1 Introduction

Predator-prey models have been studied for a long time . One of the most important problems in a Predator-prey system is the global stability of the unique positive equilibrium point . The global stability analysis for Predator-prey system without time delay has been done by many researchers . Most of them use the following methods to prove global stability of a Predator-prey system without delay . The first method is to construct a Lyapunov function [3 , 4 , 5] . The second method is to employ the Dulac Criterion to eliminate the existence of periodic orbits and then use the Poincar'e-Bendixson Theorem to analyse the global stability of the unique positive equilibrium [3 , 4 , 5 , 6] . The third method is the limit cycle stability analysis [3 , 6 , 7 , 8] . The fourth method is the comparison method [3 , 7 , 8] .

But more realistic models should include some of the past states of the population system ; that is , a real system should be modeled with time delays . In [9 , 10 , 11] , authors were to analyze the global stability of the system with time delay by constructing a Lyapunov functional .

In this thesis , we were concerned about the Leslie-Gower Predator-prey system . For this system without delay as in [12] , authors to analyze the global stability by constructing a Lyapunov function . And in [14] , authors discussed the global stability of this system with a single delay by constructing a Lyapunov functional . Now , we are to establish global stability of the Leslie-Gower Predator-prey system with a single delay with the different functional response of the predator , $p(x)$, by constructing a Lyapunov functional . In section 2 , we introduce some useful definitions and theorems . In section 3 , we analyse the global stability of the Leslie-Gower Predator-prey system with a single delay with $p(x) = cx$ in the Holling-type I , $p(x) = \frac{cx}{1+x}$ in the Holling-type II , and $p(x) = \frac{cx^2}{1+x^2}$ in the Holling-type III by constructing Lyapunov functionals . In section 4 , we illustrate our results by some examples .

2 Preliminaries

2.1 Nonlinear autonomous system

Consider the following general nonlinear autonomous system of differential equation

$$\dot{x}(t) = f(x) \quad , \quad x \in E \tag{2.1}$$

where $f \in C^1(E)$ and E is an open subset of R^n. In this thesis, we need the following definitions and theorems.

Definition 2.1 [1]

(i) A point $x_0 \in E$ is called an *equilibrium point* or *critical point* of the system (2.1) if $f(x_0) = 0$.

(ii) An equilibrium point x_0 of the system (2.1) is called a *hyperbolic equilibrium point* of the system (2.1) if none of the eigenvalues of the matrix $Df(x_0)$ have zero real part.

(iii) An equilibrium point x_0 is called a *saddle point* of the system (2.1) if it is a hyperbolic equilibrium point and $Df(x_0)$ has at least one eigenvalue with a positive real part and one with negative real part.

Definition 2.2 [1] Let E be an open subset of R^n and let $f \in C^1(E)$. For $x_0 \in E$, let $\phi(t, x_0)$ be the solution of the system (2.1) with the initial condition

$x(0) = x_0$ defined on its maximal interval of existence $I(x_0)$. Then for $t \in I(x_0)$, the set of mappings ϕ_t defined by

$$\phi_t(x_0) = \phi(t, x_0)$$

is called the *flow* of the system (2.1).

Definition 2.3 [1] Let ϕ_t denote the flow of the system (2.1) defined for all $t \in R$. An equilibrium point x_0 of the system (2.1) is *stable* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ and $t \geq 0$ we have

$$\phi_t(x) \in N_\epsilon(x_0)$$

The equilibrium point x_0 is *unstable* if it is not stable. And x_0 is *asymptotically stable* if it is stable and if there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ we have

$$\lim_{t \to \infty} \phi_t(x) = x_0$$

In order to analyse the behavior of the system (2.1) near its equilibrium points, we show that the local behavior of the nonlinear system (2.1) near a hyperbolic equilibrium point x_0 is qualitatively determined by the behavior of the linear system

$$\dot{x} = Ax \tag{2.2}$$

where the Jacobian matrix $A = Df(x_0)$. The linear function $Ax = Df(x_0)x$ is called the linear part of f at x_0.

Theorem 2.1 [1] Let E be an open subset of R^n containing x_0. Suppose that $f \in C^1(E)$ and that $f(x_0) = 0$. Suppose further that there exists a function $V \in C^1(E)$ satisfying $V(x_0) = 0$ and $V(x) > 0$ if $x \neq x_0$. Then

(a) if $\dot{V}(x) \leq 0$ for all $x \in E$, x_0 is stable.

(b) if $\dot{V}(x) < 0$ for all $x \in E - \{x_0\}$, x_0 is asymptotically stable.

(c) if $\dot{V}(x) > 0$ for all $x \in E - \{x_0\}$, x_0 is unstable.

Theorem 2.2 [1] **(The Hartman-Grobman Theorem)** Let E be an open subset of R^n containing the point x_0, let $f \in C^1(E)$, and let ϕ_t be the flow of the system (2.2). Suppose that $f(x_0) = 0$ and that the matrix $A = Df(x_0)$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for $x \in U$, there is an open interval $I(x) \subset R$ containing origin such that for all $x \in U$ and $t \in I(x)$

$$H \circ \phi_t(x) = e^{At} H(x)$$

Theorem 2.3 [1] Suppose x_0 is an equilibrium point of the system (2.1) and $A = Df(x_0)$. Let $\sigma = det(A)$ and $\gamma = trace(A)$.

(a) If $\sigma < 0$, then the system (2.1) has a saddle point at x_0.

(b) If $\sigma > 0$ and $\gamma = 0$, then the system (2.1) has a center at x_0.

(c) If $\sigma > 0$ and $\gamma^2 - 4\sigma \geq 0$, then the system (2.1) has a node at x_0; it is stable if $\gamma < 0$ and unstable if $\gamma > 0$.

(d) If $\sigma > 0$, $\gamma^2 - 4\sigma < 0$ and $\gamma \neq 0$, then the system (2.1) has a focus at x_0; it is stable if $\gamma < 0$ and unstable if $\gamma > 0$.

In order to analyze the global stability of the system (2.1), it is necessary to determine whether the closed orbit exist or not. Dulac's Criteria has established conditions under which the system (2.1) with $x \in R^2$ has no closed orbit.

Theorem 2.4 [1] **(Dulac's Criteria)** Let $f \in C^1(E)$ where E is a simply connected region in R^2. If there exists a function $H \in C^1(E)$ such that $\bigtriangledown \cdot (Hf)$ is not identically zero and does not change sign in E, then the system (2.1) has no closed orbit lying entirely in E. If A is an annular region contained in E on which $\bigtriangledown \cdot (Hf)$ does not change sign, then there is at most one limit cycle of the system (2.1) in A.

Theorem 2.5 [1] Let $f \in C^1(E)$ where E is a simply connected region in R^2. Suppose $x = x(t),\ 0 \leq t \leq T$, is a nonconstant periodic solution of preiod T of the system (2.1). If

$$\int_0^T \bigtriangledown \cdot f \ dt < 0$$

where $\bigtriangledown \cdot f$ is the divergence of the vector field f, then the $T-$periodic solution $x(t)$ is a stable limit cycle.

Definition 2.4 [1] A *periodic* or *closed orbit* of the system (2.1) is any closed solution curve of the system (2.1) which is not an equilibrium point of the system (2.1). A periodic orbit Γ is called *stable* if for each $\epsilon > 0$,there is a neighborhood U of Γ such that for all $x \in U$ and $t \geq 0$, $d(\phi(t,x),\Gamma) < \epsilon$. A periodic orbit Γ is called *unstable* if it is not stable; and Γ is called *asymptotically stable* it is stable and if for all points x in some neighborhood U of Γ

$$\lim_{t\to\infty} d(\phi(t,x),\Gamma) = 0$$

Definition 2.5 [1] A point $p \in E$ where E is an open subset of R^n is an $\omega-$*limit point* of the trajectory $\phi(\cdot,x)$ of the system (2.1) if there is a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \phi(t_n,x) = p$$

Similarly, if there is a sequence $t_n \to -\infty$ such that

$$\lim_{n\to\infty} \phi(t_n, x) = q$$

and the point $q \in E$, then the point q is called an *α-limit point* of the trajectory $\phi(\cdot, x)$ of the system (2.1). The set of all ω-limit points of a trajectory Γ is called the *ω-limit set* of Γ and it is denoted by $\omega(\Gamma)$. The set of all α-limit points of a trajectory Γ is called the *$\alpha-$limit set* of Γ and it is denoted by $\alpha(\Gamma)$. The set of all limit points of Γ, $\alpha(\Gamma) \cup \omega(\Gamma)$ is called the *limit set* of Γ.

Theorem 2.6 [1] The α and ω-limit sets of a trajectory Γ of the system (2.1), $\alpha(\Gamma)$ and $\omega(\Gamma)$, are closed subsets of E and if Γ is contained in a compact subset of R^n, then $\alpha(\Gamma)$ and $\omega(\Gamma)$, are nonempty, connected, compact subsets of E.

Definition 2.6 [1] A *limit cycle* Γ of a planar system is a cycle of the system (2.1) which is the α or $\omega-$limit set of some trajectory of the system (2.1) other than Γ. If Γ is the $\omega-$limit set of every trajectory in some neighborhood of Γ, then Γ is called an *ω-limit cycle* or *stable limit cycle*; if a cycle Γ is the α-limit set of every trajectory in some neighborhood of Γ, then Γ is called an *$\alpha -$ limit cycle* or an *unstable limit cycle*; and if Γ is the ω-limit set of the trajectory other than Γ and the α-limit set of another trajectory other than Γ, then Γ is called a *semi-stable limit cycle.*

Theorem 2.7 [1] **(The Poincaré−Bendixson Theorem)** Suppose that $f \in C^1(E)$ where E is an open subset of R^n and that the system (2.1) has a trajectory Γ contained in a compact subset F of E. Assume that the system (2.1) has only one unique equilibrium point x_0 in F, then one of the following possibilities holds.

(a) $\omega(\Gamma)$ is the equilibrium point x_0.

(b) $\omega(\Gamma)$ is a periodic orbit.

(c) $\omega(\Gamma)$ is a graphic.

2.2 Nonlinear autonomous system with delays

For ordinary differential equations, we view solutions of initial value problems as maps in Euclidean space. In order to establish a similar view for solutions of delay differential equations, we need some definitions.

We denote $\mathcal{C} \equiv C([-\tau, 0], R^n)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n with the topology of uniform convergence; That is, for $\phi \in \mathcal{C}$, the norm of ϕ is defined as $\|\phi\| = \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|$, where $|\cdot|$ is a norm in R^n. We define $x_t \in \mathcal{C}$ as $x_t(\theta) = x(t+\theta),\ \theta \in [-\tau, 0]$. Assume that Ω is a subset of $\mathcal{C}$ and $f : \Omega \to R^n$ is a given function, then we consider the following general nonlinear autonomous system of delay differential equation

$$\dot{x}(t) = f(x_t) \tag{2.3}$$

Definition 2.7 [13] Let $R^2_+ = \{x \in R^2 | x_i \geq 0\ ,\ i = 1, 2\}$. The notation $x > 0$ denotes $x \in \mathrm{Int}R^2_+$. The system (2.3) is said to be *unifromly persistent* if there exists a compact region $D \subseteq \mathrm{Int}R^2_+$ such that every solution $x(t)$ of the system (2.3) with the initial conditions eventually enters and remains in the region D.

Definition 2.8 [2] We say that $\phi \in B(0, \delta)$ if $\phi \in \mathcal{C}$ and $\|\phi\| \leq \delta$, where $\|\phi\| = \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|$.

