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Nomenclature

Symbol	Meaning
C^1	continuously differentiable of order 1
R^n	n-dimensional Enclidean space
R^2_+	$R^2_+ \equiv \{(x,y) \in R^2 \,\vert\, x \geq 0, y \geq 0\}$
int R^2_+	the interior of R^2_+
$N_\delta(x_0)$	a neighborhood of a point x_0, δ is the radius of $N_\delta(x_0)$
$B(0,\delta)$	the open ball centered at 0 of radius δ
e^x	the exponential function
$\ln x$	the natural logarithm of x
Df	the Jacobian matrix of f
$\nabla \cdot (Hf)$	$\nabla \cdot (Hf) \equiv \dfrac{\partial}{\partial x}(Hf) + \dfrac{\partial}{\partial y}(Hf)$, both H and f are functions of x and y
Γ^+	$\Gamma^+ \equiv \{\phi_t(x_0) \,\vert\, t \geq 0\}$, the positive semi-orbit
Γ^+_x	$\Gamma^+_x = \{\phi_t(x) \,\vert\, t \geq 0\}$
$\mathcal{C} \equiv C([-\tau, 0], R^n)$	the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n with the topology of uniform convergence
$\vert \cdot \vert$	the norm in R^n
$\Vert \cdot \Vert$	$\Vert\phi\Vert$, the norm of a functional ϕ which is defined as $\Vert\phi\Vert = \sup\limits_{-\tau \leq \theta \leq 0} \vert\phi(\theta)\vert$

1 Introduction

Predator-prey models have been studied for a long time. Many researchers either have no concern with time delays or tend to ignore delays in their models. But more realistic models should include some of the past states of the population systems; that is, a real system should be modeled with time delays.

In the Lotka-Volterra Model, the carrying capacity of the predator population is independent of the prey population, but in the Leslie-Gower Model, the carrying capacity of the predator population is depends on the prey population. In this paper, we consider the Leslie-Gower predator-prey system with a single discrete delay τ. The system has a unique positive equilibrium point. It is well known that if $\tau = 0$, then the unique positive equilibrium point is globally asymptotically stable. In [1] and [2], to analyze the global stability of the system without delay by constructing a Lyapunov functional or Comparison method, respectively.

The main purpose of this thesis is to establish global stability of the Leslie-Gower predator-prey system. In chapter 2, we introduce some useful definitions and theorems. In chapter 3, we analyze the global stability of the Leslie-Gower predator-prey system without time delay by using Dulac's Criterion plus Poincaré-Bendixson Theorem, or stable limit cycle analysis. In chapter 4, we analyze the global stability of the Leslie-Gower predator-prey system with a single delay by constructing a Lyapunov functional. In chapter 5, we illustrate our results by some examples.

2 Preliminaries

2.1 Nonlinear autonomous system

Consider the following general nonlinear autonomous system of differential equation

$$\dot{x}(t) = f(x) \ , \quad x \in E \tag{2.1}$$

where $f \in C^1(E)$ and E is an open subset of R^n. In this thesis, we need the following definitions and theorems.

Definition 2.1 [*5*]

(i) A point $x_0 \in E$ is called an *equilibrium point* or *critical point* of the system (2.1) if $f(x_0) = 0$.

(ii) An equilibrium point x_0 is called a *hyperbolic equilibrium point* of the system (2.1) if none of the eigenvalues of the matrix $Df(x_0)$ have zero real part.

(iii) An equilibrium point x_0 is called a *saddle point* of the system (2.1) if it is a hyperbolic equilibrium point and $Df(x_0)$ has at least one eigenvalue with a positive real part and one with a negative real part.

Definition 2.2 [*5*] Let E be an open subset of R^n and let $f \in C^1(E)$. For $x_0 \in E$, let $\phi(t, x_0)$ be the solution of the system (2.1) with the initial condition $x(0) = x_0$ defined on its maximal interval of existence $I(x_0)$. Then for $t \in I(x_0)$, the set of mappings ϕ_t defined by

$$\phi_t(x_0) = \phi(t, x_0)$$

is called the *flow* of the system (2.1).

Definition 2.3 [5] Let ϕ_t denote the flow of the system (2.1) defined for all $t \in R$. An equilibrium point x_0 of the system (2.1) is *stable* if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ and $t \geq 0$ we have

$$\phi_t(x) \in N_\varepsilon(x_0)$$

The equilibrium point x_0 is *unstable* if it is not stable. And x_0 is *asymptotically stable* if it is stable and if there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ we have

$$\lim_{t \to \infty} \phi_t(x) = x_0$$

In order to analyze the behavior of the system (2.1) near its equilibrium points, we can show that the local behavior of the nonlinear system (2.1) near a hyperbolic equilibrium point x_0 is qualitatively determined by the behavior of the linear system

$$\dot{x} = Ax$$

where the Jacobian matrix $A = Df(x_0)$. The linear function $Ax = Df(x_0)x$ is called the linear part of f at x_0.

Theorem 2.1 [5] (The Hartman-Grobman Theorem) Let E be an open subset of R^n containing the point x_0, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system (2.1). Suppose that $f(x_0) = 0$ and that the matrix $A = Df(x_0)$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x \in U$, there is an open interval $I(x) \subset R$ containing origin such that for all $x \in U$ and $t \in I(x)$

$$H \circ \phi_t(x) = e^{At} H(x)$$

Theorem 2.2 [5] Suppose that x_0 is an equilibrium point of the system (2.1) and $A = Df(x_0)$. Let $\delta = \det(\mathrm{A})$ and $\tau = \mathrm{trace}(\mathrm{A})$.

(i) If $\delta < 0$ then the system (2.1) has a saddle point at x_0.

(ii) If $\delta > 0$ and $\tau^2 - 4\delta \geq 0$ then the system (2.1) has a node at x_0; it is stable if $\tau < 0$ and unstable if $\tau > 0$.

(iii) If $\delta > 0$, $\tau^2 - 4\delta < 0$, and $\tau \neq 0$ then the system (2.1) has a focus at x_0; it is stable if $\tau < 0$ and unstable if $\tau > 0$.

(iv) If $\delta > 0$ and $\tau = 0$ then the system (2.1) has a center at x_0.

In order to analyze the global stability of the system (2.1), it is necessary to determine whether the closed orbit exists or not. Dulac's Criteria has established conditions under which the system (2.1) with $x \in R^2$ has no closed orbit.

Definition 2.4 [5] A *closed* or *periodic orbit* of the system (2.1) is any closed solution curve of the system (2.1) which is not an equilibrium point of the system (2.1). A periodic orbit Γ is called *stable* if for each $\varepsilon > 0$ there is a neighborhood U of Γ such that for all $x \in U$, $d(\Gamma_x^+, \Gamma) < \varepsilon$; i.e., if for all $x \in U$ and $t \geq 0$, $d(\phi(t,x), \Gamma) < \varepsilon$. A periodic orbit Γ is called *unstable* if it is not stable; and Γ is called *asymptotically stable* it is stable and if for all points x in some neighborhood U of Γ

$$\lim_{t\to\infty} d(\phi(t,x), \Gamma) = 0$$

Definition 2.5 [5] A point $p \in E$ where E is an open subset of R^n is an *ω-limit point* of the trajectory $\phi(\cdot, x)$ of the system (2.1) if there is a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \phi(t_n, x) = p$$

Similarly, if there is a sequence $t_n \to -\infty$ such that

$$\lim_{n\to\infty} \phi(t_n, x) = q$$

and the point $q \in E$, then the point q is called an *α-limit point* of the trajectory $\phi(\cdot, x)$ of the system (2.1). The set of all ω-limit points of a trajectory Γ is called the *ω-limit set* of Γ and it is denoted by $\omega(\Gamma)$. The set of all α-limit points of a trajectory Γ is called the *α-limit set* of Γ and it is denoted by $\alpha(\Gamma)$. The set of all limit points of Γ, $\alpha(\Gamma) \cup \omega(\Gamma)$ is called the *limit set* of Γ.

