Local Stability of the Epidemic System with Time Delays

劉耀升

Abstract

In this thesis, we analyze the dynamic bahavior of an epidemic model. Firstly, we discuss the local and global stability for the system without time delay. Secondly, we find out some sufficient conditions for local stability of the unique positive equilibrium point of the system with time-delay by constructing a Lyapunov function. Finally, we give some examples to illustrate results with pictures.

Keywords: epidemic, local stability, global stability, time-delay, Lyapunov function, switch.

Contents

Abstact **i**

1 Introduction **1**

2 Preliminaries **3**

2.1 Nonlinear autonomous system 3

2.2 Nonlinear autonomous system with delays 7

3 The model without time delay **10**

3.1 Local Stability 10

3.2 Global stability 12

4 Local stability with two time delays **15**

5 Examples **21**

References **25**

1 Introduction

Germs and virus are surround our environment and it's always cause diseases. Several years before, SARS has made a strong influence to Asia. So that we must deal with those problems, seriously. The epidemic situations can be controlled or not, that will influence our environment we lived. In [9], the author mentioned that mathematical biologist (A. J. Lotka) investigated, in a series of papers from 1912 on, a differential equation model of malarial epidemics due to Ross(1911). He examined the effect of incubation delays how that influence the populations between human and mosquito. Because most diseases relate with our life and environment, we can't neglect symptoms occur surround us.

Epidemic model has been studied by many authors. Most of them are interested in analyzing the stability of the unique positive equilibrium of the epidemic systems. Many researchers usually use the following methods to analyze the stability of the unique positive equilibrium without time delay. The first one is constructing a Lyapunov function to analyze the global stabilty of the unique positive equilibrium point. The second one is using Dulac's criterion plus Poincare'-Bendixson Theorem. The third one is the limit cycle stability analysis. The fourth one is comparison method.There are many researchers neglect the delay in epidemic systems. But more realistic model should include some of the past states of the population systems. In other words, real systems should be modified by time delays. Wendy Wang has studied local stability of the unique positive equilibrium point with time delays by using characteristic equation[8]. In this thesis, we use the same method to analyze local and global stability of the unique positive equilibrium point without time delay and then construct a Lyapunov function to prove local stability with two time delays[2][3].

In this paper, we concern the SIS model without time delay. In chapter 2, we introduce some definitions and theorems. Then consider the situation with time delays. In chapter 3, we analyze local stability of the unique positive equilibrium point by the Hartman-Grobman Theorem. Afterward we use Dulac's criterion plus Poincare'-Bexdixson Theorem to assay the global stability of the unique positive equilibrium point of the epidemic system. In chapter 4, we construct a Lyapunov function to show that the unique positive equilibrium is locally asymptotically stable. Finally, we use some examples to assay the results that we have discussed.

2 Preliminaries

2.1 Nonlinear autonomous system

Consider the following general nonlinear autonomous system of differential equation

$$\dot{x}(t) = f(x) \quad , \quad x \in E \tag{2.1}$$

where $f \in C^1(E)$ and E is an open subset of R^n. In this thesis, we need the following definitions and theorems.

Definition 2.1 [6]

(i) A point $x_0 \in E$ is called an *equilibrium point* or *critical point* of the system (2.1) if $f(x_0) = 0$.

(ii) An equilibrium point x_0 of the system (2.1) is called a *hyperbolic equilibrium point* of the system (2.1) if none of the eigenvalues of the matrix $Df(x_0)$ have zero real part.

(iii) An equilibrium point x_0 is called a *saddle point* of the system (2.1) if it is a hyperbolic equilibrium point and $Df(x_0)$ has at least one eigenvalue with a positive real part and one with negative real part.

Definition 2.2 [6] Let E be an open subset of R^n and let $f \in C^1(E)$. For $x_0 \in E$, let $\phi(t, x_0)$ be the solution of the system (2.1) with the initial condition $x(0) = x_0$ defined on its maximal interval of existence $I(x_0)$. Then for $t \in I(x_0)$, the set of mappings ϕ_t defined by

$$\phi_t(x_0) = \phi(t, x_0)$$

is called the *flow* of the system (2.1).

Definition 2.3 [6] Let ϕ_t denote the flow of the system (2.1) defined for all $t \in R$. An equilibrium point x_0 of the system (2.1) is *stable* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ and $t \geq 0$ we have

$$\phi_t(x) \in N_\epsilon(x_0)$$

The equilibrium point x_0 is *unstable* if it is not stable. And x_0 is *asymptotically stable* if it is stable and if there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ we have

$$\lim_{t \to \infty} \phi_t(x) = x_0$$

In order to analyze the behavior of the system (2.1) near its equilibrium points, we show that the local behavior of the nonlinear system (2.1) near a hyperbolic equilibrium point x_0 is qualitatively determined by the behavior of the linear system

$$\dot{x} = Ax \tag{2.2}$$

where the Jacobian matrix $A = Df(x_0)$. The linear function $Ax = Df(x_0)x$ is called the linear part of f at x_0.

Theorem 2.1 [6] **(The Hartman-Grobman Theorem)** Let E be an open subset of R^n containing the point x_0, let $f \in C^1(E)$, and let ϕ_t be the flow of the system (2.2). Suppose that $f(x_0) = 0$ and that the matrix $A = Df(x_0)$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for $x \in U$, there is an open interval $I(x) \subset R$ containing origin such that for all $x \in U$ and $t \in I(x)$

$$H \circ \phi_t(x) = e^{At} H(x)$$

Theorem 2.2 [6] Suppose x_0 is an equilibrium point of the system (2.1) and $A = Df(x_0)$. Let $\sigma = det(A)$ and $\gamma = trace(A)$.

(a) If $\sigma < 0$, then the system (2.1) has a saddle point at x_0.