(i) The solution $x = 0$ of the system (2.3) is said to be *stable* if, for any $\sigma \in R$, $\epsilon > 0$, there is a $\delta = \delta(\epsilon, \sigma)$ such that $\phi \in B(0, \delta)$ implies $x_t(\sigma, \phi) \in B(0, \epsilon)$ for $t \geq \sigma$. Otherwise, we say that $x = 0$ is *unstable.*

(ii) The solution $x = 0$ of the system (2.3) is said to be *asymptotically stable* if it is stable and there is a $b_0 = b(\sigma) > 0$ such that $\phi \in B(0, b_0)$ implies $x(\sigma, \phi)(t) \to 0$ as $t \to \infty$.

(iii) The solution $x = 0$ of the system (2.3) is said to be *uniformly stable* if the number δ in the definition of stable is independent of σ.

(iv) The solution $x = 0$ of the system (2.3) is said to be *uniformly asymptotically stable* if it is uniformly stable and there is a $b_0 > 0$ such that, for every $\eta > 0$, there is a $t_0(\eta)$ such that $\phi \in B(0, b_0)$ implies $x_t(\sigma, \phi) \in B(0, \eta)$ for $t \geq \sigma + t_0(\eta)$, for every $\sigma \in R$.

Theorem 2.8 [2] Let $u(\cdot)$ and $w(\cdot)$ be nonnegative continuous scalar functions such that, $u(0) = w(0) = 0$, $\lim_{s \to +\infty} u(s) = +\infty$, and that $V : \mathcal{C} \to R$ is continuous and satisfies

$$V(\phi) \geq u(|\phi(0)|)$$

and

$$\dot{V}(\phi) \leq -w(|\phi(0)|).$$

Then the solution $x = 0$ of $\dot{x}(t) = f(x_t)$ is uniformly stable, and every solutions is bounded. If in addition, $w(s) > 0$ for $s > 0$, then $x = 0$ is globally asymptotically stable.

3 The Model with Time Delay

Consider the Leslie-Gower predator-prey system with time delay τ modelled by

$$\begin{aligned} \dot{x}(t) &= x(t)\left\{r[1-\frac{x(t-\tau)}{K}]\right\} - p(x(t))y(t) \\ \dot{y}(t) &= y(t)\left[\delta - \beta\frac{y(t)}{x(t)}\right] \end{aligned} \tag{3.1}$$

with the initial conditions

$$\begin{aligned} & x(\theta) = \phi(\theta) \geq 0 \ , \ \theta \in [-\tau, \infty) \ , \ \phi \in C^1([-\tau,\infty),\mathbb{R}) \\ & x(0) > 0 \ \ , \ \ y(0) > 0 \end{aligned} \tag{3.2}$$

where δ , β , r , and τ are positive constants , K is defined as the prey environmental carrying capacity , x and y denote the densities of prey and predator population , respectively . Because all we want to discuss is biological population , we only consider the first quadrant in the $x-y$ plane . The following assumption is consistent with the system (3.1).

(A1) $p \in C^1([0,\infty),[0,\infty))$; $p(0) = 0$ and $p'(x) \geq 0$ for all $x > 0$.

The functional response of the predator , p(x) , has been discussed in the literature . Now , we just concern about the following $p(x)$ of this thesis , $p(x) = cx$ in the Holling-type I model , $p(x) = \frac{cx}{1+x}$ in the Holling-type II model , and $p(x) = \frac{cx^2}{1+x^2}$ in the Holling-type III model ; where c is encounter rate .

Lemma 3.1 *Every solution of the system (3.1) with the initial conditions (3.2) exists in the interval* $[0,\infty)$ *and remains positive for all* $t \geq 0$.

Proof : It is true because

$$x(t) = x(0)exp\left\{\int_0^t [r - \frac{rx(s-\tau)}{K} - \frac{p(x(s))}{x(s)}y(s)]ds\right\}$$

$$y(t) = y(0)exp\left\{\int_0^t [\delta - \beta\frac{y(s)}{x(s)}]ds\right\}$$

and $x(0), y(0) > 0$.

Lemma 3.2 *Let $(x(t), y(t))$ denote the solution of (3.1) with the initial conditions (3.2) , then*

$$0 < x(t) \leq M \ , 0 < y(t) \leq L \tag{3.3}$$

eventually for all large t,where

$$M = Ke^{r\tau} \tag{3.4}$$

$$L = \frac{\delta M}{\beta} \tag{3.5}$$

Proof : Now , we want to show that there exists a $T > 0$ such that $x(t) \leq M$ for $t > T$. By Lemma 3.1 , we know that solutions of the system (3.1) are positive , and hence , by assumption $(A1)$, and (3.1)

$$\begin{aligned} \dot{x}(t) &= x(t)\left\{r[1 - \frac{x(t-\tau)}{K}]\right\} - p(x(t))y(t) \\ &\leq rx(t)\left[1 - \frac{x(t-\tau)}{K}\right] \end{aligned} \tag{3.6}$$

Taking $M^* = K(1 + K_1)$, $0 < K_1 < e^{r\tau} - 1$. Suppose $x(t)$ is not oscillatory about M^* . That is , there exists a $T > 0$ such that either

$$x(t) > M^* \qquad for \ \ t > T_0 \tag{3.7}$$

or

$$x(t) \leq M^* \qquad for \ \ t > T_0 \tag{3.8}$$

If (3.8) holds , then for $t > T_0$

$$x(t) \leq M^* = K(1 + K_1) < Ke^{r\tau} = M$$

That is , (3.3) holds . Suppose (3.7) holds . Equation (3.6) implies that for $t > T_0 + \tau$

$$\begin{aligned} \dot{x}(t) &\leq rx(t)\left[1 - \frac{x(t-\tau)}{K}\right] \\ &< -K_1 rx(t) \end{aligned}$$

It follows that

$$\begin{aligned}\int_{T_0+\tau}^{t} \frac{\dot{x}(s)}{x(s)} ds &< \int_{T_0+\tau}^{t} -K_1 r ds \\ &= -K_1 r(t - T_0 - \tau)\end{aligned}$$

Then $0 < x(t) < x(T_0 + \tau)e^{-K_1 r(t-T_0-\tau)} \to 0$ as $t \to \infty$. That is , $\lim_{t\to\infty} x(t) = 0$ by the Squeeze Theorem . It contradicts to (3.7) . Thereore , there must exist a $T_1 > T_0$ such that $x(T_1) \leq M^*$. If $x(T_1) \leq M^*$ for all $t \geq T_1$, then (3.3) follows . If not , then there must exist a $T_2 > T_1$ such that T_2 be the first time which $x(T_2) > M^*$. Therefore , there exists a $T_3 > T_2$ such that T_3 be the first time which $x(T_3) < M^*$ by above discussion . By above , we know that $x(T_1) \leq M^*$, $x(T_2) > M^*$, and $x(T_3) \leq M^*$ where $T_1 < T_2 < T_3$. Then , by the Intermediate Value Theorem , there exists T_4 and T_5 such that

$$x(T_4) = M^* \quad , \quad T_1 \leq T_4 < T_2$$

$$x(T_5) = M^* \quad , \quad T_2 \leq T_5 < T_3$$

and $x(t) > M^*$ for $T_4 < t < T_5$. Hence there is a $T_6 \in (T_4, T_5)$ such that $x(T_6)$ is an arbitrary local maximum , and hence it follows from (3.6) that

$$0 = \dot{x}(T_6) \leq r x(T_6) \left[1 - \frac{x(T_6 - \tau)}{K}\right]$$

and this implies

$$x(T_6 - \tau) \leq K$$

Integrating both sides of (3.6) on the interval $[T_6 - \tau, T_6]$, we have

$$\ln\left[\frac{x(T_6)}{x(T_6 - \tau)}\right] = \int_{T_6-\tau}^{T_6} \frac{\dot{x}(s)}{x(s)} ds \leq \int_{T_6-\tau}^{T_6} r \left[1 - \frac{x(s - \tau)}{K}\right] ds \leq r\tau$$

It follows that

$$x(T_6) \leq x(T_6 - \tau)e^{r\tau} \leq K e^{r\tau} = M$$

Since $x(T_6)$ is local maximum of $x(t)$ and $x(T_6) \leq M$, $x(t) \leq M$ where t near T_6 . Since $x(T_6)$ is an arbitrary local maximum of $x(t)$, we can conclude that there exists a $T > 0$ such that

$$x(t) \leq M \quad for \quad t \geq T \tag{3.9}$$

Suppose $x(t)$ is oscillatory about M^* , for this case , the proof is similarly to above one . Now , we want to show that $y(t)$ is bounded above by L eventually for all large t . By (3.9) , it follows that for $t > T$

$$\begin{aligned} \dot{y}(t) &= y(t)\left[\delta - \beta\frac{y(t)}{x(t)}\right] \\ &\leq y(t)\left[\delta - \frac{\beta}{M}y(t)\right] \\ &= \delta y(t)\left[1 - \frac{\beta}{\delta M}y(t)\right] \\ &= \delta y(t)\left[1 - \frac{y(t)}{\frac{\delta M}{\beta}}\right] \end{aligned}$$

Therefore , $y(t) \leq \frac{\delta M}{\beta} = L$ for $t > T$. This completes the proof .

Lemma 3.3 *Suppose that the system (3.1) satisfies*

$$r - cL > 0 \tag{3.10}$$

where L defined by (3.5) , and c is defined in assumption $(A1)$ *. Then the system (3.1) is uniformly persistent . That is , there exists* m , l , *and* $T^* > 0$ *such that* $m \leq x \leq M$ *and* $l \leq y \leq L$ *for* $t \geq T^*$.

Proof : By Lemma 3.2 , and assumption $(A1)$, equation (3.1) follows that for $t \geq T + \tau$

$$\dot{x}(t) \geq x(t)\left[r(1 - \frac{M}{K}) - cL\right] \tag{3.11}$$

Integrating both sides of (3.11) on $[t - \tau, t]$, where $t \geq T + \tau$, then we have

$$x(t) \geq x(t-\tau)e^{[r(1-\frac{M}{K})-cL]\tau}$$

That is

$$x(t-\tau) \le x(t)e^{-[r(1-\frac{M}{K})-cL]\tau} \tag{3.12}$$

It follows from (3.1) that for $t \ge T+\tau$

$$\begin{aligned}
\dot{x}(t) &= x(t)r\left[1-\frac{x(t-\tau)}{K}\right]-p(x(t))y(t) \\
&\ge x(t)\left\{r-cL-\frac{r}{K}e^{-[r(1-\frac{M}{K})-cL]\tau}x(t)\right\} \\
&= (r-cL)x(t)\left\{1-\frac{x(t)}{\frac{K(r-cL)}{r}e^{[r(1-\frac{M}{K})-cL]\tau}}\right\}
\end{aligned}$$

It follows that

$$\liminf_{t\to\infty} x(t) \ge \frac{K(r-cL)}{r}e^{[r(1-\frac{M}{K})-cL]\tau} \equiv \overline{m}$$

and $\overline{m} > 0$ by (3.10) . So , for large t , $x(t) > \frac{\overline{m}}{2} \equiv m > 0$. It follows that

$$\begin{aligned}
\dot{y}(t) &\ge y(t)\left[\delta-\beta\frac{y(t)}{m}\right] \\
&= \delta y(t)\left[1-\frac{\beta}{\delta m}y(t)\right] \\
&= \delta y(t)\left[1-\frac{y(t)}{\frac{\delta m}{\beta}}\right]
\end{aligned}$$

Then

$$\liminf_{t\to\infty} y(t) \ge \frac{\delta m}{\beta} \equiv \overline{l}$$

So , for large t , $y(t) > \frac{\overline{l}}{2} \equiv l > 0$. Let

$$D = \{(x,y)|m \le x \le M, l \le y \le L\}$$

Then D is bounded compact region in R_+^2 that has positive distance from coordinate hyperplanes . Hence we obtain that there exists a $T^* > 0$ such that if $t \ge T^*$, then every positive solution of system (3.1) with the initial conditions (3.2) eventually enters and remains in the region D , that is , system (3.1) is uniformly persistent .