Theorem 2.3 [*5*] The α and ω-limit sets of a trajectory Γ of the system (2.1), $\alpha(\Gamma)$ and $\omega(\Gamma)$, are closed subsets of E and if Γ is contained in a compact subset of R^n, then $\alpha(\Gamma)$ and $\omega(\Gamma)$, are non-empty, connected, compact subsets of E.

Definition 2.6 [*5*] A *limit cycle* Γ of a planar system is a cycle of the system (2.1) which is the α or ω-limit set of some trajectory of the system (2.1) other than Γ. If a cycle Γ is the ω-limit set of every trajectory in some neighborhood of Γ, then Γ is called an *ω-limit cycle* or *stable limit cycle*; if a cycle Γ is the α-limit set of every trajectory in some neighborhood of Γ, then Γ is call an *α-limit cycle* or *unstable limit cycle*; and if Γ is the ω-limit set of one trajectory other than Γ and the α-limit set of another trajectory other than Γ, then Γ is called a *semi-stable limit cycle.*

Theorem 2.4 [*5*] Let E be an open subset of R^2 and suppose that $f \in C^1(E)$. Let $\gamma(t)$ be a periodic solution of the system (2.1) of period T. Then the periodic solution $\gamma(t)$ is a stable limit cycle if

$$\int_0^T \nabla \cdot f(\gamma(t))\ dt < 0$$

and it is an unstable limit cycle if

$$\int_0^T \nabla \cdot f(\gamma(t))\ dt > 0$$

Theorem 2.5 [*5*] (Dulac's Criteria) Let $f \in C^1(E)$ where E is a simply connected region in R^2. If there exists a function $H \in C^1(E)$ such that $\nabla \cdot (Hf)$ is not identically zero and does not change sign in E, then the system (2.1) has no closed orbit lying entirely in E. If A is an annular region contained in E on which $\nabla \cdot (Hf)$ does not change sign, then there is at most one limit cycle of the system (2.1) in A.

Theorem 2.6 [*5*] (The Poincaré-Bendixson Theorem) Suppose that $f \in C^1(E)$ where E is an open subset of R^2 and that the system (2.1) has a trajectory Γ with Γ^+ contained in a compact subset F of E. Assume that the system (2.1) has only a finite number of equilibrium points in F, then $\omega(\Gamma)$ is either a equilibrium point of the system (2.1), a periodic orbit of the system (2.1), or a graphic of the system (2.1).

At last, we introduce the comparison method. Let's consider the following

$$\begin{aligned} \dot{x} &= M_1(x, y) \\ \dot{y} &= N_1(x, y) \end{aligned} \tag{2.2}$$

$$\begin{aligned} \dot{x} &= M_2(x, y) \\ \dot{y} &= N_2(x, y) \end{aligned} \tag{2.3}$$

Theorem 2.7 [*4*] Suppose that two systems (2.2) and (2.3) have same unique equilibrium point E^*. If E^* is a center or globally asymptotically stable focus with respect to the system (2.3), then E^* is globally asymptotically stable with respect to the system (2.2) if and only if the following inequality holds

$$M_1(x, y) \cdot N_2(x, y) - M_2(x, y) \cdot N_1(x, y) < 0$$

2.2 Nonlinear autonomous system with delays

$\mathcal{C} \equiv C([-\tau, 0], R^n)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n with the topology of uniform convergence; i.e., for $\phi \in \mathcal{C}$, the norm of ϕ is defined as $\|\phi\| = \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|$, where $|\cdot|$ is any norm in R^n. Define $x_t \in \mathcal{C}$ as $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$. Consider the following general nonlinear autonomous system of delay differential equation

$$\dot{x}(t) = f(x_t) \tag{2.4}$$

where $f : \Omega \to R^n$ and Ω is a subset of $\mathcal{C}$. In this thesis, we need the following definitions, theorems and lemmas.

Definition 2.7 *[3]*

(i) The solution $x = 0$ of the system (2.4) is said to be *stable* if, for any $\sigma \in R$, $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, \sigma)$ such that $\phi \in B(0, \delta)$ implies $x_t(\sigma, \phi) \in B(0, \varepsilon)$ for $t \geq \sigma$. Otherwise, we say $x = 0$ is *unstable*.

(ii) The solution $x = 0$ of the system (2.4) is said to be *asymptotically stable* if it is stable and there is a $b_0 = b(\sigma) > 0$ such that $\phi \in B(0, b_0)$ implies $x(\sigma, \phi)(t) \to 0$ as $t \to \infty$.

(iii) The solution $x = 0$ of the system (2.4) is said to be *uniformly stable* if the number δ in the definition of stable is independent of σ.

(iv) The solution $x = 0$ of the system (2.4) is said to be *uniformly asymptotically stable* if it is uniformly stable and there is a $b_0 > 0$ such that, for every $\eta > 0$, there is a $t_0(\eta)$ such that $\phi \in B(0, b_0)$ implies $x_t(\sigma, \phi) \in B(0, \eta)$ for $t \geq \sigma + t_0(\eta)$, for every $\sigma \in R$.

Definition 2.8 [*6*] System (2.4) is said to be *uniformly persistent* if there exists a compact region $D \subset int\ R_+^2$ such that every solution of the system (2.4) eventually enters and remains in the region D.

Lemma 2.1 [*3*] Let $u(\cdot)$ and $w(\cdot)$ be nonnegative continuous scalar functions such that $u(0) = w(0) = 0$; $w(s) > 0$ for $s > 0$, $\lim_{s\to\infty} u(s) = +\infty$, and that $V : C \to R$ is continuous and satisfies

$$V(\phi) \geq u(|\phi(0)|)\ , \quad \dot{V}(\phi) \leq -w(|\phi(0)|).$$

Then $x = 0$ is globally asymptotically stable. That is, every solution of the system (2.4) approaches $x = 0$ as $t \to +\infty$.

3 The model without time delay

Consider the Leslie-Gower predator-prey system without time delay modelled by

$$\begin{aligned} \dot{x_1}(t) &= x_1(t)[r_1 - b_1 x_1(t) - a_1 x_2(t)] \\ \dot{x_2}(t) &= x_2(t)\left[r_2 - a_2 \frac{x_2(t)}{x_1(t)}\right] \end{aligned} \tag{3.1}$$

with the initial condition

$$x_1(0) > 0\ ,\ \ x_2(0) > 0 \tag{3.2}$$

where r_1, r_2, a_1, a_2, and b_1 are positive constants, x_1 and x_2 denote the densities of prey and predator population, respectively.

Clearly, $\widehat{E} \equiv (r_1/b_1, 0)$ is an equilibrium point and $E^* \equiv (x_1^*, x_2^*)$ is the unique positive equilibrium point in the first quadrant for the system (3.1) with the initial condition (3.2), where

$$x_1^* = \frac{r_1 a_2}{a_1 r_2 + a_2 b_1}\ ,\ \ x_2^* = \frac{r_1 r_2}{a_1 r_2 + a_2 b_1} \tag{3.3}$$

It follows from (3.3) that

$$r_2 x_1^* = a_2 x_2^*\ ,\ \ a_1 x_2^* + b_1 x_1^* = r_1 \tag{3.4}$$

Firstly, we discuss the local behavior of equilibrium points of the system (3.1) with the initial condition (3.2) by the Hartman-Grobman Theorem. The Jacobian matrix of the system (3.1) takes the form

$$J = \begin{bmatrix} r_1 - 2b_1 x_1(t) - a_1 x_2(t) & -a_1 x_1(t) \\ a_2 \dfrac{x_2^2(t)}{x_1^2(t)} & r_2 - \dfrac{2a_2 x_2(t)}{x_1(t)} \end{bmatrix}$$

The Jacobian matrix of the system (3.1) at $\widehat{E}$ is

$$\widehat{J} = \begin{bmatrix} -r_1 & -\dfrac{a_1 r_1}{b_1} \\ 0 & r_2 \end{bmatrix}$$

Since $\det(\widehat{J}) = -r_1 r_2 < 0$, the equilibrium point $\widehat{E}$ of (3.1) is a saddle point and the stable manifold is

$$\Gamma_1 = \{\, (x_1, x_2) \,|\, x_1 > 0\,,\; x_2 = 0 \,\}$$

On the other hand, the Jacobian matrix of the system (3.1) at E^* is

$$J^* = \begin{bmatrix} -b_1 x_1^* & -a_1 x_1^* \\ \\ a_2 \dfrac{(x_2^*)^2}{(x_1^*)^2} & -\dfrac{a_2 x_2^*}{x_1^*} \end{bmatrix}$$

Therefore,

$$\det(J^*) = b_1 a_2 x_2^* + a_1 a_2 \frac{(x_2^*)^2}{x_1^*}$$

$$\text{trace}(J^*) = -b_1 x_1^* - \frac{a_2 x_2^*}{x_1^*}$$

Since $\det(J^*) > 0$ and $\text{trace}(J^*) < 0$, the equilibrium point E^* of (3.1) is locally asymptotically stable.

