東海大學

工業工程與經營資訊研究所

碩士論文

於單一倉儲及多個零售商配銷系統中最佳穩定-巢狀存貨政策之研究

A New Algorithm for One-Warehouse Multi-Retailer Systems under Stationary-Nested Policy

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摘要

企業在快速交貨的時間壓縮下，企業與伙伴間的關係變得更加密切，使得供應鏈式運籌管理有日益盛行的趨勢。配銷倉儲與區域零售商的存貨政策，攸關整體供應鏈的營運績效。故本篇論文的目的是探討：在穩定-巢狀存貨政策下，如何協調集中倉儲對多個區域零售商的產品補貨時程及制定倉儲的配送批量，使供應鏈整體的平均總成本達到最小。目前，在此研究領域，尚未有學者提出一有效的解法，其能保證求得該問題的全面最佳解。因此，本研究針對此問題，剖析其最佳成本函數的最佳解結構。本研究並利用此最佳成本曲線結構的特性，推導出許多重要的理論結果，並依此理論結果設計一套有效率的最佳解搜尋演算法。再以隨機產生的實驗數據驗證之後，證實本研究的搜尋演算法不僅比文獻中其他的解法更迅速有效率，而且可以保證求得該問題的全域最佳解。

關鍵字：確定性存貨， 批量，穩定-巢狀政策，最佳解結構，搜尋演算法

A New Algorithm for One-Warehouse Multi-Retailer Systems under Stationary-Nested Policy

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Abstract

This study aims at optimally coordinating inventory among all the partners in a supply chain system with a central warehouse and multiple local retailers so as to minimize the average total costs. After reviewing the literature, we found no study proposes an efficient solution approach that guarantees to secure an optimal solution for the one-warehouse multi-retailer lot-sizing problem. The solution approaches in the literature share a common problem, namely, they do not have insights into the optimality structure of the problem. Therefore, this study first focuses on performing a full theoretical analysis on the optimality structure. Then, by utilizing our theoretical results, we derive an effective search algorithm that is able to obtain an optimal solution for the one-warehouse multi-retailer lot-sizing problem under stationary-nested policy. Based on our random experiments, we demonstrate that our search algorithm out-performs the other heuristics.

Keywords: Deterministic inventory, Lot Size, Stationary-Nested, Optimality structure, Search algorithm.

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GLOSSARY OF NOTATION

f_n	replenishment frequency
F	the set of optimal multipliers
T	a basic period in the production planning horizon
T_n	The length of time for retailer n, i.e., $T_n = f_n T$
k_0	a set-up cost for the warehouse
k_n	a set-up cost for the retailer *n*
h^n	the unit holding cost rate per unit time for product *n*
$h_n^{'}$	the unit holding cost rate per unit time at retailer *n*
d_n	the demand rate per unit time at each retailer *n*
$TC_n(T)$	the minimum cost function with respect to *T* for the retailer *n*
TC_n	the minimum cost value of $TC_n(f_n, T)$
$\lambda_n(f_n)$	the minimum basic period of $TC_n(f_n, T)$
$\Delta_n(f_n, T)$	the difference function between $TC_n(f_n / f_n + 1, T)$ and $TC_n(f_n, T)$
$\delta_n(1/f_n)$	the junction point that one should choose f_n for $T < \delta_n(1/f_n)$
T_{cc}^*	the local minimum for $f_0 = 1$ and $\{f_n = 1\}_{n=1}^N$
ω_j	the $(1/j)^{th}$ junction point of retailer *n*
TC_{opt}	the total average cost for the objective function
π	the retailer index for the retailer *n*
T_c	the current value of *T*
$F(\delta_\pi(1/f_\pi))$	$F(\delta_\pi(1/f_\pi)) \stackrel{\Delta}{=} (F(T_c) \setminus \{f_\pi\}) \cup \{f_n / f_\pi + 1\}$
$\breve{T}(F(\omega_j))$	the local minimum for the $TC_{opt}(T)$ function
β	the upper bound
T_λ^*	the last-revised upper bound, *i.e.*, $T_\lambda^* = \min\{T_\lambda^*, \beta\}$
(F^*, T^*)	the global optimal solution for the *OWMR* under *stationary-nested* policy
$TC(F^*, T^*)$	the global optimal total average cost solution
B^*	the max of three lower bounds
P^*	the lowest average cost
α	the measure of the closeness of P^* to optimality

Chapter 1 INTRODUCTION

1.1 The Motivation to Study the Lot-sizing for the One-warehouse Multi-retailer System

The coordination of inventory among the partners is one of the key factors that determine the performance of supply chains. Especially, the managers in multi-echelon supply chains address lots of their efforts to determine the optimal replenishment cycles of raw materials, the optimal production cycles of work-in-process, and the optimal batch quantities and distribution frequencies of finished products. The decision-makers are eager for a solution approach that brings an optimal lot-sizing strategy to improve the performance of the whole supply chain. Importantly, such an optimal lot-sizing strategy not only coordinate the logistics of the suppliers, the distribution centers and the retailers in unison, but also, reduce the order processing costs, the inventory holding costs and the distribution (or, the transportation) costs throughout the supply chain system.

This study aims to derive an effective search algorithm that efficiently secures an optimal lot-sizing strategy for a supply chain with one central warehouse and multiple local retailers so as to minimize the total average costs. Such a supply chain is known as a one warehouse multi-retailer (*OWMR*) system in the literature.

1.2 Statement of Scope and Purpose

The one warehouse multi-retailer (*OWMR*) lot-sizing problem is concerned with the determination of lot sizes and schedule of n retailers replenished from the central warehouse. In the *OWMR* lot-sizing problem, the warehouse holds inventory of all products and can replenish each retailer instantaneously. By applying the concept of the *OWMR* lot-sizing problem, the decision makers could effectively determine the replenishment schedule in the warehouse and lot sizes delivered from the warehouse to local retailers so as to minimize the total average costs.

In this study, we focus on obtaining the optimal lot-sizing strategy under stationary-nested policy in the *OWMR* system where stationary-nested policy assumes that the replenishment cycle of each retailer, denoted by T_n, must be an integer-ratio fraction of the replenishment cycle of the warehouse (denoted by T). That is, $T_n = f_n T$ and $f_n \in \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ for all n.

1.3 Background and Problem Description

In this section, we first introduce the decision-making scenario in the *OWMR* lot-sizing problem. The warehouse receives finished products (of a single kind) from its up-stream supplier and distributes the finished products to the local retailers in the *OWMR* system. The objective of the *OWMR* problem is to minimize the total costs incurred per unit time.

Most of assumptions in our study are the same as that defined in Schwarz's (1973) paper. Namely, no backlogging, lost sale, or transshipment is permitted anywhere in the system. Initial inventory is assumed to be zero. Customer demand occurs at each retailer at a constant rate. A holding cost is incurred for each unit of finished product per unit time stored and a setup cost is charged for each order placed at the warehouse and at each retailer. The demand rates, holding cost rates and setup costs are stationary for the warehouse and each retailer. The replenishment of orders is assumed to be instantaneous (though this assumption can be relaxed by adding lead times to the orders). In the *OWMR* system, set-up costs k_0 and k_n are incurred at the warehouse and at each retailer *n*, respectively, for every order placed.