(b) If $\sigma > 0$ and $\gamma = 0$, then the system (2.1) has a center at x_0.

(c) If $\sigma > 0$ and $\gamma^2 - 4\sigma \geq 0$, then the system (2.1) has a node at x_0; it is stable if $\gamma < 0$ and unstable if $\gamma > 0$.

(d) If $\sigma > 0$, $\gamma^2 - 4\sigma < 0$ and $\gamma \neq 0$, then the system (2.1) has a focus at x_0; it is stable if $\gamma < 0$ and unstable if $\gamma > 0$.

In order to analyze the global stability of the system (2.1), it is necessary to determine whether the closed orbit exist or not. Dulac's Criteria has established conditions under which the system (2.1) with $x \in R^2$ has no closed orbit.

Theorem 2.3 [4] **(Dulac's Criterion)** Let $f \in C^1(E)$ where E is a simply connected region in R^2. If there exists a function $H \in C^1(E)$ such that $\bigtriangledown \cdot (Hf)$ is not identically zero and does not change sign in E, then the system (2.1) has no closed orbit lying entirely in E. If A is an annular region contained in E on which $\bigtriangledown \cdot (Hf)$ does not change sign, then there is at most one limit cycle of the system (2.1) in A.

Definition 2.4 [6] A point $p \in E$ where E is an open subset of R^n is an $\omega-$*limit point* of the trajectory $\phi(\cdot, x)$ of the system (2.1) if there is a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \phi(t_n, x) = p$$

Similarly, if there is a sequence $t_n \to -\infty$ such that

$$\lim_{n\to\infty} \phi(t_n, x) = q$$

and the point $q \in E$, then the point q is called an *α-limit point* of the trajectory $\phi(\cdot, x)$ of the system (2.1). The set of all ω-limit points of a trajectory Γ is called the *ω-limit set* of Γ and it is denoted by $\omega(\Gamma)$. The set of all α-limit points of a trajectory Γ is called the *$\alpha-$limit set* of Γ and it is denoted by $\alpha(\Gamma)$. The set of all limit points of Γ, $\alpha(\Gamma) \cup \omega(\Gamma)$ is called the *limit set* of Γ.

Theorem 2.4 [6] The α and ω-limit sets of a trajectory Γ of the system (2.1), $\alpha(\Gamma)$ and $\omega(\Gamma)$, are closed subsets of E and if Γ is contained in a compact subset of R^n, then $\alpha(\Gamma)$ and $\omega(\Gamma)$, are nonempty, connected, compact subsets of E.

Definition 2.5 [6] A *limit cycle* Γ of a planar system is a cycle of the system (2.1) which is the α or $\omega-$limit set of some trajectory of the system (2.1) other than Γ. If Γ is the $\omega-$limit set of every trajectory in some neighborhood of Γ, then Γ is called an *ω-limit cycle* or *stable limit cycle*; if a cycle Γ is the α-limit set of every trajectory in some neighborhood of Γ, then Γ is called an *$\alpha -$ limit cycle* or an *unstable limit cycle*; and if Γ is the ω-limit set of the trajectory other than Γ and the α-limit set of another trajectory other than Γ, then Γ is called a *semi-stable limit cycle.*

Theorem 2.5 [7] **(The Poincaré−Bendixson Theorem)** Suppose that $f \in C^1(E)$ where E is an open subset of R^n and that the system (2.1) has a trajectory Γ contained in a compact subset F of E. Assume that the system (2.1) has only one unique equilibrium point x_0 in F, then one of the following possibilities holds.

(a) $\omega(\Gamma)$ is the equilibrium point x_0.

(b) $\omega(\Gamma)$ is a periodic orbit.

(c) $\omega(\Gamma)$ is a graphic.

2.2 Nonlinear autonomous system with delays

For ordinary differential equations, we view solutions of initial value problems as maps in Euclidean space. In order to establish a similar view for solutions of delay differential equations, we need some definitions.

We denote $\mathcal{C} \equiv C([-\tau, 0], R^n)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n with the topology of uniform convergence; That is, for $\phi \in \mathcal{C}$, the norm of ϕ is defined as $\|\phi\| = \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|$, where $|\cdot|$ is a norm in R^n. We define $x_t \in \mathcal{C}$ as $x_t(\theta) = x(t+\theta),\ \theta \in [-\tau, 0]$. Assume that Ω is a subset of $\mathcal{C}$ and $f : \Omega \to R^n$ is a given function, then we consider the following general nonlinear autonomous system of delay differential equation

$$\dot{x}(t) = f(x_t) \tag{2.3}$$

Definition 2.6 [9] A function x is called a solution of (2.3) on $[\sigma - \tau, \sigma + A)$ if $x \in C([\sigma - \tau, \sigma + A), R^n)$,$(t, x_t) \in \Omega$ and x_t satisfies the system (2.3) for $t \in [\sigma, \sigma + A)$.For given $\sigma \in R$,$\phi \in C$, we say $x(\sigma, \phi)$ is a solution of (2.3) with initial value ϕ at σ , of simply a solution through (σ, ϕ), if there is an $A > 0$ such that $x(\sigma, \phi)$ is a solution of (2.3) on $[\sigma - \tau, \sigma + A)$ and $x_\sigma(\sigma, \phi) = \phi$.

Lemma 2.1 [9] **(Barbălat's Lemma)** Let f be a nonnegative function defined on $[0, \infty)$ such that f is integrable on $[0, \infty)$ and uniformly continuous on $[0, \infty)$. Then

$$\lim_{t \to \infty} f(t) = 0.$$

We can say to show that the equilibrium points of the system (2.1) are also the equilibrium points of the system (2.3).Then define

$$\dot{u}(t) = g(u_t) \tag{2.4}$$

is the linearized form of the system (2.3), where $u(t) = x(t) - \bar{x}$.It is well known that the equilibrium point $\bar{x}$ is locally asymptotically stable for (2.3), if the trivial solution of the linearized system (2.4) is asymptotically stable.

