \\ \dot{x_2}(t) &= r_2 x_2(t)\left[1 - \frac{x_2(t-\tau_2)}{k_2}\right] - \sigma_2 x_2(t) + \sigma_1 x_1(t) \end{aligned} \tag{3.1}$$

with the initial conditions

$$\begin{aligned} & x_i(\theta) = \phi_i(\theta) > 0 \quad , \quad \theta \in [-\tau, 0] \quad , \quad \phi_i \in C\left([-\tau, 0], R_+\right) \\ & \tau = \max\{\tau_1, \tau_2\} \quad , \quad i = 1, 2 \end{aligned} \tag{3.2}$$

where r_1, r_2, k_1, k_2, σ_1, σ_2, q, and e are all assumed to be positive constants. For $i = 1, 2$, $x_i(t)$ denote the respective biomass densities of the same fish population inside the unreserved and reserved areas; r_i are the respective intrinsic growth rates of fish subpopulation inside the unreserved and reserved areas; k_i are the respective carrying capacities of fish species in the unreserved and reserved areas. σ_1 denotes the rate that the fish subpopulation of the unreserved area migrate into reserved area, σ_2 denotes the rate that the fish subpopulation of the reserved area migrate into unreserved area. Moreover e is the toatal effort applied for harvesting the fish population in the unreserved area, and q is the catahability coefficient of fish species in the unreserved area.

We note that if $\sigma_2 = 0$ and $r_1 - \sigma_1 - qe < 0$, then $\dot{x_1}(t) < 0$. Similarly, if $\sigma_1 = 0$ and $r_2 - \sigma_2 < 0$, then $\dot{x_2}(t) < 0$. Hence, throught above analysis, we assume that,

$$r_1 - \sigma_1 - qe > 0 \quad , \quad r_2 - \sigma_2 > 0 \tag{3.3}$$

3.2 Existence and Uniqueness of Positive Equilibrium Point

All we want to disscuss is the population of the ecosystem, so we just consider the first quardnt in the x_1-x_2 plane.

Clearly, we can check that the system (3.1) has only two nonnegative equilibrium points, namely $\widehat{E} \equiv (0,0)$ and $E^* \equiv (x_1^*, x_2^*)$, where x_1^* and x_2^* satisfy

$$\sigma_2 x_2^* = \frac{r_1 x_1^{*^2}}{k_1} - (r_1 - \sigma_1 - qe)x_1^* \tag{3.4}$$

$$\sigma_1 x_1^* = (\sigma_2 - r_2)x_2^* + \frac{r_2 x_2^{*^2}}{k_2} \tag{3.5}$$

Subtituting the value of x_2^* from (3.4) into (3.5), we get a cubic equation in x_1^* as

$$a x_1^{*^3} + b x_1^{*^2} + c x_1^* + d = 0$$

where

$$\begin{aligned}
a &= \frac{r_1^2 r_2}{k_1^2 k_2 \sigma_2^2} \\
b &= -\frac{2 r_1 r_2 (r_1 - \sigma_1 - qe)}{k_1 k_2 \sigma_2^2} \\
c &= \frac{r_2 (r_1 - \sigma_1 - qe)^2}{k_2 \sigma_2^2} - \frac{r_1 (r_2 - \sigma_2)}{k_1 \sigma_2} \\
d &= \frac{r_2 - \sigma_2}{\sigma_2}(r_1 - \sigma_1 - qe) - \sigma_1
\end{aligned} \tag{3.6}$$

Remark 3.1 *Let x_1^* and x_2^* satisfy the equations (3.4) and (3.5) and a, b, c and d, defined in* (3.6)*, satisfy $a x_1^{*^3} + b x_1^{*^2} + c x_1^* + d = 0$. If one of the following holds*

$$d < 0 \quad and \quad b > 0; \tag{3.7}$$

$$d < 0 \quad and \quad b < 0 \quad and \quad c - \frac{d}{b} < 0; \tag{3.8}$$

and

$$\frac{r_1 x_1^*}{k_1} > r_1 - \sigma_1 - qe \tag{3.9}$$

then $E^* \equiv (x_1^*, x_2^*)$ is the unique positive equilibrium of the system (3.1).

4 Local Stability

To investigate the local stability of the unique positive equilibrium point E^*, we linearize the system (3.1). Let $y_1(t) = x_1(t) - x_1^*$ *and* $y_2(t) = x_2(t) - x_2^*$ be the perturbed variables. After removing nonlinear terms, we obtain the linear variational system, by using equilibria conditions,

$$\begin{aligned} \dot{y_1}(t) &= \left[r_1(1 - \frac{x_1^*}{k_1}) - \sigma_1 - qe\right] y_1(t) - \left[\frac{r_1 x_1^*}{k_1}\right] y_1(t-\tau_1) + \sigma_2 y_2(t) \\ \dot{y_2}(t) &= \left[r_2(1 - \frac{x_2^*}{k_2}) - \sigma_2\right] y_2(t) - \left[\frac{r_2 x_2^*}{k_1}\right] y_2(t-\tau_2) + \sigma_1 y_1(t) \end{aligned} \tag{4.1}$$

Theorem 4.1 *Let* $E^* = (x_1^*, x_2^*)$ *be the unique positive equilibrium point of the system (3.1) and the delays* τ_1 *and* τ_2 *satisfy*

$$a_1 - a_2\tau_1 - a_3\tau_2 > 0 \tag{4.2}$$

and

$$b_1 - b_2\tau_1 - b_3\tau_2 > 0 \tag{4.3}$$

where

$$\begin{aligned} a_1 &= \frac{4r_1x_1^*}{k_1} + 2qe + \sigma_1 - \sigma_2 - 2r_1, & b_1 &= \frac{4r_2x_2^*}{k_2} - \sigma_1 + \sigma_2 - 2r_2 \\ a_2 &= \frac{2r_1x_1^*}{k_1}\left(r_1 - \sigma_1 + \frac{\sigma_2}{2} - qe + \frac{2r_1x_1^*}{k_1}\right), & b_2 &= \frac{r_1x_1^*}{k_1}\sigma_2 \\ a_3 &= \frac{r_2x_2^*}{k_2}\sigma_1, & b_3 &= \frac{2r_2x_2^*}{k_2}\left(r_2 + \frac{\sigma_1}{2} - \sigma_2 + \frac{2r_2x_2^*}{k_2}\right) \end{aligned}$$

Then the unique positive equilibrium point E^* *of the system (3.1) is locally asymptotically stable.*

Proof : The equation (4.1) can be written as

$$\begin{aligned} \frac{d}{dt}\left[y_1(t) - \frac{r_1x_1^*}{k_1}\int_{t-\tau_1}^{t} y_1(s)ds\right] &= \left[\left(r_1 - \sigma_1 - qe - \frac{2r_1x_1^*}{k_1}\right) y_1(t) + \sigma_2 y_2(t)\right] \\ \frac{d}{dt}\left[y_2(t) - \frac{r_2x_2^*}{k_2}\int_{t-\tau_2}^{t} y_2(s)ds\right] &= \left[\left(r_2 - \sigma_2 - \frac{2r_2x_2^*}{k_2}\right) y_2(t) + \sigma_1 y_1(t)\right] \end{aligned} \tag{4.4}$$

Let

$$V_{11}(y(t)) = \left[y_1(t) - \frac{r_1 x_1^*}{k_1}\int_{t-\tau_1}^{t} y_1(s)ds\right]^2 \tag{4.5}$$