Though we have only one kind of product in the *OWMR* system, it may be easier for the readers to consider the finished products stored at different retailers as different kinds of products (which are stored exclusively in their locations). We denote d_n as the demand rate per unit time at retailer *n*. And, let $h_n^{'}$ and h^n be the *holding cost rates* at retailer *n* and the warehouse, respectively. Then, the *echelon holding cost rate* at retailer *n* shall be $h_n \equiv h_n^{'} - h^n > 0$. (See Roundy's, 1985 paper.)

Under stationary-nested policy, our assumption of $T_n \le T$ facilitates us to employ the echelon method to conveniently compute the holding costs in the *OWMR* system. (Please refer to Clark and Scarf's, 1960 paper for the details.) The *system inventory* of product *n* is the sum of the inventory of product *n* at the warehouse and the inventory at retailer *n*. It is well known that the system inventory follows the familiar saw-tooth inventory pattern with an order interval of *T* (see Graves and Schwarz, 1977 and Roundy, 1985). The average holding cost of product *n* is the sum of the following two terms: (1) the product of the average system inventory of product *n* and its holding cost rate at the warehouse, and (2) the product of the average inventory at retailer *n* and its echelon holding cost rate. Therefore, the average holding cost of product *n* is given by $\frac{d_n}{2} h_n T_n + \frac{d_n}{2} h^n T$.

Assume that there are *N* local retailers in the *OWMR* system. We define a cost function $TC_n(f_n, T)$ for product *n* as

$$TC_n(f_n, T) = \frac{k_n}{T_n} + \frac{d_n h_n T_n}{2} = \frac{k_n}{f_n T} + \frac{d_n h_n f_n T}{2} \tag{1}$$

where $T_n = f_n T$, $f_n \in \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ and $n = 1, \ldots, N$. (We note that the total average costs for product n is given by $\frac{d_n h^n T}{2} + TC_n(f_n, T)$.) Then, we may formulate the *OWMR* lot-sizing problem as follows.

$$
\begin{aligned}
&\textit{Minimize}\ \ TC_{opt}(\{f_n\}, T) \equiv \frac{k_0}{f_0 T} + \sum_{n=1}^{N} \left(\frac{d_n h^n T}{2} + TC_n(f_n, T) \right) && (a) \\
&\textit{subject to}\qquad f_n \in \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\},\ f_0 = 1 && (b)
\end{aligned}
\qquad (2)
$$

The rest of thesis is organized as follows. In Chapter 2, we review the research works on the *OWMR* lot-sizing problem in the literature. In order to solve the *OWMR* lot-sizing problem, we first perform full theoretical analysis on the optimality structure of the optimal cost curve in Chapter 3. Then, in Chapter 4, we employ our theoretical results to derive a search algorithm that obtains the global optimal solution for the *OWMR* lot-sizing problem under *stationary-nested* policy. The first part of Chapter 5 presents a numerical example to demonstrate our global optimal search algorithm. Also, based on our random experiments, we demonstrate that our search algorithm out-performs the other heuristic solutions in the second part of Chapter 5. Finally, we address our concluding remarks in Chapter 6.

Chapter 2 LITERATURE SURVEY

Many researchers have been addressing their efforts to solve the optimal solution for the one-warehouse multi-retailer (*OWMR*) lot-sizing problem. Under a particular set *B* of "basic" policy, Schwarz (1973) derived an optimal policy for the stationary, continuous-time *OWMR* lot-sizing problem (with an infinite planning horizon). A basic policy is any feasible policy with the following properties:

1. Deliveries are made to the warehouse only when the warehouse has zero inventory, and at least one retailer has zero inventory.
2. Deliveries are made to any given retailer only when that retailer has zero inventory.
3. All deliveries made to any given retailer between successive deliveries to the warehouse are of equal size.

Roundy (1985) gave three terms for these important properties as follows:

1. *Zero-Inventory Ordering*: Each facility orders only when its inventory is zero.
2. *Last-Minute Ordering*: The warehouse orders only when at least one retailer orders.
3. *Stationarity-Between-Orders*: At each retailer all orders placed between two successive orders at the warehouse are of equal size.

On the other hand, the zero-inventory-ordering property also applies for a single facility (Wagner and Whitin, 1958) and for many facilities (Zangwill, 1966 and Veinott, 1969) for the finite-horizon discrete-time cases.

A policy is called *stationary* if each facility orders at equally-spaced points in time and in equal amounts. A policy is *nested* if each facility orders every time any of its immediate suppliers does, and perhaps at other times as well. Policies that are both stationary and nested are called *stationary-nested* or *single-cycle*. (One may refer to Roundy's, 1985 paper for the definitions.) Several researchers restricted their attentions to stationary-nested policy. Schwarz (1973) derived the necessary conditions for an optimal policy and some analytical bounds under stationary-nested policy. He also proposed a heuristic that usually solves a near-optimal solution for the *OWMR* lot-sizing problem. Schwarz and Schrage (1975) focused on solving the optimal lot sizes of a single product in multi-echelon assembly systems under stationary-nested policy. Graves and Schwarz (1977) investigated the characteristics of optimal continuous review policies for arborescent systems under stationary-nested policy. Maxwell and Muckstadt (1985) proposed a heuristic for complex multi-stage, multi-product systems under stationary nested policy.

Graves (1979) showed that the Joint Replenishment Problem which may be viewed as a special case of the *OWMR* lot-sizing problem. Many researchers proposed solution approaches for the Joint Replenishment Problem. One may refer to the following papers: Goyal (1974), Silver (1976), Goyal and Belton (1979), Kaspi and Rosenblatt (1983, 1991), Jackson et al. (1985), and Lee and Yao (2003).

Furthermore, Roundy (1985), and Lu and Posner (1994) solved the *OWMR* lot-sizing problem under so-called *integer-ratio* policy which restricts each retailer orders at an integer or reciprocal of an integer multiple of the warehouse order interval. Mitchell (1987) extended Roundy's (1985) results to allow backlogging and introduce a class of policies, called *nearly-integer-ratio* policies which is different from the class of integer-ratio policies by not requiring stationarity of orders placed by retailers. Anily and Federgruen (1990, 1991), and Hall (1991) added the vehicle routing costs in the *OWMR* systems.

After reviewing the literature, we found a problem that commonly shares among the solution approaches for the *OWMR* lot-sizing problem under *stationary-nested* policy. Namely, they do not have insights into the optimality structure of the problem. Therefore, this study focuses on performing a full theoretical analysis on the optimality structure of the *OWMR* lot-sizing problem under *stationary-nested* policy. Our theoretical results in this paper will lay important foundation for deriving an effective search algorithm that is able to obtain an optimal solution within a very short run time.

Chapter 3
THEORETICAL ANALYSIS
ON THE OPTIMAL COST FUNCTION

In this chapter, we present some theoretical results that provide insights into the optimality structure of the *OWMR* lot-sizing problem under *stationary-nested* policy. Let $TC_{opt}(T)$ be the optimal cost function of the *OWMR* problem with respect to *T*. Later, we will introduce the "junction points" on the curve of the $TC_{opt}(T)$ function, and also discuss some interesting properties of those junction points. These junction points assist us in securing the set of optimal multipliers for each given value of *T*, and they facilitate the design of the search algorithm presented in Chapter 4.

3.1 Some Insights into the Optimal Cost Function

Recall that $TC_n(f_n,T)$ is given by $TC_n(f_n,T)=\frac{k_n}{f_nT}+\frac{d_nh_nf_nT}{2}$ where $f_n=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right\}, n=1,\ldots,N$. The following theoretical results provide us some insights into the $TC_{opt}(T)$ function.