Theorem 3.1 *If $p(x) = cx$ in the Holling-type I model , and the delay τ satisfy*

$$r - cL > 0 \tag{3.13}$$

$$\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mr^2x^*\tau}{K^2} - \frac{Mrcy^*\tau}{2K} > 0 \tag{3.14}$$

$$\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mrcy^*\tau}{2K} > 0 \tag{3.15}$$

where m , M , and L defined in Lemmas 3.2 and 3.3 , then the unique positive equilibrium E^* of the system (3.1) is globally asymptotically stable.

Proof : Define $z(t) = (z_1(t), z_2(t))$ by

$$z_1(t) = \frac{x(t) - x^*}{x^*} \quad , \quad z_2(t) = \frac{y(t) - y^*}{y^*}$$

From (3.1) ,

$$\dot{z}_1(t) = [1 + z_1(t)] \left[-\frac{rx^*}{K} z_1(t - \tau) - cy^* z_2(t) \right] \tag{3.16}$$

$$\dot{z}_2(t) = [1 + z_2(t)] \left\{ \frac{\delta x^* z_1(t) - \beta y^* z_2(t)}{x^*[1 + z_1(t)]} \right\} \tag{3.17}$$

Let

$$V_1(z(t)) = \{z_1(t) - ln[1 + z_1(t)]\} + \{z_2(t) - ln[1 + z_2(t)]\} \tag{3.18}$$

then we have from (3.16) and (3.17) that

$$\begin{aligned}
\dot{V}_1(z(t)) &= \frac{z_1(t)\dot{z}_1(t)}{1 + z_1(t)} + \frac{z_2(t)\dot{z}_2(t)}{1 + z_2(t)} \\
&= -\frac{rx^*}{K} z_1(t) z_1(t - \tau) - cy^* z_1(t) z_2(t) + \frac{\delta}{1 + z_1(t)} z_1(t) z_2(t) \\
&\quad - \frac{\beta y^* z_2^2(t)}{x^*[1 + z_1(t)]} \\
&\leq -\frac{rx^*}{K} z_1(t) z_1(t - \tau) + \frac{\delta - cy^*[1 + z_1(t)]}{1 + z_1(t)} z_1(t) z_2(t) \\
&\quad - \frac{\beta y^* z_2^2(t)}{x^*[1 + z_1(t)]} \qquad (3.19)
\end{aligned}$$

If $\delta x^* - cy^* M > 0$, and by Lemma 3.3 , there exists a $T^* > 0$ such that $m \leq x^*[1+z_1(t)] \leq M$ and $l \leq y^*[1+z_2(t)] \leq L$ for $t > T^*$. Then (3.19) implies that

$$
\begin{aligned}
\dot{V}_1(z(t)) &\leq -\frac{rx^*}{K} z_1(t) z_1(t-\tau) \\
&\quad + \frac{\delta x^* - cy^* m}{2m}[z_1^2(t) + z_2^2(t)] - \frac{\beta y^* z_2^2(t)}{M} \\
&= -\frac{rx^*}{K} z_1(t) \left[z_1(t) - \int_{t-\tau}^{t} \dot{z}_1(s) ds \right] + (\frac{\delta x^*}{2m} - \frac{cy^*}{2}) z_1^2(t) \\
&\quad + (\frac{\delta x^*}{2m} - \frac{cy^*}{2} - \frac{\beta y^*}{M}) z_2^2(t) \\
&= -(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}) z_1^2(t) - (\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}) z_2^2(t) \\
&\quad + \frac{rx^*}{K} z_1(t) \int_{t-\tau}^{t} [1+z_1(s)][-\frac{rx^*}{K} z_1(s-\tau) - cy^* z_2(s)] ds \\
&= -(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}) z_1^2(t) - (\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}) z_2^2(t) \\
&\quad + \frac{rx^*}{K} \int_{t-\tau}^{t} [1+z_1(s)][-\frac{rx^*}{K} z_1(t) z_1(s-\tau) - cy^* z_1(t) z_2(s)] ds \\
&\leq -(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}) z_1^2(t) - (\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}) z_2^2(t) \\
&\quad + \frac{rx^*}{K} \int_{t-\tau}^{t} [1+z_1(s)][\frac{rx^*}{K} |z_1(t)||z_1(s-\tau)| + cy^* |z_1(t)||z_2(s)|] ds
\end{aligned}
\tag{3.20}
$$

Then for $t \geq T^* + \tau \equiv \widehat{T}$, we have from (3.20) that

$$
\begin{aligned}
\dot{V}_1(z(t)) &\leq -(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}) z_1^2(t) - (\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m}) z_2^2(t) \\
&\quad + \frac{Mr}{K} \int_{t-\tau}^{t} [\frac{rx^*}{K} |z_1(t)||z_1(s-\tau)| + cy^* |z_1(t)||z_2(s)|] ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq -(\frac{rx^*}{K}+\frac{cy^*}{2}-\frac{\delta x^*}{2m})z_1^2(t)-(\frac{\beta y^*}{M}+\frac{cy^*}{2}-\frac{\delta x^*}{2m})z_2^2(t)\\
&\quad +\frac{Mr}{K}\left[\frac{rx^*\tau}{2K}z_1^2(t)+\frac{rx^*}{2K}\int_{t-\tau}^{t} z_1^2(s-\tau)ds+\frac{cy^*\tau}{2}z_1^2(t)\right.\\
&\quad \left.+\frac{cy^*}{2}\int_{t-\tau}^{t} z_2^2(s)ds\right]\\
&= -(\frac{rx^*}{K}+\frac{cy^*}{2}-\frac{\delta x^*}{2m}-\frac{Mr^2x^*\tau}{2K^2}-\frac{Mrcy^*\tau}{2K})z_1^2(t)-(\frac{\beta y^*}{M}\\
&\quad +\frac{cy^*}{2}-\frac{\delta x^*}{2m})z_2^2(t)+\frac{Mr^2x^*}{2K^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds+\frac{Mrcy^*}{2K}\int_{t-\tau}^{t} z_2^2(s)ds
\end{aligned}
\tag{3.21}
$$

Let

$$
\begin{aligned}
V_2(z(t)) &= \frac{Mr^2x^*}{2K^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma-\tau)d\gamma ds\\
&\quad +\frac{Mrcy^*}{2K}\int_{t-\tau}^{t}\int_{s}^{t} z_2^2(\gamma)d\gamma ds
\end{aligned}
\tag{3.22}
$$

then

$$
\begin{aligned}
\dot{V}_2(z(t)) &= \frac{Mr^2x^*\tau}{2K^2}z_1^2(t-\tau)-\frac{Mr^2x^*}{2K^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds\\
&\quad +\frac{Mrcy^*\tau}{2K}z_2^2(t)-\frac{Mrcy^*}{2K}\int_{t-\tau}^{t} z_2^2(s)ds
\end{aligned}
\tag{3.23}
$$

and then we have from (3.21) and (3.23) that for $t \geq \widehat{T}$

$$
\begin{aligned}
\dot{V}_1(z(t))+\dot{V}_2(z(t)) &\leq -(\frac{rx^*}{K}+\frac{cy^*}{2}-\frac{\delta x^*}{2m}-\frac{Mr^2x^*\tau}{2K^2}-\frac{Mrcy^*\tau}{2K})z_1^2(t)\\
&\quad -(\frac{\beta y^*}{M}+\frac{cy^*}{2}-\frac{\delta x^*}{2m}-\frac{Mrcy^*\tau}{2K})z_2^2(t)\\
&\quad +\frac{Mr^2x^*\tau}{2K^2}z_1^2(t-\tau)
\end{aligned}
\tag{3.24}
$$

Let

$$V_3(z(t)) \;=\; \frac{Mr^2x^*\tau}{2K^2}\int_{t-\tau}^{t} z_1^2(s)ds \tag{3.25}$$

then

$$\dot{V}_3(z(t)) \;=\; \frac{Mr^2x^*\tau}{2K^2}z_1^2(t) - \frac{Mr^2x^*\tau}{2K^2}z_1^2(t-\tau) \tag{3.26}$$

Now define a Lyapunov functional $V(z(t))$ as

$$V(z(t)) \;=\; V_1(z(t)) + V_2(z(t)) + V_3(z(t)) \tag{3.27}$$

then we have from (3.24) and (3.26) that for $t \geq \widehat{T}$

$$\begin{aligned}\dot{V}(z(t)) \;&\leq\; -(\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mr^2x^*\tau}{K^2} - \frac{Mrcy^*\tau}{2K})z_1^2(t)\\ &\quad -(\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mrcy^*\tau}{2K})z_2^2(t)\\ &\equiv\; -\zeta z_1^2(t) - \eta z_2^2(t)\end{aligned} \tag{3.28}$$

Then it follows from (3.14) and (3.15) that $\zeta > 0$ and $\eta > 0$. Let $w(s) = \widehat{N}s^2$ where $\widehat{N} = \min\{\zeta, \eta\}$, then w is nonnegative continuous on $[0,\infty)$, $w(0) = 0$, and $w(s) > 0$ for $s > 0$. It follows from (3.28) that for $t \geq \widehat{T}$

$$\dot{V}(z(t)) \leq -\widehat{N}\;[z_1^2(t) + z_2^2(t)] = -\widehat{N}\;\|z(t)\|^2 = -w(\|z(t)\|) \tag{3.29}$$

Now , we want to find a function u such that $V(z(t)) \geq u(\|z(t)\|)$. It follows from (3.18) , (3.22) , and (3.25) that

$$V(z(t)) \geq \{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\} \tag{3.30}$$

By the Taylor Theorem , we have that

$$z_i(t) - \ln[1 + z_i(t)] = \frac{z_i^2(t)}{2[1 + \theta_i(t)]^2} \tag{3.31}$$

where $\theta_i(t) \in (0, z_i(t))$ or $(z_i(t), 0)$ for $i = 1, 2$.