Then

$$\begin{aligned}
\dot{V}_{11}(y(t)) &= 2\left[y_1(t) - \frac{r_1 x_1^*}{k_1}\int_{t-\tau_1}^{t} y_1(s)ds\right]\left[\left(r_1 - \sigma_1 - qe - \frac{2r_1 x_1^*}{k_1}\right)y_1(t)\right. \\
&\quad \left. + \sigma_2 y_2(t)\right] \\
&= 2\left(r_1 - \sigma_1 - qe - \frac{2r_1 x_1^*}{k_1}\right)y_1^2(t) + 2\sigma_2 y_1(t)y_2(t) \\
&\quad - \frac{2r_1 x_1^*}{k_1}\left(r_1 - \sigma_1 - qe - \frac{2r_1 x_1^*}{k_1}\right)\int_{t-\tau_1}^{t} y_1(t)y_1(s)ds \\
&\quad - \frac{2r_1 x_1^*}{k_1}\sigma_2\int_{t-\tau_1}^{t} y_2(t)y_1(s)ds \\
&\le 2\left(r_1 - \sigma_1 - qe - \frac{2r_1 x_1^*}{k_1}\right)y_1^2(t) + 2\sigma_2|y_1(t)||y_2(t)| \\
&\quad + \frac{2r_1 x_1^*}{k_1}\left(r_1 - \sigma_1 - qe + \frac{2r_1 x_1^*}{k_1}\right)\int_{t-\tau_1}^{t} |y_1(t)||y_1(s)|ds \\
&\quad + \frac{2r_1 x_1^*}{k_1}\sigma_2\int_{t-\tau_1}^{t} |y_2(t)||y_1(s)|ds \\
&\le 2\left(r_1 - \sigma_1 - qe - \frac{2r_1 x_1^*}{k_1}\right)y_1^2(t) + \sigma_2 y_1^2(t) + \sigma_2 y_2^2(t) \\
&\quad + \frac{r_1 x_1^*}{k_1}\left(r_1 - \sigma_1 - qe + \frac{2r_1 x_1^*}{k_1}\right)\left[\tau_1 y_1^2(t) + \int_{t-\tau_1}^{t} y_1^2(s)ds\right] \\
&\quad + \frac{r_1 x_1^*}{k_1}\sigma_2\left[\tau_1 y_2^2(t) + \int_{t-\tau_1}^{t} y_1^2(s)ds\right] \qquad (4.6)
\end{aligned}$$

Now, let

$$V_{12}(y(t)) = \frac{r_1 x_1^*}{k_1}\left(r_1 - \sigma_1 + \sigma_2 - qe + \frac{2r_1 x_1^*}{k_1}\right)\int_{t-\tau_1}^{t}\int_{s}^{t} y_1^2(u)duds \tag{4.7}$$

then

$$\begin{aligned}\dot{V}_{12}(y(t)) &= \frac{r_1 x_1^*}{k_1}\left(r_1 - \sigma_1 + \sigma_2 - qe + \frac{2r_1 x_1^*}{k_1}\right)\tau_1 y_1^2(t) \\ &\quad -\frac{r_1 x_1^*}{k_1}\left(r_1 - \sigma_1 + \sigma_2 - qe + \frac{2r_1 x_1^*}{k_1}\right)\int_{t-\tau_1}^{t} y_1^2(s)ds \end{aligned} \tag{4.8}$$

We define

$$V_1(y(t)) = V_{11}(y(t)) + V_{12}(y(t)) \tag{4.9}$$

then

$$\begin{aligned}\dot{V}_1(y(t)) &\leq \left[2\left(r_1 - \sigma_1 - qe - \frac{2r_1 x_1^*}{k_1}\right) + \sigma_2\right] y_1^2(t) \\ &\quad +\frac{2r_1 x_1^*}{k_1}\left(r_1 - \sigma_1 + \frac{\sigma_2}{2} - qe + \frac{2r_1 x_1^*}{k_1}\right)\tau_1 y_1^2(t) \\ &\quad +\sigma_2 y_2^2(t) + \frac{r_1 x_1^*}{k_1}\sigma_2 \tau_1 y_2^2(t) \end{aligned} \tag{4.10}$$

Similarly, let

$$V_{21}(y(t)) = \left[y_2(t) - \frac{r_2 x_2^*}{k_2}\int_{t-\tau_2}^{t} y_2(s)ds\right]^2 \tag{4.11}$$

Then

$$\begin{aligned}\dot{V}_{21}(y(t)) &= 2\left[y_2(t) - \frac{r_2 x_2^*}{k_2}\int_{t-\tau_2}^{t} y_2(s)ds\right]\left[\left(r_2 - \sigma_2 - \frac{2r_2 x_2^*}{k_2}\right)y_2(t) + \sigma_1 y_1(t)\right] \\ &= 2\left(r_2 - \sigma_2 - \frac{2r_2 x_2^*}{k_2}\right)y_2^2(t) + 2\sigma_1 y_1(t) y_2(t) \\ &\quad -\frac{2r_2 x_2^*}{k_2}\left(r_2 - \sigma_2 - \frac{2r_2 x_2^*}{k_2}\right)\int_{t-\tau_2}^{t} y_2(t) y_2(s)ds \end{aligned}$$

$$
\begin{aligned}
&-\frac{2r_2x_2^*}{k_2}\sigma_1\int_{t-\tau_2}^{t} y_1(t)y_2(s)ds \\
\leq\ & 2\left(r_2-\sigma_2-\frac{2r_2x_2^*}{k_2}\right)y_2^2(t)+2\sigma_1|y_1(t)||y_2(t)| \\
&+\frac{2r_2x_2^*}{k_2}\left(r_2-\sigma_2+\frac{2r_2x_2^*}{k_2}\right)\int_{t-\tau_2}^{t}|y_2(t)||y_2(s)|ds \\
&+\frac{2r_2x_2^*}{k_2}\sigma_1\int_{t-\tau_2}^{t}|y_1(t)||y_2(s)|ds \\
\leq\ & 2\left(r_2-\sigma_2-\frac{2r_2x_2^*}{k_2}\right)y_2^2(t)+\sigma_1y_1^2(t)+\sigma_1y_2^2(t) \\
&+\frac{r_2x_2^*}{k_2}\left(r_2-\sigma_2+\frac{2r_2x_2^*}{k_2}\right)\left[\tau_2y_2^2(t)+\int_{t-\tau_2}^{t}y_2^2(s)ds\right] \\
&+\frac{r_2x_2^*}{k_2}\sigma_1\left[\tau_2y_1^2(t)+\int_{t-\tau_2}^{t}y_2^2(s)ds\right] \qquad (4.12)
\end{aligned}
$$

Now, let

$$
V_{22}(y(t))=\frac{r_2x_2^*}{k_2}\left(r_2+\sigma_1-\sigma_2+\frac{2r_2x_2^*}{k_2}\right)\int_{t-\tau_2}^{t}\int_{s}^{t}y_2^2(u)duds \qquad (4.13)
$$

then

$$
\begin{aligned}
\dot{V}_{22}(y(t)) &= \frac{r_2x_2^*}{k_2}\left(r_2+\sigma_1-\sigma_2+\frac{2r_2x_2^*}{k_2}\right)\tau_2y_2^2(t) \\
&-\frac{r_2x_2^*}{k_2}\left(r_2+\sigma_1-\sigma_2+\frac{2r_2x_2^*}{k_2}\right)\int_{t-\tau_2}^{t}y_2^2(s)ds \qquad (4.14)
\end{aligned}
$$

We define

$$
V_2(y(t))=V_{21}(y(t))+V_{22}(y(t)) \qquad (4.15)
$$

then

$$
\dot{V}_2(y(t)) \leq \left[2\left(r_2-\sigma_2-\frac{2r_2x_2^*}{k_2}\right)+\sigma_1\right]y_2^2(t)
$$

$$+\frac{2r_2x_2^*}{k_2}\left(r_2+\frac{\sigma_1}{2}-\sigma_2+\frac{2r_2x_2^*}{k_2}\right)\tau_2 y_2^2(t)$$