For a given *T*, one may obtain the optimal multiplier f_n so as to minimize $TC_n(f_n,T)$. We denote it as $\underline{TC}_n(T)$, the minimum cost function with respect to *T* for retailer *n*, *i.e.*,

$$\underline{TC}_n(T)=\min_{f_n\in\frac{1}{p},p\in N^+}\{TC_n(f_n,T)\} \qquad (3).$$

Then, the following lemma holds for each retailer *n*.

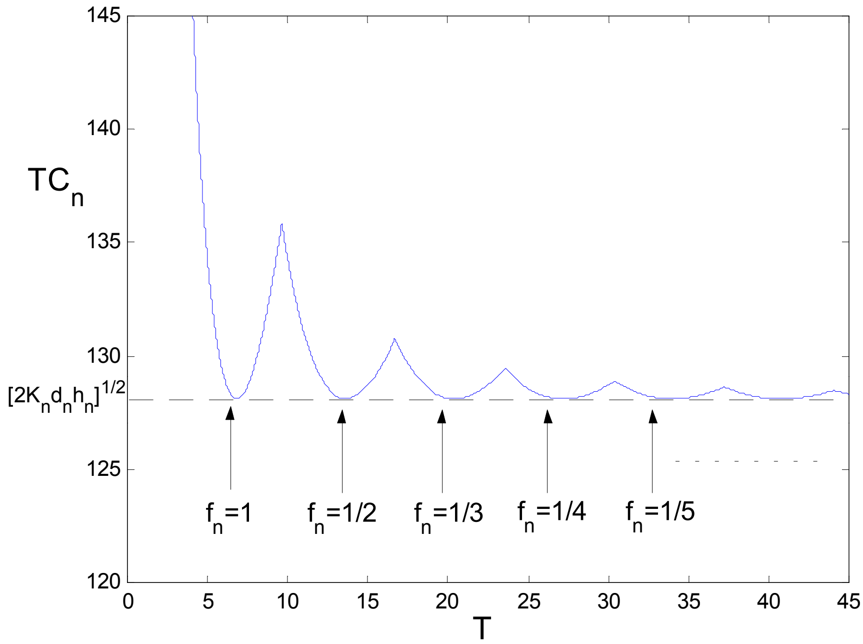

Figure 1: The Piece-wise Convex Curve for $\underline{TC}_n(T)$

Lemma 1 $\underline{TC}_n(T)$ *is a* piece-wise convex *function with respect to T in Figure 1. Also, for each* f_n*, one can secure the local minima for* $\underline{TC}_n(T)$ *at*

$$\lambda_n(f_n) = \frac{1}{f_n}\sqrt{\frac{2k_n}{d_n h_n}} \tag{4}$$

with the minimum cost of $TC_n(T) = \sqrt{2k_n d_n h_n}$ (Schwarz, 1973).

The following proposition shows the optimality structure of the *OWMR* problem.

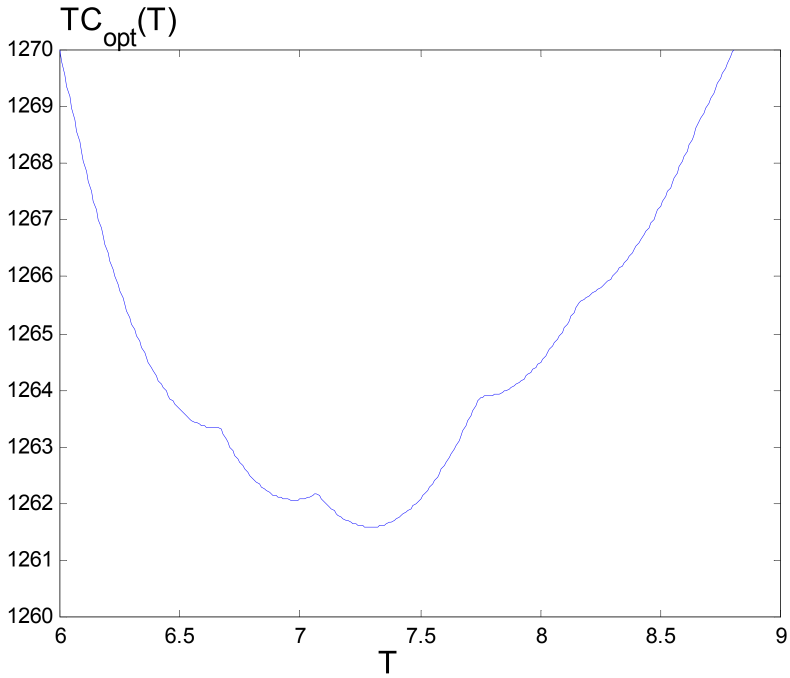

Figure 2: The Piece-wise Convex Curve for the Optimal-cost Function.

Proposition 1 *The* $TC_{opt}(T)$ *function is* piece-wise convex *with respect to T.*

Proof. At a given *T*, the optimal value $TC_{opt}(T)$ is given by $TC_{opt}(\{f_n, T\}) \equiv k_0 / T + \sum_{n>1}\left(h^n T + TC_n(f_n, T)\right)$ where $\frac{k_0}{T}$ and $\sum_{n>1} h^n T$ are *convex* functions and each $\underline{TC}_n(T)$ function is *piece-wise convex* with respect to T by Lemma 1. Since $TC_{opt}(T)$ is the sum of convex functions and a *piece-wise* convex function, it is surely a *piece-wise* convex function. ■

3.2 The Junction Points

Next, we introduce the "junction points" on the curve of the $TC_{opt}(T)$ function. Recall that the $\underline{TC}_n(T)$ function is piece-wise convex. We define a junction point for the $\underline{TC}_n(T)$ function as a particular value of T

where two consecutive convex curves concatenate. These junction points determine at '*what value of T'* where one should change the multiplier of retailer *n* from f_n to $f_n/(f_n+1)$ so as to secure the minimum value for the $\underline{TC}_n(T)$ function.

We first derive a closed-form for the location of the junction points for retailer *n* as follows. We define the difference function $\Delta_n(f_n,T)$ by

$$\Delta_n(f_n,T)\overset{\Delta}{=}TC_n(\frac{f_n}{f_n+1},T)-TC_n(f_n,T)=\frac{k_n}{T}-\frac{d_nh_nT}{2}\left(\frac{f_n^2}{f_n+1}\right) \quad (5).$$

We note that $\Delta_n(f_n,T)$ is the cost difference between using f_n and f_n/f_n+1 as its multiplier for $\underline{TC}_n(\cdot,T)$. Since the function $\Delta_n(f_n,T)$ is an increasing function with respect to *T*. Suppose that the search algorithm proceeds from a lower bound toward larger values of *T*, we evaluate $\Delta_n(f_n,T)$ from positive values, to zero and finally, to negative values. Let ω be the point where $\Delta_n(f_n,T)$ reaches zero. Assume that f_n is the optimal multiplier for retailer *n* for $T<\omega$. This scheme implies that one should keep using f_n until it meets ω. From the point ω onwards, the value of $\underline{TC}_n(\cdot,T)$ can be improved by using $f_n/(f_n+1)$ as its optimal multiplier. We note that ω is the point where two neighboring convex curves $TC_n(f_n,T)$ and $TC_n(f_n/(f_n+1),T)$ meet. Importantly, such a junction point ω provides us with the information not only on "*which retailer n*" to modify, but also on "*where on the T-axis*" to replace f_n with $f_n/(f_n+1)$. By eq. (5), we identify a junction point for retailer *n* by

$$\delta_n(1/f_n)=\sqrt{\frac{1+f_n}{f_n^2}}\sqrt{\frac{2k_n}{d_nh_n}} \quad (6).$$

More specifically, $\delta_n(1/f_n = 1/j)$ is the $(1/j)^{th}$ junction point of retailer *n* where $\frac{1}{j} \in N^+$. Therefore, the junction point $\delta_n(1/j)$ provides us the information that one should choose $f_n = j$ for $T < \delta_n(1/j)$ and choose $f_n = j/(j+1)$, *vice versa*, to obtain the lowest value for the $\underline{TC}_n(T)$ function.