In order to analyze the local stability of the system (2.4), we need the following definitions and theorems.

Definition 2.7 [9] We say that $\phi \in B(0, \delta)$ if $\phi \in \mathcal{C}$ and $\|\phi\| \leq \delta$, where $\|\phi\| = \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|$.

(i) The solution $x = 0$ of the system (2.3) is said to be *stable* if, for any $\sigma \in R$, $\epsilon > 0$, there is a $\delta = \delta(\epsilon, \sigma)$ such that $\phi \in B(0, \delta)$ implies $x_t(\sigma, \phi) \in B(0, \epsilon)$ for $t \geq \sigma$. Otherwise, we say that $x = 0$ is *unstable*.

(ii) The solution $x = 0$ of the system (2.3) is said to be *asymptotically stable* if it is stable and there is a $b_0 = b(\sigma) > 0$ such that $\phi \in B(0, b_0)$ implies $x(\sigma, \phi)(t) \to 0$ as $t \to \infty$.

(iii) The solution $x = 0$ of the system (2.3) is said to be *uniformly stable* if the number δ in the definition of stable is independent of σ.

(iv) The solution $x = 0$ of the system (2.3) is said to be *uniformly asymptotically stable* if it is uniformly stable and there is a $b_0 > 0$ such that, for every $\eta > 0$, there is a $t_0(\eta)$ such that $\phi \in B(0, b_0)$ implies $x_t(\sigma, \phi) \in B(0, \eta)$ for $t \geq \sigma + t_0(\eta)$, for every $\sigma \in R$.

Theorem 2.6 [9] If

$$sup\{Re\lambda : det\Delta(\lambda) = 0\} < 0$$

where $det\Delta(\lambda) = 0$ is the characteristic equation of (2.4),then the zero solution of (2.4)is uniformly asymptotically stable. If $Re\lambda > 0$ for some λ satisfying $det\Delta(\lambda) =$

0, then the system (2.4) is unstable.Moreover, if $det\Delta(\lambda) = 0$ has a nonsimple pure root,then the system (2.4) is also unstable.

Furthermore,we consider the special case of the linearized system (2.4) as the following general linear real scalar neutral differential difference equation with a single delay $\tau(\tau > 0)$

$$\sum_{k=0}^{n} a_k y^{(k)} + \sum_{k=0}^{n} b_k y^{(k)}(t - \tau) \tag{2.5}$$

where $y^{(0)} \equiv y(t)$.Thus the stability analysis of (2.5) is equivalent to the problem of determining conditions under which all roots of its characteristic equation

$$\sum_{k=0}^{n} a_k \lambda^k + (\sum_{k=0}^{n} b_k \lambda^k) e^{-\lambda\tau} = 0 \tag{2.6}$$

lie in the left of the complex plane and are uniformly bounded away from the imaginary axis.Without loss of generality, we assume $a_n = 1$

Theorem 2.7 [9] If $|b_n| > 1$, then, for all $\tau > 0$, there is an infinite number of roots of (2.6) whose real parts are positive.

An immediate consequence of this theorem is the following:

Theorem 2.8 [9] If $|b_n| > 1$, then the trivial solution of (2.5) is unstable for all $\tau > 0$.

3 The model without time delay

Consider the SIS epidemic model without time delay modeled by

$$\begin{aligned} \dot{s} &= b - \lambda si + \beta i - bs \\ \dot{i} &= \lambda \alpha si - (\gamma + b)i \end{aligned} \tag{3.1}$$

with the initial condition

$$s(0) \geq 0 \quad , \quad i(0) \geq 0 \tag{3.2}$$

where b , α , β , λ and γ are all positive constants. s and i denote the susceptible and infectious individuals , respectively. All we want to discuss is biological population , so we just consider the first quadrant in the xy-plane.

3.1 Local Stability

Clearly, $\bar{E} = (1, 0)$ is the equilibrium point. And $E^* = (s^*, i^*)$ is the unique positive equilibrium point in the first quadrant of the system (3.1) with initial conditions (3.2) where

$$s^* = \frac{\gamma + b}{\lambda \alpha} \quad , \quad i^* = \frac{b(\lambda \alpha - b - \gamma)}{\lambda(\gamma + b - \alpha \beta)} \tag{3.3}$$

where

$$[\lambda \alpha - (b + \gamma)][(b + \gamma) - \alpha \beta] > 0 \tag{3.4}$$

Lemma 3.1 *If the equation (3.4) holds, then $E^* = (s^*, i^*)$ is the unique positive equilibrium point of the system (3.1).*

Lemma 3.2 *All solutions (s(t),i(t)) of the system (3.1) with initial conditions (3.2) are bounded.*

Proof: Since $s(t)+e(t)+i(t)+r(t)\equiv 1$, we have $s(t)+i(t)\leq 1$. This can illustrate that all solutions $(s(t), i(t))$ are bounded.

Now let us analyze the local behavior of the system (3.1) at the equilibrium $\bar{E} = (1, 0)$ and $E^* = (s^*, i^*)$, where s^*, i^* as given by (3.3).