$$+\sigma_1 y_1^2(t)+\frac{r_2x_2^*}{k_2}\sigma_1\tau_2 y_1^2(t) \tag{4.16}$$

Now we define a Lyapunov functional $V(y(t))$ as

$$V(y(t)) \;=\; V_1(y(t))+V_2(y(t)) \tag{4.17}$$

then we have from (4.10) and (4.16) that

$$\begin{aligned}
\dot V(y(t)) &= \dot V_1(y(t))+\dot V_2(y(t))\\
&\le \left\{\left[2\left(r_1-\sigma_1-qe-\frac{2r_1x_1^*}{k_1}\right)+\sigma_1+\sigma_2\right]\right.\\
&\quad \left.+\left[\frac{2r_1x_1^*}{k_1}\left(r_1-\sigma_1+\frac{\sigma_2}{2}-qe+\frac{2r_1x_1^*}{k_1}\right)\right]\tau_1+\left[\frac{r_2x_2^*}{k_2}\sigma_1\right]\tau_2\right\}y_1^2(t)\\
&\quad +\left\{\left[2\left(r_2-\sigma_2-\frac{2r_2x_2^*}{k_2}\right)+\sigma_1+\sigma_2\right]\right.\\
&\quad \left.+\left[\frac{r_1x_1^*}{k_1}\sigma_2\right]\tau_1+\left[\frac{2r_2x_2^*}{k_2}\left(r_2+\frac{\sigma_1}{2}-\sigma_2+\frac{2r_2x_2^*}{k_2}\right)\right]\tau_2\right\}y_2^2(t)\\
&= -\left\{\left[\frac{4r_1x_1^*}{k_1}+2qe+\sigma_1-\sigma_2-2r_1\right]\right.\\
&\quad \left.-\left[\frac{2r_1x_1^*}{k_1}\left(r_1-\sigma_1+\frac{\sigma_2}{2}-qe+\frac{2r_1x_1^*}{k_1}\right)\right]\tau_1-\left[\frac{r_2x_2^*}{k_2}\sigma_1\right]\tau_2\right\}y_1^2(t)\\
&\quad -\left\{\left[\frac{4r_2x_2^*}{k_2}-\sigma_1+\sigma_2-2r_2\right]\right.\\
&\quad \left.-\left[\frac{r_1x_1^*}{k_1}\sigma_2\right]\tau_1-\left[\frac{2r_2x_2^*}{k_2}\left(r_2+\frac{\sigma_1}{2}-\sigma_2+\frac{2r_2x_2^*}{k_2}\right)\right]\tau_2\right\}y_2^2(t)\\
&\equiv -\zeta_1 y_1^2(t)-\zeta_2 y_2^2(t)
\end{aligned} \tag{4.18}$$

Clearly, (4.2) and (4.3) implies that $\zeta_1 > 0$ and $\zeta_2 > 0$. Denote $\zeta = \min\{\zeta_1, \zeta_2\}$, then (4.18) leads to

$$V(t) + \zeta \int_T^t [y_1^2(s) + y_2^2(s)]ds \leq V(T) \quad \text{for} \quad t \geq T \tag{4.19}$$

which in turn implies $y_1^2(t)+y_2^2(t) \in L_1[T,\infty)$. We can see from (4.1) and the boundedness of $y(t)$ that $y_1^2(t) + y_2^2(t)$ is uniformly continuous. By using lemma 2.1, we conclude that $\lim_{t\to\infty}[y_1^2(t)+y_2^2(t)] = 0$. Therefore the zero solution of (4.1) is asymptotically stable and this completes the proof. ■

5 Uniform Persistence

The system (3.1) has a unique positive equilibrium point if (3.7) or (3.8) and (3.9) holds. In the following, we always assume that such a positive equilibrium exists and denotes it by $E^*(x_1^*, x_2^*)$. The following lemmas are elementary which are concerned with the qualitative nature of solutions of the system (3.1).

Lemma 5.1 *All solutions of the system* (3.1) *with the initial conditions* (3.2) *are positive for all* $t \geq 0$.

Proof : It is true because

$$\begin{aligned} x_1(t) &= x_1(0)exp\left\{\int_0^t \left[r_1\left(1 - \frac{x_1(s-\tau_1)}{k_1}\right) - \sigma_1 + \sigma_2\frac{x_2}{x_1} - qe\right]ds\right\} \\ x_2(t) &= x_2(0)exp\left\{\int_0^t \left[r_2\left(1 - \frac{x_2(s-\tau_2)}{k_2}\right) - \sigma_2 + \sigma_1\frac{x_1}{x_2}\right]ds\right\} \end{aligned} \tag{5.1}$$

and $x_i(0) > 0 (i = 1, 2)$. Therefore, all solutions $(x_1(t), x_2(t))$ of the system (3.1) with the initial conditions (3.2) are positive. ■

Lemma 5.2 *Let* $y(t)$ *be the positive solution of the system*

$$\dot{y}(t) \leq ry(t)\left[1 - \frac{y(t-\tau)}{k}\right] \tag{5.2}$$

with the initial condition

$$y(\theta) = \phi_i(\theta) > 0 \quad , \quad \theta \in [-\tau, 0] \quad , \quad \phi \in C\left([-\tau, 0], R_+\right) \tag{5.3}$$

then

$$0 < y(t) \leq ke^{r\tau} \tag{5.4}$$

eventually for all large t.

Proof : Suppose $y(t)$ is not oscillatory about k. That is, there exists a $\bar{t} > 0$ such that either

$$y(t) \leq k \qquad\qquad \text{for} \quad t > \bar{t} \tag{5.5}$$

or

$$y(t) > k \qquad\qquad \text{for} \quad t > \bar{t} \tag{5.6}$$

If (5.5) holds for $t > \bar{t}$, it follows

$$y(t) \leq k \leq ke^{r\tau}$$

then $y(t) \leq ke^{r\tau}$ follows. If (5.6) holds for $t > \bar{t}$, inequality (5.2) implies that for $t > \bar{t} + \tau$ and for some constant $a > 0$

$$\begin{aligned} \dot{y}(t) &\leq ry(t)\left[1 - \frac{y(t-\tau)}{k}\right] \\ &\leq -ary(t) \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\bar{t}+\tau}^{t} \frac{\dot{y}(s)}{y(s)} ds &\leq \int_{\bar{t}+\tau}^{t} -ar ds \\ &= -ar(t - \bar{t} - \tau) \end{aligned}$$

Then $0 \leq y(t) \leq y(\bar{t} + \tau)e^{-ar(t-\bar{t}-\tau)}$. Thus, $\lim_{t\to\infty} y(t) = 0$ by Squeeze Theorem, which contradics (5.6). Therefore case (5.6) fails.

Suppose now $y(t)$ is oscillatory about k. Let $y(t^*)$ denote any arbitrary local maximum of $y(t)$, then it follows from (5.2) that

$$0 = \dot{y}(t^*) \leq ry(t^*)\left[1 - \frac{y(t^* - \tau)}{k}\right] \tag{5.7}$$

and this implies

$$y(t^* - \tau) \leq k$$

By the continuity of $y(t)$, we conclude that there exists a $\hat{t} \in [t^* - \tau, t^*)$ such that $y(\hat{t}) = k$. It follows that,

$$\begin{aligned} \int_{\hat{t}}^{t^*} \frac{\dot{y}(s)}{y(s)} ds &\leq \int_{\hat{t}}^{t^*} \left[r(1 - \frac{y(s-\tau)}{k})\right] ds \\ &\leq r(t^* - \hat{t}) \leq r\tau \end{aligned} \tag{5.8}$$

which implies that

$$y(t^*) \leq ke^{r\tau}$$

Since $y(t^*)$ is an arbitrary local maximum of $y(t)$, we can conclude that there exists some $\tilde{t} > 0$ such that $y(t) \leq ke^{r\tau}$ for $t > \tilde{t}$. ■

Theorem 5.1 *Suppose that the system* (3.1) *satisfies* (3.3). *Then the system* (3.1) *is uniformly persistent, that is, there exist* m_1, m_2, M *and* $T^* > 0$ *such that* $m_1 \leq x_1(t) \leq M$ *and* $m_2 \leq x_2(t) \leq M$ *for* $t > T^*$, *where*

$$M = k_1 e^{r_1\tau_1} + k_2 e^{r_2\tau_2} \tag{5.9}$$

$$m_1 = \frac{k_1(r_1 - \sigma_1 - qe)}{1.1r_1} \exp\left(\left[(r_1 - \sigma_1 - qe) - \frac{r_1 M}{k_1}\right]\tau_1\right) \tag{5.10}$$

$$m_2 = \frac{k_2(r_2 - \sigma_2)}{1.1r_2} \exp\left(\left[(r_2 - \sigma_2) - \frac{r_2 M}{k_2}\right]\tau_2\right) \tag{5.11}$$

Proof : By Lemma 5.1, we know that all solutions of the system (3.1) with the intial conditions (3.2) are positive, and by (3.1) we have

$$\begin{aligned}
\dot{x_1}(t) + \dot{x_2}(t) &= r_1 x_1(t)\left[1 - \frac{x_1(t-\tau_1)}{k_1}\right] + r_2 x_2(t)\left[1 - \frac{x_2(t-\tau_2)}{k_2}\right] \\
&\quad - qe x_1(t) \\
&\leq r_1 x_1(t)\left[1 - \frac{x_1(t-\tau_1)}{k_1}\right] - \frac{qe x_1(t)}{4} \\
&\quad + r_2 x_2(t)\left[1 - \frac{x_2(t-\tau_2)}{k_2}\right] - \frac{qe x_1(t)}{4} \\
&\equiv \dot{y_1}(t) + \dot{y_2}(t)
\end{aligned} \tag{5.12}$$