Case1 : If $0 < \theta_i(t) < z_i(t)$ for $i = 1, 2$, then

$$\frac{z_i^2(t)}{[1 + z_i(t)]^2} < \frac{z_i^2(t)}{[1 + \theta_i(t)]^2} < z_i^2(t) \tag{3.32}$$

By Lemma 3.3 , it follows that for $t \geq T^*$

$$m \leq x^*[1+z_1(t)] = x(t) \leq M$$

$$l \leq y^*[1+z_2(t)] = y(t) \leq L \tag{3.33}$$

Then (3.32) implies that

$$\left(\frac{x^*}{M}\right)^2 z_1^2(t) \leq \frac{z_1^2(t)}{[1+\theta_1(t)]^2} < z_1^2(t)$$

$$\left(\frac{y^*}{L}\right)^2 z_2^2(t) \leq \frac{z_2^2(t)}{[1+\theta_2(t)]^2} < z_2^2(t) \tag{3.34}$$

It follows that (3.30) , (3.31) , and (3.34) that for $t \geq T^*$

$$\begin{aligned} V(z(t)) &\geq \frac{1}{2}\frac{z_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2}\frac{z_2^2(t)}{[1+\theta_2(t)]^2} \\ &\geq \frac{1}{2}\left(\frac{x^*}{M}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{y^*}{L}\right)^2 z_2^2(t) \\ &\geq \min\left\{\frac{1}{2}\left(\frac{x^*}{M}\right)^2, \frac{1}{2}\left(\frac{y^*}{L}\right)^2\right\} [z_1^2(t)+z_2^2(t)] \\ &\equiv \widetilde{N}\, \|z(t)\|^2 \end{aligned}$$

Case2 : If $-1 < z_i(t) < \theta_i(t) < 0$ for $i = 1, 2$, then

$$z_i^2(t) < \frac{z_i^2(t)}{[1+\theta_i(t)]^2} < \frac{z_i^2(t)}{[1+z_i(t)]^2} \tag{3.35}$$

By (3.33) , (3.35) implies that

$$z_1^2(t) < \frac{z_1^2(t)}{[1+\theta_1(t)]^2} \leq \left(\frac{x^*}{m}\right)^2 z_1^2(t)$$

$$z_2^2(t) < \frac{z_2^2(t)}{[1+\theta_2(t)]^2} \leq \left(\frac{y^*}{l}\right)^2 z_2^2(t) \tag{3.36}$$

It follows that (3.30) , (3.31) , and (3.36) that for $t \geq T^*$

$$
\begin{aligned}
V(z(t)) &\geq \frac{1}{2}\frac{z_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2}\frac{z_2^2(t)}{[1+\theta_2(t)]^2} \\
&> \frac{1}{2}z_1^2(t) + \frac{1}{2}z_2^2(t) \\
&\geq \frac{1}{2}\left(\frac{x^*}{M}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{y^*}{L}\right)^2 z_2^2(t) \\
&\geq \widetilde{N}\ [z_1^2(t) + z_2^2(t)] \\
&= \widetilde{N}\ \|z(t)\|^2
\end{aligned}
$$

Case3 : If $0 < \theta_1(t) < z_1(t)$ and $-1 < z_2(t) < \theta_2(t) < 0$, then it follows that (3.30) , (3.31) , (3.34) and (3.36) that for $t \geq T^*$

$$
\begin{aligned}
V(z(t)) &\geq \frac{1}{2}\frac{z_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2}\frac{z_2^2(t)}{[1+\theta_2(t)]^2} \\
&> \frac{1}{2}\left(\frac{x^*}{M}\right)^2 z_1^2(t) + \frac{1}{2}z_2^2(t) \\
&\geq \frac{1}{2}\left(\frac{x^*}{M}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{y^*}{L}\right)^2 z_2^2(t) \\
&\geq \widetilde{N}\ [z_1^2(t) + z_2^2(t)] \\
&= \widetilde{N}\ \|z(t)\|^2
\end{aligned}
$$

Case4 : If $-1 < z_1(t) < \theta_1(t) < 0$ and $0 < \theta_2(t) < z_2(t)$, then it follows that (3.30) , (3.31) , (3.34) and (3.36) that for $t \geq T^*$

$$
\begin{aligned}
V(y(t)) &\geq \frac{1}{2}\frac{z_1^2(t)}{[1+\theta_1(t)]^2} + \frac{1}{2}\frac{z_2^2(t)}{[1+\theta_2(t)]^2} \\
&> \frac{1}{2}z_1^2(t) + \frac{1}{2}\left(\frac{y^*}{L}\right)^2 z_2^2(t)
\end{aligned}
$$

$$\geq \frac{1}{2}\left(\frac{x^*}{M}\right)^2 z_1^2(t) + \frac{1}{2}\left(\frac{y^*}{L}\right)^2 z_2^2(t)$$

$$\geq \widetilde{N}\ [z_1^2(t) + z_2^2(t)]$$

$$= \widetilde{N}\ \|z(t)\|^2$$

Let $u(s) = \widetilde{N}s^2$, then u is nonnegative continuous on $[0, \infty)$, $u(0) = 0$, $u(s) > 0$ for $s > 0$, and $\lim_{s\to\infty} u(s) = +\infty$. So , by case1 $\sim$ case4 , we have

$$V(z(t)) \geq u(\|z(t)\|) \qquad for \quad t \geq T^* \tag{3.37}$$

So the equilibrium point E^* of the system (3.1) is globally asymptotically stable with $p(x) = cx$. ■

Theorem 3.2 *If* $p(x) = \dfrac{cx}{1+x}$ *in the Holling-type II model , and the delay* τ *satisfy*

$$r - cL > 0 \tag{3.38}$$

$$\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)} - \frac{rx^*(x^*+M)}{K(1+m)}$$

$$- \frac{Mr^2x^*\tau(1+x^*)(K+3x^*+1+M)}{K^2(1+m)^2} - \frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2} > 0 \tag{3.39}$$

$$\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*(1+M)}{2m(1+m)} - \frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2} > 0 \tag{3.40}$$

where m *,* M *, and* L *defined in Lemmas 3.2 and 3.3 , then the unique positive equilibrium* E^* *of the system (3.1) is globally asymptotically stable.*

Proof : Define $z(t) = (z_1(t), z_2(t))$ by

$$z_1(t) = \frac{x(t) - x^*}{x^*} \quad , \quad z_2(t) = \frac{y(t) - y^*}{y^*}$$

From (3.1) ,

$$\begin{aligned}
\dot{z}_1(t) &= [1 + z_1(t)] \left\{ \frac{rx^*z_1(t)}{1 + x^*[1 + z_1(t)]} - \frac{rx^{*2}z_1(t)}{K\{1 + x^*[1 + z_1(t)]\}} \right. \\
&- \frac{rx^*z_1(t-\tau)}{K\{1 + x^*[1 + z_1(t)]\}} - \frac{rx^{*2}z_1(t-\tau)}{K\{1 + x^*[1 + z_1(t)]\}} - \frac{rx^{*2}z_1(t-\tau)z_1(t)}{K\{1 + x^*[1 + z_1(t)]\}} \\
&- \left. \frac{cy^*z_2(t)}{1 + x^*[1 + z_1(t)]} \right\} \qquad (3.41)
\end{aligned}$$

$$\dot{z}_2(t) = [1 + z_2(t)] \left\{ \frac{\delta x^*z_1(t) - \beta y^*z_2(t)}{x^*[1 + z_1(t)]} \right\} \tag{3.42}$$

Let

$$V_1(z(t)) = \{z_1(t) - ln[1 + z_1(t)]\} + \{z_2(t) - ln[1 + z_2(t)]\} \tag{3.43}$$

then we have from (3.41) and (3.42) that

$$
\begin{aligned}
\dot{V}_1(z(t)) &= \frac{\dot{z}_1(t)z_1(t)}{1+z_1(t)} + \frac{\dot{z}_2(t)z_2(t)}{1+z_2(t)} \\
&= \frac{rx^*z_1^2(t)}{1+x^*[1+z_1(t)]} - \frac{rx^{*2}z_1^2(t)}{K\{1+x^*[1+z_1(t)]\}} - \frac{rx^*z_1(t)z_1(t-\tau)}{K\{1+x^*[1+z_1(t)]\}} \\
&- \frac{rx^{*2}z_1(t)z_1(t-\tau)}{K\{1+x^*[1+z_1(t)]\}} - \frac{rx^{*2}z_1^2(t)z_1(t-\tau)}{K\{1+x^*[1+z_1(t)]\}} - \frac{\beta y^*z_2^2(t)}{x^*[1+z_1(t)]} \\
&+ \frac{\delta\{1+x^*[1+z_1(t)]\} - cy^*[1+z_1(t)]}{\{1+x^*[1+z_1(t)]\}[1+z_1(t)]} z_1(t)z_2(t) \\
&\leq \frac{rx^*z_1^2(t)}{1+x^*[1+z_1(t)]} - \frac{rx^{*2}z_1^2(t)}{K\{1+x^*[1+z_1(t)]\}} + \frac{rx^{*2}z_1^2(t)|z_1(t-\tau)|}{K\{1+x^*[1+z_1(t)]\}} \\
&- \frac{\beta y^*z_2^2(t)}{x^*[1+z_1(t)]} + \frac{\delta\{1+x^*[1+z_1(t)]\} - cy^*[1+z_1(t)]}{\{1+x^*[1+z_1(t)]\}[1+z_1(t)]} z_1(t)z_2(t) \\
&- \left\{ \frac{rx^*}{K\{1+x^*[1+z_1(t)]\}} + \frac{rx^{*2}}{K\{1+x^*[1+z_1(t)]\}} \right\} z_1(t)z_1(t-\tau)
\end{aligned}
\tag{3.44}
$$

If $\delta x^*(1+m) - cy^*M > 0$, and by Lemma 3.3 , there exists a $T^* > 0$ such that $m \leq x^*[1+z_1(t)] \leq M$ and $l \leq y^*[1+z_2(t)] \leq L$ for $t > T^*$. Then (3.44) implies that

$$
\begin{aligned}
\dot{V}_1(z(t)) &\leq \frac{rx^*z_1^2(t)}{1+m} - \frac{rx^{*2}z_1^2(t)}{K(1+M)} + \frac{rx^{*2}z_1^2(t)|z_1(t-\tau)|}{K(1+m)} \\
&- \frac{\beta y^*z_2^2(t)}{M} - \frac{cy^*z_1^2(t)}{2(1+m)} - \frac{cy^*z_2^2(t)}{2(1+m)} + \frac{\delta x^*(1+M)z_1^2(t)}{2m(1+m)} + \frac{\delta x^*(1+M)z_2^2(t)}{2m(1+m)}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{rx^*(1+x^*)}{K\{1+x^*[1+z_1(t)]\}} z_1(t) \left[z_1(t) - \int_{t-\tau}^{t} \dot{z}_1(s) ds \right] \\
&= -\left[\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)} \right] z_1^2(t) \\
&- \left[\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*(1+M)}{2m(1+m)} \right] z_2^2(t) + \frac{rx^{*2}}{K(1+m)} z_1^2(t) |z_1(t-\tau)| \\
&+ \frac{rx^*(1+x^*)}{K\{1+x^*[1+z_1(t)]\}} \int_{t-\tau}^{t} [1+z_1(s)] \left\{ \frac{rx^* z_1(t) z_1(s)}{1+x^*[1+z_1(s)]} \right. \\
&- \frac{rx^{*2} z_1(t) z_1(s)}{K\{1+x^*[1+z_1(s)]\}} - \frac{rx^* z_1(t) z_1(s-\tau)}{K\{1+x^*[1+z_1(s)]\}} - \frac{rx^{*2} z_1(t) z_1(s-\tau)}{K\{1+x^*[1+z_1(s)]\}} \\
&- \left. \frac{rx^{*2} z_1(t) z_1(s) z_1(s-\tau)}{K\{1+x^*[1+z_1(s)]\}} - \frac{cy^* z_1(t) z_2(s)}{1+x^*[1+z_1(s)]} \right\} ds \\
&\leq -\left[\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)} \right] z_1^2(t) \\
&- \left[\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*(1+M)}{2m(1+m)} \right] z_2^2(t) + \frac{rx^{*2}}{K(1+m)} z_1^2(t) |z_1(t-\tau)| \\
&+ \frac{rx^*(1+x^*)}{K\{1+x^*[1+z_1(t)]\}} \int_{t-\tau}^{t} [1+z_1(s)] \left\{ \frac{rx^* |z_1(t)||z_1(s)|}{1+x^*[1+z_1(s)]} \right. \\
&+ \frac{rx^{*2} |z_1(t)||z_1(s)|}{K\{1+x^*[1+z_1(s)]\}} + \frac{rx^* |z_1(t)||z_1(s-\tau)|}{K\{1+x^*[1+z_1(s)]\}} \\
&+ \left. \frac{rx^{*2} |z_1(t)||z_1(s-\tau)|}{K\{1+x^*[1+z_1(s)]\}} + \frac{cy^* |z_1(t)||z_2(s)|}{1+x^*[1+z_1(s)]} \right\} ds
\end{aligned}
$$