Let

$$\begin{aligned} \dot{s} &= b - \lambda si + \beta i - bs \equiv f_1(s,i) \\ \dot{i} &= \lambda\alpha si - (\gamma + b)i \quad \equiv f_2(s,i) \end{aligned} \tag{3.5}$$

The Jacobian matrix of the system (3.1) takes the form

$$J \equiv \begin{bmatrix} \frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial i} \\ \frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial i} \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda i - b & -\lambda s + \beta \\ \lambda\alpha i & -\lambda\alpha s - \gamma - b \end{bmatrix}$$

The Jacobian matrix of the system (3.1) at $\bar{E}$ is

$$\bar{J} = \begin{bmatrix} -b & -\lambda + \beta \\ 0 & -\lambda\alpha - \gamma - b \end{bmatrix}$$

Therefore

$$det(\bar{J}) = b(\lambda\alpha + \gamma + b)$$

$$trace(\bar{J}) = -b + (-\lambda\alpha - \gamma - b)$$

Since $det(\bar{J}) > 0$ and $trace(\bar{J}) < 0$, the equilibrium $\bar{E}$ is locally asymptotically stable.

Theorem 3.1 *Let E^* be the unique positive equilibrium point. If $\beta/\lambda < s^* < 1$ holds, then E^* is locally asymptotically stable.*

Proof: The Jacobian matrix of the system (3.1) at $E^*$is

$$J^* = \begin{bmatrix} -\lambda i^* - b & -\lambda s^* + \beta \\ \\ \lambda\alpha i^* & -\lambda\alpha s^* - \gamma - b \end{bmatrix}$$

Therefore

$$det(J^*) = (\lambda i^* + b)(\lambda\alpha s^* + \gamma + b) + (\lambda s^* - \beta)(\lambda\alpha i^*)$$

$$trace(J^*) = -\lambda i^* - b - \lambda\alpha s^* - \gamma - b$$

Since $det(J^*) > 0$ and $trace(J^*) < 0$, the unique positive equilibrium point E^* is locally asymptotically stable.

Remark 3.1 *We know the equilibrium $\bar{E} = (1, 0)$ is locally asymptotically stable. It means that if we can restrain disease from certain of range then it will be controlled.*

Remark 3.2 *In Theorem 3.1 , if susceptible population s^* greater than the number β/λ then the equilibrium point E^* is locally asymptotically stable.*

3.2 Global stability

In this section , we use the following two different methods to analyze the global stability of the unique positive equilibrium point E^* of the system (3.1):

(1) Dulac's Criterion plus Poincare'-Bendixson Theorem

(2) Stable limit cycle analysis

Theorem 3.2 *Let E^* be the unique positive equilibrium point of the system (3.1). If $\beta/\lambda < s^* < 1$ then E^* of the system (3.1) is globally asymptotically stable.*

Proof: We use Dulac's Criterion plus Poincare'-Bendixson Theorem to analyze the system (3.1). Consider

$$H(s,i) = \frac{1}{si} \tag{3.6}$$

where $s > 0$, $i > 0$. Then

$$\begin{aligned}
\nabla \cdot (Hf) &= \frac{\partial}{\partial s}(H \cdot f_1) + \frac{\partial}{\partial i}(H \cdot f_2) \\
&= \frac{\partial}{\partial s}\left[\frac{1}{si}(b - \lambda si + \beta i - bs)\right] + \frac{\partial}{\partial i}\left[\frac{1}{si}(\lambda \alpha si - \gamma i - bi)\right] \\
&= \frac{\partial}{\partial s}\left[\frac{b}{si} - \lambda + \frac{\beta}{s} - \frac{b}{i}\right] + \frac{\partial}{\partial i}\left[\lambda\alpha - \frac{(\gamma + b)}{s}\right] \\
&= -(\frac{b}{is^2} + \frac{\beta}{s^2}) < 0
\end{aligned}$$

Hence by the Dulac's Criterion, there is no closed orbit in the first quadrant. From Theorem 3.1, we know the equilibrium point E^* is locally asymptotically stable. By the Lemma 3.2 and the Poincare'-Bendixson Theorem, it suffices to show that the unique positive equilibrium E^* of the system (3.1)is globally asymptotically stable.

Secondly, we want to analyze the global stability of the system (3.1) by using the method (2). Now, we want to show that the system (3.1) has no closed orbit $\Gamma = \{(s(t), i(t)) | \; 0 \geq t \geq T\}$ in the first quadrant, then

$$\begin{aligned}
\Delta &= \int_\Gamma \left(\frac{\partial f_1}{\partial s} + \frac{\partial f_2}{\partial i}\right) ds \\
&= \int_0^T [-\lambda i(t) - b]dt + \int_0^T [\lambda \alpha s(t) - (\gamma + b)]dt \\
&= \int_0^T [-\lambda i(t) - b]dt + \int_0^T \frac{\dot{i}(t)}{i(t)}\, dt
\end{aligned}$$

$$= \int_0^T [-\lambda i(t) - b]dt + \int_{i(0)}^{i(T)} \frac{1}{i}\, di$$

Since Γ is a T-periodic,

$$\int_{i(0)}^{i(T)} \frac{1}{i}\, di = 0$$

Hence we obtain that

$$\begin{aligned} \Delta &= -\int_0^T [\lambda i(t) + b]dt \\ &< 0 \end{aligned}$$

This indicates that all closed orbits of the system (3.1) in the first quadrant are orbitally stable. Since every closed orbit is orbitally stable and then there is an unique stable limit cycle in the first quadrant. That is, the unique positive equilibrium point E^* is unstable. However, by Theorem 3.1, E^* is locally asymptotically stable. Thus there is no close orbit in the first quadrant. By Lemma 3.2 and the Poincare'-Bendixson Theorem, it suffices to show that the unique positive equilibrium point E^* of the system (3.1) is globally asymptotically stable in the first quadrant.

