Abstract

For two vertices u and v in an oriented graph D, a u-v geodesic is a directed path of minimum length from u to v. Let $I(u, v)$ denote the set of all vertices lying on a u-v geodesic or a v-u geodesic. If A is a subset of $V(D)$, then $I(A)$ is the union of all $I(u, v)$ for $u, v \in A$. The geodetic number $g(D)$ is the minimum cardinality of the subset A of $V(D)$ with $I(A) = V(D)$. For a nontrivial connected graph G, the geodetic spectrum of G is the set of geodetic numbers among the all orientations of G and the strong geodetic spectrum of G is the set of geodetic numbers among the all strongly connected orientations of G. In this paper, we investigate geodetic spectra and strong geodetic spectra of graphs. For the geodetic spectra of graphs, we demonstrate that for every two integers n and m with $1 \leq n-1 \leq m \leq \binom{n}{2}$, there exists a connected graph G of order n and size m such that $S(G) = \{2, 3, \ldots, n\}$. We also determine the geodetic spectra of complete r-partite graphs, cycles and trees. These results provide answers to a conjecture and two problems given by Chartrand and Zhang [?]. For the strong geodetic spectra of graphs, we show that the strong geodetic spectrum of each graph is the subset of $\{2, 3, \cdots, n-2\}$ and for every positive integers n, m, k, with $2 \leq k \leq n-3$, and $n+k \leq m \leq \binom{n}{2} - k$, we can construct a strongly connected digraph D of order n and size m such that $g(D) = k$.

The study of convex set is a fundamental and significant topic in geometric, topology, and functional analysis, see [1]. Usually, a set C of points in a metric space (X, d) is called *convex* if for every two points x and y in C, all points of any geodesic (shortest arc, curve, or path) from x to y are contained in C.

In graph theory, the best-know metric space is $(V(G), d)$, where $V(G)$ is the vertex set of a graph G and the *distance* $d_G(u, v)$ between two vertices u and v is the minimum number of edges of a u-v path. Convexity in graphs is discussed in the book by Buckley and Harary [**?**] and studied by Harary and Neimenen [**?**].

In a connected graph G, the *convex hull* of a vertex subset A is the smallest convex set containing A. The *hull number* of G is the smallest cardinality of a set whose convex hull is $V(G)$. The concept of the hull number of a graph was introduced by Everett and Seidman [**?**] and studied further by [**?**, **?**, **?**].

Chartrand and Zhang [**?**] extended the study of geodetic number to oriented graphs. They showed that there is an oriented graph of order n and geodetic number k with $2 \leq k \leq n$ as the following theorem.

Theorem 1.1 [9] *For every two integers k and n with $2 \leq k \leq n$, there exists an oriented graph of order n and geodetic number k.*

They considered the lower geodetic number and the upper geodetic number of some special graphs(tree, cycle, and complete bipartite graphs) in [**?**].

Theorem 1.2 [9]

(a) *If T is a tree of order $n \geq 2$ with exactly k end-vertices, then $g^-(T) = k$ and $g^+(T) = n$.*

$g^+(C_n) = n - 1$.

(c) *For two integers r and s with $2 \le r \le s$, $g^-(K_{r,s}) = 2$ and $g^+(K_{r,s}) = r + s$.*

For integers n and m with $1 \le n-1 \le m \le \binom{n}{2}$, Chartrand and Zhang [**?**] gave a connected graph G_1 (G_2) of order n and size m such that $g^-(G_1) = 2$ ($g^+(G_2) = n$).

Theorem 1.3 [9] *For every two integers n and m with $1 \le n-1 \le m \le \binom{n}{2}$, there exist two connected graphs G^+ and G^- of order n and size m such that $g^+(G^+) = n$ and $g^-(G^-) = 2$.*

In this thesis, we focus on the study of the geodetic spectra of graphs. First, we demonstrate that for every two integers n and m with $1 \le n-1 \le m \le \binom{n}{2}$, there exists a connected graph G of order n and size m such that the geodetic spectrum $S(G) = \{2, 3, \ldots, n\}$. This result generalizes Theorem 1.3 and provide the positive answer for the conjecture and problem given by Chartrand and Zhang [**?**]. Second, we confirm the geodetic spectra of trees, cycles, and complete r-partite graphs. The geodetic spectrum of cycle C_n is $\{3\} \cup \{2s : 2 \le 2s \le n\}$. This implies that the geodetic spectrum of a connected graph is not forever consecutive. Finally, we stretch our study on the strong geodetic spectra of graphs. We prove that for any triple n, m, k of integers with $2 \le k \le n-3$ and $n+k \le m \le \binom{n}{2} - k$, there exists a strongly connected graph D_m of order n, size m and $g(D) = k$.