$$- \frac{rx^*(1+x^*)}{K\{1+x^*[1+z_1(t)]\}} \int_{t-\tau}^{t} \frac{rx^{*2} z_1(t) z_1(s-\tau)}{K\{1+x^*[1+z_1(s)]\}} [1+z_1(s)]^2 ds$$

$$+ \frac{rx^*(1+x^*)}{K\{1+x^*[1+z_1(t)]\}} \int_{t-\tau}^{t} \frac{rx^{*2} |z_1(t)||z_1(s-\tau)|}{K\{1+x^*[1+z_1(s)]\}} [1+z_1(s)] ds$$

$$(3.45)$$

Then for $t \geq T^* + \tau \equiv \widehat{T}$, we have from (3.45) that

$$\begin{aligned}
\dot{V}_1(z(t)) \quad \leq \quad & - \left[\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)} \right] z_1^2(t) \\
- \quad & \left[\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*(1+M)}{2m(1+m)} \right] z_2^2(t) + \frac{rx^{*2}}{K(1+m)} (1 + \frac{M}{x^*}) z_1^2(t) \\
+ \quad & \frac{Mr(1+x^*)}{K(1+m)} \int_{t-\tau}^{t} \left[\frac{rx^*|z_1(t)||z_1(s)|}{1+m} + \frac{rx^{*2}|z_1(t)||z_1(s)|}{K(1+m)} \right. \\
+ \quad & \left. \frac{rx^*|z_1(t)||z_1(s-\tau)|}{K(1+m)} + \frac{rx^{*2}|z_1(t)||z_1(s-\tau)|}{K(1+m)} + \frac{cy^*|z_1(t)||z_2(s)|}{1+m} \right] ds \\
+ \quad & \frac{M^2 r(1+x^*)}{Kx^*(1+m)} \int_{t-\tau}^{t} \frac{rx^{*2}|z_1(t)||z_1(s-\tau)|}{K(1+m)} ds \\
+ \quad & \frac{Mr(1+x^*)}{K(1+m)} \int_{t-\tau}^{t} \frac{rx^{*2}|z_1(t)||z_1(s-\tau)|}{K(1+m)} ds \\
\leq \quad & - \left[\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)} \right. \\
- \quad & \frac{rx^*(x^*+M)}{K(1+m)} - \frac{Mr(1+x^*)\tau}{K(1+m)^2} (\frac{rx^*}{2} + \frac{3rx^{*2}}{2K} + \frac{rx^*}{2K} + \frac{cy^*}{2})
\end{aligned}$$

$$
\begin{aligned}
&- \quad \frac{M^2r^2x^*(1+x^*)\tau}{2K^2(1+m)^2}\Bigg]\, z_1^2(t) - \left[\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*}{2m}\right] z_2^2(t) \\
&+ \quad \frac{Mr^2x^*(1+x^*)(K+x^*)}{2K^2(1+m)^2}\int_{t-\tau}^{t} z_1^2(s)ds \\
&+ \quad \frac{Mr^2x^*(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds \\
&+ \quad \frac{Mrcy^*(1+x^*)}{2K(1+m)^2}\int_{t-\tau}^{t} z_2^2(s)ds
\end{aligned}
\tag{3.46}
$$

Let

$$
\begin{aligned}
V_2(z(t)) &= \frac{Mr^2x^*(1+x^*)(K+x^*)}{2K^2(1+m)^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma)d\gamma ds \\
&+ \frac{Mr^2x^*(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma-\tau)d\gamma ds \\
&+ \frac{Mrcy^*(1+x^*)}{2K(1+m)^2}\int_{t-\tau}^{t}\int_{s}^{t} z_2^2(\gamma)d\gamma ds
\end{aligned}
\tag{3.47}
$$

then

$$
\begin{aligned}
\dot{V}_2(z(t)) &= \frac{Mr^2x^*\tau(1+x^*)(K+x^*)}{2K^2(1+m)^2} z_1^2(t) \\
&- \frac{Mr^2x^*(1+x^*)(K+x^*)}{2K^2(1+m)^2}\int_{t-\tau}^{t} z_1^2(s)ds \\
&+ \frac{Mr^2x^*\tau(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2} z_1^2(t-\tau)
\end{aligned}
$$

$$- \frac{Mr^2x^*(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds$$

$$+ \frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2}z_2^2(t) - \frac{Mrcy^*(1+x^*)}{2K(1+m)^2}\int_{t-\tau}^{t} z_2^2(s)ds$$

(3.48)

and then we have from (3.46) and (3.48) that for $t \geq \widehat{T}$

$$\begin{aligned}
\dot{V}_1(z(t)) + \dot{V}_2(z(t)) \quad \leq \quad & - \Bigg[\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)} \\
& - \frac{rx^*(x^*+M)}{K(1+m)} - \frac{Mr(1+x^*)\tau}{K(1+m)^2}(\frac{rx^*}{2} + \frac{3rx^{*2}}{2K} + \frac{rx^*}{2K} + \frac{cy^*}{2}) \\
& - \frac{M^2r^2x^*\tau(1+x^*)}{2K^2(1+m)^2} - \frac{Mr^2x^*\tau(1+x^*)(K+x^*)}{2K^2(1+m)^2}\Bigg] z_1^2(t) \\
& - \Bigg[\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*(1+M)}{2m(1+m)} - \frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2}\Bigg] z_2^2(t) \\
& + \frac{Mr^2x^*\tau(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}z_1^2(t-\tau)
\end{aligned}$$

(3.49)

Let

$$V_3(z(t)) = \frac{Mr^2x^*\tau(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}\int_{t-\tau}^{t} z_1^2(s)ds \quad (3.50)$$

then

$$\begin{aligned}
\dot{V}_3(z(t)) \quad = \quad & \frac{Mr^2x^*\tau(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}z_1^2(t) \\
& - \frac{Mr^2x^*\tau(1+x^*)(M+1+2x^*)}{2K^2(1+m)^2}z_1^2(t-\tau)
\end{aligned} \quad (3.51)$$

Now define a Lyapunov functional $V(z(t))$ as

$$V(z(t)) \quad = \quad V_1(z(t)) + V_2(z(t)) + V_3(z(t)) \tag{3.52}$$

then we have from (3.49) and (3.51) that for $t \geq \widehat{T}$

$$\begin{aligned}
\dot{V}(z(t)) \quad \leq \quad & -\left[\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)}\right. \\
& - \frac{rx^*(x^*+M)}{K(1+m)} - \frac{Mr^2x^*\tau(1+x^*)(K+3x^*+1+M)}{K^2(1+m)^2} \\
& - \left.\frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2}\right] z_1^2(t) \\
& - \left[\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*(1+M)}{2m(1+m)} - \frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2}\right] z_2^2(t) \\
\equiv \quad & -\zeta z_1^2(t) - \eta z_2^2(t) \qquad (3.53)
\end{aligned}$$

Then it follows from (3.39) and (3.40) that $\zeta > 0$ and $\eta > 0$. Let $w(s) = \widehat{N}s^2$ where $\widehat{N} = \min\{\zeta, \eta\}$, then w is nonnegative continuous on $[0, \infty)$, $w(0) = 0$, and $w(s) > 0$ for $s > 0$. It follows from (3.53) that for $t \geq \widehat{T}$

$$\dot{V}(z(t)) \leq -\widehat{N}\ [z_1^2(t) + z_2^2(t)] = -\widehat{N}\ \|z(t)\|^2 = -w(\|z(t)\|) \tag{3.54}$$

Now , we want to find a function u such that $V(z(t)) \geq u(\|z(t)\|)$. It follows from (3.43) , (3.47) , and (3.50) that

$$V(z(t)) \geq \{z_1(t) - \ln[1 + z_1(t)]\} + \{z_2(t) - \ln[1 + z_2(t)]\} \tag{3.55}$$

then by Theorem 3.1 , we have that $u(s) = \widetilde{N}s^2$ and $V(z(t)) \geq u(\|z(t)\|)$. So the equilibrium point E^* of the system (3.1) is globally asymptotically stable with $p(x) = \frac{cx}{1+x}$. ■

Theorem 3.3 *If* $p(x) = \dfrac{cx^2}{1+x^2}$ *in the Holling-type III model , and the delay* τ *satisfy*

$$r - cL > 0 \tag{3.56}$$

$$\frac{rx^*(1+3x^{*2})}{K(1+M^2)} + \frac{cx^*y^*}{2(1+m^2)} + \frac{cx^*y^*}{1+M^2} - \frac{rx^*(M+3x^*)}{1+m^2}$$

$$- \frac{3rx^{*2}(M+x^*)}{K(1+m^2)} - \frac{rx^*(M+x^*)^2}{K(1+m^2)} - \frac{\delta x^*(1+M^2)}{2m(1+m^2)} - \frac{cx^*(L+y^*)}{1+m^2}$$

$$- \frac{Mr^2x^*\tau(1+x^{*2})(3Kx^*+9x^{*2}+MK+5Mx^*+1+M^2)}{K^2(1+m^2)^2}$$

$$- \frac{Mrcy^*\tau(1+x^{*2})(4x^*+M)}{2K(1+m^2)^2} > 0 \tag{3.57}$$

$$\frac{\beta y^*}{M} + \frac{cx^*y^*}{2(1+m^2)} - \frac{\delta x^*(1+M^2)}{2m(1+m^2)} - \frac{Mrcy^*\tau(1+x^{*2})(2x^*+M)}{2K(1+m^2)^2} > 0 \tag{3.58}$$

where m *,* M *, and* L *defined in Lemmas 3.2 and 3.3 , then the unique positive equilibrium* E^* *of the system (3.1) is globally asymptotically stable.*

Proof : Define $z(t) = (z_1(t), z_2(t))$ by

$$z_1(t) = \frac{x(t)-x^*}{x^*} \quad , \quad z_2(t) = \frac{y(t)-y^*}{y^*}$$

From (3.1) ,

$$\begin{aligned}
\dot{z}_1(t) &= [1+z_1(t)]\left\{\frac{2rx^{*2}z_1(t)}{1+x^{*2}[1+z_1(t)]^2} + \frac{rx^{*2}z_1^2(t)}{1+x^{*2}[1+z_1(t)]^2} - \frac{2rx^{*3}z_1(t)}{K\{1+x^{*2}[1+z_1(t)]^2\}}\right. \\
&- \frac{rx^{*3}z_1^2(t)}{K\{1+x^{*2}[1+z_1(t)]^2\}} - \frac{rx^*z_1(t-\tau)}{K\{1+x^{*2}[1+z_1(t)]^2\}} - \frac{rx^{*3}z_1(t-\tau)}{K\{1+x^{*2}[1+z_1(t)]^2\}} \\
&- \frac{2rx^{*3}z_1(t)z_1(t-\tau)}{K\{1+x_1^{*2}[1+z_1(t)]^2\}} - \frac{rx^{*3}z_1^2(t)z_1(t-\tau)}{K\{1+x^{*2}[1+z_1(t)]^2\}} - \frac{cx^*y^*z_1(t)}{1+x^{*2}[1+z_1(t)]^2}
\end{aligned}$$

$$- \quad \frac{cx^*y^*z_2(t)}{1+x^{*2}[1+z_1(t)]^2} - \frac{cx^*y^*z_1(t)z_2(t)}{1+x^{*2}[1+z_1(t)]^2}\Bigg\} \tag{3.59}$$

$$\dot{z}_2(t) = [1+z_2(t)]\left\{\frac{\delta x^* z_1(t) - \beta y^* z_2(t)}{x^*[1+z_1(t)]}\right\} \tag{3.60}$$