Remark 3.3 *In theorem 3.2, we know that the equilibrium point E^* is globally asymptotically stable. It means that disease will be controlled and even disappear, finally.*

4 Local stability with two time delays

Consider the SIS epidemic model with time delay τ_1 and τ_2 modeled by

$$\begin{aligned} \dot{s}(t) &= b - \lambda s(t)i(t) + \beta i(t-\tau_1) - bs(t) \\ \dot{i}(t) &= \lambda\alpha s(t-\tau_2)i(t-\tau_2) - (\gamma+b)i(t) \end{aligned} \tag{4.1}$$

with the initial conditions

$$\begin{aligned} s(t) = \phi_1(\theta) > 0 \ , \ i(t) = \phi_2(\theta) > 0 \ , \ \theta \in [-\tau, 0] \\ \tau = max\{\tau_1, \tau_2\} \ , \ \phi_i \in C([-\tau, 0], R) \ , i = 1, 2 \end{aligned} \tag{4.2}$$

where b , α , β , λ and γ are all positive constants. $\phi_i(t)$ (i=1,2) are continuous bounded functions on the interval $[-\tau, 0]$. $s(t)$ and $i(t)$ denote the susceptible and infectious individuals, respectively.

Lemma 4.1 *All solutions (s(t),i(t)) of the system (3.1) with initial conditions (3.2) are positive for all* $t \geq 0$.

Proof: It is true because

$$\begin{aligned} s(t) &= s(0)exp\left\{\int_0^t \left[\frac{b}{s(s)} - \lambda i(s) + \frac{\beta i(s-\tau_1)}{s(s)} - b\right]\right\} ds \\ i(t) &= i(0)exp\left\{\int_0^t \left[\frac{\lambda\alpha s(s-\tau_2)i(s-\tau_2)}{i(s)} - (\gamma+b)\right]\right\} ds \end{aligned} \tag{4.3}$$

Thus , all solutions $(s(t), i(t))$ of the system (3.1) with initial conditions (3.2) are positive.

To investigate the local stability of the equlibrium point E^* we linearize the system (4.1). Let $y_1(t) = s(t) - s^*$,$y_2(t) = i(t) - i^*$ be the perturbed variables. After removing nonlinear terms, we obtain the linear variational system, by using equilibria conditions as

$$\begin{aligned}
\dot{y_1}(t) &= (-\lambda i^* - b)y_1(t) - \lambda s^* y_2(t) + \beta y_2(t-\tau_1) \\
\dot{y_2}(t) &= -(\gamma + b)y_2(t) + \lambda\alpha s^* y_2(t-\tau_2) + \lambda\alpha i^* y_1(t-\tau_2)
\end{aligned} \tag{4.4}$$

Theorem 4.1 *Let E^* be positive equilibrium point. And the delays τ_1 and τ_2 satisfy*

$$\lambda s^* > \beta \tag{4.5}$$

$$\alpha_1 - \alpha_2\tau_1 - \alpha_3\tau_2 > 0 \tag{4.6}$$

$$\beta_1 - \beta_2\tau_1 - \beta_3\tau_2 > 0 \tag{4.7}$$

where

$$\begin{aligned}
\alpha_1 &= (2\lambda i^* + 2b + \beta) - (\lambda s^* + \lambda\alpha i^*) \\
\alpha_2 &= \beta(\lambda i^* + b) \\
\alpha_3 &= 2(\lambda\alpha i^*)^2 + 2(\lambda\alpha)^2 s^* i^* + \lambda\alpha i^*(\gamma + b) \\
\beta_1 &= 2(\gamma + b - \lambda\alpha s^*) - (\lambda s^* - \beta + \lambda\alpha i^*) \\
\beta_2 &= \beta(2\lambda s^* + \lambda i^* + b - 2\beta) \\
\beta_3 &= 2(\lambda\alpha)^2 s^* i^* + 2(\lambda\alpha s^*)(\gamma + b) + 2(\lambda\alpha s^*)^2 + \lambda\alpha i^*(\gamma + b)
\end{aligned}$$

then the unique positive equilibrium point E^* of the system (4.1) is locally asymptotically stable.

Proof: Let

$$
\begin{aligned}
W_1(y(t)) &= \left(y_1(t) + \beta \int_{t-\tau_1}^{t} y_2(s)ds\right)^2 \\
&+ \left(y_2(t) + \lambda\alpha i^* \int_{t-\tau_2}^{t} y_1(s)ds + \lambda\alpha s^* \int_{t-\tau_2}^{t} y_2(s)ds\right)^2
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
\dot{W}_1(y(t)) &= 2\left(y_1(t) + \beta \int_{t-\tau_1}^{t} y_2(s)ds\right)[\dot{y}_1(t) + \beta(y_2(t) - y_2(t-\tau_1))] \\
&+2\left(y_2(t) + \lambda\alpha i^* \int_{t-\tau_2}^{t} y_1(s)ds + \lambda\alpha s^* \int_{t-\tau_2}^{t} y_2(s)ds\right) \\
&[\dot{y}_2(t) + \lambda\alpha i^*(y_1(t) - y_1(t-\tau_2)) + \lambda\alpha s^*(y_2(t) - y_2(t-\tau_2))] \\
&= 2\left(y_1(t) + \beta \int_{t-\tau_1}^{t} y_2(s)ds\right) \\
&\cdot[(-\lambda i^* - b)y_1(t) + (-\lambda s^* + \beta)y_2(t)] \\
&+2\left(y_2(t) + \lambda\alpha i^* \int_{t-\tau_2}^{t} y_1(s)ds + \lambda\alpha s^* \int_{t-\tau_2}^{t} y_2(s)ds\right) \\
&\cdot[-(\gamma + b)y_2(t) + \lambda\alpha i^* y_1(t) + \lambda\alpha s^* y_2(t)] \\
&= -2(\lambda i^* + b)y_1^2(t) - 2(\lambda s^* - \beta)y_1(t)y_2(t) - \beta(\lambda i^* + b)\int_{t-\tau_1}^{t} 2y_1(t)y_2(s)ds \\
&-\beta(\lambda s^* - \beta)\int_{t-\tau_1}^{t} 2y_2(t)y_2(s)ds - 2(\gamma + b)y_2^2(t) + 2\lambda\alpha i^* y_1(t)y_2(t) \\
&+2\lambda\alpha s^* y_2^2(t) - \lambda\alpha i^*(\gamma + b)\int_{t-\tau_2}^{t} 2y_1(s)y_2(t)ds + (\lambda\alpha i^*)^2 \int_{t-\tau_2}^{t} 2y_1(t)y_1(s)ds \\
&+(\lambda\alpha)^2 s^* i^* \int_{t-\tau_2}^{t} 2y_1(s)y_2(t)ds - \lambda\alpha s^*(\gamma + b)\int_{t-\tau_2}^{t} 2y_2(t)y_2(s)ds
\end{aligned}
$$