This implies

$$\dot{x_1}(s) + \dot{x_2}(s) \leq \dot{y_1}(s) + \dot{y_2}(s) \tag{5.13}$$

where

$$\begin{aligned}
\dot{y_1}(s) &= r_1 y_1(s)\left[1 - \frac{y_1(s-\tau_1)}{k_1}\right] - \frac{qe y_1(s)}{4} \\
\dot{y_2}(s) &= r_2 y_2(s)\left[1 - \frac{y_2(s-\tau_2)}{k_2}\right] - \frac{qe y_1(s)}{4}
\end{aligned} \tag{5.14}$$

Integrating (5.13) over $[0, t]$ and taking $x_1(0) + x_2(0) = y_1(0) + y_2(0)$, we have

$$x_1(t) + x_2(t) \leq y_1(t) + y_2(t) \tag{5.15}$$

From (5.14), it follows

$$\begin{aligned}
\dot{y_1}(s) &\leq r_1 y_1(s)\left[1 - \frac{y_1(s-\tau_1)}{k_1}\right] \\
\dot{y_2}(s) &\leq r_2 y_2(s)\left[1 - \frac{y_2(s-\tau_2)}{k_2}\right]
\end{aligned} \tag{5.16}$$

By Lemma 5.2, there exit T_1 and T_2 such that

$$y_1(t) \quad \leq \quad k_1 e^{r_1\tau_1} \qquad \text{for} \quad t > T_1 \tag{5.17}$$

and

$$y_2(t) \quad \leq \quad k_2 e^{r_2\tau_2} \qquad \text{for} \quad t > T_2 \tag{5.18}$$

By (5.15), we have

$$x_1(t) + x_2(t) \leq k_1 e^{r_1\tau_1} + k_2 e^{r_2\tau_2} \tag{5.19}$$

for $t > T = max\{T_1, T_2\}$. Hence,

$$x_1(t) \leq k_1 e^{r_1\tau_1} + k_2 e^{r_2\tau_2} \equiv M$$

$$x_2(t) \leq k_1 e^{r_1\tau_1} + k_2 e^{r_2\tau_2} \equiv M \tag{5.20}$$

for $t > T$. Now we want to show that $x_i(t) \geq m_i$ for all large t, $i = 1, 2$. From (3.1), we have for $t > T + \tau_1$

$$\begin{aligned} \dot{x_1}(t) &= x_1(t)\left[(r_1 - \sigma_1 - qe) - \frac{r_1}{k_1}x_1(t-\tau_1) + \sigma_2\frac{x_2(t)}{x_1(t)}\right] \\ &\geq x_1(t)\left[(r_1 - \sigma_1 - qe) - \frac{r_1 M}{k_1}\right] \end{aligned} \tag{5.21}$$

Integrating (5.21) over $[t - \tau_1, t]$, where $t \geq T + 2\tau_1$, we have

$$\begin{aligned} \ln\left(\frac{x_1(t)}{x_1(t-\tau_1)}\right) &\geq \int_{t-\tau_1}^{t}\left[(r_1 - \sigma_1 - qe) - \frac{r_1 M}{k_1}\right] ds \\ &= \left(r_1 - \sigma_1 - qe - \frac{r_1 M}{k_1}\right)\tau_1 \end{aligned} \tag{5.22}$$

which implies that, for $t \geq T + 2\tau_1$

$$x_1(t) \geq x_1(t-\tau_1)\exp\left(\left[(r_1 - \sigma_1 - qe) - \frac{r_1 M}{k_1}\right]\tau_1\right) \tag{5.23}$$

It follows from (5.21) that for $t \geq T + 2\tau_1$

$$\begin{aligned}
\dot{x_1}(t) &= x_1(t)\left[(r_1 - \sigma_1 - qe) - \frac{r_1}{k_1} x_1(t-\tau_1) + \sigma_2 \frac{x_2}{x_1}\right] \\
&\geq x_1(t)\left[(r_1 - \sigma_1 - qe) - \frac{r_1}{k_1} x_1(t-\tau_1)\right] \\
&\geq x_1(t)\left\{(r_1 - \sigma_1 - qe) - \frac{r_1}{k_1} \exp\left(\left[\frac{r_1 M}{k_1} - (r_1 - \sigma_1 - qe)\right]\tau_1\right) x_1(t)\right\}
\end{aligned} \tag{5.24}$$

It follows that

$$\begin{aligned}
\liminf_{t\to\infty} x_1(t) &\geq \frac{k_1(r_1 - \sigma_1 - qe)}{r_1} \exp\left(\left[(r_1 - \sigma_1 - qe) - \frac{r_1 M}{k_1}\right]\tau_1\right) \\
&\equiv \overline{m}_1
\end{aligned} \tag{5.25}$$

and $\overline{m}_1 > 0$ by (3.3). So, for large t, $x_1(t) > \overline{m}_1/1.1 \equiv m_1 > 0$. Similarly, From (3.1), we have, for $t > T^* + \tau_2$

$$\begin{aligned}
\dot{x_2}(t) &= x_2(t)\left[(r_2 - \sigma_1) - \frac{r_2}{k_2} x_2(t-\tau_2) + \sigma_1 \frac{x_1(t)}{x_2(t)}\right] \\
&\geq x_2(t)\left[(r_2 - \sigma_2) - \frac{r_2 M}{k_2}\right]
\end{aligned} \tag{5.26}$$

Integrating (5.26) over $[t - \tau_2, t]$, where $t \geq T + 2\tau_2$, we have

$$\begin{aligned}
\ln\left(\frac{x_2(t)}{x_2(t-\tau_2)}\right) &\geq \int_{t-\tau_2}^{t} \left[(r_2 - \sigma_2) - \frac{r_2 M}{k_2}\right] ds \\
&= \left(r_2 - \sigma_2 - \frac{r_2 M}{k_2}\right)\tau_2
\end{aligned} \tag{5.27}$$

which implies that, for $t \geq T + 2\tau_2$

$$x_2(t) \geq x_2(t-\tau_2) \exp\left(\left[(r_2 - \sigma_2) - \frac{r_2 M}{k_2}\right]\tau_2\right) \tag{5.28}$$

It follows from (5.26) that, for $t \geq T + 2\tau_2$

$$\begin{aligned}
\dot{x_2}(t) &= x_2(t)\left[(r_2-\sigma_2) - \frac{r_2}{k_2}x_2(t-\tau_2) + \sigma_1\frac{x_1}{x_2}\right] \\
&\geq x_2(t)\left[(r_2-\sigma_2) - \frac{r_2}{k_2}x_2(t-\tau_2)\right] \\
&\geq x_2(t)\left\{(r_2-\sigma_2) - \frac{r_2}{k_2}\exp\left(\left[\frac{r_2M}{k_2} - (r_2-\sigma_2)\right]\tau_2\right)x_2(t)\right\}
\end{aligned} \tag{5.29}$$

It follows that

$$\begin{aligned}
\liminf_{t\to\infty} x_2(t) &\geq \frac{k_2(r_2-\sigma_2)}{r_2}\exp\left(\left[(r_2-\sigma_2) - \frac{r_2M}{k_2}\right]\tau_2\right) \\
&\equiv \overline{m}_2
\end{aligned} \tag{5.30}$$

and $\overline{m}_2 > 0$ by (3.3). So, for large t, $x_2(t) > \overline{m}_2/1.1 \equiv m_2 > 0$. Let

$$\mathcal{D} = \{(x_1(t), x_2(t)) | m_1 \leq x_1(t) \leq M, m_2 \leq x_2(t) \leq M\}.$$

Then $\mathcal{D}$ is a bounded compact region in R_+^2 that positive distance from coordinate hyperplancds. Hence we obtain that there exists a $T^* > 0$ such that if $t \geq T^*$, then every positive solution of system (3.1) with the initial conditions (3.2) eventually enters and remains in the region $\mathcal{D}$, that is, system (3.1) is uniformly persistent. ■

6 Global Asymptotical Stability

In this section, we derive sufficient conditions which guarantee that the positive equilibrium point E^* of the system (3.1) is globally asymptotically stable. Our method in the proof is to construct a suitable Lyapunov functinal.