Lemma 3.1 *All solutions* $(x_1(t), x_2(t))$ *of the system* (3.1) *with the initial condition* (3.2) *are positive and bounded.*

Proof. Firstly, we want to show that all solutions $(x_1(t), x_2(t))$ of the system (3.1) with the initial condition (3.2) are positive. That is, if $(x_1(0), x_2(0))$ is in the first quadrant, then $(x_1(t), x_2(t))$ is also in the first quadrant for all $t \geq 0$. Let's divide the first quadrant into four regions I-IV which are defined below:

$$\begin{aligned}
\text{I} &= \{(x_1, x_2) \,|\, r_1 - b_1 x_1 - a_1 x_2 > 0, r_2 x_1 - a_2 x_2 > 0, x_1 > 0, x_2 > 0\} \\
\text{II} &= \{(x_1, x_2) \,|\, r_1 - b_1 x_1 - a_1 x_2 < 0, r_2 x_1 - a_2 x_2 > 0, x_1 > 0, x_2 > 0\} \\
\text{III} &= \{(x_1, x_2) \,|\, r_1 - b_1 x_1 - a_1 x_2 < 0, r_2 x_1 - a_2 x_2 < 0, x_1 > 0, x_2 > 0\} \\
\text{IV} &= \{(x_1, x_2) \,|\, r_1 - b_1 x_1 - a_1 x_2 > 0, r_2 x_1 - a_2 x_2 < 0, x_1 > 0, x_2 > 0\}
\end{aligned}$$

See Figure 3.1(a). Consider the following two cases:

(a) $(x_1(0), x_2(0))$ is near the positive x_1-axis;

(b) $(x_1(0), x_2(0))$ is near the positive x_2-axis;

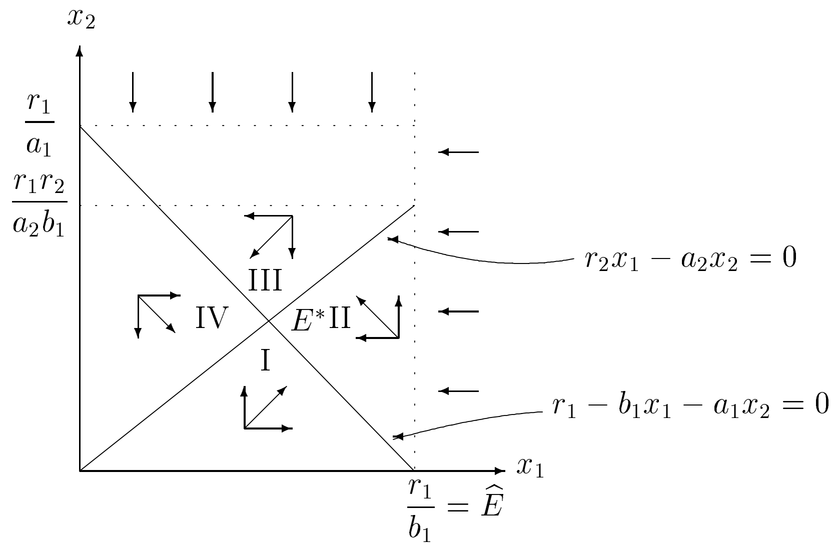
In case (a), the initial point $(x_1(0), x_2(0))$ will be in the region I or II. Since $\dot{x}_2$ is positive in the region I or II, the solution $(x_1(t), x_2(t))$ with the initial point $(x_1(0), x_2(0))$ will run away the positive x_1-axis. In case (b), the initial point $(x_1(0), x_2(0))$ will be in the region III or IV. Since $\dot{x}_1$ is positive in the region IV, the solution $(x_1(t), x_2(t))$ with the initial point $(x_1(0), x_2(0))$ will run away the positive x_2-axis. Now, we want to show that if the initial point $(x_1(0), x_2(0))$ starts in III, then the trajectory of the solution $(x_1(t), x_2(t))$ will go into the region IV. That is, the trajectory of the solution $(x_1(t), x_2(t))$ will not stay in the region III or not go to x_2-axis. Suppose that the trajectory finally stays at some point $(\overline{x}_1, \overline{x}_2)$ in the region III, then $(\overline{x}_1, \overline{x}_2)$ will be an equilibrium point of the system (3.1). It is contradictory. Therefore any solution $(x_1(t), x_2(t))$ start in the region III won't stay in it. On the other hand, if the trajectory in the region III approaches to x_2-axis, then $\dot{x}_1 \to 0$ and $\dot{x}_2 \to -\infty$ as $x_1 \to 0$. Hence there is a $t_1 > 0$ such that $(x_1(t), x_2(t))$ is in the region IV whenever $t \geq t_1$. Therefore, by above discussion, we know that all solutions $(x_1(t), x_2(t))$ are positive.

Secondly, we want to show that all solutions $(x_1(t), x_2(t))$ of the system (3.1) with the initial condition (3.2) are bounded. We know $\dot{x_1} < 0$ for $x_1 \geq r_1/b_1$ and $x_2 > 0$. Hence solutions $(x_1(t), x_2(t))$ of the system (3.1) with the initial point $(x_1(0), x_2(0))$ and $x_1(0) \geq r_1/b_1$, there exists a $T_1 > 0$ such that $x_1(t) < r_1/b_1$ for $t > T_1$. Suppose that $x_2 \geq L \equiv \max\{r_1/a_1\,,\, r_1 r_2/a_2 b_1\}$ and $x_1 < r_1/b_1$. Now we want to show that there exists a $T_2 > 0$ such that $x_2(t) < L$ for $t > T_2$ whenever $x_1(0) < r_1/b_1$ and $x_2(0) \geq L$. If $L = r_1/a_1$, then $x_2 \geq r_1/a_1 > r_1 r_2/a_2 b_1$ and

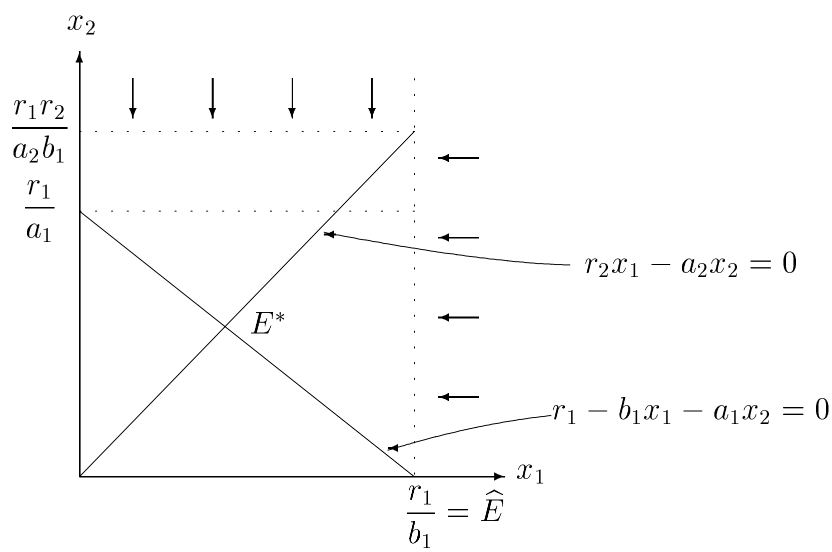
$$\begin{aligned} \dot{x_2} &= x_2 \left[r_2 - \frac{a_2 x_2}{x_1} \right] \\ &\leq x_2 \left[r_2 - \frac{r_1 r_2}{b_1 x_1} \right] \end{aligned}$$

$$= \quad x_2 \left[\frac{r_2(b_1 x_1 - r_1)}{b_1 x_1} \right] < 0$$

See Figure 3.1(a). On the other hand, if $L = r_1 r_2 / a_2 b_1$, then $x_2 \geq r_1 r_2 / a_2 b_1$, and then $\dot{x}_2 < 0$. See Figure 3.1(b). Hence, by above discussion, we know solutions $(x_1(t), x_2(t))$ of the system (3.1) with the initial point $(x_1(0), x_2(0))$ and $x_1(0) < r_1/b_1$, $x_2(0) \geq L$, there exists a $T_2 > 0$ such that $x_2(t) < L$ for $t > T_2$. So $x_1(t) < r_1/b_1$ and $x_2(t) < L$ for $t > T \equiv \max\{T_1, T_2\}$. That is, all solutions $(x_1(t), x_2(t))$ are bounded. ■