$$
\begin{aligned}
& +(\lambda\alpha)^2 s^* i^* \int_{t-\tau_2}^{t} 2y_1(t)y_2(s)ds + (\lambda\alpha s^*)^2 \int_{t-\tau_2}^{t} 2y_2(t)y_2(s)ds \\
\leq\ & -2(\lambda i^* + b)y_1^2(t) + 2(\lambda s^* - \beta)|y_1(t)||y_2(t)| + \beta(\lambda i^* + b) \int_{t-\tau_1}^{t} 2|y_1(t)||y_2(s)|ds \\
& +\beta(\lambda s^* - \beta) \int_{t-\tau_1}^{t} 2|y_2(t)||y_2(s)|ds - 2(\gamma + b - \lambda\alpha s^*)y_2^2(t) + 2\lambda\alpha i^*|y_1(t)||y_2(t)| \\
& +\lambda\alpha i^*(\gamma + b) \int_{t-\tau_2}^{t} 2|y_1(s)||y_2(t)|ds + (\lambda\alpha i^*)^2 \int_{t-\tau_2}^{t} 2|y_1(t)||y_1(s)|ds \\
& +(\lambda\alpha)^2 s^* i^* \int_{t-\tau_2}^{t} 2|y_1(s)||y_2(t)|ds + \lambda\alpha s^*(\gamma + b) \int_{t-\tau_2}^{t} 2|y_2(t)||y_2(s)|ds \\
& +(\lambda\alpha)^2 s^* i^* \int_{t-\tau_2}^{t} 2|y_1(t)||y_2(s)|ds + (\lambda\alpha s^*)^2 \int_{t-\tau_2}^{t} 2|y_2(t)||y_2(s)|ds \\
\leq\ & -2(\lambda i^* + b)y_1^2(t) + (\lambda s^* - \beta)[y_1^2(t) + y_2^2(t)] \\
& +\beta(\lambda i^* + b)[y_1^2(t)\tau_1 + \int_{t-\tau_1}^{t} y_2^2(s)ds] + \beta(\lambda s^* - \beta)[y_2^2(t)\tau_1 + \int_{t-\tau_1}^{t} y_2^2(s)ds] \\
& -2(\gamma + b - \lambda\alpha s^*)y_2^2(t) + \lambda\alpha i^*[y_1^2(t) + y_2^2(t)] \\
& +\lambda\alpha i^*(\gamma + b)[y_2^2(t)\tau_2 + \int_{t-\tau_2}^{t} y_1^2(s)ds] + (\lambda\alpha i^*)^2[y_1^2(t)\tau_2 + \int_{t-\tau_2}^{t} y_1^2(s)ds] \\
& +(\lambda\alpha)^2 s^* i^*[y_2^2(t)\tau_2 + \int_{t-\tau_2}^{t} y_1^2(s)ds] + \lambda\alpha s^*(\gamma + b)[y_2^2(t)\tau_2 + \int_{t-\tau_2}^{t} y_2^2(s)ds] \\
& +(\lambda\alpha)^2 s^* i^*[y_1^2(t)\tau_2 + \int_{t-\tau_2}^{t} y_2^2(s)ds] + (\lambda\alpha s^*)^2[y_2^2(t)\tau_2 + \int_{t-\tau_2}^{t} y_2^2(s)ds] \\
=\ & \left[-2(\lambda i^* + b) + \lambda s^* - \beta + \beta(\lambda i^* + b)\tau_1 + \lambda\alpha i^* + (\lambda\alpha i^*)^2\tau_2 + (\lambda\alpha)^2 s^* i^* \tau_2\right] y_1^2(t) \\
& + [(\lambda s^* - \beta) + \beta(\lambda s^* - \beta)\tau_1 - 2(\gamma + b - \lambda\alpha s^*) + \lambda\alpha i^* + \lambda\alpha i^*(\gamma + b)\tau_2 \\
& +(\lambda\alpha)^2 s^* i^* \tau_2 + \lambda\alpha s^*(\gamma + b)\tau_2 + (\lambda\alpha s^*)^2\tau_2\left.\right] y_2^2(t)
\end{aligned}
$$

$$+[\beta(\lambda i^* + b) + \beta(\lambda s^* - \beta)] \int_{t-\tau_1}^{t} y_2^2(s)ds$$

$$+[\lambda\alpha i^*(\gamma + b) + (\lambda\alpha i^*)^2 + (\lambda\alpha)^2 s^* i^*] \int_{t-\tau_2}^{t} y_1^2(s)ds$$

$$+[\lambda\alpha s^*(\gamma + b) + (\lambda\alpha)^2 s^* i^* + (\lambda\alpha s^*)^2] \int_{t-\tau_2}^{t} y_2^2(s)ds \quad (4.9)$$