Theorem 6.1 *Let E^* be the unique equilibrium point of the system (3.1) and the delays τ_1 and τ_2 satisfy*

$$\alpha_1 - \alpha_2\tau_1 - \alpha_3\tau_2 > 0 \quad \textit{and} \quad \beta_1 - \beta_2\tau_1 - \beta_3\tau_2 > 0 \tag{6.1}$$

where $M_i (i = 1, 2)$ are defined by (5.9), and

$$\alpha_1 = \frac{r_1 x_1^*}{k_1} + \frac{\sigma_2 x_2^*}{M} - \frac{\sigma_1 m_1 x_1^* + \sigma_2 m_2 x_2^*}{2m_1 m_2}$$

$$\alpha_2 = \frac{r_1^2 M x_1^*}{k_1^2} + \frac{3r_1\sigma_2 M x_2^*}{2m_1 k_1}$$

$$\alpha_3 = \frac{r_2\sigma_1 M x_1^*}{2m_2 k_2}$$

$$\beta_1 = \frac{r_2 x_2^*}{k_2} + \frac{\sigma_1 x_1^*}{M} - \frac{\sigma_1 m_1 x_1^* + \sigma_2 m_2 x_2^*}{2m_1 m_2}$$

$$\beta_2 = \frac{r_1\sigma_2 M x_2^*}{2m_1 k_1}$$

$$\beta_3 = \frac{r_2^2 M x_2^*}{k_2^2} + \frac{3r_2\sigma_1 M x_1^*}{2m_2 k_2}$$

then the unique positive equilibrium point E^* of the system (3.1) is globally asymptotically stable.

Proof : Define

$$z(t) \;=\; (z_1(t), z_2(t))$$

by

$$x_1(t) = x_1^*(1 + z_1(t)),\ x_2(t) = x_2^*(1 + z_2(t)).$$

From(3.1), we have

$$\begin{aligned}\frac{dz_1(t)}{dt} &= (1+z_1(t))\left\{-\frac{r_1x_1^*}{k_1}z_1(t-\tau_1)+\frac{\sigma_2x_2^*}{x_1^*}\frac{z_2(t)}{1+z_1(t)}-\frac{\sigma_2x_2^*}{x_1^*}\frac{z_1(t)}{1+z_1(t)}\right\}\\ \frac{dz_2(t)}{dt} &= (1+z_2(t))\left\{-\frac{r_2x_2^*}{k_2}z_2(t-\tau_2)+\frac{\sigma_1x_1^*}{x_2^*}\frac{z_1(t)}{1+z_2(t)}-\frac{\sigma_1x_1^*}{x_2^*}\frac{z_2(t)}{1+z_2(t)}\right\}\end{aligned} \tag{6.2}$$

Let

$$V_1(z_t) = \{z_1(t)-\ln[1+z_1(t)]\}+\{z_2(t)-\ln[1+z_2(t)]\} \tag{6.3}$$

then we have from (6.2) that

$$\begin{aligned}\dot{V}_1(z_t) &= \frac{z_1(t)\dot{z}_1(t)}{1+z_1(t)}+\frac{z_2(t)\dot{z}_2(t)}{1+z_2(t)}\\ &= -\frac{r_1x_1^*}{k_1}z_1(t)z_1(t-\tau_1)+\frac{\sigma_2x_2^*}{x_1^*(1+z_1(t))}z_1(t)z_2(t)\\ &\quad -\frac{\sigma_2x_2^*}{x_1^*(1+z_1(t))}z_1^2(t)-\frac{r_2x_2^*}{k_2}z_2(t)z_2(t-\tau_2)\\ &\quad +\frac{\sigma_1x_1^*}{x_2^*(1+z_2(t))}z_1(t)z_2(t)-\frac{\sigma_1x_1^*}{x_2^*(1+z_2(t))}z_2^2(t)\end{aligned} \tag{6.4}$$

By Theorem 5.1, there exist a $T^*>0$ such that $m_1 \le x_1^*[1+z_1(t)] \le M$, and $m_2 \le x_2^*[1+z_2(t)] \le M_2$ for $t > T^*$. Then (6.4) implies that

$$\begin{aligned}\dot{V}_1(z_t) &\le -\frac{r_1x_1^*}{k_1}z_1(t)\left(z_1(t)-\int_{t-\tau_1}^{t}\dot{z}_1(s)ds\right)+\frac{\sigma_2x_2^*}{x_1^*(1+z_1(t))}z_1(t)z_2(t)\\ &\quad -\frac{\sigma_2x_2^*}{M}z_1^2(t)-\frac{r_2x_2^*}{k_2}z_2(t)\left(z_2(t)-\int_{t-\tau_2}^{t}\dot{z}_2(s)ds\right)\\ &\quad +\frac{\sigma_1x_1^*}{x_2^*(1+z_2(t))}z_1(t)z_2(t)-\frac{\sigma_1x_1^*}{M}z_2^2(t)\end{aligned}$$

$$
\begin{aligned}
\leq\ & -\frac{r_1x_1^*}{k_1}z_1^2(t)+\frac{r_1x_1^*}{k_1}z_1(t)\int_{t-\tau_1}^{t}\dot{z}_1(s)ds+\frac{\sigma_2x_2^*}{m_1}|z_1(t)||z_2(t)| \\
& -\frac{\sigma_2x_2^*}{M}z_1^2(t)+-\frac{r_2x_2^*}{k_2}z_2^2(t)+\frac{r_2x_2^*}{k_2}z_2(t)\int_{t-\tau_2}^{t}\dot{z}_2(s)ds \\
& +\frac{\sigma_1x_1^*}{m_2}|z_1(t)||z_2(t)|-\frac{\sigma_1x_1^*}{M}z_2^2(t) \\
=\ & -\frac{r_1x_1^*}{k_1}+\frac{\sigma_2x_2^*}{M}z_1^2(t)-\frac{r_2x_2^*}{k_2}+\frac{\sigma_1x_1^*}{M}z_2^2(t) \\
& +\frac{\sigma_2x_2^*}{2m_1}z_1^2(t)+\frac{\sigma_2x_2^*}{2m_1}z_2^2(t)+\frac{\sigma_1x_1^*}{2m_2}z_1^2(t)+\frac{\sigma_1x_1^*}{2m_2}z_2^2(t) \\
& +\frac{r_1x_1^*}{k_1}z_1(t)\left\{\int_{t-\tau_1}^{t}(1+z_1(s))\left[-\frac{r_1x_1^*}{k_1}z_1(s-\tau_1)\right.\right. \\
& \left.\left.+\frac{\sigma_2x_2^*}{x_1^*}\frac{z_2(s)}{1+z_1(s)}-\frac{\sigma_2x_2^*}{x_1^*}\frac{z_1(s)}{1+z_1(s)}\right]ds\right\} \\
& +\frac{r_2x_2^*}{k_2}z_2(t)\left\{\int_{t-\tau_2}^{t}(1+z_2(s))\left[-\frac{r_2x_2^*}{k_2}z_2(s-\tau_2)\right.\right. \\
& \left.\left.+\frac{\sigma_1x_1^*}{x_2^*}\frac{z_1(s)}{1+z_2(s)}-\frac{\sigma_1x_1^*}{x_2^*}\frac{z_2(s)}{1+z_2(s)}\right]ds\right\} \\
\leq\ & -\left(\frac{r_1x_1^*}{k_1}+\frac{\sigma_2x_2^*}{M}-\frac{\sigma_1m_1x_1^*+\sigma_2m_2x_2^*}{2m_1m_2}\right)z_1^2(t) \\
& -\left(\frac{r_2x_2^*}{k_2}+\frac{\sigma_1x_1^*}{M}-\frac{\sigma_1m_1x_1^*+\sigma_2m_2x_2^*}{2m_1m_2}\right)z_2^2(t) \\
& +\frac{r_1x_1^*}{k_1}\int_{t-\tau_1}^{t}(1+z_1(s))\left[\frac{r_1x_1^*}{k_1}|z_1(t)||z_1(s-\tau_1)|+\frac{\sigma_2x_2^*}{m_1}|z_1(t)||z_2(s)|\right.
\end{aligned}
$$

$$+\frac{\sigma_2 x_2^*}{m_1}|z_1(t)||z_1(s)|\bigg] ds$$

$$+\frac{r_2 x_2^*}{k_2}\int_{t-\tau_2}^{t}(1+z_2(s))\left[\frac{r_2 x_2^*}{k_2}|z_2(t)||z_2(s-\tau_2)|+\frac{\sigma_1 x_1^*}{m_2}|z_2(t)||z_1(s)|\right.$$