A *simple graph* G consist of an order pair $(V(G), E(G))$ which $V(G)$ is a finite nonempty of vertices and $E(G)$ is a pair of distinct vertices of $V(G)$. Denote an *oriented graph* (or *digraph*) D to be an ordered pair $(V(D), A(D))$ with a finite nonempty set $V(D)$ of vertices and a finite set $A(D)$ of arcs which are ordered pairs of vertices of $V(D)$. A *directed path* (or *dipath*) P=(v_1, v_2, $v_3, ..., v_n$) of D is a subgraph of G with the vertex set $\{v_1,\ v_2,\ v_3, ..., v_n\}$ and the arc set $\{(v_1,\ v_2),\ (v_2,\ v_3), ..., (v_{n-1},\ v_n)\} \subseteq A(D)$. For two vertices u, v of D, a u-v *geodesic* is a directed path of minimum length from u to v. Denote $I(u, v)$ to be the set of all vertices lying on a u-v geodesic or v-u geodesic. For $A \subseteq V(D)$, we define $I(A) = \underset{u,v \in A}{\cup} I(u, v)$. A set A is called a *geodetic set* of D if $I(A) = V(D)$. The minimum cardinality of geodetic set in D is the *geodetic number* $g(D)$ of D. An oriented graph received by directing each edge of a graph G is called an *orientation* of G. The *geodetic spectrum* $S(G)$ of a graph G is the set of geodetic number $g(D)$ for all orientations of G, i.e.,

$$S(G) = \{g(D) : D \text{ is an orientation of } G\}.$$

The *lower geodetic number* $g^-(G)$ is the minimum number of $S(G)$ and The *upper geodetic number* $g^+(G)$ is the maximum number of $S(G)$, that is,

$$g^-(G) = \min S(G) \qquad \textit{and} \qquad g^+(G) = \max S(G).$$

An orientation D of a graph G is called a *strongly connected orientation* of G if for any two vertices u, v of G, there is a u-v dipath and a v-u dipath in D. The *strong geodetic spectrum* $S(G)$ of a graph G is the set of geodetic number $g(D)$ for all strongly connected orientations of G, i.e.,

$$S(G) = \{g(D) : D \text{ is a strongly connected orientation of } G\}.$$

In this chapter, we mainly discuss the geodetic spectra of connected graphs. First, we construct a connected graph G of order n and size m with $S(G) = \{2, 3, \cdots, n\}$ and $n-1 \leq m \leq \binom{n}{2}$. Second, we determine geodetic spectra of complete r-partite graphs, cycles and trees.

3.1 Preliminary properties

It is clear that for any two graphs G and H,

$$S(G \cup H) = \{a + b : a \in S(G) \text{ and } b \in S(H)\},$$

where $G \cup H$ is the union of G and H. Hence, in this paper we only study the geodetic spectra of connected graphs. First, if G is a connected graph of order $n \geq 2$, then

$$S(G) \subseteq \{2, 3, \ldots, n\}. \tag{1}$$

The attempt of this section is to determine the geodetic spectrum for several classes of graphs.

A *source* (respectively, *sink*) of an oriented graph is a vertex of in-degree (respectively, out-degree) zero. Notice that a source or a sink can not be an interior vertex of a geodesic. Hence, we have

Proposition 3.1.1 *In any oriented graph, sources and sinks all lie in any geodetic set.*

The following lemmas are useful in our studies for the geodetic spectra of graphs, as shown in the consequent sections.

partitioned into $\{x\} = V_0, V_1, \ldots, V_r = \{y\}$ *such that every vertex of the graph is in an* x-y *path* $x = x_0, x_1, \cdots, x_r = y$ *with* $x_i \in V_i$ *for* $0 \le i \le r$, *then* $2 \in S(G)$.