Now, we let

$$\begin{aligned} W_2(y(t)) &= [\beta(\lambda i^* + b) + \beta(\lambda s^* - \beta)] \int_{t-\tau_1}^{t} \int_{s}^{t} y_2^2(\rho)d\rho ds \\ &+[\lambda\alpha i^*(\gamma + b) + (\lambda\alpha i^*)^2 + (\lambda\alpha)^2 s^* i^*] \int_{t-\tau_2}^{t} \int_{s}^{t} y_1^2(\rho)d\rho ds \\ &+[\lambda\alpha s^*(\gamma + b) + (\lambda\alpha)^2 s^* i^* + (\lambda\alpha s^*)^2] \int_{t-\tau_2}^{t} \int_{s}^{t} y_2^2(\rho)d\rho ds \qquad (4.10) \end{aligned}$$

then

$$\begin{aligned} \dot{W}_2(y(t)) &= [\beta(\lambda i^* + b) + \beta(\lambda s^* - \beta)]\tau_1 y_2^2(t) - [\beta(\lambda i^* + b) + \beta(\lambda s^* - \beta)] \int_{t-\tau_1}^{t} y_2^2(s)ds \\ &+[\lambda\alpha i^*(\gamma + b) + (\lambda\alpha i^*)^2 + (\lambda\alpha)^2 s^* i^*]\tau_2 y_1^2(t) \\ &-[\lambda\alpha i^*(\gamma + b) + (\lambda\alpha i^*)^2 + (\lambda\alpha)^2 s^* i^*] \int_{t-\tau_2}^{t} y_1^2(s)ds \\ &+[\lambda\alpha s^*(\gamma + b) + (\lambda\alpha)^2 s^* i^* + (\lambda\alpha s^*)^2]\tau_2 y_2^2(t) \\ &-[\lambda\alpha s^*(\gamma + b) + (\lambda\alpha)^2 s^* i^* + (\lambda\alpha s^*)^2] \int_{t-\tau_2}^{t} y_2^2(s)ds \qquad (4.11) \end{aligned}$$

Now we define a Lyapunov functional $W(y(t))$ as

$$W(y(t)) = W_1(y(t)) + W_2(y(t)) \qquad (4.12)$$

Now we have from (4.12) and (4.14) that

$$
\begin{aligned}
\frac{dW(y(t))}{dt} &= \frac{dW_1(y(t))}{dt} + \frac{dW_2(y(t))}{dt} \\
&\le -\{[(2\lambda i^* + 2b + \beta) - (\lambda s^* + \lambda\alpha i^*)] - [\beta(\lambda i^* + b)]\tau_1 \\
&\quad -[2(\lambda\alpha i^*)^2 + 2(\lambda\alpha)^2 s^* i^* + \lambda\alpha i^*(\gamma + b)]\tau_2\} \, y_1^2(t) \\
&\quad -\{[2(\gamma + b - \lambda\alpha s^*) - (\lambda s^* - \beta + \lambda\alpha i^*)] \\
&\quad -[\beta(2\lambda s^* + \lambda i^* + b - 2\beta)]\,\tau_1 - \left[2(\lambda\alpha)^2 s^* i^* + 2(\lambda\alpha s^*)(\gamma + b)\right. \\
&\quad \left. +2(\lambda\alpha s^*)^2 + \lambda\alpha i^*(\gamma + b)\right]\tau_2\}\, y_2^2(t) \\
&\equiv -\eta_1 y_1^2(t) - \eta_2 y_2^2(t) \qquad (4.13)
\end{aligned}
$$

Clearly, (4.8), (4.9), (4.10)and implies that $\eta_1 > 0$ and $\eta_2 > 0$. Denote $\eta = min\{\eta_1, \eta_2\}$, then (4.15) leads to

$$
W(t) + \eta \int_T^t [y_1^2(s) + y_2^2(s)]ds \le W(T) \quad for \quad t \ge T \qquad (4.14)
$$

and which implies $y_1^2(t) + y_2^2(t) \in L_1[T, \infty)$. We can see from (4.1) and the boundedness of y(t) that $y_1^2(t) + y_2^2(t)$ is uniform continuous and then, using Barbalat's Lemma, we can conclude that $\lim_{t\to\infty}[y_1^2(t) + y_2^2(t)] = 0$. Therefore the zero solution of (4.1) is asymptotically stable and this completes the proof.

In this section,we present simple examples to illustrate the procedures of applying our results.

5 Examples

Example 5.1

Consider the following system:

$$\begin{aligned} \dot{s}(t) &= 0.5 - 2s(t)i(t) + 0.005i(t) - 0.5s(t) \\ \dot{i}(t) &= 2(0.4)s(t)i(t) - (0.001 + 0.5)i(t) \end{aligned} \tag{5.1}$$

Comparing the system (5.1) with the system (3.1),we get $b = 0.5$; $\lambda = 2$; $\alpha = 0.4$; $\beta = 0.005$; and $\gamma = 0.001$. So the system(5.1) has an unique positive equilibrium point $E^* = (0.63, 0.15)$
And

$$s^* > \frac{\beta}{\lambda} = 0.0025$$

Then we conclude that the unique positive equilibrium point E^* of the system (5.1) is locally asymptotically stable by Theorem 3.1.The trajectory of the system (5.1) is depicted in Figure 5.1.