$$\left.+\frac{\sigma_1 x_1^*}{m_2}|z_2(t)||z_2(s)|\right] ds \tag{6.5}$$

For $t \geq T^* + \tau \equiv \tilde{T}$, we have from (6.5) that

$$\begin{aligned}
\dot{V}_1(t) \ \leq\ & -\left(\frac{r_1 x_1^*}{k_1}+\frac{\sigma_2 x_2^*}{M}-\frac{\sigma_1 m_1 x_1^*+\sigma_2 m_2 x_2^*}{2m_1 m_2}\right) z_1^2(t) \\
& -\left(\frac{r_2 x_2^*}{k_2}+\frac{\sigma_1 x_1^*}{M}-\frac{\sigma_1 m_1 x_1^*+\sigma_2 m_2 x_2^*}{2m_1 m_2}\right) z_2^2(t) \\
& +\frac{r_1 M}{k_1}\int_{t-\tau_1}^{t}\left[\frac{r_1 x_1^*}{k_1}|z_1(t)||z_1(s-\tau_1)|+\frac{\sigma_2 x_2^*}{m_1}|z_1(t)||z_2(s)|\right. \\
& \left.+\frac{\sigma_2 x_2^*}{m_1}|z_1(t)||z_1(s)|\right] ds \\
& +\frac{r_2 M}{k_2}\int_{t-\tau_2}^{t}\left[\frac{r_2 x_2^*}{k_2}|z_2(t)||z_2(s-\tau_2)|+\frac{\sigma_1 x_1^*}{m_2}|z_2(t)||z_1(s)|\right. \\
& \left.+\frac{\sigma_1 x_1^*}{m_2}|z_2(t)||z_2(s)|\right] ds \\
\leq\ & -\left(\frac{r_1 x_1^*}{k_1}+\frac{\sigma_2 x_2^*}{M}-\frac{\sigma_1 m_1 x_1^*+\sigma_2 m_2 x_2^*}{2m_1 m_2}\right) z_1^2(t) \\
& -\left(\frac{r_2 x_2^*}{k_2}+\frac{\sigma_1 x_1^*}{M}-\frac{\sigma_1 m_1 x_1^*+\sigma_2 m_2 x_2^*}{2m_1 m_2}\right) z_2^2(t) \\
& +\frac{r_1 M}{k_1}\left\{\left[\frac{r_1 x_1^*}{2k_1}+\frac{\sigma_2 x_2^*}{m_1}\right]\tau_1 z_1^2(t)+\int_{t-\tau_1}^{t}\frac{r_1 x_1^*}{2k_1} z_1^2(s-\tau_1) ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t-\tau_1}^{t}\frac{\sigma_2 x_2^*}{2m_1}z_1^2(s)ds+\int_{t-\tau_1}^{t}\frac{\sigma_2 x_2^*}{2m_1}z_2^2(s)ds\bigg\}\\
&+\frac{r_2 M}{k_2}\left\{\left[\frac{r_2 x_2^*}{2k_2}+\frac{\sigma_1 x_1^*}{m_2}\right]\tau_2 z_2^2(t)+\int_{t-\tau_2}^{t}\frac{r_2 x_2^*}{2k_2}z_2^2(s-\tau_2)ds\right.\\
&\left.+\int_{t-\tau_2}^{t}\frac{\sigma_1 x_1^*}{2m_2}z_1^2(s)ds+\int_{t-\tau_2}^{t}\frac{\sigma_1 x_1^*}{2m_2}z_2^2(s)ds\right\}\\
= &-\left[\left(\frac{r_1 x_1^*}{k_1}+\frac{\sigma_2 x_2^*}{M}-\frac{\sigma_1 m_1 x_1^*+\sigma_2 m_2 x_2^*}{2m_1 m_2}\right)\right.\\
&\left.-\left(\frac{r_1^2 M x_1^*}{2k_1^2}+\frac{r_1\sigma_2 M x_2^*}{m_1 k_1}\right)\tau_1\right]z_1^2(t)\\
&-\left[\left(\frac{r_2 x_2^*}{k_2}+\frac{\sigma_1 x_1^*}{M}-\frac{\sigma_1 m_1 x_1^*+\sigma_2 m_2 x_2^*}{2m_1 m_2}\right)\right.\\
&\left.-\left(\frac{r_2^2 M x_2^*}{2k_2^2}+\frac{r_2\sigma_1 M x_1^*}{m_2 k_2}\right)\tau_2\right]z_2^2(t)\\
&+\int_{t-\tau_1}^{t}\frac{r_1^2 M x_1^*}{2k_1^2}z_1^2(s-\tau_1)ds+\int_{t-\tau_1}^{t}\frac{r_1\sigma_2 M x_2^*}{2m_1 k_1}z_1^2(s)ds\\
&+\int_{t-\tau_1}^{t}\frac{r_1\sigma_2 M x_2^*}{2m_1 k_1}z_2^2(s)ds+\int_{t-\tau_2}^{t}\frac{r_2^2 M x_2^*}{2k_2^2}z_2^2(s-\tau_2)ds\\
&+\int_{t-\tau_2}^{t}\frac{r_2\sigma_1 M x_1^*}{2m_2 k_2}z_1^2(s)ds+\int_{t-\tau_2}^{t}\frac{r_2\sigma_1 M x_1^*}{2m_2 k_2}z_2^2(s)ds
\end{aligned}
\tag{6.6}
$$

Let

$$
\begin{aligned}
V_2(z_t) = &\frac{r_1^2 M x_1^*}{2k_1^2}\int_{t-\tau_1}^{t}\int_{s}^{t}z_1^2(\eta-\tau_1)d\eta ds+\frac{r_1\sigma_2 M x_2^*}{2m_1 k_1}\int_{t-\tau_1}^{t}\int_{s}^{t}z_1^2(\eta)d\eta ds\\
&+\frac{r_1\sigma_2 M x_2^*}{2m_1 k_1}\int_{t-\tau_1}^{t}\int_{s}^{t}z_2^2(\eta)d\eta ds+\frac{r_2^2 M x_2^*}{2k_2^2}\int_{t-\tau_2}^{t}\int_{s}^{t}z_2^2(\eta-\tau_2)d\eta ds
\end{aligned}
$$

$$+\frac{r_2\sigma_1 M x_1^*}{2m_2k_2}\int_{t-\tau_2}^{t}\int_{s}^{t} z_1^2(\eta)d\eta ds+\frac{r_2\sigma_1 M x_1^*}{2m_2k_2}\int_{t-\tau_2}^{t}\int_{s}^{t} z_2^2(\eta)d\eta ds \tag{6.7}$$

Then

$$\begin{aligned}
\dot{V}_2(z_t)dt \;=\; & \frac{r_1^2 M x_1^*\tau_1}{2k_1^2}z_1^2(t-\tau_1)-\frac{r_1^2 M x_1^*}{2k_1^2}\int_{t-\tau_1}^{t} z_1^2(s-\tau_1)ds \\
& +\frac{r_1\sigma_2 M x_2^*\tau_1}{2m_1k_1}z_1^2(t)-\frac{r_1\sigma_2 M x_2^*}{2m_1k_1}\int_{t-\tau_1}^{t} z_1^2(s)ds \\
& +\frac{r_1\sigma_2 M x_2^*\tau_1}{2m_1k_1}z_2^2(t)-\frac{r_1\sigma_2 M x_2^*}{2m_1k_1}\int_{t-\tau_1}^{t} z_2^2(s)ds \\
& +\frac{r_2^2 M x_2^*\tau_2}{2k_2^2}z_2^2(t-\tau_2)-\frac{r_2^2 M x_2^*}{2k_2^2}\int_{t-\tau_2}^{t} z_2^2(s-\tau_2)ds \\
& +\frac{r_2\sigma_1 M x_1^*\tau_2}{2m_2k_2}z_1^2(t)-\frac{r_2\sigma_1 M x_1^*}{2m_2k_2}\int_{t-\tau_2}^{t} z_1^2(s)ds \\
& +\frac{r_2\sigma_1 M x_1^*\tau_2}{2m_2k_2}z_2^2(t)-\frac{r_2\sigma_1 M x_1^*}{2m_2k_2}\int_{t-\tau_2}^{t} z_2^2(s)ds
\end{aligned} \tag{6.8}$$