Proof. Consider the orientation D of G given by orienting an edge uv of G (where $u \in V_i$ and $v \in V_j$ with $i \le j$) from u to v if $j = i + 1$, and from v to u otherwise. Then $d_D(x, y) = r$ and every vertex of D is in an x-y geodesic. Thus $\{x, y\}$ is a geodetic set of D, and so $g(D) = 2 \in S(G)$. ∎

Corollary 3.1.3 *If a connected graph* G *of order at least two has a hamiltonian path, then* $2 \in S(G)$.

Lemma 3.1.4 *If* D *is an oriented graph obtained from another oriented graph* D' *by adding a set* X *of pairwise non-adjacent new vertices each joining to all vertices in* D', *then* $g(D) = g(D') + |X|$.

Proof. If A is a minimum geodetic set of D', then $A \cup X$ is a geodetic set of D since a geodesic in D' is also a geodesic in D. Then

$$g(D) \le |A \cup X| = |A| + |X| = g(D') + |X|.$$

On the other hand, as all vertices in X are sources of D, a minimum geodetic set of D is of the form $A \cup X$. Notice that A is a geodetic set of D' since a geodesic of D is either a geodesic of D' or a directed path of the form x, y with $x \in X$ and $y \in V(D')$. Consequently,

$$g(D') \le |A| = |A \cup X| - |X| = g(D) - |X|.$$

Therefore $g(D) = g(D') + |X|$. ∎

order at least two, with a specified vertex x', by adding a set X of pairwise non-adjacent new vertices each joining to x'. If x' is a source of D', then $g(D) = g(D') + |X| - 1$. If x' is a sink of D', then $g(D) = g(D') + |X|$.

Proof. Suppose x' is a source of D'. If A is a minimum geodetic set of D', then $x' \in A$ by Proposition 3.1.1. Since a geodesic in D' is also one in D and a geodesic in D starting from a vertex in X has x' as its second vertex, $(A - \{x'\}) \cup X$ is a geodetic set of D. Thus,

$$g(D) \leq |(A - \{x'\}) \cup X| = |A| + |X| - 1 = g(D) + |X| - 1.$$

On the other hand, suppose a minimum geodetic set of D is of the form $A \cup X$. Since a geodesic in D starting from a vertex in X having x' as its second vertex, A does not contains x' otherwise the removing of x' from $A \cup X$ gives a geodetic set of D of smaller size. Consequently, $A \cup \{x'\}$ is a geodetic set of D' and so

$$g(D') \leq |A \cup \{x'\}| = |A \cup X| - |X| + 1 = g(D) - |X| + 1.$$

These prove that $g(D) = g(D') + |X| - 1$.

The proof for the case when x is a sink is similar, except that x' is still a sink and so remains in a geodetic set of D. This gives $g(D) = g(D') + |X|$. ∎

3.2 Graphs with geodetic spetra $\{2, 3, \ldots, n\}$

In this section, we study graphs G for which the equality holds in Relation (1), i.e., $S(G) = \{2, 3, \ldots, n\}$. Particular examples of such graphs are complete graphs, complete graphs with an edge deleted, and complete r-partite graphs.

Theorem 3.2.1 *For every two integers n and m with $1 \leq n - 1 \leq m \leq \binom{n}{2}$, there exists a connected graph G of order n and size m such that $S(G) = \{2, 3, \ldots, n\}$.*

m with $1 \le n-1 \le m \le \binom{n}{2}$ there exists a connected graph G of order n and size m, with a specific vertex x, such that the following two conditions hold.

(C1) The graph G has a hamiltonian path starting from x.

(C2) For $3 \le k \le n$, G has an orientation D using x as a source and $g(D) = k$.

The assertion is clearly true for $n \le 3$. Suppose $n \ge 4$ and assertion is true for $n-1$. We consider the following two cases.

Case 1. $2n-3 \le m \le \binom{n}{2}$.

In this case, $(n-1)-1 \le m-(n-1) \le \binom{n-1}{2}$. By the induction hypothesis, there exist a connected graph G' of order $n-1$ and size $m-(n-1)$ with a vertex x' satisfying the following conditions (C1$'$) and (C2$'$).

(C1$'$) The graph G' has a hamiltonian path starting from x'.

(C2$'$) For $3 \le k' \le n-1$, G' has an orientation D' using x' as a source and $g(D') = k'$.