The following theoretical results on the junction points provide strengthen foundation for such a search scheme.

Lemma 2 *Suppose that* $f_n^{(L)}$ *and* $f_n^{(R)}$, *respectively*, *are the optimal multipliers of the left-side and right-side convex curves with regard to a junction point of the* $\underline{TC}_n(T)$ *function. Then,* $f_n^{(R)} = \dfrac{f_n^{(L)}}{f_n^{(L)}+1}$.

Proposition 2 *All the junction points for each individual retailer n, will be inherited by the* $TC_{opt}(T)$ *curve.*

Proof. The proof is presented in Appendix A.1. ■

In other words, Proposition 2 asserts that if a junction point ω shows on one piece-wise convex curve $\underline{TC}_n(T)$, then, ω must also show on the piece-wise convex curve of the $TC_{opt}(T)$ function as a junction point. Let *F*(*T*) be the set of optimal multipliers at a given *T*, *i.e.*, $F(T) = \{f_n^*(T)\}$. The following theorem is an immediate result of Lemma 2 and Proposition 2.

Theorem 1 *Suppose that* $F^{(L)} = \{f_n^{(L)}\}$ *and* $F^{(R)} = \{f_n^{(R)}\}$, *respectively, are the set of optimal multipliers for the left-side and right-side convex curves with regard to a junction point in the plot of the* $TC_{opt}(T)$ *function. Then,* $F^{(R)}$ *is secured from* $F^{(L)}$ *by changing at least one of its optimal* *multiplier by* $f_n^{(R)} = \frac{f_n^{(L)}}{f_n^{(L)}+1}$.

Usually, only one f_n changes at a junction point except for some extreme cases in which two retailers share the same junction point.

The following corollary is also a by-product of Lemma 2 and Proposition 2, and it provides an easier way to secure each $f_n^*(T) \in F(T)$.

Corollary 1 *For any given T, one can secure each* $f_n^*(T) \in F(T)$ *by*

$$f_n^*(T) = \left\{ \begin{array}{ll} 1, & T < 2\sqrt{\frac{k_n}{d_n h_n}} \\ m, & \sqrt{\frac{1-m}{m^2}}\sqrt{\frac{2k_n}{d_n h_n}} < T \le \sqrt{\frac{1+m}{m^2}}\sqrt{\frac{2k_n}{d_n h_n}} \end{array} \right\} \quad (7).$$

The following corollary is important for the design of the proposed search algorithm.

Corollary 2 *Let* ω_1 *and* ω_2 *be two neighboring junction points for the function* $TC_{opt}(T)$, *and* $\omega_1 < \omega_2$. *Then, the set of optimal multipliers for the* $TC_{opt}(T)$ *function is invariant in* (ω_1, ω_2).

Proof. It is obvious by Theorem 1. We know that $F^{(\omega_2)}$ is secured from $F^{(\omega_1)}$ by changing at least one of its optimal multiplier by $f_n^{(\omega_2)} = \frac{f_n^{(\omega_1)}}{f_n^{(\omega_1)}+1}$.

Thus, the set of optimal multipliers for the $TC_{opt}(T)$ function is invariant in (ω_1, ω_2). ■

Chapter 4
A GLOBAL OPTIMAL SEARCH ALGORITHM

In this Chapter, we present a search scheme, which secures a global optimal solution for the *OWMR* problem under *stationary-nested* policy.

Recall that we assert that the $TC_{opt}(T)$ function is piece-wise convex with respect to T in Chapter 3. Also, some interesting properties on the junction points reveal the optimality structure of the $TC_{opt}(T)$ function. These theoretical results encourage us to solve the *OWMR* problem by searching along the T-axis.

To design such a search algorithm, we first need to define the search range by a lower and an upper bound on the T-axis, which are denoted by T_L and T_λ, respectively. We note that the bounds T_L and T_λ are derived by asserting that the best local minimum in $[T_L, T_\lambda]$ must be no worst than any solution outside of $[T_L, T_\lambda]$. Naively, one can secure a global optimal solution for the *OWMR* problem by a small-step search algorithm which enumerates $T \in [T_L, T_\lambda]$ and using a vary small step-size $\Delta T \to 0$. But, this is neither efficient nor accurate, since the step-size determines its performance. Also, the run time of the search algorithm may be extremely long if the search range $[T_L, T_\lambda]$ is wide.

In order to propose an efficient search algorithm, we must utilize our theoretical results on the optimality structure, especially, the properties of the junction points on the $TC_{opt}(T)$ function. By Lemma 2 and Proposition 2, we can easily secure all of the junction points within any search range $[T_L, T_\lambda]$ by eq. (6). Corollary 2 asserts that the set of optimal multipliers for $TC_{opt}(T)$ is invariant in any convex interval between two neighboring junction points. These theoretical results lead us to the following idea: if we are able to obtain all of the local minima for each convex curve in $[T_L, T_\lambda]$, we surely can secure a global optimal solution by picking the one with the lowest objective value.

In the following sections, we first derive a lower bound on the search range. Then, we demonstrate how to use the junction points to proceed with the search. Also, we propose an approach to secure and revise the upper bound on the search range. Finally, we summarize our proposed search algorithm.

4.1 A Lower Bound

In this section, we derive a lower bound on the search range by the *Common Cycle (CC)* approach in which it requires that $f_0 = 1$ and $f_n = 1$ for all n, *i.e.*, all the retailers share the same replenishment cycle.

Denote as T_{cc}^* the optimal replenishment cycle for the CC approach. Then, one may easily secure T_{cc}^* by the following expression.

$$T_{cc}^* = \sqrt{\frac{2\left(k_0 + \sum_{n>1} k_n\right)}{\sum_{n>1} d_n \left(h^n + h_n\right)}} \tag{8}$$

Proposition 3 asserts that the search scheme may skip the range $\left(\infty, T_{cc}^*\right)$. Consequently, we may set T_{cc}^* in eq. (8) as a lower bound of the search range.

Proposition 3 *For the* $TC_{opt}(T)$ *function, there exist no local minima for* $T < T_{cc}^*$.

Proof. Proposition 1 asserts that $TC_{opt}(T)$ function is *piece-wise convex*. It implies that the global optimal solution must be one of its local minima. The local minimum for any set of $\left\{ f_n : f_n \in \frac{1}{P}, P \in N^+ \right\}$ is expressed in eq. (10). By eq. (10), it is obvious that $\breve{T}(\{f_n\}) \geq T_{cc}^*$ since $f_n \leq 1$ for all *n*. Therefore, there exists no local minima for $T < T_{cc}^*$. ■

Proposition 3 also implies that we may employ T_{cc}^* as an initial point for the search algorithm to start the search from T_{cc}^* toward higher values of *T* (until it meets an upper bound T_λ).