Let

$$V_1(z(t)) = \{z_1(t) - ln[1+z_1(t)]\} + \{z_2(t) - ln[1+z_2(t)]\} \tag{3.61}$$

then we have from (3.59) and (3.60) that

$$\begin{aligned}
\dot{V}_1(z(t)) &= \frac{\dot{z}_1(t)z_1(t)}{1+z_1(t)} + \frac{\dot{z}_2(t)z_2(t)}{1+z_2(t)} \\
&= \frac{2rx^{*2}z_1^2(t)}{1+x^{*2}[1+z_1(t)]^2} + \frac{rx^{*2}z_1^3(t)}{1+x^{*2}[1+z_1(t)]^2} - \frac{2rx^{*3}z_1^2(t)}{K\{1+x_1^{*2}[1+z_1(t)]^2\}} \\
&- \frac{rx^{*3}z_1^3(t)}{K\{1+x_1^{*2}[1+z_1(t)]^2\}} - \frac{rx^*z_1(t)z_1(t-\tau)}{K\{1+x^{*2}[1+z_1(t)]^2\}} - \frac{rx^{*3}z_1(t)z_1(t-\tau)}{K\{1+x^{*2}[1+z_1(t)]^2\}} \\
&- \frac{2rx^{*3}z_1^2(t)z_1(t-\tau)}{K\{1+x^{*2}[1+z_1(t)]^2\}} - \frac{rx^{*3}z_1^3(t)z_1(t-\tau)}{K\{1+x^{*2}[1+z_1(t)]^2\}} - \frac{cx^*y^*z_1^2(t)}{1+x^{*2}[1+z_1(t)]^2} \\
&- \frac{cx^*y^*z_1^2(t)z_2(t)}{1+x^{*2}[1+z_1(t)]^2} + \frac{\delta\{1+x^{*2}[1+z_1(t)]^2\} - cx^*y^*[1+z_1(t)]}{\{1+x^{*2}[1+z_1(t)]^2\}[1+z_1(t)]}z_1(t)z_2(t) \\
&- \frac{\beta y^* z_2^2(t)}{x^*[1+z_1(t)]}
\end{aligned} \tag{3.62}$$

If $\delta(1+m^2) - cy^*M > 0$, and by Lemma 3.3 , there exists a $T^* > 0$ such that $m \leq x^*[1+z_1(t)] \leq M$ and $l \leq y^*[1+z_2(t)] \leq L$ for $t > T^*$. Then (3.62) implies that

$$\begin{aligned}
\dot{V}_1(z(t)) &\leq \frac{2rx^{*2}z_1^2(t)}{1+x^{*2}[1+z_1(t)]^2} - \frac{2rx^{*3}z_1^2(t)}{K\{1+x_1^{*2}[1+z_1(t)]^2\}} - \frac{cx^*y^*z_1^2(t)}{1+x^{*2}[1+z_1(t)]^2} \\
&+ \frac{rx^{*2}z_1^2(t)|z_1(t)|}{1+x^{*2}[1+z_1(t)]^2} + \frac{rx^{*3}z_1^2(t)|z_1(t)|}{K\{1+x_1^{*2}[1+z_1(t)]^2\}} + \frac{2rx^{*3}z_1^2(t)|z_1(t-\tau)|}{K\{1+x^{*2}[1+z_1(t)]^2\}}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{cx^*y^*z_1^2(t)|z_2(t)|}{1+x^{*2}[1+z_1(t)]^2} + \frac{rx^{*3}z_1^2(t)|z_1(t)||z_1(t-\tau)|}{K\{1+x^{*2}[1+z_1(t)]^2\}} - \frac{\beta y^* z_2^2(t)}{x^*[1+z_1(t)]} \\
&+ \quad \frac{\delta\{1+x^{*2}[1+z_1(t)]^2\} - cx^*y^*[1+z_1(t)]}{\{1+x^{*2}[1+z_1(t)]^2\}[1+z_1(t)]}|z_1(t)||z_2(t)| \\
&- \quad \frac{rx^*z_1(t)z_1(t-\tau)}{K\{1+x^{*2}[1+z_1(t)]^2\}} - \frac{rx^{*3}z_1(t)z_1(t-\tau)}{K\{1+x^{*2}[1+z_1(t)]^2\}} \\
&\leq \quad \frac{2rx^{*2}}{1+m^2}z_1^2(t) - \frac{2rx^{*3}}{K(1+M^2)}z_1^2(t) - \frac{cx^*y^*}{1+M^2}z_1^2(t) + \frac{rx^{*2}}{1+m^2}z_1^2(t)|z_1(t)| \\
&+ \quad \frac{rx^{*3}}{K(1+m^2)}z_1^2(t)|z_1(t)| + \frac{2rx^{*3}}{K(1+m^2)}z_1^2(t)|z_1(t-\tau)| + \frac{cx^*y^*}{1+m^2}z_1^2(t)|z_2(t)| \\
&+ \quad \frac{rx^{*3}}{K(1+m^2)}z_1(t)^2|z_1(t)||z_1(t-\tau)| - \frac{\beta y^*}{M}z_2^2(t) + \frac{\delta x^*(1+M^2)}{2m(1+m^2)}[z_1^2(t)+z_2^2(t)] \\
&- \quad \frac{cx^*y^*}{2(1+m^2)}[z_1^2(t)+z_2^2(t)] - \frac{rx^*(1+x^{*2})}{K\{1+x^{*2}[1+z_1(t)]^2\}}z_1(t)\left[z_1(t) - \int_{t-\tau}^{t}\dot{z}_1(s)ds\right] \\
&\leq \quad \frac{2rx^{*2}}{1+m^2}z_1^2(t) - \frac{2rx^{*3}}{K(1+M^2)}z_1^2(t) - \frac{cx^*y^*}{1+M^2}z_1^2(t) + \frac{rx^*(M+x^*)}{1+m^2}z_1^2(t) \\
&+ \quad \frac{rx^{*2}(M+x^*)}{K(1+m^2)}z_1^2(t) + \frac{cx^*(L+y^*)}{(1+m^2)}z_1^2(t) - \frac{\beta y^*}{M}z_2^2(t) + \frac{\delta x^*(1+M^2)}{2m(1+m^2)}z_1^2(t) \\
&- \quad \frac{cx^*y^*}{2(1+m^2)}z_1^2(t) + \frac{\delta x^*(1+M^2)}{2m(1+m^2)}z_2^2(t) - \frac{cx^*y^*}{2(1+m^2)}z_2^2(t) - \frac{rx^*(1+x^{*2})}{K(1+M^2)}z_1^2(t) \\
&+ \quad \frac{2rx^{*3}}{K(1+m^2)}z_1^2(t)|z_1(t-\tau)| + \frac{rx^{*3}}{K(1+m^2)}z_1(t)^2|z_1(t)||z_1(t-\tau)|
\end{aligned}
$$

$$+ \quad \frac{rx^*(1+x^{*2})}{K\{1+x^{*2}[1+z_1(t)]^2\}} \int_{t-\tau}^{t} [1+z_1(s)]$$

$$\times \quad \left\{ \frac{2rx^{*2}z_1(t)z_1(s)}{1+x^{*2}[1+z_1(s)]^2} + \frac{rx^{*2}z_1(t)z_1^2(s)}{1+x^{*2}[1+z_1(s)]^2} - \frac{2rx^{*3}z_1(t)z_1(s)}{K\{1+x^{*2}[1+z_1(s)]^2\}} \right.$$

$$- \quad \frac{rx^{*3}z_1(t)z_1^2(s)}{K\{1+x^{*2}[1+z_1(s)]^2\}} - \frac{rx^*z_1(t)z_1(s-\tau)}{K\{1+x^{*2}[1+z_1(s)]^2\}} - \frac{rx^{*3}z_1(t)z_1(s-\tau)}{K\{1+x^{*2}[1+z_1(s)]^2\}}$$

$$- \quad \frac{2rx^{*3}z_1(t)z_1(s)z_1(s-\tau)}{K\{1+x^{*2}[1+z_1(s)]^2\}} - \frac{rx^{*3}z_1(t)z_1^2(s)z_1(s-\tau)}{K\{1+x^{*2}[1+z_1(s)]^2\}} - \frac{cx^*y^*z_1(t)z_1(s)}{1+x^{*2}[1+z_1(s)]^2}$$

$$\left. - \quad \frac{cx^*y^*z_1(t)z_2(s)}{1+x^{*2}[1+z_1(s)]^2} - \frac{cx^*y^*z_1(t)z_1(s)z_2(s)}{1+x^{*2}[1+z_1(s)]^2} \right\} ds \qquad (3.63)$$

Then for $t \geq T^* + \tau \equiv \widehat{T}$, we have from (3.63) that

$$\dot{V}_1(z(t)) \quad \leq \quad -\frac{cx^*y^*}{1+M^2}z_1^2(t) + \frac{rx^*(M+3x^*)}{1+m^2}z_1^2(t) + \frac{3rx^{*2}(M+x^*)}{K(1+m^2)}z_1^2(t) + \frac{cx^*(L+y^*)}{(1+m^2)}z_1^2(t)$$

$$+ \quad \frac{rx^*(M+x^*)^2}{K(1+m^2)}z_1(t)^2 - \frac{\beta y^*}{M}z_2^2(t) + \frac{\delta x^*(1+M^2)}{2m(1+m^2)}z_1^2(t) - \frac{cx^*y^*}{2(1+m^2)}z_1^2(t)$$

$$- \quad \frac{rx^*(1+3x^{*2})}{K(1+M^2)}z_1^2(t) + \frac{\delta x^*(1+M^2)}{2m(1+m^2)}z_2^2(t) - \frac{cx^*y^*}{2(1+m^2)}z_2^2(t)$$

$$+ \quad \frac{Mr(1+x^{*2})}{K(1+m^2)^2}\int_{t-\tau}^{t} \left(2rx^{*2}|z_1(t)||z_1(s)| + \frac{2rx^{*3}}{K}|z_1(t)||z_1(s)| \right.$$

$$+ \quad \frac{rx^*}{K}|z_1(t)||z_1(s-\tau)| + cx^*y^*|z_1(t)||z_2(s)| + cx^*y^*|z_1(t)||z_1(s)|$$