(a) $L = \frac{r_1}{a_1}$



(b) $L = \frac{r_1 r_2}{a_2 b_1}$

Figure 3.1 : Schematic diagram for the proof of Lemma 3.1

Theorem 3.1 *The unique positive equilibrium point E^* of the system* (3.1) *is globally asymptotically stable.*

Proof. Now, we want to use the following two methods to analyze the global stability of the system (3.1) with the initial condition (3.2):

(i) Dulac's Criterion plus Poincaré-Bendixson Theorem

(ii) Stable limit cycle analysis

Firstly, we use the method (i) to analyze the system (3.1). Consider

$$H(x_1, x_2) = \frac{1}{x_1 x_2} \qquad x_1 > 0\,,\; x_2 > 0$$

Then

$$\begin{aligned}
\nabla \cdot (Hf) &= \frac{\partial}{\partial x_1}\left\{H \cdot [x_1 (r_1 - b_1 x_1 - a_1 x_2)]\right\} + \frac{\partial}{\partial x_2}\left\{H \cdot \left[x_2 \left(r_2 - a_2 \frac{x_2}{x_1}\right)\right]\right\} \\
&= -\frac{x_1}{x_1^2 x_2}(r_1 - b_1 x_1 - a_1 x_2) + \frac{1}{x_1 x_2}(r_1 - 2b_1 x_1 - a_1 x_2) \\
&\quad -\frac{x_2}{x_1 x_2^2}\left(r_2 - a_2 \frac{x_2}{x_1}\right) + \frac{1}{x_1 x_2}\left(r_2 - \frac{2a_2 x_2}{x_1}\right) \\
&= -\frac{b_1}{x_2} - \frac{a_2}{x_1^2} < 0
\end{aligned}$$

Hence by the Dulac's Criterion, there is no closed orbit in the first quadrant. From above, we see that E^* is locally asymptotically stable. By Lemma 3.1 and Poincaré-Bendixson theorem, it suffices to show that the unique positive equilibrium point E^* is globally asymptotically stable in the first quadrant.

Secondly, we use the method (ii) to analyze the system (3.1). Now, we want to show that the system (3.1) has no closed orbit in the first quadrant. If not, there is a T-periodic orbit $\Gamma = \{\,(x_1(t), x_2(t))\,|\, 0 \le t \le T\,\}$ in the first quadrant. Compute

$$\Delta = \int_0^T \{\frac{\partial}{\partial x_1}[x_1(t)(r_1 - b_1 x_1(t) - a_1 x_2(t))] + \frac{\partial}{\partial x_2}[x_2(t)(r_2 - a_2 \frac{x_2(t)}{x_1(t)})]\}\; dt$$

$$= \int_0^T \{[r_1 - 2b_1x_1(t) - a_1x_2(t)] + [r_2 - 2a_2\frac{x_2(t)}{x_1(t)}]\}\ dt$$

$$= \int_0^T [\frac{\dot{x_1}(t)}{x_1(t)} - b_1x_1(t) + \frac{\dot{x_2}(t)}{x_2(t)} - a_2\frac{x_2(t)}{x_1(t)}]\ dt$$

$$= \ln\left(\frac{x_1(T)}{x_1(0)}\right) + \ln\left(\frac{x_2(T)}{x_2(0)}\right) - \int_0^T \left[b_1x_1(t) + a_2\frac{x_2(t)}{x_1(t)}\right]\ dt$$

$$= -\int_0^T \left[b_1x_1(t) + a_2\frac{x_2(t)}{x_1(t)}\right]\ dt < 0$$

So all closed orbits of the system (3.1) in the first quadrant are orbitally stable. Since every closed orbit is orbitally stable and then there is an unique stable limit cycle in the first quadrant, E^* is unstable. However, E^* is locally asymptotically stable. Thus there is no closed orbit in the first quadrant. By Lemma 3.1 and Poincaré-Bendixson theorem, it suffices to show that the unique positive equilibrium point E^* is globally asymptotically stable in the first quadrant. ■

Remark 3.1 *In [2] the same result with Theorem 3.1 was obtained via the Lyapunov functional*

$$V(x_1, x_2) = \ln\frac{x_1}{x_1^*} + \frac{x_1^*}{x_1} + \frac{a_1x_1^*}{a_2}\left(\ln\frac{x_2}{x_2^*} + \frac{x_2^*}{x_2}\right)$$

Remark 3.2 *Analyze the system* (3.1) *by using Comparison method is not covered yet in this thesis.*

4 The model with time delay

Consider the Leslie-Gower predator-prey system with time delay τ modelled by

$$\begin{aligned} \dot{x_1}(t) &= x_1(t)[r_1 - b_1 x_1(t-\tau) - a_1 x_2(t)] \\ \dot{x_2}(t) &= x_2(t)\left[r_2 - a_2 \frac{x_2(t)}{x_1(t)}\right] \end{aligned} \tag{4.1}$$

with the initial conditions

$$\begin{aligned} x_1(\theta) &= \phi(\theta) \geq 0, \theta \in [-\tau, 0], \phi \in C([-\tau, 0], R) \\ x_1(0) &> 0\ ,\ \ x_2(0) > 0 \end{aligned} \tag{4.2}$$

where r_1, r_2, a_1, a_2, b_1, and τ are positive constants, x_1 and x_2 denote the densities of prey and predator population, respectively.

Lemma 4.1 *Every solutions of the system* (4.1) *with the initial conditions* (4.2) *exists in the interval* $[0, \infty)$ *and remains positive for all* $t \geq 0$.

Proof. It is true because

$$x_1(t) = x_1(0) \exp\left\{\int_0^t [r_1 - b_1 x_1(s-\tau) - a_1 x_2(s)]\ ds\right\}$$

$$x_2(t) = x_2(0) \exp\left\{\int_0^t \left[r_2 - a_2 \frac{x_2(s)}{x_1(s)}\right]\ ds\right\}$$

and $x_i(0) > 0$ for $i = 1, 2$.