4.2 Proceeding with the Search by Junction Points

By utilizing the theoretical properties of the junction points, we show how to proceed with the search from our initial point T_{cc}^* in this subchapter.

Before proceeding with the search, we first secure $F\left(T_{cc}^*\right)$, *i.e.*, the set of optimal multipliers at T_{cc}^* by Corollary 1.

Next, we show how to proceed with the search by utilizing a sequence of (sorted) junction points. By Lemma 2 and Proposition 2, each junction point $\{\delta_n(1/f_n)\}$ provides the information that one should change the optimal multiplier of retailer *n* from f_n to $\frac{f_n}{f_n+1}$ at $\delta_n(1/f_n)$ to secure the optimal value for the $TC_{opt}(T)$ function. Therefore, during the search, we need to keep an *n-dimensional* array $\{\delta_n(1/f_n) | n = 1,\ldots,N\}$ in which each value of $\delta_n(1/f_n)$ indicates the location of the *next* junction point of each retailer *n* where the optimal multiplier of retailer *n* should be changed. Since the algorithm searches toward higher values of *T*, one shall change the multiplier for the particular retailer *n* with the *smallest* value of $\delta_n(1/f_n)$ to correctly update the set of optimal multipliers. Let T_c be the current value of T where the search algorithm reaches. Denote as π the retailer index for the retailer n with the *smallest* value of $\delta_n(1/f_n)$, *i.e.*, $\pi = \arg\min_n \{\delta_n(1/f_n) > T_c\}$. To proceed with the search form T_c, by Theorem 1, we need to update the set of optimal multipliers at $\delta_n(1/f_n)$ by

$$F(\delta_\pi(1/f_\pi)) \overset{\Delta}{=} (F(T_c) \setminus \{f_\pi\}) \cup \{f_n/f_\pi + 1\} \tag{9}$$

where '\' denotes set subtraction.

Let $\{\omega_j\}$ be the sequence of the points that the algorithm reaches. Also, by the definition, we have $\omega_0 \overset{\Delta}{=} T_{cc}^*$. From another point of view, the algorithm searches along $\{\omega_j\}$, a (sorted) sequence of junction points from T_{cc}^*, where $\omega_{j+1} \geq \omega_j$, j=0, 1, 2,…. Note that this array is sorted on the location of the junction points in ascending order except that the initial point T_{cc}^* may not be a junction point. Importantly, Corollary 2 asserts that the set of optimal multipliers for the $TC_{opt}(T)$ function is invariant between ω_{j+1} and ω_j. Therefore, $F(\omega_j)$ is the set of optimal multipliers for the $TC_{opt}(T)$ function in the interval $(\omega_{j+1}, \omega_j]$. Denote as $\breve{T}(F(\omega_j))$ the minimum for the set of multipliers $\breve{T}(F(\omega_j))$. The following proposition indicates the existing condition and the location of a local minimum for the $TC_{opt}(T)$ function.

Proposition 4 *Let*

$$\breve{T}(F(\omega_j)) = \sqrt{\frac{2\left(k_0 + \sum_{n>1} \frac{k_n}{f_n^*(\omega_j)}\right)}{\sum_{n>1} d_n \left[h^n + h_n f_n^*(\omega_j)\right]}} \tag{10}$$

$\breve{T}(F(\omega_j))$ *is a local minimum for the* $TC_{opt}(T)$ *function if* $\breve{T}(F(\omega_j)) \in (\omega_{j+1}, \omega_j]$ *where* $f_n^*(\omega_j) \in F(\omega_j)$, $\forall n$.

Proof. For any given set of $\{f_n\}$, one may secure its local minimum, $\breve{T}(\{f_n\})$, by securing the derivative of the $TC_{opt}(T)$ function with respect to *T,* and equating it to zero. By Corollary 2, $\breve{T}(\{f_n\})$ becomes a local minimum for the $TC_{opt}(T)$ function when $\breve{T}(F(w_j)) \in (\omega_{j+1}, \omega_j]$. ■

4.3 Secure and Revise the Upper Bound

Recall that in order to secure a global optimal solution, the search scheme needs to secure all the local minima in $[T_{cc}^*, T_\lambda]$. Therefore, we derive an upper bound T_λ in this section. Also, in order to shorten the search range, we revise the best-on-hand upper bound during the search process.

Let T^* and F^* be the best-on-hand local minimum and its corresponding set of optimal multipliers, respectively. After obtaining a new local minimum $\breve{T}$, we first check if it secures an objective value lower than the best-on-hand solution. If $TC(F(\breve{T}), \breve{T}) < TC(F^*, T^*)$, we should revise the best-on-hand solution by $T^* = \breve{T}$, and $F^* = F(\breve{T})$. Meanwhile, one should try to revise the best-on-hand (*i.e.*, smallest-valued) upper bound, denoted by T_λ^*, to shorten the search range.

We derive another upper bound β in Lemma 3. We note that our upper bound β is derived by asserting that there exists no any solution which secures a lower objective value than $TC(F(\breve{T}), \breve{T})$ for $T < \beta$.

Lemma 3 *At a local minimum* $\breve{T}$, *one may secure an upper bound* β *on the search range by*

$$\beta = \frac{\left(\frac{k_0}{\breve{T}} + \sum_{N>1}\phi_n + \frac{\breve{T}}{2}\sum_{N>1} d_n h^n\right) + \sqrt{\left(\frac{k_0}{\breve{T}} + \sum_{n>1}\phi_n + \frac{\breve{T}}{2}\sum_{n>1} d_n h^n\right)^2 - 2k_0\left(\sum_{n>1} d_n h^n\right)}}{\sum_{N>1} d_n h^n} \quad (11)$$

where

$$\phi_n\left(f_n^*(\breve{T}),\breve{T}\right) = \begin{cases} \frac{k_n}{\breve{T}} + \frac{d_n h_n \breve{T}}{2} - \sqrt{2k_n d_n h_n}, & f_n^*(\breve{T}) = 1 \\ \sqrt{2k_n d_n h_n}\left[\frac{1}{2}\left(\frac{2+f_n}{\sqrt{1+f_n}}\right) - 1\right], & f_n^*(\breve{T}) < 1 \end{cases} \quad (12).$$

Proof. The proof is presented in Appendix A.6. ■

We summarize the revision of T_λ^* at the newly-obtained, best-on-hand local minimum in the following proposition.

Proposition 5 *After securing a new local minimum* $\breve{T}$, *if* $TC\left(F(\breve{T}),\breve{T}\right) < TC\left(F^*,T^*\right)$, *then one should revise* T_λ^* *by* $T_\lambda^* = \min\{T_\lambda^*, \beta\}$ *where* β *is obtained from eq. (11).*

4.4 The Algorithm

We are now ready to enunciate the proposed search algorithm. Recall that the algorithm searches form T_{cc}^* toward higher values of *T* until it meets the last-revised upper bound T_λ^*. In the search process, we use a sequence of (sorted) junction points as the backbone and secure all the local minima of the $TC_{opt}(T)$ function between $[T_{cc}^*, T_\lambda^*]$. Denote as T^* and F^* the value of T and the corresponding set of optimal multipliers for the global optimal solution. We summarize the step-by-step procedure of the proposed search algorithm as follows.