$$\left. + \quad \frac{rx^{*3}}{K}|z_1(t)||z_1(s-\tau)| \right) ds$$

$$
\begin{aligned}
+ &\quad \frac{rx^*(1+x^{*2})}{K\{1+x^{*2}[1+z_1(t)]^2\}}\int_{t-\tau}^{t}[1+z_1(s)] \\
\times &\quad \left\{\left[\frac{rx^{*2}z_1(t)z_1(s)}{1+x^{*2}[1+z_1(s)]^2}-\frac{rx^{*3}z_1(t)z_1(s)}{K\{1+x^{*2}[1+z_1(s)]^2\}}-\frac{2rx^{*3}z_1(t)z_1(s-\tau)}{K\{1+x^{*2}[1+z_1(s)]^2\}}\right.\right. \\
- &\quad \left.\frac{cx^*y^*z_1(t)z_2(s)}{1+x^{*2}[1+z_1(s)]^2}\right][1+z_1(s)]-[\frac{rx^{*2}z_1(t)z_1(s)}{1+x^{*2}[1+z_1(s)]^2}-\frac{rx^{*3}z_1(t)z_1(s)}{K\{1+x^{*2}[1+z_1(s)]^2\}} \\
- &\quad \left.\frac{2rx^{*3}z_1(t)z_1(s-\tau)}{K\{1+x^{*2}[1+z_1(s)]^2\}}-\frac{cx^*y^*z_1(t)z_2(s)}{1+x^{*2}[1+z_1(s)]^2}]\right\}ds \\
- &\quad \frac{rx^*(1+x^{*2})}{K\{1+x^{*2}[1+z_1(t)]^2\}}\int_{t-\tau}^{t}[1+z_1(s)]\left\{\frac{rx^{*3}z_1(t)z_1(s-\tau)}{K\{1+x^{*2}[1+z_1(s)]^2\}}[1+z_1(s)]^2\right. \\
- &\quad \left.\frac{rx^{*3}z_1(t)z_1(s-\tau)}{K\{1+x^{*2}[1+z_1(s)]^2\}}-\frac{2rx^{*3}z_1(t)z_1(s)z_1(s-\tau)}{K\{1+x^{*2}[1+z_1(s)]^2\}}\right\}ds \\
\leq &\quad -\frac{cx^*y^*}{1+M^2}z_1^2(t)+\frac{rx^*(M+3x^*)}{(1+m^2)}z_1^2(t)+\frac{3rx^{*2}(M+x^*)}{K(1+m^2)}z_1^2(t)+\frac{cx^*(L+y^*)}{(1+m^2)}z_1^2(t) \\
+ &\quad \frac{rx^*(M+x^*)^2}{K(1+m^2)}z_1(t)^2-\frac{\beta y^*}{M}z_2^2(t)+\frac{\delta x^*(1+M^2)}{2m(1+m^2)}z_1^2(t)-\frac{cx^*y^*}{2(1+m^2)}z_1^2(t) \\
- &\quad \frac{rx^*(1+3x^{*2})}{K(1+M^2)}z_1^2(t)+\frac{\delta x^*(1+M^2)}{2m(1+m^2)}z_2^2(t)-\frac{cx^*y^*}{2(1+m^2)}z_2^2(t) \\
+ &\quad \frac{Mr(1+x^{*2})}{K(1+m^2)^2}\left[rx^{*2}\tau z_1^2(t)+rx^{*2}\int_{t-\tau}^{t}z_1^2(s)ds+\frac{rx^{*3}\tau}{K}z_1^2(t)\right. \\
+ &\quad \frac{rx^{*3}}{K}\int_{t-\tau}^{t}z_1^2(s)ds+\frac{rx^*\tau}{2K}z_1^2(t)+\frac{rx^*}{2K}\int_{t-\tau}^{t}z_1^2(s-\tau)ds+\frac{rx^{*3}\tau}{2K}z_1^2(t)
\end{aligned}
$$

$$+\quad \frac{rx^{*3}}{2K}\int_{t-\tau}^{t} z_1^2(s-\tau)ds + \frac{cx^*y^*\tau}{2}z_1^2(t) + \frac{cx^*y^*}{2}\int_{t-\tau}^{t} z_1^2(s)ds$$

$$+\quad \frac{cx^*y^*\tau}{2}z_1^2(t) + \frac{cx^*y^*}{2}\int_{t-\tau}^{t} z_2^2(s)ds\Big]$$

$$+\quad \frac{M^2r(1+x^{*2})}{Kx^*(1+m^2)^2}\int_{t-\tau}^{t}\Big[rx^{*2}|z_1(t)||z_1(s)| + \frac{rx^{*3}}{K}|z_1(t)||z_1(s)|$$

$$+\quad \frac{2rx^{*3}}{K}|z_1(t)||z_(s-\tau)| + cx^*y^*|z_1(t)||z_2(s)|\Big]ds$$

$$+\quad \frac{Mr(1+x^{*2})}{K(1+m^2)^2}\int_{t-\tau}^{t}[rx^{*2}|z_1(t)||z_1(s)| + \frac{rx^{*3}}{K}|z_1(t)||z_1(s)|$$

$$+\quad \frac{2rx^{*3}}{K}|z_1(t)||z_(s-\tau)| + cx^*y^*|z_1(t)||z_2(s)|]ds$$

$$+\quad \frac{M^3r^2x^*(1+x^{*2})}{K^2(1+m^2)^2}\int_{t-\tau}^{t}|z_1(t)||z_1(s-\tau)|ds$$

$$+\quad \frac{Mr^2x^{*3}(1+x^{*2})}{K^2(1+m^2)^2}\int_{t-\tau}^{t}|z_1(t)||z_1(s-\tau)|ds$$

$$+\quad \frac{2Mr^2x^{*3}(1+x^{*2})}{K^2(1+m^2)^2}\int_{t-\tau}^{t}\{|z_1(t)||z_1(s-\tau)|[1+z_1(s)] + |z_1(t)||z_1(s-\tau)|\}ds$$

$$\leq\quad -\Big[\frac{rx^*(1+3x^{*2})}{K(1+M^2)} + \frac{cx^*y^*}{2(1+m^2)} + \frac{cx^*y^*}{1+M^2} - \frac{rx^*(M+3x^*)}{1+m^2}$$

$$-\frac{3rx^{*2}(M+x^*)}{K(1+m^2)} - \frac{rx^*(M+x^*)^2}{K(1+m^2)} - \frac{\delta x^*(1+M^2)}{2m(1+m^2)} - \frac{cx^*(L+y^*)}{1+m^2}$$

$$
\begin{aligned}
&- \frac{Mr(1+x^{*2})\tau}{K(1+m^2)^2}\left(\frac{3rx^{*2}}{2}+\frac{9rx^{*3}}{2K}+\frac{rx^*}{2K}+\frac{3cx^*y^*}{2}\right)\\
&- \frac{M^2r(1+x^{*2})\tau}{Kx^*(1+m^2)^2}\left(\frac{rx^{*2}}{2}+\frac{5rx^{*3}}{2K}+\frac{cx^*y^*}{2}\right)\\
&- \left.\frac{M^3r^2x^*(1+x^{*2})\tau}{2K^2(1+m^2)^2}\right] z_1^2(t) - \left[\frac{\beta y^*}{M}+\frac{cx^*y^*}{2(1+m^2)}-\frac{\delta x^*(1+M^2)}{2m(1+m^2)}\right] z_2^2(t)\\
&+ \frac{Mr(1+x^{*2})}{K(1+m^2)^2}\left[(\frac{3rx^{*2}}{2}+\frac{3rx^{*3}}{2K}+\frac{cx^*y^*}{2})\int_{t-\tau}^{t} z_1^2(s)ds\right.\\
&+ \left. cx^*y^*\int_{t-\tau}^{t} z_2^2(s)ds + (\frac{rx^*}{2K}+\frac{3rx^{*3}}{K})\int_{t-\tau}^{t} z_1^2(s-\tau)ds\right]\\
&+ \frac{M^2r(1+x^{*2})}{Kx^*(1+m^2)^2}\left[(\frac{rx^{*2}}{2}+\frac{rx^{*3}}{2K})\int_{t-\tau}^{t} z_1^2(s)ds\right.\\
&+ \left.\frac{cx^*y^*}{2}\int_{t-\tau}^{t} z_2^2(s)ds + \frac{2rx^{*3}}{K}\int_{t-\tau}^{t} z_1^2(s-\tau)ds\right]\\
&+ \frac{M^3r^2x^*(1+x^{*2})}{2K^2(1+m^2)^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds
\end{aligned}
\tag{3.64}
$$

Let

$$
\begin{aligned}
V_2(z(t)) &= \frac{Mrx^*(1+x^{*2})(3Krx^*+3rx^{*2}+Kcy^*+MrK+Mrx^*)}{2K^2(1+m^2)^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma)d\gamma ds\\
&+ \frac{Mrcy^*(1+x^{*2})(2x^*+M)}{2K(1+m^2)^2}\int_{t-\tau}^{t}\int_{s}^{t} z_2^2(\gamma)d\gamma ds
\end{aligned}
$$

$$+ \quad \frac{Mr^2x^*(1+x^{*2})(1+6x^{*2}+4Mx^*+M^2)}{2K^2(1+m^2)^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma-\tau)d\gamma ds$$

(3.65)

then

$$\dot{V}_2(z(t)) = \frac{Mrx^*\tau(1+x^{*2})(3Krx^*+3rx^{*2}+Kcy^*+MrK+Mrx^*)}{2K^2(1+m^2)^2}z_1^2(t)$$

$$- \quad \frac{Mrx^*(1+x^{*2})(3Krx^*+3rx^{*2}+Kcy^*+MrK+Mrx^*)}{2K^2(1+m^2)^2}\int_{t-\tau}^{t} z_1^2(s)ds$$

$$+ \quad \frac{Mrcy^*\tau(1+x^{*2})(2x^*+M)}{2K(1+m^2)^2}z_2^2(t)$$

$$- \quad \frac{Mrcy^*(1+x^{*2})(2x^*+M)}{2K(1+m^2)^2}\int_{t-\tau}^{t} z_2^2(s)ds$$

$$+ \quad \frac{Mr^2x^*\tau(1+x^{*2})(1+6x^{*2}+4Mx^*+M^2)}{2K^2(1+m^2)^2}z_1^2(t-\tau)$$

$$- \quad \frac{Mr^2x^*(1+x^{*2})(1+6x^{*2}+4Mx^*+M^2)}{2K^2(1+m^2)^2}\int_{t-\tau}^{t} z_1^2(s-\tau)ds$$

(3.66)

and then we have from (3.64) and (3.66) that for $t \geq \widehat{T}$

$$\dot{V}_1(z(t)) + V_2(\dot{z}(t)) \leq -\left[\frac{rx^*(1+3x^{*2})}{K(1+M^2)} + \frac{cx^*y^*}{2(1+m^2)} + \frac{cx^*y^*}{1+M^2} - \frac{rx^*(M+3x^*)}{1+m^2}\right.$$

$$- \frac{3rx^{*2}(M+x^*)}{K(1+m^2)} - \frac{rx^*(M+x^*)^2}{K(1+m^2)} - \frac{\delta x^*(1+M^2)}{2m(1+m^2)} - \frac{cx^*(L+y^*)}{1+m^2}$$

$$- \frac{Mrx^*\tau(1+x^{*2})(3Krx^*+3rx^{*2}+Kcy^*+MrK+Mrx^*)}{2K^2(1+m^2)^2}$$

$$-\frac{Mr(1+x^{*2})\tau}{K(1+m^2)^2}\left(\frac{3rx^{*2}}{2}+\frac{9rx^{*3}}{2K}+\frac{rx^*}{2K}+\frac{3cx^*y^*}{2}\right)$$

$$-\frac{M^2r(1+x^{*2})\tau}{Kx^*(1+m^2)^2}\left(\frac{rx^{*2}}{2}+\frac{5rx^{*3}}{2K}+\frac{cx^*y^*}{2}\right)-\left.\frac{M^3r^2x^*(1+x^{*2})\tau}{2K^2(1+m^2)^2}\right]z_1^2(t)$$

$$-\left[\frac{\beta y^*}{M}+\frac{cx^*y^*}{2(1+m^2)}-\frac{\delta x^*(1+M^2)}{2m(1+m^2)}-\frac{Mrcy^*\tau(1+x^{*2})(2x^*+M)}{2K(1+m^2)^2}\right]z_2^2(t)$$

$$+\frac{Mr^2x^*\tau(1+x^{*2})(1+6x^{*2}+4Mx^*+M^2)}{2K^2(1+m^2)^2}z_1^2(t-\tau) \tag{3.67}$$

Let

$$V_3(z(t)) \;=\; \frac{Mr^2x^*\tau(1+x^{*2})(1+6x^{*2}+4Mx^*+M^2)}{2K^2(1+m^2)^2}\int_{t-\tau}^{t} z_1^2(s)ds \tag{3.68}$$

then

$$\dot{V}_3(z(t)) \;=\; \frac{Mr^2x^*\tau(1+x^{*2})(1+6x^{*2}+4Mx^*+M^2)}{2K^2(1+m^2)^2}z_1^2(t)$$

$$-\;\frac{Mr^2x^*\tau(1+x^{*2})(1+6x^{*2}+4Mx^*+M^2)}{2K^2(1+m^2)^2}z_1^2(t-\tau) \tag{3.69}$$