From (6.6) and (6.8), we have, for $t \geq \tilde{T}$

$$\begin{aligned}
\frac{dV_1(z_t)}{dt}+\frac{dV_2(z_t)}{dt} \;\leq\; & -\Bigg\{\left[\frac{r_1x_1^*}{k_1}+\frac{\sigma_2x_2^*}{M}-\frac{\sigma_1m_1x_1^*+\sigma_2m_2x_2^*}{2m_1m_2}\right] \\
& -\left[\frac{r_1^2Mx_1^*}{2k_1^2}+\frac{r_1\sigma_2Mx_2^*}{m_1k_1}\right]\tau_1\Bigg\}z_1^2(t) \\
& -\Bigg\{\left[\frac{r_2x_2^*}{k_2}+\frac{\sigma_1x_1^*}{M}-\frac{\sigma_1m_1x_1^*+\sigma_2m_2x_2^*}{2m_1m_2}\right] \\
& -\left[\frac{r_2^2Mx_2^*}{2k_2^2}+\frac{r_2\sigma_1Mx_1^*}{m_2k_2}\right]\tau_2\Bigg\}z_2^2(t) \\
& +\frac{r_1^2Mx_1^*\tau_1}{2k_1^2}z_1^2(t-\tau_1)+\frac{r_1\sigma_2Mx_2^*\tau_1}{2m_1k_1}z_1^2(t)
\end{aligned}$$

$$+\frac{r_1\sigma_2 M x_2^*\tau_1}{2m_1k_1}z_2^2(t) + \frac{r_2^2 M x_2^*\tau_2}{2k_2^2}z_2^2(t-\tau_2)$$

$$+\frac{r_2\sigma_1 M x_1^*\tau_2}{2m_2k_2}z_1^2(t) + \frac{r_2\sigma_1 M x_1^*\tau_2}{2m_2k_2}z_2^2(t) \tag{6.9}$$

Let

$$V_3(z_t) = \frac{r_1^2 M x_1^*\tau_1}{2k_1^2}\int_{t-\tau_1}^{t} z_1^2(s)ds + \frac{r_2^2 M x_2^*\tau_2}{2k_2^2}\int_{t-\tau_2}^{t} z_2^2(s)ds \tag{6.10}$$

then

$$\frac{dV_3(z_t)}{dt} = \frac{r_1^2 M x_1^*\tau_1}{2k_1^2}z_1^2(t) - \frac{r_1^2 M x_1^*\tau_1}{2k_1^2}z_1^2(t-\tau_1)$$

$$+\frac{r_2^2 M x_2^*\tau_2}{2k_2^2}z_2^2(t) - \frac{r_2^2 M x_2^*\tau_2}{2k_2^2}z_2^2(t-\tau_2) \tag{6.11}$$

Now define a Lyapunov functional $V(z_t)$ as

$$V(z_t) = V_1(z_t) + V_2(z_t) + V_3(z_t) \tag{6.12}$$

Then we have from (6.9) and (6.11) that for $t \geq \widetilde{T}$

$$\dot{V}(z_t) = \dot{V}_1(z_t) + \dot{V}_2(z_t) + \dot{V}_3(z_t)$$

$$\leq -\left\{\left[\frac{r_1x_1^*}{k_1} + \frac{\sigma_2 x_2^*}{M} - \frac{\sigma_1 m_1 x_1^* + \sigma_2 m_2 x_2^*}{2m_1m_2}\right] - \left[\frac{r_1^2 M x_1^*}{2k_1^2} + \frac{r_1\sigma_2 M x_2^*}{m_1k_1}\right]\tau_1\right\} z_1^2(t)$$

$$-\left\{\left[\frac{r_2x_2^*}{k_2} + \frac{\sigma_1 x_1^*}{M} - \frac{\sigma_1 m_1 x_1^* + \sigma_2 m_2 x_2^*}{2m_1m_2}\right] - \left[\frac{r_2^2 M x_2^*}{2k_2^2} + \frac{r_2\sigma_1 M x_1^*}{m_2k_2}\right]\tau_2\right\} z_2^2(t)$$

$$+\frac{r_1^2 M x_1^*\tau_1}{2k_1^2}z_1^2(t) + \frac{r_2^2 M x_2^*\tau_2}{2k_2^2}z_2^2(t) + \frac{r_1\sigma_2 M x_2^*\tau_1}{2m_1k_1}z_1^2(t)$$

$$+\frac{r_1\sigma_2 M x_2^*\tau_1}{2m_1k_1}z_2^2(t) + \frac{r_2\sigma_1 M x_1^*\tau_2}{2m_2k_2}z_1^2(t) + \frac{r_2\sigma_1 M x_1^*\tau_2}{2m_2k_2}z_2^2(t)$$

$$
\begin{aligned}
= \; & -\left\{\left[\frac{r_1 x_1^*}{k_1} + \frac{\sigma_2 x_2^*}{M} - \frac{\sigma_1 m_1 x_1^* + \sigma_2 m_2 x_2^*}{2m_1 m_2}\right] - \left[\frac{r_1^2 M x_1^*}{k_1^2} + \frac{3r_1\sigma_2 M x_2^*}{2m_1 k_1}\right]\tau_1 \right. \\
& \left. - \left[\frac{r_2\sigma_1 M x_1^*}{2m_2 k_2}\right]\tau_2\right\} z_1^2(t) \\
& -\left\{\left[\frac{r_2 x_2^*}{k_2} + \frac{\sigma_1 x_1^*}{M} - \frac{\sigma_1 m_1 x_1^* + \sigma_2 m_2 x_2^*}{2m_1 m_2}\right] - \left[\frac{r_2^2 M x_2^*}{k_2^2} + \frac{3r_2\sigma_1 M x_1^*}{2m_2 k_2}\right]\tau_2 \right. \\
& \left. - \left[\frac{r_1\sigma_2 M x_2^*}{2m_1 k_1}\right]\tau_1\right\} z_2^2(t) \\
\equiv \; & -\psi_1 z_1^2(t) - \psi_2 z_2^2(t) \qquad (6.13)
\end{aligned}
$$

It follows from (6.1) that $\psi_1 > 0$ and $\psi_2 > 0$. Let $w(s) = \psi s^2$, where $\psi = min\{\psi_1, \psi_2\}$, then w is nonnegative continuous on $[0, \infty), w(0) = 0$, and $w(s) > 0$ for $s > 0$. It follows from (6.13) that for $t \geq \widetilde{T}$

$$
\dot{V}(z_t) \leq -\psi[z_1^2(t) + z_2^2(t)] = -\psi|z(t)|^2 = -w(|z(t)|) \qquad (6.14)
$$

Now, we want to find a function u such that $V(z_t) \geq u(|z(t)|)$. It follows from $(6.3), (6.7)$, and (6.10) that

$$
V(z_t) \;\; \geq \;\; \{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\} \qquad (6.15)
$$

By the Taylor Theorem, we have that

$$
z_i(t) - ln[1 + z_i(t)] = \frac{z_i^2(t)}{2[1 + \theta_i(t)]^2} \qquad (6.16)
$$

where $\theta_i(t) \in (0, z_i(t))$ or $(z_i(t), 0)$ for $i = 1, 2$.