Lemma 4.2 *Let* $(x_1(t), x_2(t))$ *denote the solution of* (4.1) *with the initial condition* (4.2)*, then*

$$0 < x_i(t) \leq M_i\ ,\ \ i = 1, 2 \tag{4.3}$$

eventually for all large t, where

$$M_1 = \frac{r_1}{b_1} e^{r_1 \tau} \tag{4.4}$$

$$M_2 = \frac{r_2}{a_2} M_1 \tag{4.5}$$

Proof. Now, we want to show that there exists a $T > 0$ such that $x_1(t) \le M_1$ for $t > T$. By Lemma 4.1, we know that solutions of the system (4.1) with the initial condition (4.2) are positive, and hence, by (4.1),

$$\begin{aligned} \dot{x_1}(t) &= x_1(t)[r_1 - b_1 x_1(t-\tau) - a_1 x_2(t)] \\ &\le x_1(t)[r_1 - b_1 x_1(t-\tau)] \end{aligned} \tag{4.6}$$

Taking $M_1^* = r_1(1+k_1)/b_1$, $0 < k_1 < e^{r_1\tau} - 1$. Suppose $x_1(t)$ is not oscillatory about M_1^*. That is, there exists a $T_0 > 0$ such that either

$$x_1(t) > M_1^* \qquad for \quad t > T_0 \tag{4.7}$$

or

$$x_1(t) \le M_1^* \qquad for \quad t > T_0 \tag{4.8}$$

If (4.8) holds, then for $t > T_0$

$$x_1(t) \le M_1^* = \frac{r_1(1+k_1)}{b_1} < \frac{r_1}{b_1} e^{r_1\tau} = M_1$$

That is, (4.3) holds. Suppose (4.7) holds. Equation (4.6) implies that for $t > T_0 + \tau$

$$\begin{aligned} \dot{x}_1(t) &\le x_1(t)[r_1 - b_1 x_1(t-\tau)] \\ &< -k_1 r_1 x_1(t) \end{aligned}$$

It follows that

$$\int_{T_0+\tau}^{t} \frac{\dot{x}_1(s)}{x_1(s)}\, ds < \int_{T_0+\tau}^{t} -k_1 r_1 \, ds = -k_1 r_1 (t - T_0 - \tau)$$

Then $0 < x_1(t) < x_1(T_0+\tau)\, e^{-k_1 r_1 (t - T_0 - \tau)} \to 0$ as $t \to \infty$. That is, $\lim_{t\to\infty} x_1(t) = 0$ by the Squeeze Theorem. It contradicts to (4.7). Therefore, there must exist a $T_1 > T_0$

such that $x_1(T_1) \le M_1^*$. If $x_1(t) \le M_1^*$ for all $t \ge T_1$, then (4.3) follows. If not, then there must exist a $T_2 > T_1$ such that T_2 be the first time which $x_1(T_2) > M_1^*$. Therefore, there exists a $T_3 > T_2$ such that T_3 be the first time which $x_1(T_3) < M_1^*$ by above discussion. By above, we know that $x_1(T_1) \le M_1^*$, $x_1(T_2) > M_1^*$, and $x_1(T_3) \le M_1^*$ where $T_1 < T_2 < T_3$. Then, by the Intermediate Value Theorem, there exists T_4 and T_5 such that

$$x_1(T_4) = M_1^* \ , \quad T_1 \le T_4 < T_2$$

$$x_1(T_5) = M_1^* \ , \quad T_2 \le T_5 < T_3$$

and $x_1(t) > M_1^*$ for $T_4 < t < T_5$. Hence there is a $T_6 \in (T_4, T_5)$ such that $x_1(T_6)$ is an arbitrary local maximum, and hence it follows from (4.6) that

$$0 = \dot{x}_1(T_6) \le x_1(T_6)[r_1 - b_1 x_1(T_6 - \tau)]$$

and this implies

$$x_1(T_6 - \tau) \le \frac{r_1}{b_1}$$

Integrating both sides of (4.6) on the interval $[T_6 - \tau, T_6]$, we have

$$\ln\left[\frac{x_1(T_6)}{x_1(T_6 - \tau)}\right] = \int_{T_6-\tau}^{T_6} \frac{\dot{x}_1(s)}{x_1(s)}\, ds \le \int_{T_6-\tau}^{T_6} [r_1 - b_1 x_1(s-\tau)]\, ds \le r_1\tau$$

It follows that

$$x_1(T_6) \le x_1(T_6 - \tau)\ e^{r_1\tau} \le \frac{r_1}{b_1} e^{r_1\tau} = M_1$$

Since $x_1(T_6)$ is local maximum of $x_1(t)$ and $x_1(T_6) \le M_1$, $x_1(t) \le M_1$ where t near T_6. Since $x_1(T_6)$ is an arbitrary local maximum of $x_1(t)$, we can conclude that there exists a $T > 0$ such that

$$x_1(t) \le M_1 \qquad for \quad t \ge T \tag{4.9}$$

Suppose $x_1(t)$ is oscillatory about M_1^*, for this case, the proof is similarly to above one. Now, we want to show that $x_2(t)$ is bounded above by M_2 eventually for all large t. By (4.9), it follows that for $t > T$

$$\begin{aligned} \dot{x}_2(t) &= x_2(t)[r_2 - a_2\frac{x_2(t)}{x_1(t)}] \\ &\leq x_2(t)[r_2 - \frac{a_2}{M_1}x_2(t)] \\ &= r_2x_2(t)[1 - \frac{a_2}{r_2M_1}x_2(t)] \\ &= r_2x_2(t)[1 - \frac{x_2(t)}{\frac{r_2M_1}{a_2}}] \end{aligned}$$

Therefore, $x_2(t) \leq r_2M_1/a_2 = M_2$ for $t > T$. This completes the proof. ■

Lemma 4.3 *Suppose that the system* (4.1) *satisfies*

$$r_1 - a_1M_2 > 0 \tag{4.10}$$

where M_2 defined by (4.5)*. Then the system* (4.1) *is uniformly persistent. That is, there exists m_1, m_2, and $T^* > 0$ such that $m_i \leq x_i(t) \leq M_i$ for $t \geq T^*$, $i = 1, 2$.*

Proof. By Lemma 4.2, equation (4.1) follows that for $t \geq T + \tau$

$$\dot{x_1}(t) \geq x_1(t)[r_1 - b_1M_1 - a_1M_2] \tag{4.11}$$

Integrating both sides of (4.11) on $[t-\tau, t]$, where $t \geq T + \tau$, then we have

$$x_1(t) \geq x_1(t-\tau)\ e^{(r_1-b_1M_1-a_1M_2)\tau}$$

That is,

$$x_1(t-\tau) \leq x_1(t)\ e^{-(r_1-b_1M_1-a_1M_2)\tau} \tag{4.12}$$

It follows from (4.1) that for $t \geq T + \tau$

$$\begin{aligned}
\dot{x_1}(t) &= x_1(t)[r_1 - b_1 x_1(t-\tau) - a_1 x_2(t)] \\
&\geq x_1(t)[r_1 - a_1 M_2 - b_1 e^{-(r_1 - b_1 M_1 - a_1 M_2)\tau}\ x_1(t)] \\
&= (r_1 - a_1 M_2)\ x_1(t)\ [1 - \frac{b_1 e^{-(r_1 - b_1 M_1 - a_1 M_2)\tau}}{r_1 - a_1 M_2}\ x_1(t)] \\
&= (r_1 - a_1 M_2)\ x_1(t)\ [1 - \frac{x_1(t)}{\frac{r_1 - a_1 M_2}{b_1}\ e^{(r_1 - b_1 M_1 - a_1 M_2)\tau}}]
\end{aligned}$$

It follows that

$$\liminf_{t\to\infty} x_1(t) \geq \frac{r_1 - a_1 M_2}{b_1}\ e^{(r_1 - b_1 M_1 - a_1 M_2)\tau} \equiv \overline{m_1}$$

and $\overline{m_1} > 0$ by (4.10). So, for large t, $x_1(t) > \overline{m_1}/2 \equiv m_1 > 0$. It follows that

$$\begin{aligned}
\dot{x_2}(t) &\geq x_2(t)[r_2 - \frac{a_2}{m_1} x_2(t)] \\
&= r_2 x_2(t)[1 - \frac{a_2}{r_2 m_1} x_2(t)] \\
&= r_2 x_2(t)[1 - \frac{x_2(t)}{\frac{r_2 m_1}{a_2}}]
\end{aligned}$$

Then

$$\liminf_{t\to\infty} x_2(t) \geq \frac{r_2 m_1}{a_2} \equiv \overline{m_2}$$

So, for large t, $x_2(t) > \overline{m_2}/2 \equiv m_2 > 0$. Let

$$D = \{(x_1, x_2) \,|\, m_1 \leq x_1 \leq M_1\,,\ m_2 \leq x_2 \leq M_2\}$$

Then D is a bounded compact region in R_+^2 that has positive distance from coordinate hyperplanes. Hence we obtain that there exists a $T^* > 0$ such that if $t \geq T^*$, then every positive solution of system (4.1) with the initial conditions (4.2) eventually enters and remains in the region D, that is, system (4.1) is uniformly persistent. ■

Theorem 4.1 *If the delay τ satisfy*

$$r_1 - a_1 M_2 > 0 \tag{4.13}$$

$$b_1 M_1^2 \tau < 2x_1^* \tag{4.14}$$

$$b_1 m_1 M_1 (r_1 + b_1 x_1^*)\tau < 2x_1^*(b_1 m_1 - a_1 M_2 - a_1 x_2^*) \tag{4.15}$$

where $m_1, M_1,$ and M_2 defined in Lemmas 4.2 and 4.3, then the unique positive equilibrium E^* of the system (4.1) is globally asymptotically stable.