Let G be the graph obtained from G' by adding a new vertex x joining to all vertices in G'. Then G is a connected graph of order n and size m. Also, (C1$'$) for G' implies (C1) for G. And, for any $3 \le k \le n$, we have $2 \le k-1 \le n-1$. By (C1$'$) and (C2$'$) and Corollary 3.1.3, G' has an orientation D' with $g(D') = k-1$. D' together with the directed edges from x to all vertices in D' forms an orientation D of G with x as a source. According to Lemma 3.1.4, $g(D) = g(D') + 1 = k$.

Case 2. $n-1 \le m \le 2n-4$.

In this case, $(n-1)-1 \le m-1 \le 2n-5 \le \binom{n-1}{2}$. By the induction hypotheses, there is a connected graph G' of order $n-1$ and size $m-1$, with a vertex x', satisfying conditions (C1$'$) and (C2$'$). Let G be the graph obtained from G' by adding a new vertex x joining to x'. Then G is a connected graph of order n and size m. Also,

$3 \le k \le n-1$, by (C2′), G' has an orientation D' using x' as source and $g(D') = k$. Then D' together with the directed edge xx' forms an orientation D of G using x as a source. By Lemma 3.1.5, $g(D) = g(D') + |\{x\}| - 1 = k$. For the case when $k = n$, we have $3 \le k-1 \le n-1$. By (C2′), G' has an orientation D' using x' as source and $g(D') = k-1$. Reverse the direction of each edge in D' and direct x to x' to get an orientation D of G using x as a source. By Lemma 3.1.5, $g(D) = g(D') + |\{x\}| = k$.

The assertion then follows from induction. ∎

As a consequence, we have

Corollary 3.2.2 *For any integer $n \ge 2$, we have $S(K_n) = S(K_n - \{e\}) = \{2, 3, \dots, n\}$.*

Proof. The corollary follows from the fact that for $m = \binom{n}{2}$ (respectively, $m = \binom{n}{2} - 1$) the only graph G in Theorem 3.2.1 is K_n (respectively, $K_n - \{e\}$). ∎

Theorem 3.2.1 provides positive answers to the following conjecture and problem given by Chartrand and Zhang [?].

Conjecture 1 ([9]) *For any triple n, m, k of integers with $1 \le n-1 \le m \le \binom{n}{2}$ and $2 \le k \le n$, there exists a connected oriented graph of order n, size m and geodetic number k.*

Problem 1 ([9]) *For every two integers n and m with $1 \le n-1 \le m \le \binom{n}{2}$, does there exist a connected graph G of order n and size m such that*

$$g^-(G) \le g^+(H) \text{ and } g^-(H) \le g^+(G)$$

for every connected graph H of order n and size m?

empty sets $V_1, V_2, \ldots, V_r$ such that for any two vertices $x \in V_i$ and $y \in V_j$, we have x is adjacent to y if and only if $i \neq j$. We denote the complete r-partite graph by $K_{n_1,n_2,\ldots,n_r}$ when $|V_i| = n_i$ for $1 \leq i \leq r$. Let $n = n_1 + n_2 + \ldots n_r$.

Theorem 3.2.3 *If $G = K_{n_1,n_2,\ldots,n_r}$ is a complete r-partite graph of order n in which every vertex is of degree at least two, then $S(G) = \{2, 3, \ldots, n\}$.*

Proof. We shall prove the theorem by induction on n. For the case when $n \leq 3$, $G = K_3$ and so the theorem follows from Corollary 3.2.2. Suppose $n \geq 4$ and the theorem is true for all $n' < n$. For any $2 \leq k \leq n$, we consider the following three cases. Without loss of generality, we may assume that $n_1 \geq n_2 \geq \ldots \geq n_r$.

Case 1. $r = 2$ and $k \geq n_1 + n_2 - 1$.

For the case when $k = n_1 + n_2$, orient all edges of the graph from V_1 to V_2 gives an orientation with n_1 sources and n_2 sinks. Then $k = n_1 + n_2 \in S(G)$.

For the case when $k = n_1 + n_2 - 1$, orient all edges of the graph from V_1 to V_2 except one edge yx with $y \in V_2$ and $x \in V_1$. Then all vertices in $V_1 - \{x\}$ are sources and in $V_2 - \{y\}$ are sinks. While $(V_1 - \{x\}) \cup (V_2 - \{y\})$ is not a geodetic set, the set $(V_1 - \{x\}) \cup (V_2)$ is. Thus, $k \in S(G)$.

Case 2. $r = 2$ with $k \leq n_1 + n_2 - 2$ or $r \geq 3$ with $k \leq n_1 + n_2 - 1$.