Step 1: The Initialization.

(a) Utilize T_{cc}^* in eq. (8) as an initial point and let lower bound $T_\lambda^* = \infty$.

(b) Calculate and sort all the junction points in eq. (6) and $w_1 = \delta_\pi(1/f_\pi)$ by $\pi = \arg\min_n \{\delta_n(1/f_n) > T_c\}$ where $T_c = T_{cc}^*$, respectively.

(c) Use Corollary 1 to obtain $F(T_{cc}^*)$.

(d) Check by Proposition 4: if $\breve{T}(F(T_{cc}^*)) \in [T_{cc}^*, w_1)$, set $T^* = \breve{T}(F(T_{cc}^*))$ and $F^* = F(T_{cc}^*)$, calculate $TC(F^*, T^*)$, and calculate the best-on-hand upper bound T_λ^* by Proposition 5; otherwise, let $T_\lambda^* = \infty$ and $TC(F^*, T^*) = \infty$.

(e) Set $j = 1$ and $T_c = w_j$.

Step 2: The Search Procedure.

(a) Obtain $F(w_j)$ by $F(w_j) \equiv (F(T_c) \setminus \{f_\pi\}) \cup \{f_n / f_n + 1\}$ and $w_{j+1} = \delta_\pi(1/f_\pi)$ by $\pi = \arg\min_n \{\delta_n(1/f_n) > T_c\}$.

(b) Check by Proposition 4; if $\breve{T}(F(w_j)) \in [w_j, w_{j+1})$, calculate $TC(F(w_j), \breve{T}(F(w_j)))$ and try to revise the best-on-hand upper bound T_λ^* by Proposition 5.

(c) If $TC(F(w_j), \breve{T}(F(w_j))) < TC(F^*, T^*)$, set $T^* = \breve{T}(F(w_j))$ and

$$F^* = F\left(w_j\right).$$

Step 3: The Termination Condition of the Search Algorithm.

(a) If $w_{j+1} > T_\lambda^*$, output T^*, F^*, $TC\left(F^*, T^*\right)$ and the algorithm will terminate.

(b) Otherwise, set $j = j+1$ and $T_c = w_j$. Go to Step 2.

Chapter 5
NUMERICAL EXPERIMENTS

In this chapter, we first present a numerical example to demonstrate the implementation of the proposed search algorithm. In the second part of this chapter, we will compare the solutions solved the proposed search algorithm with Schwarz's (1973) lower bound and the solutions by Schwarz's (1973) heuristic.

5.1 A Demonstrative Example

The numerical example presented in this section demonstrates the implementation of the proposed search algorithm. Table 1 presents the set of parameters used in this numerical example.

The search process is summarized as follows.

1. We first secure $T_{cc}^{*} = 0.0541$, and let $T_c = T_{cc}^{*}$. Also, we secure the set of optimal multipliers at T_{cc}^{*}, *i.e.*, $F(T_{cc}^{*})$, by $\{1/3, 1/2, 1/7, 1/2, 1, 1/2, 1, 1, 1, 1/2\}$.

Table 1: The Set of Parameters Used in the Demonstrative Example.

Retailer	1	2	3	4	5	6	7	8	9	10
Setup cost	15	50	2	32	38	48	46	42	12	20
Holding cost	1.3	1.57	1.65	1.62	0.52	1.85	0.3	0.21	0.56	0.6
Demand rate	95200	49550	48500	45500	93550	42500	44500	35500	18850	60000
Warehouse										
Setup cost	500									
Holding cost	0.13	0.157	0.165	0.162	0.052	0.185	0.03	0.021	0.056	0.06

2. We obtain $\omega_1 = \delta_3(1/f_3 = 7) = 0.0558$ by $\pi = \arg\min_n \{\delta_n(1/f_n) > T_c\} = 3$. By Proposition 4, we have local minimum $\breve{T}(F(T_{cc}^*))$ by 0.0823; but, obviously, $\breve{T}(F(T_{cc}^*)) \notin [T_{cc}^*, \omega_1)$, we set $T_\lambda^* = \infty$ and $TC(F^*, T^*) = \infty$.
3. Next, we move to ω_1 and obtain the set of optimal multipliers $F(\omega_1)$ by $F(\omega_1) \equiv F(T_{cc}^*) \setminus \{f_3 = 1/7\} \cup \{f_3 = 1/8\}$, which is given by $\{1/3, 1/2, 1/8, 1/2, 1, 1/2, 1, 1, 1, 1/2\}$. Then, we let $T_c = \omega_1$ and secure $\omega_2 = \delta_1(1/f_1 = 3) = 0.057$ by $\pi = \arg\min_n \{\delta_n(1/f_n) > T_c\} = 1$. We obtain the local minimum $\breve{T}(F(\omega_1))$ by 0.0826. Therefore, $\breve{T}(F(\omega_1)) \notin [\omega_1, \omega_2)$. Next, we move to ω_2 and get the set of optimal multipliers by $F(\omega_2) \equiv F(\omega_1) \setminus \{f_1 = 1/3\} \cup \{f_1 = 1/4\}$, which is given by $\{1/4, 1/2, 1/8, 1/2, 1, 1/2, 1, 1, 1, 1/2\}$.
4. We continue the search, but find no local minimum in $[T_{cc}^*, \omega_{21})$. The first local minimum is secured in the interval $[\omega_{21}, \omega_{22}) = [0.1155, 0.1217)$. We have $F^* = F(\omega_{21}) = \{1/7, 1/3, 1/16, 1/4, 1/3, 1/3, 1, 1, 1/2, 1/3\}$, $T^* = \breve{T}(F(\omega_{21})) = 0.1197$ and $TC(F^*, T^*) = 22475.31$. Also, we secure the first upper bound $T_\lambda^* = 0.1718$ by proposition 5.
5. For the entire search process, we secure totally 6 local minima for this example. All of the local minima, their corresponding set of optimal multipliers, objective values and their revised upper bound are summarized in Table 2.
6. When the search algorithm secured the second local minimum $\breve{T} = 0.123$, it tried to revise the best-on-hand upper bound $T_\lambda^* = \min\{T_\lambda^*, \beta\} = \min\{0.1718, 0.1703\} = 0.1703$ by Lemma 3 and Proposition 5. Until we obtain the sixth local minimum $\breve{T} = 0.1453$,

and revise the best-on-hand upper bound $T_\lambda^* = 0.1612$. Therefore, the search algorithm stops when it encounters the smallest junction point that is larger than T_λ^*, that is $\omega_{40} = 0.1647$. In this example, before the search algorithm terminates, it visits totally 40 junction points.

7. The global optimal solution and the set of optimal multipliers are given by $T^* = 0.1417$ (*i.e.,* the forth local minimum) and $F^* = \{1/9, 1/4, 1/19, 1/5, 1/3, 1/4, 1/2, 1, 1/3, 1/4\}$, respectively. The optimal average total cost is $22,422.18.

Table 2: The Local Minima Secured in the Search Process of the Global Optimum Search Algorithm.