Proof. Define $y(t) = (y_1(t), y_2(t))$ by

$$y_1(t) = \frac{x_1(t) - x_1^*}{x_1^*}\,,\; y_2(t) = \frac{x_2(t) - x_2^*}{x_2^*}$$

From (4.1),

$$\dot{y_1}(t) = [1 + y_1(t)][-b_1 x_1^* y_1(t-\tau) - a_1 x_2^* y_2(t)] \tag{4.16}$$

$$\dot{y_2}(t) = [1 + y_2(t)]\left[\frac{r_2 x_1^* y_1(t) - a_2 x_2^* y_2(t)}{x_1^*(1 + y_1(t))}\right] \tag{4.17}$$

Let

$$V_1(y(t)) = \frac{1}{a_1 x_1^* x_2^*}\{y_1(t) - \ln[1 + y_1(t)]\} + \frac{1}{r_2 x_1^*}\{y_2(t) - \ln[1 + y_2(t)]\} \tag{4.18}$$

then we have from (4.16) and (4.17) that

$$\begin{aligned}
\dot{V}_1(y(t)) &= \frac{1}{a_1 x_1^* x_2^*}\cdot\frac{y_1(t)\dot{y}_1(t)}{1 + y_1(t)} + \frac{1}{r_2 x_1^*}\cdot\frac{y_2(t)\dot{y}_2(t)}{1 + y_2(t)} \\
&= -\frac{b_1}{a_1 x_2^*} y_1(t) y_1(t-\tau) - \frac{1}{x_1^*} y_1(t) y_2(t) + \frac{y_1(t) y_2(t)}{x_1^*[1 + y_1(t)]} - \frac{y_2^2(t)}{x_1^*[1 + y_1(t)]} \\
&= -\frac{b_1}{a_1 x_2^*} y_1(t) y_1(t-\tau) - \frac{y_1^2(t) y_2(t)}{x_1^*[1 + y_1(t)]} - \frac{y_2^2(t)}{x_1^*[1 + y_1(t)]} \\
&\le -\frac{b_1}{a_1 x_2^*} y_1(t) y_1(t-\tau) + \frac{|y_2(t)| y_1^2(t)}{x_1^*[1 + y_1(t)]} - \frac{y_2^2(t)}{x_1^*[1 + y_1(t)]}
\end{aligned} \tag{4.19}$$

By Lemma 4.3, there exists a $T^* > 0$ such that $m_i \leq x_i^*[1 + y_i(t)] = x_i(t) \leq M_i$ for $t > T^*$, $i = 1, 2$. Then (4.19) implies that

$$
\begin{aligned}
\dot{V}_1(y(t)) &\leq -\frac{b_1}{a_1 x_2^*} y_1(t) y_1(t-\tau) + \frac{1}{m_1} |y_2(t)| y_1^2(t) - \frac{1}{M_1} y_2^2(t) \\
&\leq -\frac{b_1}{a_1 x_2^*} y_1(t) y_1(t-\tau) + \frac{1}{m_1} \left(1 + \frac{M_2}{x_2^*}\right) y_1^2(t) - \frac{1}{M_1} y_2^2(t) \\
&= -\frac{b_1}{a_1 x_2^*} y_1(t) [y_1(t) - \int_{t-\tau}^{t} \dot{y}_1(s)\, ds] + \frac{1}{m_1} \left(1 + \frac{M_2}{x_2^*}\right) y_1^2(t) - \frac{1}{M_1} y_2^2(t) \\
&= -\left(\frac{b_1}{a_1 x_2^*} - \frac{M_2}{m_1 x_2^*} - \frac{1}{m_1}\right) y_1^2(t) - \frac{1}{M_1} y_2^2(t) \\
&\quad + \frac{b_1}{a_1 x_2^*} y_1(t) \int_{t-\tau}^{t} [1 + y_1(s)][-b_1 x_1^* y_1(s-\tau) - a_1 x_2^* y_2(s)]\, ds \\
&= -\left(\frac{b_1}{a_1 x_2^*} - \frac{M_2}{m_1 x_2^*} - \frac{1}{m_1}\right) y_1^2(t) - \frac{1}{M_1} y_2^2(t) \\
&\quad + \frac{b_1}{a_1 x_2^*} \int_{t-\tau}^{t} [1 + y_1(s)][-b_1 x_1^* y_1(t) y_1(s-\tau) - a_1 x_2^* y_1(t) y_2(s)]\, ds \\
&\leq -\left(\frac{b_1}{a_1 x_2^*} - \frac{M_2}{m_1 x_2^*} - \frac{1}{m_1}\right) y_1^2(t) - \frac{1}{M_1} y_2^2(t) \\
&\quad + \frac{b_1}{a_1 x_2^*} \int_{t-\tau}^{t} [1 + y_1(s)][b_1 x_1^* |y_1(t) y_1(s-\tau)| + a_1 x_2^* |y_1(t) y_2(s)|]\, ds \quad (4.20)
\end{aligned}
$$

Then for $t \geq T^* + \tau \equiv \widehat{T}$, we have from (4.20) that

$$
\begin{aligned}
\dot{V}_1(y(t)) &\leq -\left(\frac{b_1}{a_1 x_2^*} - \frac{M_2}{m_1 x_2^*} - \frac{1}{m_1}\right) y_1^2(t) - \frac{1}{M_1} y_2^2(t) \\
&\quad + \frac{b_1 M_1}{a_1 x_1^* x_2^*} \int_{t-\tau}^{t} [b_1 x_1^* |y_1(t)||y_1(s-\tau)| + a_1 x_2^* |y_1(t)||y_2(s)|]\, ds \\
&\leq -\left(\frac{b_1}{a_1 x_2^*} - \frac{M_2}{m_1 x_2^*} - \frac{1}{m_1}\right) y_1^2(t) - \frac{1}{M_1} y_2^2(t) + \frac{b_1 M_1}{a_1 x_1^* x_2^*} \left[\frac{b_1 x_1^* \tau}{2} y_1^2(t)\right. \\
&\quad \left. + \frac{b_1 x_1^*}{2} \int_{t-\tau}^{t} y_1^2(s-\tau)\, ds + \frac{a_1 x_2^* \tau}{2} y_1^2(t) + \frac{a_1 x_2^*}{2} \int_{t-\tau}^{t} y_2^2(s)\, ds\right]
\end{aligned}
$$

$$= -\left(\frac{b_1}{a_1x_2^*} - \frac{M_2}{m_1x_2^*} - \frac{b_1^2M_1\tau}{2a_1x_2^*} - \frac{b_1M_1\tau}{2x_1^*} - \frac{1}{m_1}\right) y_1^2(t) - \frac{1}{M_1}y_2^2(t)$$

$$+\frac{b_1^2M_1}{2a_1x_2^*}\int_{t-\tau}^{t} y_1^2(s-\tau)\ ds + \frac{b_1M_1}{2x_1^*}\int_{t-\tau}^{t} y_2^2(s)\ ds \tag{4.21}$$