In this case, we can properly choose nonempty subset $A \subseteq V_1$ and $B \subseteq V_2$ with $|A| + |B| = k$ such that $V(G) - (A \cup B)$ can be partition into nonempty set A' and B' with the property that all vertices of $A \cup A'$ are adjacent to all vertices of $B \cup B'$. Notice that there may have other edges when $r \geq 3$. Also choose $x \in A$ and $y \in B$. Orient the graph so that all vertices in A are sources, all vertices in B are sinks, and any vertex $b' \in B'$ is toward to any vertex $a' \in A'$. After that, reverse

yb' for those edges $b'y$ with $b' \in B'$. Then, all vertices in $A - \{x\}$ are sources and in $B - \{y\}$ are sinks. While $(A - \{x\}) \cup (B - \{y\})$ is not a geodetic set, the set $A \cup B$ is geodetic set since $d_D(x, y) = 3$ and x, b', a', y is a geodesic for all $b' \in B'$ and $a' \in A'$. Suppose $A \cup B$ is not a minimum geodetic set. Then a geodetic set is of the form $C = (A - \{x\}) \cup (B - \{y\}) \cup \{z\}$ for some z.

If $z = x$, then the only geodesic containing vertices not in C is a, a', x with $a \in A - \{x\}$ and $a' \in \{y\} \cup (A' - V_1)$. In this case, $B' \bigcap I(C) = \emptyset$, a contradiction. Similarly, we have that $z \neq y$. If $z \in A'$, then the only geodesic containing vertices not in C is a, b', z with $a \in A - \{x\}$, $b' \in B'$ and $z \in A' \bigcap V_1$. In this case, $x \notin I(C)$, a contradiction. Similarly, $z \notin B'$.

Hence, $k = |A| + |B| \in S(G)$.

Case 3. $r \geq 3$ and $k \geq n_1 + n_2$

We may assume that $n_1 \geq 2$ for otherwise $G \cong K_n$ and so the theorem follows from Corollary 3.2.2. In this case, $k \geq n_2 + 2 \geq n_r + 2$.

If $G - V_r$ has a vertex of degree less than 2, then $r = 3$ and $n_2 = n_3 = 1$. Thus $k \geq n_1 + 2 = n$ and so $k = n$. As $G - V_1 = K_2$ has an orientation D' with $g(D') = 2$, by Lemma 3.1.4 using $X = V_1$, graph G then has an orientation D with $g(D) = 2 + |X| = n = k$.

If any vertex in $G - V_r$ is of degree at least two, as $2 \leq k - n_r \leq n - n_r$, by the induction hypothesis, $G - V_r$ has an orientation D' with $g(D') = k - n_r$. By Lemma 3.1.4 using $X = V_r$, G has an orientation D with $g(D) = k$. ∎

3.3 Geodetic spetra of trees and cycles

This section studies geodetic spectra of trees and cycles. The geodetic spectra of the graphs given in the last section are all of the form $\{2, 3, \ldots, n\}$. This is not always

Theorem 3.3.1 *If T is a tree with $n \geq 2$ vertices and ℓ leaves, then $S(T) = \{\ell, \ell+1, \ldots, n\}$.*

Proof. First, any leaf of T is a source or a sink in any orientation of T. So, the leaves are in any geodetic set of any orientation. Thus we only need to prove that for any k with $\ell \leq k \leq n$, there exists some orientation D of T with $g(D) = k$, i.e. $k \in S(T)$. We shall prove this by induction.

When T is a star with center x_1 and the other vertices $x_2, x_3, \ldots, x_n$, we have $\ell = \max\{2, n-1\}$. The orientation D_1 of T with $E(D_1) = \{x_1x_2, x_1x_3, \cdots, x_1x_n\}$ gives $n \in S(T)$ by Lemma 3.1.2. The orientation D_2 of T with $E(D_2) = \{x_2x_1, x_1x_3, x_1x_4, \cdots, x_1x_n\}$ gives $\ell = n-1 \in S(T)$. So the theorem hold.

When T is not a star, we have $n \geq 4$. Choose a longest path in the tree whose second vertex is x. Then all neighbors of x except exactly one are leaves. Let the set of all these leaves is X. Then $T - X$ is a tree with $n - |X| \geq 2$ vertices and $\ell - |X| + 1 \geq 2$ leaves.