$\lfloor\omega_j, \omega_{j+1})$	$[0.1155, 0.1217)$	$[0.1229, 0.1231)$	$[0.1309, 0.1378)$	$[0.1393, 0.1443)$	$[0.1443, 0.1453)$	$[0.1455, 0.1527)$
f_1	1/7	1/8	1/8	1/9	1/9	1/9
f_2	1/3	1/3	1/4	1/4	1/4	1/4
f_3	1/16	1/17	1/18	1/19	1/19	1/20
f_4	1/4	1/4	1/4	1/5	1/5	1/5
f_5	1/3	1/3	1/3	1/3	1/4	1/4
f_6	1/3	1/3	1/4	1/4	1/4	1/4
f_7	1	1	1/2	1/2	1/2	1/2
f_8	1	1	1	1	1	1
f_9	1/2	1/2	1/3	1/3	1/3	1/3
f_{10}	1/3	1/4	1/4	1/4	1/4	1/4
$\breve{T}_j$	0.1197	0.123	0.1373	0.1417	0.1451	0.1453
$TC_{opt}(\breve{T}_j)$	22475.31	22479.84	22424.99	22422.18*	22425.55	22425.56
β	0.1718	0.1703	0.1612	0.1622	0.1631	0.1632
T_λ^*	0.1718	0.1703	0.1612	0.1612	0.1612	0.1612

5.2 Random Examples

In this section, we present our random experiments to show the proposed search algorithm out-performs Schwarz's heuristic.

The experimental settings are the same as that shown in van Eij's (1993) paper excepted holing cost rate for the warehouse. The annual demand rate d_n, the holding cost rate for the retailers $h_n^{'}$, and the setup cost for the retailers k_n were randomly generated from uniform distributions UNIF[100-100,000], UNIF[0.2-2], and UNIF[1-51], respectively. And, the holding cost rate h^n for the warehouse were selected at random from ($h^n / h_n^{'} = 0.2, 0.4, 0.6, 0.8$) using a uniform probability distribution. We tested four levels of the number of retailers (n=5, 10, 20, 100) and eight levels of setup cost for the warehouse (k_0=1, 5, 10, 50, 100, 300, 500, 1000). For each combination of n and k_0, we generated 1,000 examples and solved each of them by the proposed search algorithm and Schwarz's heuristic solutions. We summarize the comparison of two solution approaches in Table 3.

In Schwarz's (1973) paper, he also derived three analytical lower bounds. Let P^* be the objective value obtained by either of the solution approaches. Also, denote B^* as the best of three lower bounds, *i.e.*, B^* = max [Bound1, Bound2, Bound3]. Let α = $100[(P^* - B^*)/B^*]$ which is an error measure of P^*. And, $\alpha_{\max}$ is the maximum error, $\alpha_{\min}$ is the minimum error and $\overline{\alpha}$ is the average error.

In Table 3, we can obtain some information. First, when n is increasing (*i.e.*, n = 100), it is shown that the proposed search algorithm is more efficient in its run time than Schwarz' heuristic. Second, the error decreases as k_0 increases. Third, the values of α (*i.e.*, $\alpha_{\max}$, $\alpha_{\min}$ or $\overline{\alpha}$)

of proposed search algorithm are less than (or equal to) the solution by Schwarz' heuristic.

Based on our random experiments, we conclude that the proposed search algorithm out-performs Schwarz's heuristic.

Table 3: The Comparison between Wang et al.'s and Schwarz's heuristic.

	Average run time		**Schwarz' heuristic**			**Wang et al.'s search algorithm**		
n=5	**Schwarz**	**Wang et al.**	α_{max}	α_{min}	$\bar{\alpha}$	α_{max}	α_{min}	$\bar{\alpha}$
$k_0=1$	0.023547	0.008016	83.80%	0.16%	11.75%	80.45%	0.16%	11.21%
$k_0=5$	0.022703	0.010344	68.18%	0.11%	9.82%	65.19%	0.11%	9.37%
$k_0=10$	0.02275	0.010312	57.79%	0.12%	8.40%	56.86%	0.12%	8.07%
$k_0=50$	0.01875	0.010453	31.33%	0.12%	4.97%	31.32%	0.09%	4.86%
$k_0=100$	0.015813	0.010546	20.82%	0.03%	3.75%	20.82%	0.02%	3.65%
$k_0=300$	0.01	0.01086	8.84%	0.02%	1.49%	8.79%	0.02%	1.44%
$k_0=500$	0.007281	0.011109	5.08%	0.00%	0.75%	5.07%	0.00%	0.73%
$k_0=1000$	0.00425	0.011485	2.61%	0.00%	0.29%	2.58%	0.00%	0.28%
	Average run time		**Schwarz' heuristic**			**Wang et al.'s search algorithm**		
n=10	**Schwarz**	**Wang et al.**	α_{max}	α_{min}	$\bar{\alpha}$	α_{max}	α_{min}	$\bar{\alpha}$
$k_0=1$	0.045891	0.016547	43.88%	1.20%	12.55%	43.88%	1.20%	12.05%
$k_0=5$	0.043859	0.016594	41.63%	1.08%	11.31%	41.63%	1.08%	10.85%
$k_0=10$	0.044828	0.016594	39.47%	0.98%	10.36%	39.47%	0.98%	9.98%
$k_0=50$	0.040406	0.01675	31.40%	0.92%	7.92%	31.37%	0.92%	7.81%
$k_0=100$	0.036484	0.017015	28.04%	0.56%	6.85%	28.01%	0.55%	6.77%
$k_0=300$	0.027438	0.017407	18.93%	0.24%	3.66%	18.93%	0.24%	3.61%
$k_0=500$	0.021984	0.017906	12.06%	0.09%	2.02%	12.06%	0.09%	1.99%
$k_0=1000$	0.014766	0.018344	5.47%	0.05%	0.79%	5.47%	0.05%	0.78%
	Average run time		**Schwarz' heuristic**			**Wang et al.'s search algorithm**		
n=20	**Schwarz**	**Wang et al.**	α_{max}	α_{min}	$\bar{\alpha}$	α_{max}	α_{min}	$\bar{\alpha}$
$k_0=1$	0.091157	0.030047	32.93%	2.64%	13.10%	32.88%	2.52%	12.65%
$k_0=5$	0.090875	0.03036	30.98%	2.49%	12.33%	30.93%	2.36%	11.89%
$k_0=10$	0.090375	0.031704	29.34%	2.38%	11.75%	29.30%	2.23%	11.35%
$k_0=50$	0.085844	0.030282	24.71%	2.33%	10.31%	24.67%	2.12%	10.10%
$k_0=100$	0.080234	0.030781	23.92%	1.85%	9.50%	23.89%	1.83%	9.42%
$k_0=300$	0.064531	0.032688	20.32%	0.68%	6.66%	20.31%	0.68%	6.61%
$k_0=500$	0.057562	0.032031	15.29%	0.32%	4.62%	15.29%	0.31%	4.60%
$k_0=1000$	0.043859	0.033296	7.57%	0.13%	2.09%	7.57%	0.13%	2.07%
	Average run time		**Schwarz' heuristic**			**Wang et al.'s search algorithm**		
n=100	**Schwarz**	**Wang et al.**	α_{max}	α_{min}	$\bar{\alpha}$	α_{max}	α_{min}	$\bar{\alpha}$
$k_0=1$	0.455125	0.179	24.98%	7.88%	13.55%	24.97%	7.58%	13.15%
$k_0=5$	0.452391	0.17836	24.63%	7.73%	13.34%	24.62%	7.49%	12.94%
$k_0=10$	0.452	0.178656	24.33%	7.66%	13.19%	24.32%	7.43%	12.79%
$k_0=50$	0.448406	0.17936	23.37%	7.62%	12.81%	23.37%	7.42%	12.45%
$k_0=100$	0.443453	0.180515	23.12%	7.52%	12.63%	23.12%	7.08%	12.33%
$k_0=300$	0.421453	0.181875	22.64%	6.15%	11.52%	22.63%	5.99%	11.37%
$k_0=500$	0.400297	0.184328	21.50%	4.40%	10.36%	21.49%	4.31%	10.29%
$k_0=1000$	0.361297	0.190031	19.53%	2.25%	8.31%	19.53%	2.23%	8.29%