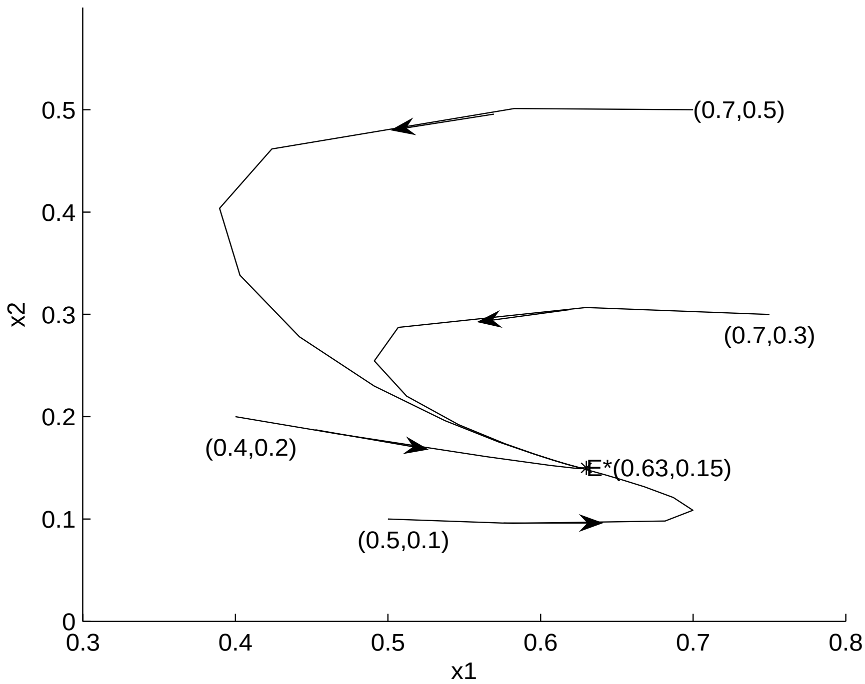

Figure5.1: The local trajectories of the system (5.1)

Example 5.2

Consider the following system:

$$\begin{aligned}\dot{s}(t) &= 0.7 - 5s(t)i(t) + 0.02i(t) - 0.7s(t) \\ \dot{i}(t) &= 5(0.4)s(t)i(t) - (0.3 + 0.7)i(t)\end{aligned} \tag{5.2}$$

Comparing the system (5.2) with the system (3.1),we get $b = 0.7$; $\lambda = 5$; $\alpha = 0.4$; $\beta = 0.02$; and $\gamma = 0.3$. So the system(5.1) has an unique positive equilibrium point $E^* = (0.5, 0.14)$

And

$$s^* > \frac{\beta}{\lambda} = 0.004$$

Then we conclude that the unique positive equilibrium point E^* of the system (5.2) is globally asymptotically stable by Theorem 3.2.The trajectory of the system (5.2) is depicted in Figure 5.2.

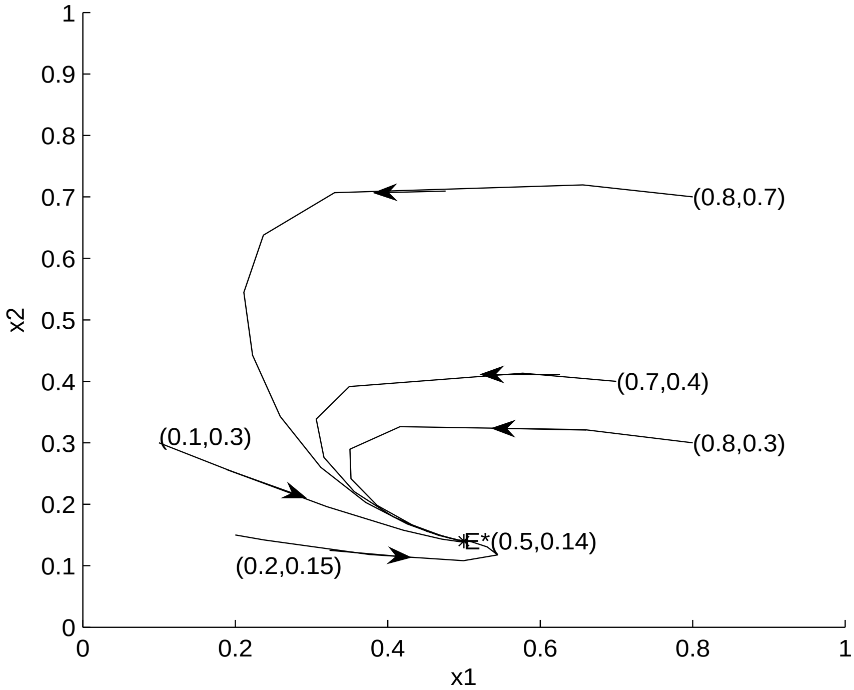

Figure5.2: The global trajectories of the system (5.2)

References

[1] C.P. Ho and Y.L. Ou , The Influence of Time Delay on Local Stability for a Predator-Prey System , Tunghai Science: vol 4 , pp47-62 , 2002

[2] C.P. Ho and H.T. Wang , Local and Global Stability for a Predator-Prey System with Time-Delays , Submit , 2004

[3] C.P. Ho and C.M. Lin , Local and Global Stability for a Predator-Prey Model of Modified Leslie-Gower and Holling-Type Ⅱ with Time-Delay , Submit , 2004

[4] J.K. Hale ; Ordinary Differential Equations , Wiley Interscience , New York , 1969.

[5] K. Gopalsamy , Stability and Oscillations in Delay Differential Equations of Population Dynamics , Kluwer Academic Publisher , Dordrecht , 1992.

[6] L. Perko , Differential Equations and Dynamical Systems , Springer-Verlag New York , Inc. 1991.

[7] S. Wiggins ; Introduction to Applied Nonlinear Dynamical Systems and Chaos , Springer-Verlag New York , Inc. 1990.

[8] Wendy Wang , Global Behavior of an SEIRS Epidemic Model with Time Delays , Applied Mathematics Letters 15 , pp423-428 , P.R. China 2002.

[9] Y. Kuang , Delay Differential Equations with Applications in Population Dynamics , Academic Press , Boston , 1993.

[10] 何肇寶，動態系統及應用， 全華科技圖書股份有限公司，台北，1997.

[11] 何肇寶，數理生態， 全華科技圖書股份有限公司，台北，2004.