Case 1: If $0 < \theta_i(t) < z_i(t)$ for $i = 1, 2$, then

$$
\frac{z_i^2(t)}{[1 + z_i(t)]^2} < \frac{z_i^2(t)}{[1 + \theta_i(t)]^2} < z_i^2(t) \qquad (6.17)
$$

By Theorem 5.1, it follows that for $t \geq T^*$

$$
m_i \leq x_i^*[1 + z_i(t)] = x_i(t) \leq M_i, \quad \text{for} \quad i = 1, 2 \qquad (6.18)
$$

Then (6.17) implies that

$$(\frac{x_1^*}{M_1})^2 z_1^2(t) \leq \frac{z_1^2(t)}{[1+\theta_1(t)]^2} < z_1^2(t)$$

$$(\frac{x_2^*}{M_2})^2 z_2^2(t) \leq \frac{z_2^2(t)}{[1+\theta_2(t)]^2} < z_2^2(t) \tag{6.19}$$

It follows from (6.15), (6.16) and (6.19) that for $t \geq T^*$

$$\begin{aligned} V(z_t) &\geq \frac{z_1^2(t)}{2[1+\theta_1(t)]^2} + \frac{z_2^2(t)}{2[1+\theta_2(t)]^2} \\ &\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq min\left\{\frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2, \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2\right\}\left[z_1^2(t)+z_2^2(t)\right] \\ &\equiv \widetilde{m}|z(t)|^2 \end{aligned} \tag{6.20}$$

Case 2 : If $-1 < z_i(t) < \theta_i(t) < 0$ for $i = 1, 2$, then

$$z_i^2(t) < \frac{z_i^2(t)}{[1+\theta_i(t)]^2} < \frac{z_i^2(t)}{[1+z_i(t)]^2} \tag{6.21}$$

By (6.18), (6.21) implies that

$$z_1^2(t) < \frac{z_1^2(t)}{[1+\theta_1(t)]^2} \leq \left(\frac{x_1^*}{m_1}\right)^2 z_1^2(t)$$

$$z_2^2(t) < \frac{z_2^2(t)}{[1+\theta_2(t)]^2} \leq \left(\frac{x_2^*}{m_2}\right)^2 z_2^2(t) \tag{6.22}$$

It follows from (6.15), (6.16) and (6.22) that for $t \geq T^*$

$$\begin{aligned} V(z_t) &\geq \frac{z_1^2(t)}{2[1+\theta_1(t)]^2} + \frac{z_2^2(t)}{2[1+\theta_2(t)]^2} \\ &> \frac{1}{2}z_1^2(t) + \frac{1}{2}z_2^2(t) \\ &\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq \widetilde{m}\left[z_1^2(t)+z_2^2(t)\right] \\ &= \widetilde{m}|z(t)|^2 \end{aligned} \tag{6.23}$$

Case 3 : If $0 < \theta_1(t) < z_1(t)$ and $-1 < z_2(t) < \theta_2(t) < 0$, then it follows from (6.15), (6.16), (6.19) and (6.22) that for $t \geq T^*$

$$\begin{aligned} V(z_t) &\geq \frac{z_1^2(t)}{2[1+\theta_1(t)]^2} + \frac{z_2^2(t)}{2[1+\theta_2(t)]^2} \\ &> \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2} z_2^2(t) \\ &\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq \widetilde{m}\left[z_1^2(t) + z_2^2(t)\right] \\ &= \widetilde{m}|z(t)|^2 \end{aligned} \tag{6.24}$$

Case 4 : If $-1 < z_1(t) < \theta_1(t) < 0$ and $0 < \theta_2(t) < z_2(t)$, then it follows from (6.15),(6.16),(6.19) and (6.22) that for $t \geq T^*$

$$\begin{aligned} V(z_t) &\geq \frac{z_1^2(t)}{2[1+\theta_1(t)]^2} + \frac{z_2^2(t)}{2[1+\theta_2(t)]^2} \\ &> \frac{1}{2} z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq \frac{1}{2}\left(\frac{x_1^*}{M_1}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{x_2^*}{M_2}\right)^2 z_2^2(t) \\ &\geq \widetilde{m}\left[z_1^2(t) + z_2^2(t)\right] \\ &= \widetilde{m}|z(t)|^2 \end{aligned} \tag{6.25}$$

Let $u(s) = \widetilde{m}s^2$, then u is nonnegative continuous on $[0, \infty)$, $u(0) = 0, u(s) > 0$ for $s > 0$, and $\lim_{s\to\infty} u(s) = +\infty$. So, by Case 1 to Case 4, we have

$$V(z_t) \geq u(|z(t)|) \quad \text{for} \quad t \geq T^* \tag{6.26}$$

This shows that the unique equilibrium point E^* of the system (3.1) is globally asymptotically stable. ■

7 Example

In this section, we present a simple example to illustrate the procedures of applying our results consisting of local and global stability.

Example 7.1

Consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= 10x_1(t)\left[1 - \frac{x_1(t-0.0001)}{100}\right] - 5.2x_1(t) + 5x_2(t) - \frac{1}{100}5x_1(t) \\ \dot{x}_2(t) &= 10x_2(t)\left[1 - \frac{x_2(t-0.0001)}{100}\right] + 5.2x_1(t) - 5x_2(t) \end{aligned} \tag{7.1}$$

Comparing the system (7.1) with the system (3.1), we get $r_1 = 10,\ r_2 = 10,\ k_1 = 100,\ k_2 = 100,\ \sigma_1 = 5.2,\ \sigma_2 = 5,\ q = 0.01,\ e = 5,\ \tau_1 = 0.0001$, and $\tau_2 = 0.0001$. So the system (7.1) has a unique positive equilibrium point $E^* = (98.63, 100.85)$.

Since

$$a_1 - a_2\tau_1 - a_3\tau_2 = 19.694 > 0$$

$$b_1 - b_2\tau_1 - b_3\tau_2 = 20.08 > 0,$$

The unique positive equilibrium point E^* of the system (7.1) is local asymptotically stable by Theorem 4.1. Some local trajectories of the system (7.1) are depicted in Figure 7.1.

Since

$$\alpha_1 - \alpha_2\tau_1 - \alpha_3\tau_2 = 0.28 > 0$$

$$\beta_1 - \beta_2\tau_1 - \beta_3\tau_2 = 1.08 > 0,$$

Then we conclude that the unique positive equilibrium point E^* of the system (7.1) is global asymptotically stable by Theorem 6.1. Some global trajectories of the system (7.1) are depicted in Figure 7.2.

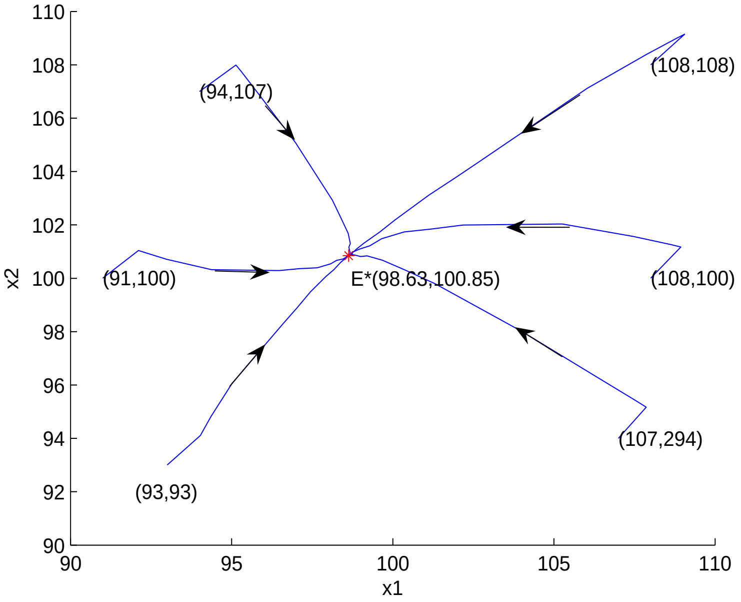

Figure 7.1: The local trajectories of the system (7.1).

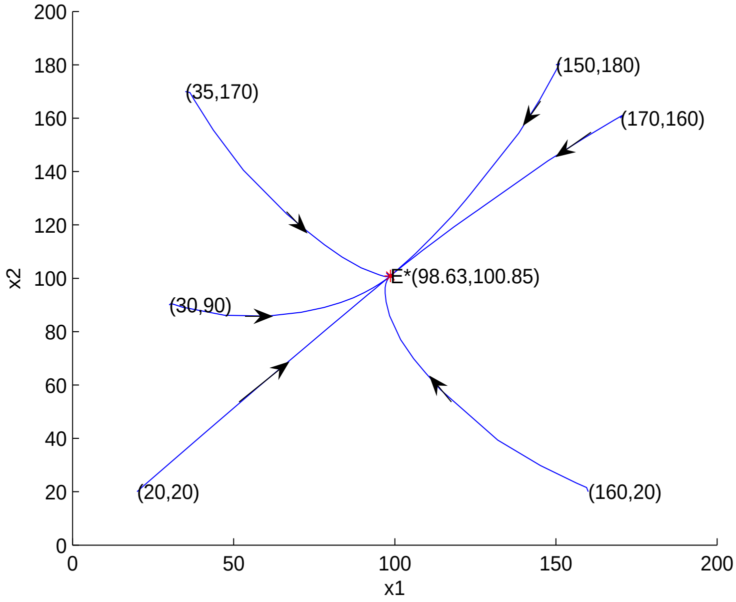

Figure 7.2: The global trajectories of the system (7.1).

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