Let

$$V_2(y(t)) = \frac{b_1^2M_1}{2a_1x_2^*}\int_{t-\tau}^{t}\int_{s}^{t} y_1^2(\gamma-\tau)\ d\gamma\ ds + \frac{b_1M_1}{2x_1^*}\int_{t-\tau}^{t}\int_{s}^{t} y_2^2(\gamma)\ d\gamma\ ds \tag{4.22}$$

then

$$\dot{V}_2(y(t)) = \frac{b_1^2M_1\tau}{2a_1x_2^*}y_1^2(t-\tau) - \frac{b_1^2M_1}{2a_1x_2^*}\int_{t-\tau}^{t} y_1^2(s-\tau)\ ds + \frac{b_1M_1\tau}{2x_1^*}y_2^2(t) - \frac{b_1M_1}{2x_1^*}\int_{t-\tau}^{t} y_2^2(s)\ ds \tag{4.23}$$

and then we have from (4.21) and (4.23) that for $t \geq \widehat{T}$

$$\dot{V}_1(y(t)) + \dot{V}_2(y(t)) \leq -\left(\frac{b_1}{a_1x_2^*} - \frac{M_2}{m_1x_2^*} - \frac{b_1^2M_1\tau}{2a_1x_2^*} - \frac{b_1M_1\tau}{2x_1^*} - \frac{1}{m_1}\right) y_1^2(t) - \left(\frac{1}{M_1} - \frac{b_1M_1\tau}{2x_1^*}\right) y_2^2(t) + \frac{b_1^2M_1\tau}{2a_1x_2^*}y_1^2(t-\tau) \tag{4.24}$$

Let

$$V_3(y(t)) = \frac{b_1^2M_1\tau}{2a_1x_2^*}\int_{t-\tau}^{t} y_1^2(s)\ ds \tag{4.25}$$

then

$$\dot{V}_3(y(t)) = \frac{b_1^2M_1\tau}{2a_1x_2^*}y_1^2(t) - \frac{b_1^2M_1\tau}{2a_1x_2^*}y_1^2(t-\tau) \tag{4.26}$$

Now define a Lyapunov functional $V(y(t))$ as

$$V(y(t)) = V_1(y(t)) + V_2(y(t)) + V_3(y(t)) \tag{4.27}$$

then we have from (4.24) and (4.26) that for $t \geq \widehat{T}$

$$\begin{aligned}
\dot{V}(y(t)) &\leq -\left(\frac{b_1}{a_1x_2^*} - \frac{M_2}{m_1x_2^*} - \frac{b_1^2M_1\tau}{a_1x_2^*} - \frac{b_1M_1\tau}{2x_1^*} - \frac{1}{m_1}\right)y_1^2(t) \\
&\quad -\left(\frac{1}{M_1} - \frac{b_1M_1\tau}{2x_1^*}\right)y_2^2(t) \\
&= -\frac{2x_1^*(b_1m_1 - a_1M_2 - a_1x_2^*) - b_1m_1M_1(r_1 + b_1x_1^*)\tau}{2a_1m_1x_1^*x_2^*}y_1^2(t) \\
&\quad -\frac{2x_1^* - b_1M_1^2\tau}{2x_1^*M_1}y_2^2(t) \\
&\equiv -\alpha\ y_1^2(t) - \beta\ y_2^2(t) \qquad (4.28)
\end{aligned}$$

Then it follows from (4.14) and (4.15) that $\alpha > 0$ and $\beta > 0$. Let $w(s) = \widehat{N}s^2$ where $\widehat{N} = \min\{\alpha, \beta\}$, then w is nonnegative continuous on $[0, \infty]$, $w(0) = 0$, and $w(s) > 0$ for $s > 0$. It follows from (4.28) that for $t \geq \widehat{T}$

$$\dot{V}(y(t)) \leq -\widehat{N}\ [y_1^2(t) + y_2^2(t)] = -\widehat{N}\ |y(t)|^2 = -w(|y(t)|) \qquad (4.29)$$

Now, we want to find a function u such that $V(y(t)) \geq u(|y(t)|)$. It follows from (4.18), (4.22), and (4.25) that

$$V(y(t)) \geq \frac{1}{a_1x_1^*x_2^*}\{y_1(t) - \ln[1 + y_1(t)]\} + \frac{1}{r_2x_1^*}\{y_2(t) - \ln[1 + y_2(t)]\} \qquad (4.30)$$

By the Taylor Theorem, we have that

$$y_i(t) - \ln[1 + y_i(t)] = \frac{y_i^2(t)}{2[1 + \theta_i(t)]^2} \qquad (4.31)$$

where $\theta_i(t) \in (0, y_i(t))$ or $(y_i(t), 0)$ for $i = 1, 2$.

Case1 : If $0 < \theta_i(t) < y_i(t)$ for $i = 1, 2$, then

$$\frac{y_i^2(t)}{[1 + y_i(t)]^2} < \frac{y_i^2(t)}{[1 + \theta_i(t)]^2} < y_i^2(t) \qquad (4.32)$$

By Lemma 4.3, it follows that for $t \geq T^*$

$$m_i \leq x_i^*[1 + y_i(t)] = x_i(t) \leq M_i\ ,\quad i = 1, 2 \qquad (4.33)$$

Then (4.32) implies that

$$\left(\frac{x_i^*}{M_i}\right)^2 y_i^2(t) \le \frac{y_i^2(t)}{[1+\theta_i(t)]^2} < y_i^2(t) \quad , \quad i = 1,2 \tag{4.34}$$

It follows that (4.30), (4.31), and (4.34) that for $t \ge T^*$

$$\begin{aligned} V(y(t)) &\ge \frac{1}{2a_1x_1^*x_2^*}\frac{y_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2r_2x_1^*}\frac{y_2^2(t)}{[1+\theta_2(t)]^2} \\ &\ge \frac{1}{2a_1x_1^*x_2^*}\left(\frac{x_1^*}{M_1}\right)^2 y_1^2(t) + \frac{1}{2r_2x_1^*}\left(\frac{x_2^*}{M_2}\right)^2 y_2^2(t) \\ &\ge \min\left\{\frac{1}{2a_1x_1^*x_2^*}\left(\frac{x_1^*}{M_1}\right)^2 , \frac{1}{2r_2x_1^*}\left(\frac{x_2^*}{M_2}\right)^2\right\} [y_1^2(t)+y_2^2(t)] \\ &\equiv \widetilde{N}\ |y(t)|^2 \end{aligned}$$

Case2 : If $-1 < y_i(t) < \theta_i(t) < 0$ for $i = 1,2$, then

$$y_i^2(t) < \frac{y_i^2(t)}{[1+\theta_i(t)]^2} < \frac{y_i^2(t)}{[1+y_i(t)]^2} \tag{4.35}$$

By (4.33), (4.35) implies that

$$y_i^2(t) < \frac{y_i^2(t)}{[1+\theta_i(t)]^2} \le \left(\frac{x_i^*}{m_i}\right)^2 y_i^2(t) \quad , \quad i = 1,2 \tag{4.36}$$

It follows that (4.30), (4.31), and (4.36) that for $t \ge T^*$

$$\begin{aligned} V(y(t)) &\ge \frac{1}{2a_1x_1^*x_2^*}\frac{y_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2r_2x_1^*}\frac{y_2^2(t)}{[1+\theta_2(t)]^2} \\ &> \frac{1}{2a_1x_1^*x_2^*}y_1^2(t) + \frac{1}{2r_2x_1^*}y_2^2(t) \\ &\ge \frac{1}{2a_1x_1^*x_2^*}\left(\frac{x_1^*}{M_1}\right)^2 y_1^2(t) + \frac{1}{2r_2x_1^*}\left(\frac{x_2^*}{M_2}\right)^2 y_2^2(t) \\ &\ge \widetilde{N}\ [y_1^2(t)+y_2^2(t)] \\ &= \widetilde{N}\ |y(t)|^2 \end{aligned}$$

Case3 : If $0 < \theta_1(t) < y_1(t)$ and $-1 < y_2(t) < \theta_2(t) < 0$, then it follows that (4.30), (4.31), (4.34) and (4.36) that for $t \geq T^*$

$$\begin{aligned} V(y(t)) &\geq \frac{1}{2a_1x_1^*x_2^*}\frac{y_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2r_2x_1^*}\frac{y_2^2(t)}{[1+\theta_2(t)]^2} \\ &> \frac{1}{2a_1x_1^*x_2^*}\left(\frac{x_1^*}{M_1}\right)^2 y_1^2(t) + \frac{1}{2r_2x_1^*}y_2^2(t) \\ &\geq \frac{1}{2a_1x_1^*x_2^*}\left(\frac{x_1^*}{M_1}\right)^2 y_1^2(t) + \frac{1}{2r_2x_1^*}\left(\frac{x_2^*}{M_2}\right)^2 y_2^2(t) \\ &\geq \widetilde{N}\ [y_1^2(t) + y_2^2(t)] \\ &= \widetilde{N}\ |y(t)|^2 \end{aligned}$$