For the case when $\ell \leq k \leq n-1$, we have $\ell - |X| + 1 \leq k - |X| + 1 \leq n - |X|$. By the induction hypothesis, $T - X$ has an orientation D_3 such that $g(D_3) = k - |X| + 1$. As x is a leaf in $T - X$, we may assume that x is a source of D_3 by reverse all the arcs in D_3 if necessary. D_3 together with the edges directed from all vertices of X to x results an orientation D of T. By Lemma 3.1.5, we have $g(D) = g(D_3) + |X| - 1 = k$.

For the case when $k = n$, we have $\ell - |X| + 1 \leq k - |X| \leq n - |X|$. By the induction hypothesis, $T - X$ has an orientation D_4 such that $g(D_4) = k - |X|$. As x is a leaf in $T - X$, we may assume that x is a sink of D_4. Then D_4 together with the edges directed from all vertices of X to x results an orientation D of T. By Lemma

The graphs given above all have spectra consisting of consecutive integers. This is not always the case as shown by the cycles.

The *n-cycle* is the graph C_n with vertex set $V(C_n) = \{0, 1, \ldots, n-1\}$ and edge set $E(C_n) = \{(0,1), (1,2), \ldots, (n-2, n-1), (n-1, 0)\}$. The vertices of C_n are assume to be taken modulo n. For instance, $(n-1)+1$ is 0 and $0-1$ is $n-1$.

Theorem 3.3.2 *For $n \geq 3$, we have $S(C_n) = \{3\} \bigcup \{2s : 2 \leq 2s \leq n\}$.*

Proof. Suppose D is an orientation of C_n with s sources $a_1, a_2, \ldots, a_s$ and hence s sinks $b_1, b_2, \ldots, b_s$.

For the case when $s = 0$, we have $g(D) = 2$ since any two vertices form a geodetic set of D. To consider the case of $s \geq 1$, we may assume that the sources and the sinks are alternatively clockwise in the cycles as $a_1, b_1, a_2, b_2, \cdots, a_s, b_s$.

For the case when $s = 1$ with $d_D(a_1, b_1) = n/2$, we have that $\{a_1, b_1\}$ is a geodetic set and so $g(D) = 2$. For the case when $s = 1$ but $d_D(a_1, b_1) < n/2$, the set $\{a_1, b_1\}$ is not a geodetic set, while either $\{a_1, a_1 + 1, b_1\}$ or $\{a_1, a_1 - 1, b_1\}$ is a geodetic set. Thus, $g(D) = 3$.

For the case when $s \geq 2$, the geodesics from a_i to b_i and from a_i to b_{i-1} $(1 \leq i \leq s)$, where $b_0 = b_s$, cover all vertices of the cycle. This together with Proposition 3.1.1 gives that $\{a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_s\}$ is a minimum geodetic set of D and so $g(D) = 2s$.

On the other hand, it is easy to orient C_n into D such that D has exactly s sources and exactly s sinks with $0 \leq s \leq \lfloor n/2 \rfloor$. The lemma then follows. ∎

Theorem 3.3.2 gives as negative answer to the following problem proposed by Chartrand and Zhang [?].

$k \leq g^+(G)$, does there exist an orientation D with $g(D) = k$?

It is well-known that a graph has a strongly connected orientation if and only if it is bridgeless and connected. As C_n has only one strongly connected orientation in which there is no sources or sinks, $S^*(C_n) = \{2\}$ for any integer $n \geq 3$. In general we have

Theorem 4.1 *If G is a bridgeless connected graph of order $n \geq 4$, then $S^*(G) \subseteq \{2, 3, \ldots, n-2\}$.*

Proof. We shall prove that for any strongly connected orientation D of G, it is always the case that $g(D) \leq n-2$. We first consider the case when $G \ncong K_n$. In this case, there are two vertices x and y such that $d_D(x, y) \geq 2$ and $d_D(y, x) \geq 2$. Choose an x-y geodesic P and a y-x geodesic Q. Note that $|(V(P) \cup V(Q)) - \{x, y\}| \geq 2$ and so $(V(D) - V(P) - V(Q)) \cup \{x, y\}$ is a geodetic set of D of size at most $n-2$.