Chapter 6
CONCLUDING REMARKS

This study fills two research gaps in the literature of the One-warehouse Multi-retailers (*OWMR*) problem under *stationary-nested* policy. First, our study presents several important theoretical results on the optimality structure of the *OWMR* problem under *stationary-nested* policy. For instance, Proposition 1 asserts that $TC_{opt}(T)$ function is a *piece-wise convex* function of *T*. Also, we have thorough discussion on the properties of the junction points on the $TC_{opt}(T)$ function in Chapter 3.

Second, by utilizing these theoretical results, we propose an efficient search algorithm that always secures the global optimal solution for the *OWMR* problem under *stationary-nested* policy in Chapter 4. In our search algorithm, we employ tight bounds that significantly shorten the search range and our search algorithm effectively gets global optimum in a very short run time. Based on our random experiments in Chapter 5, we demonstrate that our search algorithm out-perform Schwarz's heuristic. The proposed search algorithm is the first solution approach in literature that always secures the global optimal solution for the *OWMR* under *stationary-nested* policy. Our theoretical results in this study shall establish an important foundation for other lot sizing and scheduling problems.

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APPENDIX

A.1 Proof for Proposition 2

Proof. Recall that $TC_{opt}(T) \equiv k_0/T + \sum_{n>1}\left(h^n T + TC_n(T)\right)$ is separable.

Assume that ω is a junction point for retailer *n*, but not for the other (*n*-1) retailers. Then, there must exist $\varepsilon > 0$ such that

1. the curve for $k_0/T + \sum_{n>1} h^n T + \sum_{j \neq n} \underline{TC}_j(T)$ is convex in the interval of $[\omega - \varepsilon, \omega + \varepsilon]$ since each one of $\underline{TC}_j(T)$ where $j \neq n$ is convex in $[\omega - \varepsilon, \omega + \varepsilon]$, and
2. $\underline{TC}_n(T)$ is convex in the intervals of $[\omega, \omega + \varepsilon]$.

Except at the junction point ω, $TC_{opt}(T) = \frac{k_0}{T} + \sum_{n>1} h^n T + \sum_{j \neq n} \underline{TC}_j(T) + \sum_n \underline{TC}_n(T)$ is still convex in the intervals $[\omega - \varepsilon, \omega]$ and $[\omega, \omega + \varepsilon]$. Therefore, ω becomes a junction point of $TC_{opt}(T)$. ■

A.2 Proof for Lemma 3

Proof. We note that the function $\phi_n\left(f_n^*(\breve{T}), \breve{T}\right)$ indicates the maximum magnitude of decrement in $TC_n(f_n, T)$ from $\breve{T}$ to any value of $T > \breve{T}$ for retailer *n*. Recall that Lemma 1 asserts that the function $TC_n(f_n, T)$ is bounded from below by $\sqrt{2k_n d_n h_n}$. If the optimal multiplier for retailer *n* is $f_n^*(\breve{T}) = 1$, then the maximum magnitude of decrement in $TC_n(f_n, T)$ from $\breve{T}$ to any value of $T > \breve{T}$ is bounded by $\frac{k_n}{\breve{T}} + \frac{d_n h_n \breve{T}}{2} - \sqrt{2k_n d_n h_n}$. If $f_n^*(\breve{T}) < 1$, then

$$TC_n\left(f_n^*\left(\breve{T}\right),\breve{T}\right) \le \max\left\{TC_n\left(f_n^*\left(\breve{T}\right),\delta_n\left(\frac{1+f_n^*\left(\breve{T}\right)}{f_n^*\left(\breve{T}\right)}\right)\right),TC_n\left(f_n^*\left(\breve{T}\right),\delta_n\left(1/f_n^*\left(\breve{T}\right)\right)\right)\right\}$$

by the piece-wise convexity of the $TC_n\left(f_n^*\left(\breve{T}\right),\breve{T}\right)$ function. In fact, $TC_n\left(f_n^*\left(\breve{T}\right),\breve{T}\right) \le TC_n\left(f_n^*\left(\breve{T}\right),\delta_n\left(1/f_n^*\left(\breve{T}\right)\right)\right)$, since one can easily prove that $TC_n\left(f_n^*\left(\breve{T}\right),\delta_n\left(1/f_n^*\left(\breve{T}\right)\right)\right) > TC_n\left(f_n^*\left(\breve{T}\right),\delta_n\left(\frac{1+f_n^*\left(\breve{T}\right)}{f_n^*\left(\breve{T}\right)}\right)\right)$. By plugging eq. (6), we have the following concise expression for $TC_n\left(f_n^*\left(\breve{T}\right),\delta_n\left(f_n^*\left(\breve{T}\right)\right)\right)$ after some simplification.

$$TC_n\left(f_n^*\left(\breve{T}\right),\delta_n\left(1/f_n^*\left(\breve{T}\right)\right)\right) = \sqrt{2k_n d_n h_n}\left[\frac{1}{2}\left(\frac{2+f_n}{\sqrt{1+f_n}}\right)\right] \qquad (13).$$

In other words, if $f_n^*\left(\breve{T}\right) < 1$, the maximum magnitude of decrement in $TC_n\left(f_n, T\right)$ from $\breve{T}$ to any value of $T > \breve{T}$ is bounded by $\sqrt{2k_n d_n h_n}\left[\frac{1}{2}\left(\frac{2+f_n}{\sqrt{1+f_n}}\right)-1\right]$. Also, a set-up cost k_0 at warehouse would decrease from $\frac{k_0}{\breve{T}}$ to $\frac{k_0}{T}$ from $\breve{T}$ to any value of $T > \breve{T}$.

On the other hand, the sum of holding cost for product *n* at the warehouse would increase from $\frac{\breve{T}\sum_{n>1} d_n h^n}{2}$ to $\frac{T\sum_{n>1} d_n h^n}{2}$ from $\breve{T}$ to any value of $T > \breve{T}$.

The upper bound is derived by asserting that for $T \ge \beta$, the increment in the sum of holding cost for product *n* at the warehouse, *i.e.*, $\frac{T\sum_{n>1} d_n h^n}{2} - \frac{\breve{T}\sum_{n>1} d_n h^n}{2}$, must exceed the maximum magnitude of decrement, *i.e.*, $\frac{k_0}{\breve{T}} - \frac{k_0}{T} + \sum_{n>1}\phi_n\left(f_n^*\left(\breve{T}\right),\breve{T}\right)$; or,

$$\frac{\sum_{n>1} d_n h_n}{2}\left(T-\breve{T}\right) \geq \frac{k_0}{\breve{T}} - \frac{k_0}{T} + \sum_{n>1} \phi_n\left(f_n^*\left(\breve{T}\right), \breve{T}\right),$$ which gives exactly eq. (11).

■

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