Case4 : If $-1 < y_1(t) < \theta_1(t) < 0$ and $0 < \theta_2(t) < y_2(t)$, then it follows that (4.30), (4.31), (4.34) and (4.36) that for $t \geq T^*$

$$\begin{aligned} V(y(t)) &\geq \frac{1}{2a_1x_1^*x_2^*}\frac{y_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2r_2x_1^*}\frac{y_2^2(t)}{[1+\theta_2(t)]^2} \\ &> \frac{1}{2a_1x_1^*x_2^*}y_1^2(t) + \frac{1}{2r_2x_1^*}\left(\frac{x_2^*}{M_2}\right)^2 y_2^2(t) \\ &\geq \frac{1}{2a_1x_1^*x_2^*}\left(\frac{x_1^*}{M_1}\right)^2 y_1^2(t) + \frac{1}{2r_2x_1^*}\left(\frac{x_2^*}{M_2}\right)^2 y_2^2(t) \\ &\geq \widetilde{N}\ [y_1^2(t) + y_2^2(t)] \\ &= \widetilde{N}\ |y(t)|^2 \end{aligned}$$

Let $u(s) = \widetilde{N}s^2$, then u is nonnegative continuous on $[0,\infty)$, $u(0) = 0$, $u(s) > 0$ for $s > 0$, and $\lim_{s\to\infty} u(s) = +\infty$. So, by case1 $\sim$ case4, we have

$$V(y(t)) \geq u(|y(t)|) \qquad for \quad t \geq T^* \tag{4.37}$$

So the equilibrium point E^* of the system (4.1) is globally asymptotically stable by Lemma 2.1. ■

5 Examples

We present below two simple examples to illustrate the procedures of applying our results.

Example 5.1 Consider the system

$$\begin{aligned} \dot{x_1}(t) &= x_1(t)[1 - 10x_1(t) - x_2(t)] \\ \dot{x_2}(t) &= x_2(t)\left[1 - \frac{2x_2(t)}{x_1(t)}\right] \end{aligned} \tag{5.1}$$

where $r_1 = r_2 = 1$, $a_1 = 1$, $a_2 = 2$, $b_1 = 10$, and $E^* = (2/21, 1/21)$. Then we conclude that the unique positive equilibrium point E^* of the system (5.1) is globally asymptotically stable by Theorem 3.1. The trajectory of the system (5.1) is depicted in Figure 5.1.

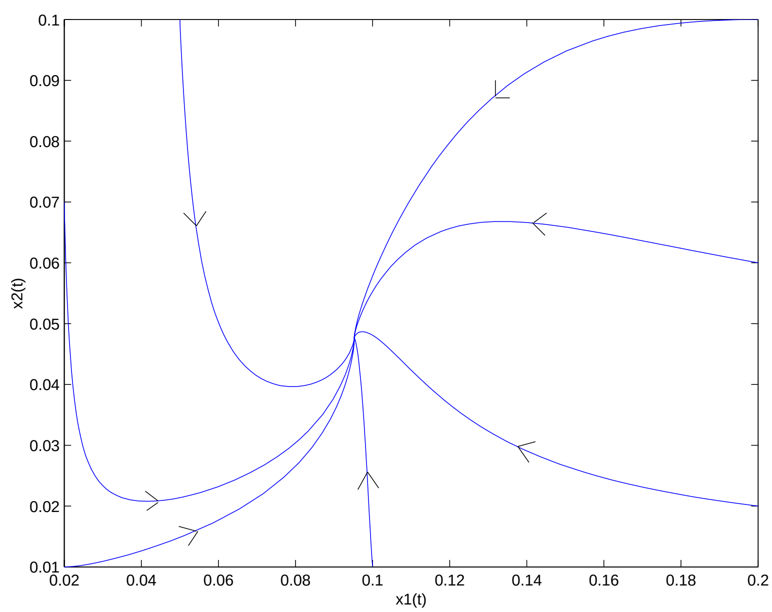


Figure 5.1 : The trajectory of the system (5.1)

Example 5.2 Consider the system

$$\begin{aligned} \dot{x_1}(t) &= x_1(t)[1 - 10x_1(t-\tau) - x_2(t)] \\ \dot{x_2}(t) &= x_2(t)\left[1 - \frac{2x_2(t)}{x_1(t)}\right] \end{aligned} \tag{5.2}$$

where $r_1 = r_2 = 1$, $a_1 = 1$, $a_2 = 2$, $b_1 = 10$, and $E^* = (2/21, 1/21)$. Then

$$r_1 - a_1 M_2 = 0.9325 > 0$$

$$2x_1^* - b_1 M_1^2 \tau = 0.1358 > 0$$

$$2x_1^*(b_1 m_1 - a_1 M_2 - a_1 x_2^*) - b_1 m_1 M_1 (r_1 + b_1 x_1^*)\tau = 0.0239 > 0$$

whenever $\tau = 0.3$. Consequently, by Theorem 4.1, we conclude that the unique positive equilibrium point E^* of the system (5.2) is globally asymptotically stable. The trajectory of the system (5.2) is depicted in Figure 5.2.

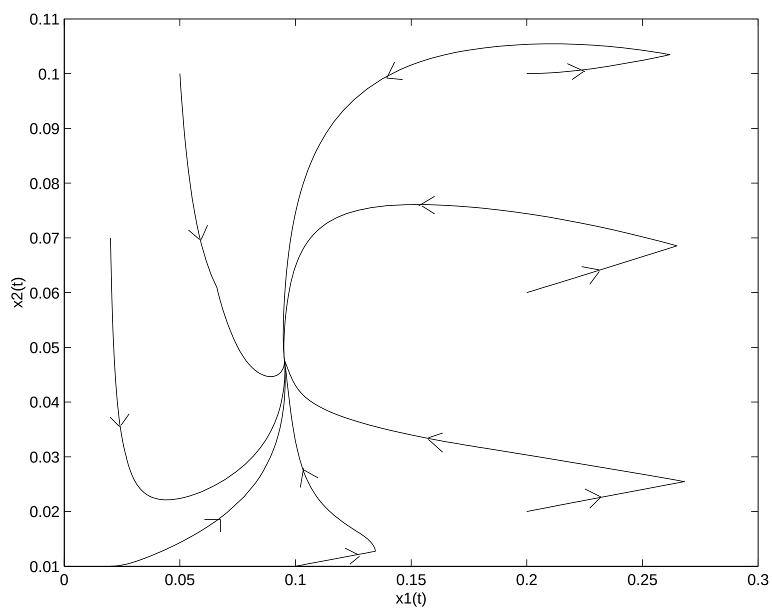


Figure 5.2 : The trajectory of the system (5.2) with $\tau = 0.3$

6 Conclusions

In this thesis, we obtain that a sufficient condition for the global stability of the Leslie-Gower predator-prey system with time delay. We believe that the Leslie-Gower predator-prey system with time delay as follows will be an important topic for future study.

$$\begin{aligned} \dot{x_1}(t) &= x_1(t)[r_1 - b_1 x_1(t-\tau) - a_1 x_2(t)] \\ \dot{x_2}(t) &= x_2(t)\left[r_2 - a_2 \frac{x_2(t)}{x_1(t-\tau)}\right] \end{aligned} \tag{6.1}$$

$$\begin{aligned} \dot{x_1}(t) &= x_1(t)[r_1 - b_1 x_1(t-\tau_1) - a_1 x_2(t-\tau_2)] \\ \dot{x_2}(t) &= x_2(t)\left[r_2 - a_2 \frac{x_2(t-\tau_2)}{x_1(t-\tau_1)}\right] \end{aligned} \tag{6.2}$$

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