Next, we consider the case when $G \cong K_n$. Suppose to the contrary that $g(D) \geq n-1$. For any geodesic $x_0, x_1, \ldots, x_\ell$ in D, $V(D) - \{x_1, x_2, \ldots, x_{\ell-1}\}$ is a geodetic set of size $n - \ell + 1$. Then $g(D) \geq n-1$ implies $n - \ell + 1 \geq n - 1$ and so $\ell \leq 2$. That is, $d_D(u, v) \leq 2$ for any two vertices u and v. For any vertex x, let $x_1, x_2, \ldots, x_r$ be its out-neighbors and $y_1, y_2, \ldots, y_s$ its in-neighbors. Then $d_D(x, y_i) = 2$ and so there is some j_i with (x_{j_i}, y_i) an arc in D for all $1 \leq i \leq s$. Consequently, $\{x, x_1, x_2, \ldots, x_r\}$ is a geodetic set of D. And so $n - s = r + 1 \geq n - 1$ or $s \leq 1$. Thus each vertex is of in-degree 1. This would imply that $n = \binom{n}{2}$ and so $n = 3$, a contradiction. ∎

Theorem 4.2 *For any triple n, m, k of integers with $2 \leq k \leq n-3$ and $n + k \leq m \leq \binom{n}{2} - k$, there exists a strongly connected orientation D of order n, size m and $g(D) = k$.*

arcs are described as follows. First, it contains the following $n+k$ arcs .

$$\{(1,i),(i,k+2):2\leq i\leq k+1\}\cup\{1,k+2\}$$
$$\cup\{(j,j+1):k+2\leq j\leq n-1\}\cup\{(n,1)\}. \qquad (2)$$

If $n+k<m$, then we add arcs one by one to D according to the following order.

$$(i,i') \quad \text{for} \quad 2\leq i\leq k+1 \text{ and } i+1\leq i'\leq k+1; \qquad (3)$$

$$(1,j) \quad \text{for} \quad k+3\leq j\leq n-1; \qquad (4)$$

$$(j',j) \quad \text{for} \quad k+2\leq j\leq n-2 \text{ and } j+2\leq j'\leq n; \qquad (5)$$

$$(i,j) \quad \text{for} \quad 2\leq i\leq k+1 \text{ and } k+3\leq j\leq n-1. \qquad (6)$$

In the case when D contains arcs in Formula (6) and the last such arcs is $(k+1,j)$, we reverse the arcs $(j+1,x)$ for $k+2\leq x\leq j-1$. It is then easy to see the following properties.

(P1) D is strongly connected as it has the arcs in Formula (2).

(P2) If $1\leq i\leq i'\leq k+1$ and $i'x$ is an arc, then ix is an arc.

(P3) For $2\leq i'\leq k+1$, the possible in-neighbors of i' are those $i<i'$.

From (P2) and (P3), we have that if $2\leq i'\leq k+1$, then any geodesic containing i' can not use i' as an interior vertex, for otherwise it is of the form $\cdots,i,i',x,\cdots$ with $i<i'$ and ix an arc, which is a contradiction. Then any geodetic set of D include the set $S=\{2,3,\cdots,k+1\}$.

On the other hand, we claim that S is a geodetic set of D. For the case when D contains no arc of the form $(k+1,j)$ from Formula (6), the only $(k+1)$-2 geodesic is $k+1,k+2,k+3,\cdots,n,1,2$, and hence $I(S)=V(D)$. On the other hand, suppose $(k+1,j)$ is the last arc added into D. Then a $(k+1)$-2 geodesic is of the form $k+1,x,j+1,j+2,\cdots,n,1,2$ for some $k+2\leq x\leq j$. Again $I(S)=V(D)$.

For the study of the geodetic spectrum, we construct a connected graph G with n vertices, m edges, and $S(G) = \{2, 3, \ldots, n\}$ for n and m being positive integers and $n - 1 \leq m \leq \binom{n}{2}$. This result implies that a conjecture of Chartrand and Zhang [**?**] is true. On the other hand, we determine the geodetic spectra of complete r-partite graphs, cycles, trees. Those solutions give answers of two problems of Chartrand and Zhang [**?**].

For every positive integers n, m, k, with $2 \leq k \leq n-3$, and $n+k \leq m \leq \binom{n}{2} - k$, we can construct a strongly directed graph D of order n and size m such that $g(D) = k$. In future, we are interesting in finding a connected graph G with n vertices, m edges, and the strong geodetic spectrum $\{2, 3, \ldots, n-2\}$ for every positive integers n, m, with $n + 1 \leq m \leq \binom{n}{2}$.

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