Now define a Lyapunov functional $V(z(t))$ as

$$V(z(t)) \;=\; V_1(z(t))+V_2(z(t))+V_3(z(t)) \tag{3.70}$$

then we have from (3.67) and (3.69) that for $t \geq \widehat{T}$

$$\dot{V}(z(t)) \;\leq\; -\left[\frac{rx^*(1+3x^{*2})}{K(1+M^2)}+\frac{cx^*y^*}{1+M^2}+\frac{cx^*y^*}{2(1+m^2)}-\frac{rx^*(M+3x^*)}{1+m^2}\right.$$

$$
\begin{aligned}
& - \quad \frac{3rx^{*2}(M+x^*)}{K(1+m^2)} - \frac{rx^*(M+x^*)^2}{K(1+m^2)} - \frac{\delta x^*(1+M^2)}{2m(1+m^2)} - \frac{cx^*(L+y^*)}{1+m^2} \\
& - \quad \frac{Mr^2x^*\tau(1+x^{*2})(3Kx^*+9x^{*2}+MK+5Mx^*+1+M^2)}{K^2(1+m^2)^2} \\
& - \quad \left. \frac{Mrcy^*\tau(1+x^{*2})(4x^*+M)}{2K(1+m^2)^2} \right] z_1^2(t) \\
& - \quad \left[\frac{\beta y^*}{M} + \frac{cx^*y^*}{2(1+m^2)} - \frac{\delta x^*(1+M^2)}{2m(1+m^2)} - \frac{Mrcy^*\tau(1+x^{*2})(2x^*+M)}{2K(1+m^2)^2} \right] z_2^2(t) \\
& \equiv \quad -\zeta z_1^2(t) - \eta z_2^2(t) \qquad (3.71)
\end{aligned}
$$

Then it follows from (3.57) and (3.58) that $\zeta > 0$ and $\eta > 0$. Let $w(s) = \widehat{N}s^2$ where $\widehat{N} = \min\{\zeta, \eta\}$, then w is nonnegative continuous on $[0, \infty)$, $w(0) = 0$, and $w(s) > 0$ for $s > 0$. It follows from (3.71) that for $t \geq \widehat{T}$

$$
\dot{V}(z(t)) \leq -\widehat{N}\ [z_1^2(t) + z_2^2(t)] = -\widehat{N}\ \|z(t)\|^2 = -w(\|z(t)\|) \qquad (3.72)
$$

Now , we want to find a function u such that $V(z(t)) \geq u(\|z(t)\|)$. It follows from (3.61) , (3.65) , and (3.68) that

$$
V(z(t)) \geq \{z_1(t) - \ln[1+z_1(t)]\} + \{z_2(t) - \ln[1+z_2(t)]\} \qquad (3.73)
$$

then by Theorem 3.1 , we have that $u(s) = \widetilde{N}s^2$ and $V(z(t)) \geq u(\|z(t)\|)$. So the equilibrium point E^* of the system (3.1) is globally asymptotically stable with $p(x) = \frac{cx^2}{1+x^2}$. ■

Remark 3.1 By Theorem 3.1 , Theorem 3.2 , and Theorem 3.3 we can assume that

$$
\begin{aligned}
V(z(t)) &= \{z_1(t) - ln[1+z_1(t)]\} + \{z_2(t) - ln[1+z_2(t)]\} \\
&+a(p(x))\frac{Mrx^*}{2K^2}\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma)d\gamma ds \\
&+b(p(x))\frac{Mr^2x^*}{2K^2}\left[\int_{t-\tau}^{t}\int_{s}^{t} z_1^2(\gamma-\tau)d\gamma ds + \tau\int_{t-\tau}^{t} z_1^2(s)ds\right] \\
&+c(p(x))\frac{Mrcy^*}{2K}\int_{t-\tau}^{t}\int_{s}^{t} z_2^2(\gamma)d\gamma ds
\end{aligned}
$$

Then

(i) for $p(x) = cx$ in the Holling-Type I model

$$a(p(x)) = 0 \quad , \quad b(p(x)) = 1 \quad , \quad c(p(x)) = 1$$

(ii) for $p(x) = \frac{cx}{1+x}$ in the Holling-Type II model

$$
\begin{aligned}
a(p(x)) &= \frac{r(1+x^*)}{(1+m)^2} \\
b(p(x)) &= \frac{(1+x^*)(M+1+2x^*)}{(1+m)^2} \\
c(p(x)) &= \frac{1+x^*}{(1+m)^2}
\end{aligned}
$$

(iii) for $p(x) = \frac{cx^2}{1+x^2}$ *in the Holling-Type III model*

$$a(p(x)) \quad = \quad \frac{(1+x^{*2})(3Krx^* + 3rx^{*2} + Kcy^* + MKr + Mrx^*)}{(1+m^2)^2}$$

$$b(p(x)) \quad = \quad \frac{(1+x^{*2})(1+6x^{*2} + 4Mx^* + M^2)}{(1+m^2)^2}$$

$$c(p(x)) \quad = \quad \frac{(1+x^{*2})(2x^* + M)}{(1+m^2)^2}$$

4 Examples

Example 4.1 *In the Holling-Type I we consider the system*

$$\begin{aligned}\dot{x}(t) &= x(t)\left[3 - 10x(t-\tau) - 15y(t)\right] \\ \dot{y}(t) &= y(t)\left[1 - 6\frac{y(t)}{x(t)}\right]\end{aligned} \tag{4.1}$$

where $r = 3$, $K = \frac{3}{10}$, $c = 15$, $\delta = 1$, $\beta = 6$, and $E^* = (\frac{6}{25}, \frac{1}{25})$. Then

$$r - cL = 2.24246 > 0$$

$$\delta x^* - cy^* M = 0.0582 > 0$$

$$\frac{rx^*}{K} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mr^2x^*\tau}{K^2} - \frac{Mrcy^*\tau}{2K} = 1.5940 > 0$$

$$\frac{\beta y^*}{M} + \frac{cy^*}{2} - \frac{\delta x^*}{2m} - \frac{Mrcy^*\tau}{2K} = 0.0103 > 0$$

whenever $\tau = \frac{1}{300}$. Consequently , by Theorem 3.1 , we conclude that the unique positive equilibrium point E^* of the system (4.1) is globally asymptotically stable . The trajectory of the system (4.1) is dipicted in Figure 4.1 .

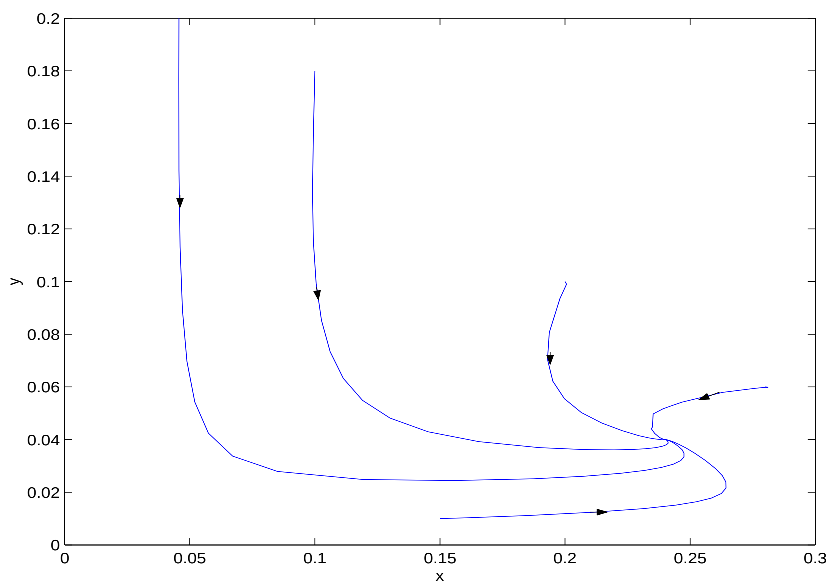

Figure 4.1: The trajectory of the system (4.1) with $\tau = \frac{1}{300}$.

Example 4.2 *In the Holling-Type II we consider the system*

$$\begin{aligned}\dot{x}(t) &= x(t)\left[3 - 10x(t-\tau) - 15\frac{y(t)}{1+x(t)}\right] \\ \dot{y}(t) &= y(t)\left[1 - 6\frac{y(t)}{x(t)}\right]\end{aligned} \tag{4.2}$$

where $r = 3$, $K = \frac{3}{10}$, $c = 15$, $\delta = 1$, $\beta = 6$, and $E^* = (\frac{1}{4}, \frac{1}{24})$. Then

$$r - cL = 2.24246 > 0$$

$$\delta x^*(1+m) - cy^* M = 0.08843 > 0$$

$$\frac{rx^*(1+2x^*)}{K(1+M)} + \frac{cy^*}{2(1+m)} - \frac{rx^*}{1+m} - \frac{\delta x^*(1+M)}{2m(1+m)} - \frac{rx^*(x^*+M)}{K(1+m)} -$$

$$\frac{Mr^2x^*\tau(1+x^*)(K+3x^*+1+M)}{K^2(1+m)^2} - \frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2} = 0.05316 > 0$$

$$\frac{\beta y^*}{M} + \frac{cy^*}{2(1+m)} - \frac{\delta x^*(1+M)}{2m(1+m)} - \frac{Mrcy^*\tau(1+x^*)}{2K(1+m)^2} = 0.00475 > 0$$

whenever $\tau = \frac{1}{300}$. Consequently , by Theorem 3.2 , we conclude that the unique positive equilibrium point E^* of the system (4.2) is globally asymptotically stable . The trajectory of the system (4.2) is dipicted in Figure 4.2 .

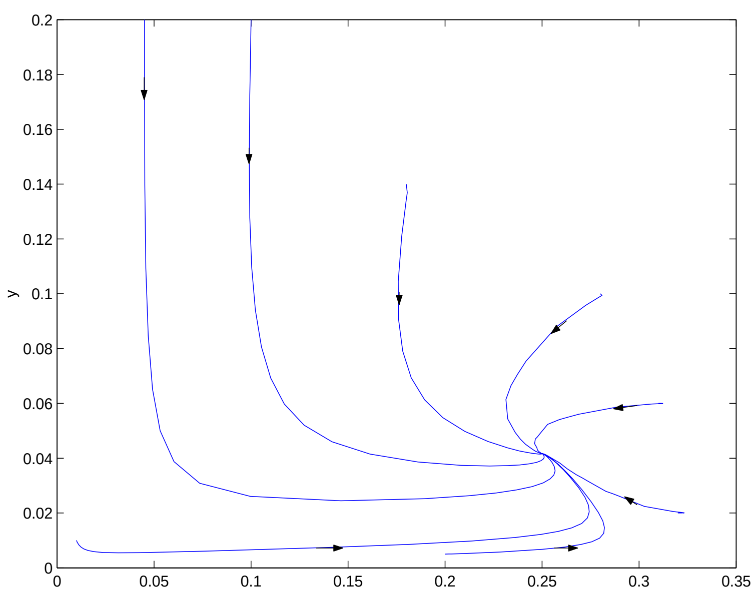

Figure 4.2: The trajectory of the system (4.2) with $\tau = \frac{1}{300}$.

5 Conclusion

In this thesis , we obtain that the sufficient condition for the global stability of the Leslie-Gower predator-prey system in Holling-Type I , Holling-Type II , and Holling-Type III models with time delay , respectively . But we believe that the global stability of the predator-prey model with time delay with all different functional response of the predator , $p(x)$, for instance , $p(x) = mx$, $p(x) = \frac{mx}{a+x}$, $p(x) = \frac{mx^2}{a+x^2}$, $p(x) = mx^c, 0 < c \leq 1$, or $p(x) = m(1 - e^{-cx})$ will be an important topic for future study .

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